Math. 412 (Fulling) 19 October 2012

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

1. *(30 pts.)* Solve by separation of variables or an equivalent transform technique:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < \infty, \quad 0 < y < \pi),$$

$$u(0, y) = 0 \quad (0 < y < \pi),$$

$$u(x, 0) = 0\,, \quad u(x, \pi) = f(x) \quad (0 < x < \infty).$$

(Consider only bounded solutions.)

Transform method: We have data on an infinite half-line, so a Fourier sine or cosine transform seems appropriate. The Dirichlet condition at $x = 0$ calls for a sine transform. Let's write $\tilde{g}(k)$ for the sine transform of a function $g(x)$. Transforming each equation in the problem, we get

$$-k^2 \tilde{u}(k, y) + \frac{\partial^2 \tilde{u}}{\partial y^2} = 0\,, \quad \tilde{u}(k, 0) = 0\,, \quad \tilde{u}(k, \pi) = \tilde{f}(k)\,.$$

From the first two equations, $\tilde{u}(k, y) = C(k)\sinh(ky)$, and so $C(k) = \tilde{f}(k)/\sinh(\pi k)$ from the third equation. Thus

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh(ky)}{\sinh(k\pi)} \sin(kx)\tilde{f}(k)\,dk\,, \qquad \tilde{f}(k) \equiv \int_0^\infty f(x)\sin(kx)\,dx\,.$$

Separation method: We look for solutions $u_{\text{sep}}(x, y) = X(x)Y(y)$ and get

$$\frac{Y''}{Y} = -\frac{X''}{X} = \lambda\,, \quad X(0) = 0\,, \quad Y(0) = 0\,.$$

Again looking ahead at the data, we set $\lambda = +k^2$ to get oscillatory solutions in x. Then $X(x) = \sin(kx)$ and $Y(y) = \sinh(ky)$. So the full solution is

$$u(x, y) = \int_0^\infty C(k)\sin(kx)\sinh(ky)\,dk\,.$$

Finally, $f(x) = \int_0^\infty C(k)\sin(kx)\sinh(k\pi)\,dk$, so

$$C(k) = \frac{2}{\pi}\int_0^\infty \frac{f(x)\sin(kx)}{\sinh(k\pi)}\,dx\,.$$

2. *(20 pts.) (regular)* Solve

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad (0 < x < \infty, \quad 0 < y < \pi),$$

$$w(0, y) = h(y) \quad (0 < y < \pi),$$

$$w(x, 0) = 0\,, \quad w(x, \pi) = 0 \quad (0 < x < \infty).$$

(Consider only bounded solutions.)

This time we have data on the finite interval of length π, with Dirichlet boundary conditions, so we expect a Fourier sine series (with $\pi/L = 1$).

Separation method: Write

$$-\frac{Y''}{Y} = \frac{X''}{X} = +k^2\,, \qquad Y(0) = 0\,, \quad Y(\pi) = 0\,.$$

So $Y(y) = \sin(ny)$, and $X(x) = e^{-nx}$ since the solution must be bounded as $x \to 0$. Thus the full solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-nx} \sin(ny)\,.$$

Finally,

$$b_n = \frac{2}{\pi} \int_0^{\pi} h(y) \sin(ny)\, dy\,.$$

"Transform" method: The analogue of the transform is the calculation of the Fourier coefficients. Let's write

$$\tilde{g}_n \equiv \frac{2}{\pi} \int_0^{\pi} g(y) \sin(ny)\, dy\,.$$

Then

$$\frac{\partial^2 \tilde{w}_n}{\partial x^2} - n^2 \tilde{w}_n(x) = 0\,, \quad \tilde{w}_n(0) = \tilde{h}_n\,.$$

Therefore, $\tilde{w}_n(x) = \tilde{h}_n e^{-nx}$. Finally,

$$u(x, y) = \sum_{n=1}^{\infty} \tilde{h}_n e^{-nx} \sin(ny)\,.$$

2. *(20 pts.) (honors)* Try to find any solution of

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} \quad (0 < t < \infty, \quad 0 < y < \pi),$$

$$v(t, 0) = g(t)\,, \quad v(t, \pi) = 0 \quad (0 < t < \infty).$$

You won't succeed, but go as far as you can. (Assume that $g(t) \to 0$ as $t \to +\infty$.)

This is a variation on the "vegetable cellar" problem. Take a Laplace transform in t:

$$sU(s,y) = \frac{\partial^2 U}{\partial y^2}\,, \quad U(s,0) = G(s)\,, \quad U(s,\pi) = 0\,.$$

Thus

$$U(s,y) = G(s)\,\frac{\sinh[\sqrt{s}(\pi - y)]}{\sinh(\sqrt{s}\pi)}\,.$$

Now "all" we need to do is to find the inverse Laplace transform of this function.

3. *(10 pts.)* Assume that you have a solution of the honors version of Question 2. Explain how you would use it to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \quad (0 < t < \infty, \quad 0 < y < \pi),$$

$$u(0,y) = f(y) \quad (0 < y < \pi),$$

$$u(t,0) = g(t)\,, \quad u(t,\pi) = 0 \quad (0 < t < \infty).$$

Let $w = u - v$, so that $u = v + w$. Then

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial y^2}\,, \quad w(t,0) = 0 = w(t,\pi)\,, \quad w(0,y) = f(y) - v(0,y) \equiv h(y)\,.$$

Finding w is a routine heat flow problem (covered last month, and similar to the regular version of Question 2).

4. *(15 pts.)*

(a) Justify the claim that, when x and y are positive,

$$\frac{2}{\pi}\int_0^\infty \cos(kx)\cos(ky)\,dk = \delta(x-y)\,.$$

Substitute one of the Fourier cosine formulas into the other:

$$\tilde{g}(k) \equiv \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(kz)g(x)\,dx\,, \quad g(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(kz)\tilde{g}(k)\,dk\,.$$

$$g(x) = \frac{2}{\pi}\int_0^\infty dz \int_0^\infty dk\cos(kx)\cos(kz)g(z)\,dz\,.$$

Thus the given integral acts as a delta function.

Alternative method: Write out everything in terms of $e^{\pm ikx}$ and use $\int_{-\infty}^\infty e^{ikz} = 2\pi\delta(z)$. This way you will get the answer to (b) first.

(b) What happens when x is allowed to be negative?

$$\frac{2}{\pi}\int_0^\infty \cos(kx)\cos(ky)\,dk = \delta(x-y) + \delta(x+y)\,.$$

There are several arguments to justify this conclusion. (1) The object must be even in x because $\cos(kx)$ is. (2) The cosine transform is equivalent to the ordinary (exponential) Fourier transform applied to even extensions, so every feature on the positive axis has a mirror image on the negative axis.

5. *(25 pts.)* Construct the Green function that solves

$$y'' - 9y = f(x) \quad (0 < x < 1),$$
$$y(0) = 0 = y(1)\,.$$

Clearly state the formula for calculating y from G and f.

Hint: $\sinh a \cosh b - \cosh a \sinh b = \sinh(a-b)$.

The formula will be

$$y(x) = \int_0^1 G(x,z) f(z)\,dz\,.$$

The Green function must satisfy

$$\frac{\partial^2 G}{\partial x^2} - 9G = \delta(x-z)\,, \quad G(0,z) = 0 = G(1,z)\,.$$

The differential equation is interpreted as

$$\frac{\partial^2 G}{\partial x^2} - 9G = 0 \quad \text{for } x < z \text{ and } x > z\,.$$

$$G(z^+,z) = G(z^-,z)\,, \quad \frac{\partial G}{\partial x}(z^+,z) - \frac{\partial G}{\partial x}(z^-,z) = 1\,.$$

Therefore, in view of the boundary conditions,

$$G(x,z) = \begin{cases} A(z)\sinh(3x) & \text{for } 0 < x < z, \\ B(z)\sinh(3(x-1)) & \text{for } z < x < 1. \end{cases}$$

The jump conditions give

$$A(z)\sinh(3z) = B(z)\sinh(3(z-1))\,, \quad 3A(z)\cosh(3z) - 3B(z)\cosh(3(z-1)) = -1\,.$$

After some algebra using the hint, you get

$$A(z) = \frac{\sinh(3(z-1))}{3\sinh 3}\,, \quad B(z) = \frac{\sinh(3z)}{3\sinh 3}\,.$$

Alternative method: Call the left-hand and right-side solutions $u_-(x) = \sinh(3x)$, $u_+(x) = \sinh(3(x-1))$. The general formula that comes up in all such problems is

$$G(x,z) = \frac{u_-(x_-)u_+(x_+)}{W(u_-,u_+)}\,,$$

where the Wronskian is (for any x)

$$W(u_-,u_+) = u_1 u_+' = u_+ u_-' = 3\sinh(3x)\cosh(3(x-1)) - 3\cosh(3x)\sinh(3(x-1)) = 3\sinh 3\,.$$

So

$$G(x,z) = \frac{\sinh(3x_-)\sinh(3(x_+ - 1))}{3\sinh 3}\,.$$