Convergence theorems

So far we've seen that we can solve the heat equation with homogenized Dirichlet boundary conditions and arbitrary initial data (on the interval $[0, \pi]$), provided that we can express an arbitrary function g (on that interval) as an infinite linear combination of the eigenfunctions $\sin(nx)$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Furthermore, we saw that if such a series exists, its coefficients must be given by the formula

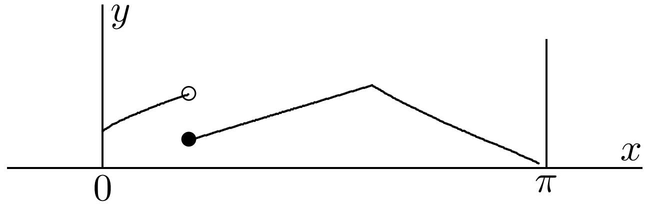
$$b_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx \, dx.$$

So the burning question of the hour is: Does this Fourier sine series really converge to $g(x)$?

No mathematician can answer this question without first asking, "What kind of convergence are you talking about? And what technical conditions does g satisfy?" There are three standard convergence theorems, each of which states that certain technical conditions are sufficient to guarantee a certain kind of convergence. Generally speaking,

$$\begin{aligned} &\text{more smoothness in } g \\ \iff\quad &\text{more rapid decrease in } b_n \text{ as } n \to \infty \\ \iff\quad &\text{better convergence of the series.} \end{aligned}$$

Definition: g is *piecewise smooth* if its derivative is piecewise continuous. That is, $g'(x)$ is defined and continuous at all but a finite number of points (in the domain $[0, \pi]$, or whatever finite interval is relevant to the problem), and at those bad points g' has finite one-sided limits. (At such a point g itself is allowed to be discontinuous, but only the "finite jump" type of discontinuity is allowed.)



This class of functions is singled out, not only because one can rather easily prove convergence of their Fourier series (see next theorem), but also because they are a natural type of function to consider in engineering problems. (Think of electrical voltages under the control of a switch, or applied forces in a mechanical problem.)

Pointwise Convergence Theorem: If g is continuous and piecewise smooth, then its Fourier sine series converges at each x in $(0, \pi)$ to $g(x)$. If g

is piecewise smooth but not necessarily continuous, then the series converges to

$$\tfrac{1}{2}[g(x^-) + g(x^+)]$$

(which is just $g(x)$ if g is continuous at x). [Note that at the endpoints the series obviously converges to 0, regardless of the values of $g(0)$ and $g(\pi)$. This zero is simply $\frac{1}{2}[g(0^+) + g(0^-)]$ or $\frac{1}{2}[g(\pi^+) + g(\pi^-)]$ for the odd extension!]

Uniform Convergence Theorem: If g is both continuous and piecewise smooth, and $g(0) = g(\pi) = 0$, then its Fourier sine series converges *uniformly* to g throughout the interval $[0, \pi]$.

Remarks:

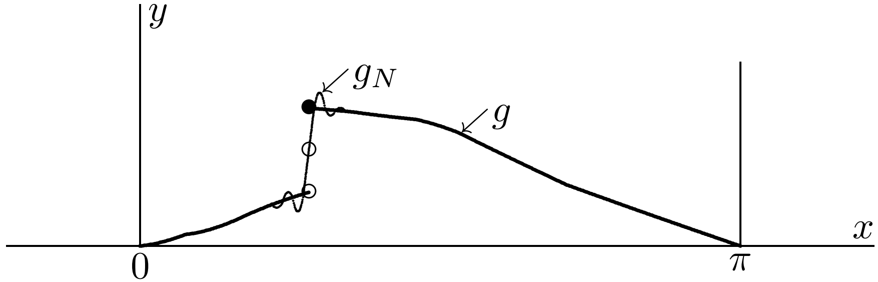
1. *Uniform convergence* means: For every ϵ we can find an N so big that the

partial sum

$$g_N(x) \equiv \sum_{n=1}^{N} b_n \sin(nx)$$

approximates $g(x)$ to within an error ϵ everywhere in $[0, \pi]$. The crucial point is that the same N works for *all* x; in other words, you can draw a horizontal line, $y = \epsilon$, that lies completely above the graph of $|g(x) - g_N(x)|$.

2. In contrast, if the convergence is nonuniform (merely pointwise), then for each x we can take enough terms to get the error $|g(x) - g_N(x)|$ smaller than ϵ, but the N may depend on x as well as ϵ. It is easy to see that if g is discontinuous, then uniform convergence is impossible, because the approximating functions g_N need a finite "time" to jump across the gap. There will always be points near the jump point where the approximation is bad.



It turns out that g_N develops "ears" or "overshoots" right next to the jump. This is called the *Gibbs phenomenon*.

3. For the same reason, the sine series can't converge uniformly near an endpoint where g doesn't vanish. An initial-value function which violated the condition $g(0) = g(\pi) = 0$ would be rather strange from the point of view of the Dirichlet boundary value problem that gave rise to the sine series, since there we want $u(0,t) = u(\pi,t) = 0$ and also $u(x,0) = g(x)$!

4. If g is piecewise continuous, it can be proved that $b_n \to 0$ as $n \to \infty$. (This is one form of the *Riemann–Lebesgue theorem*.) This is a key step in proving the pointwise convergence theorem.

   If g satisfies the conditions of the uniform convergence theorem, then integration by parts shows that

$$b_n = \frac{2}{n\pi} \int_0^\pi g'(x) \cos{(nx)}\, dx,$$

   and by another version of the Riemann–Lebesgue theorem this integral also approaches 0 when n is large, so that b_n *falls off at* ∞ *faster than* n^{-1}. This additional falloff is "responsible" for the uniform convergence of the series. (This remark is as close as we'll come in this course to proofs of the convergence theorems.)

5. There are continuous (but not piecewise smooth) functions whose Fourier

series do *not* converge, but it is hard to construct an example! (See Appendix B.)

The third kind of convergence is related to ...

Parseval's Equation: $$\int_0^\pi |g(x)|^2\,dx = \frac{\pi}{2}\sum_{n=1}^\infty |b_n|^2.$$

(In particular, the integral converges if and only if the sum does.)

"Proof": Taking convergence for granted, let's calculate the integral. (I'll assume that $g(x)$ and b_n are real, although I've written the theorem so that it applies also when things are complex.)

$$\begin{aligned}
\int_0^\pi |g(x)|^2\,dx &= \int_0^\pi \sum_{n=1}^\infty \sum_{m=1}^\infty b_n \sin(nx)\, b_m \sin(mx)\,dx \\
&= \int_0^\pi \sum_{n=1}^\infty {b_n}^2 \sin^2 nx\,dx \\
&= \frac{\pi}{2} \sum_{n=1}^\infty {b_n}^2 .
\end{aligned}$$

(The integrals have been evaluated by the orthogonality relations stated earlier. Only terms with $m = n$ contribute, because of the orthogonality of the sine functions. The integral with $m = n$ can be evaluated by a well known rule of thumb: The integral of $\sin^2 \omega x$ over any integral number of quarter-cycles of the trig function is half of the integral of $\sin^2 \omega x + \cos^2 \omega x$ — namely, the length of the interval, which is π in this case.)

There are similar Parseval equations for Fourier cosine series and for the full Fourier series on interval $(-\pi, \pi)$. In addition to its theoretical importance, which we can only hint at here, Parseval's equation can be used to evaluate certain numerical infinite sums, such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

(Work it out for $g(x) = x$.)

Definition: g is *square-integrable* on $[0, \pi]$ if the integral in Parseval's equation converges:

$$\int_0^{\pi} |g(x)|^2\, dx < \infty.$$

L^2 (or Mean) Convergence Theorem: If g is square-integrable, then the series converges *in the mean*:

$$\int_0^\pi |g(x) - g_N(x)|^2 \, dx \to 0 \quad \text{as } N \to \infty.$$

Remarks:

1. Recalling the formulas for the length and distance of vectors in 3-dimensional space,

$$|\vec{x}|^2 \equiv \sum_{n=1}^{3} {x_n}^2, \qquad |\vec{x} - \vec{y}|^2 \equiv \sum_{n=1}^{3} (x_n - y_n)^2,$$

   we can think of the Parseval integral as a measure of the "length" of g, and the integral in the theorem as a measure of the "distance" between g and g_N. (This geometrical way of thinking becomes very valuable when we consider general orthogonal basis functions later on.)

2. A function can be square-integrable without being piecewise smooth, or even bounded. Example:

$$g(x) \equiv \left(x - \tfrac{1}{2}\right)^{-\frac{1}{3}}.$$

Also (cf. Remark 5 above) a series can converge in the mean without converging pointwise (not to mention uniformly). This means that the equation

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

must not be taken too literally in such a case — such as by writing a computer program to add up the terms for a fixed value of x. (The series will converge (pointwise) for "almost" all x, but there may be special values where it doesn't.)

Prior to Fall 2000 this course spent about three weeks *proving* the convergence theorems and covering other aspects of the theory of Fourier series. (That material has been removed to make room for more information about PDEs, notably Green functions and the classification of PDEs as elliptic, hyperbolic, or parabolic.) Notes for those three weeks are attached as Appendix B.