Delta "functions"

The PDE problem defining any Green function is most simply expressed in terms of the *Dirac delta function.* This, written $\delta(x-z)$ (also sometimes written $\delta(x,z)$, $\delta_z(x)$, or $\delta_0(x-z)$), is a make-believe function with these properties:

1. $\delta(x-z) = 0$ for all $x \neq z$, and

$$\int_{-\infty}^{\infty} \delta(x-z)\,dx = 1.$$

2. **The key property:** For all continuous functions f,

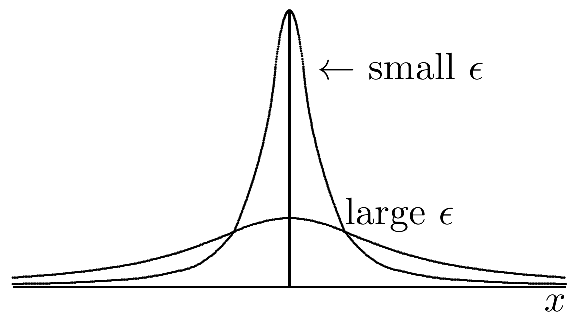
$$\int_{-\infty}^{\infty} \delta(x-z)\,f(x)\,dx = f(z).$$

Also,

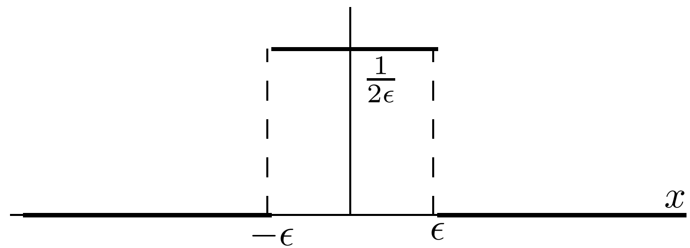
$$\int_a^b \delta(x-z)\, f(x)\, dx = \begin{cases} f(z) & \text{if } z \in (a,b), \\ 0 & \text{if } z \notin [a,b]. \end{cases}$$

3. $\delta(x)$ is the limit of a family of increasingly peaked functions, each with integral 1:

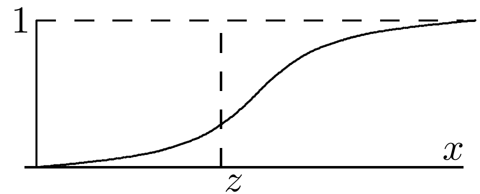
$$\begin{aligned} \delta(x) &= \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2} \\ &\text{or } \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2} \\ &\text{or } \lim_{\epsilon \downarrow 0} d_\epsilon(x), \end{aligned}$$



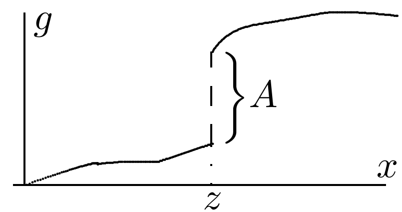
where d_ϵ is a step function of the type drawn here:



4. $\delta(x-z) = \dfrac{d}{dx}h(x-z)$, where $h(w)$ is the unit step function, or Heaviside function (equal to 1 for $w > 0$ and to 0 for $w < 0$). Note that $h(t-z)$ is the limit as $\epsilon \downarrow 0$ of a family of functions of this type:



GENERALIZATION OF 4: If $g(x)$ has a jump discontinuity of size A at $x = z$, then its "derivative" contains a term $A\,\delta(x - z)$. (A may be negative.)

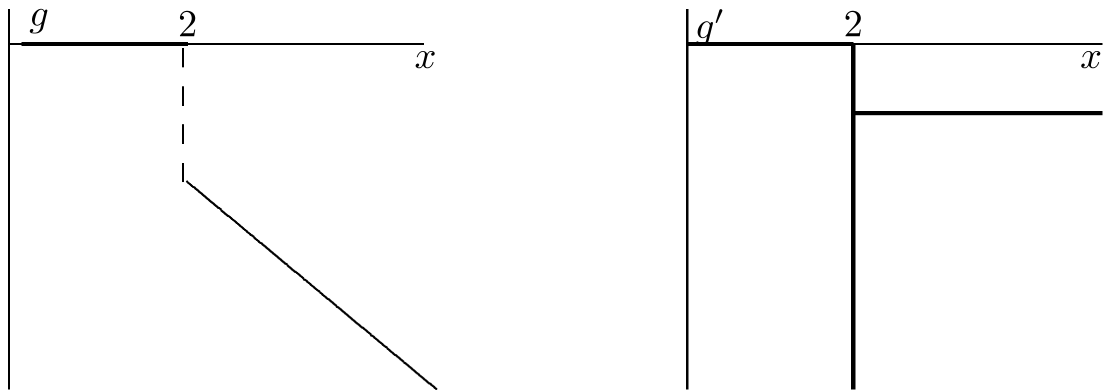


Example:

$$g(x) = \left\{ \begin{array}{rl} 0 & \text{for } x < 2, \\ -x & \text{for } x \geq 2 \end{array} \right\} = -x\,h(x-2).$$

Then

$$\begin{aligned} g'(x) &= -h(x-2) - x\,h'(x-2) \\ &= -h(x-2) - 2\,\delta(x-2). \end{aligned}$$



Interpretation of differential equations involving δ

Consider

$$y'' + p(x)y' + q(x)y = A\,\delta(x - z).$$

We expect the solution of this equation to be the limit of the solution of an equation whose source term is a finite but very narrow and hard "kick" at $x = z$. The δ equation is easier to solve than one with a finite peak.

The equation is taken to mean:

$$y'' + py' + qy = 0 \quad \text{for } x < z. \tag{1}$$

$$y'' + py' + qy = 0 \quad \text{for } x > z. \tag{2}$$

$$(3) \qquad y \text{ is continuous at } z\text{:} \quad \lim_{x\downarrow z} y(x) = \lim_{x\uparrow z} y(x).$$

[*Notational remarks:* $\lim_{x\downarrow z}$ means the same as $\lim_{x\to z^+}$; $\lim_{x\uparrow z}$ means $\lim_{x\to z^-}$. Also, $\lim_{x\downarrow z} y(x)$ is sometimes written $y(z^+)$, and so on.]

$$(4) \qquad \lim_{x\downarrow z} y'(x) = \lim_{x\uparrow z} y'(x) + A.$$

Conditions (3) and (4) tell us how to match solutions of (1) and (2) across the joint. Here is the reasoning behind them:

Assume (3) for the moment. Integrate the ODE from $x = z - \epsilon$ to $x = z + \epsilon$ (where ϵ is very small):

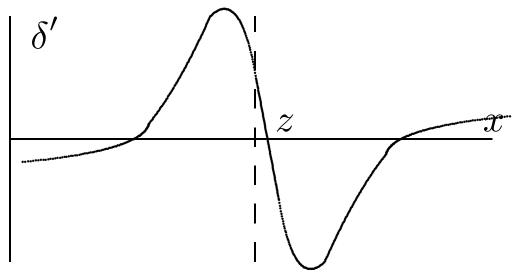
$$\int_{z-\epsilon}^{z+\epsilon} y''\,dx + \int_{z-\epsilon}^{z+\epsilon} (py' + qy)\,dx = A\int_{z-\epsilon}^{z+\epsilon} \delta(x-z)\,dx$$

That is,

$$y'(z+\epsilon) - y'(z-\epsilon) + \text{small term } (\to 0 \text{ as } \epsilon \downarrow 0) = A.$$

In the limit $\epsilon \to 0$, (4) follows.

Now if y *itself* had a jump at z, then y' would contain $\delta(x-z)$, so y'' would contain $\delta'(x-z)$, which is a singularity "worse" than δ. (It is a limit of functions like the one in the graph shown here.) Therefore, (3) is necessary.



We can solve such an equation by finding the general solution on the interval to the left of z and the general solution to the right of z, and then matching the function and its derivative at z by rules (3) and (4) to determine the undetermined coefficients.

Consider the example

$$y'' = \delta(x - 1), \qquad y(0) = 0, \quad y'(0) = 0.$$

For $x < 1$, we must solve the equation $y'' = 0$. The general solution is $y = Ax+B$, and the initial conditions imply then that

$$y = 0 \quad \text{for } x < 1.$$

For $x > 1$, we again must have $y'' = 0$ and hence $y = Cx + D$ (different constants this time). On this interval we have $y' = C$. To find C and D we have to apply rules (3) and (4):

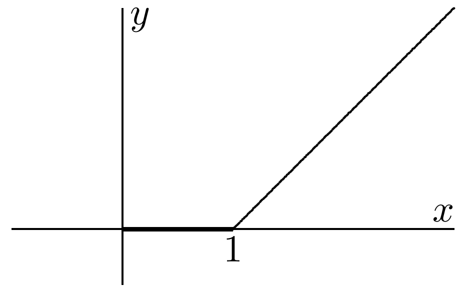
$$\begin{aligned} 0 = y(1^-) &= y(1^+) = C + D, \\ 0 + 1 = y'(1^-) + 1 &= y'(1^+) = C. \end{aligned}$$

That is,

$$\begin{aligned} C + D &= 0, \\ C \quad\quad &= 1. \end{aligned}$$

Therefore, $C = 1$ and $D = -1$. Thus $y(x) = x - 1$ for $x > 1$. The complete solution is therefore

$$y(x) = (x - 1)\, h(x - 1).$$



Delta functions and Green functions

For lack of time, in this course we won't devote much attention to *nonhomogeneous* partial differential equations. (Haberman, however, discusses them extensively.) So far our nonhomogeneities have been initial or boundary data, not terms in the PDE itself. But problems like

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \rho(t, x)$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = j(x,y),$$

where ρ and j are given functions, certainly do arise in practice. Often transform techniques or separation of variables can be used to reduce such PDEs to nonhomogeneous *ordinary* differential equations (a single ODE in situations of extreme symmetry, but more often an infinite family of ODEs).

Here I will show how the delta function and the concept of a Green function can be used to solve nonhomogeneous ODEs.

Example 1: *The Green function for the one-dimensional Dirichlet problem.* Let's start with an equation containing our favorite linear differential operator:

$$\frac{d^2 y}{dx^2} + \omega^2 y = f(x). \qquad (*)$$

We require that

$$y(0) = 0, \qquad y(\pi) = 0.$$

Here ω is a positive constant, and f is a "known" but arbitrary function. Thus our solution will be a formula for y in terms of f. In fact, it will be given by a Green-function integral:

$$y(x) = \int_0^\pi G_\omega(x, z)\, f(z)\, dz,$$

where G is independent of f — but, of course, depends on the left-hand side of the differential equation $(*)$ and on the boundary conditions.

We can solve the problem for general f by studying the equation

$$\frac{d^2 y}{dx^2} + \omega^2 y = \delta(x - z) \qquad (*_z)$$

(with the same boundary conditions). We will give the solution of $(*_z)$ the name $G_\omega(x, z)$. Since

$$f(x) = \int_0^\pi \delta(x - z)\, f(z)\, dz$$

(for x in the interval $(0, \pi)$) and since the operator on the left-hand side of $(*)$ is linear, we expect that

$$y(x) \equiv \int_0^\pi G_\omega(x - z)\, f(z)\, dz$$

will be the solution to our problem! That is, since the operator is linear, it can be moved inside the integral (which is a limit of a sum) to act directly on the

Green function:

$$\begin{aligned}\frac{d^2y}{dx^2} + \omega^2 y &= \int_0^\pi \left(\frac{d^2}{dx^2} + \omega^2\right) G_\omega(x-z)\, f(z)\, dz \\ &= \int_0^\pi \delta(x-z)\, f(z)\, dz \\ &= f(x),\end{aligned}$$

as desired. Furthermore, since G vanishes when $x = 0$ or π, so does the integral defining y; so y satisfies the right boundary conditions.

Therefore, the only task remaining is to solve $(*_z)$. We go about this with the usual understanding that

$$\delta(x-z) = 0 \quad \text{whenever } x \neq z.$$

Thus $(*_z)$ implies

$$\frac{d^2 G_\omega(x,z)}{dx^2} + \omega^2 G_\omega(x,z) = 0 \quad \text{if } x \neq z.$$

Therefore, for some constants A and B,

$$G_\omega(x,z) = A\cos\omega x + B\sin\omega x \quad \text{for } x < z,$$

and, for some constants C and D,

$$G_\omega(x,z) = C\cos\omega x + D\sin\omega x \quad \text{for } x > z.$$

We do *not* necessarily have $A = C$ and $B = D$, because the homogeneous equation for G is not satisfied when $x = z$; that point separates the interval into two disjoint subintervals, and we have a *different* solution of the homogeneous equation on each. Note, finally, that the four unknown "constants" are actually functions of z: there is no reason to expect them to turn out the same for all z's.

We need four equations to determine these four unknowns. Two of them are the boundary conditions:

$$0 = G_\omega(0,z) = A, \qquad 0 = G_\omega(\pi,z) = C\cos\omega\pi + D\sin\omega\pi.$$

The third is that G is continuous at z:

$$A \cos \omega z + B \sin \omega z = G_\omega(z, z) = C \cos \omega z + D \sin \omega z.$$

The final condition is the one we get by integrating $(*_z)$ over a small interval around z:

$$\frac{\partial}{\partial x} G_\omega(z^+, z) - \frac{\partial}{\partial x} G_\omega(z^-, z) = 1.$$

(Notice that although there is no variable "x" left in this equation, the partial derivative with respect to x is still meaningful: it means to differentiate with respect to the first argument of G (*before* letting that argument become equal to the second one).) This last condition is

$$-\omega C \sin \omega z + \omega D \cos \omega z + \omega A \sin \omega z - \omega B \cos \omega z = 1.$$

One of the equations just says that $A = 0$. The others can be rewritten

$$\begin{aligned} C\cos\omega\pi + \quad D\sin\omega\pi &= 0, \\ B\sin\omega z - C\cos\omega z - D\sin\omega z &= 0, \\ -\omega B\cos\omega z - \omega C\sin\omega z + \omega D\cos\omega z &= 1. \end{aligned}$$

This system can be solved by Cramer's rule: After a grubby calculation, too long to type, I find that the determinant is

$$\begin{vmatrix} 0 & \cos\omega\pi & \sin\omega\pi \\ \sin\omega z & -\cos\omega z & -\sin\omega z \\ -\omega\cos\omega z & -\omega\sin\omega z & \omega\cos\omega z \end{vmatrix} = -\omega\sin\omega\pi.$$

If ω is *not* an integer, this is nonzero, and so we can go on through additional grubby calculations to the answers,

$$B(z) = \frac{\sin\omega(z-\pi)}{\omega\sin\omega\pi},$$

$$C(z) = -\frac{\sin \omega z}{\omega},$$

$$D(z) = \frac{\cos \omega\pi \sin \omega z}{\omega \sin \omega\pi}.$$

Thus

$$G_\omega(x,z) = \frac{\sin \omega x \sin \omega(z-\pi)}{\omega \sin \omega\pi} \quad \text{for } x < z,$$

$$G_\omega(x,z) = \frac{\sin \omega z \sin \omega(x-\pi)}{\omega \sin \omega\pi} \quad \text{for } x > z.$$

(Reaching the last of these requires a bit more algebra and a trig identity.)

So we have found the Green function! Notice that it can be expressed in the unified form

$$G_\omega(x,z) = \frac{\sin \omega x_< \sin \omega(x_> - \pi)}{\omega \sin \omega\pi},$$

where

$$x_< \equiv \min(x,z), \qquad x_> \equiv \max(x,z).$$

This symmetrical structure is very common in such problems.

Finally, if ω *is* an integer, it is easy to see that the system of three equations in three unknowns has no solutions. It is no accident that these are precisely the values of ω for which $(*)$'s corresponding *homogeneous* equation,

$$\frac{d^2y}{dx^2} + \omega^2 y = 0,$$

has solutions satisfying the boundary conditions. If the homogeneous problem has solutions (other than the zero function), then the solution of the nonhomogeneous problem (if it exists) must be nonunique, and we have no right to expect to find a formula for it! In fact, the *existence* of solutions to the nonhomogeneous problem also depends upon whether ω is an integer (and also upon f), but we don't have time to discuss the details here.

Remark: The algebra in this example could have been reduced by writing the solution for $x > z$ as

$$G_\omega(x,z) = E \sin \omega(x-\pi).$$

(That is, we build the boundary condition at π into the formula by a clever choice of basis solutions.) Then we would have to solve merely two equations in two unknowns (B and E) instead of a 3×3 system.

Example 2: *Variation of parameters in terms of delta and Green functions.* Let's go back to the general second-order linear ODE,

$$y'' + p(x)y' + q(x)y = f(x),$$

and construct the solution satisfying

$$y(0) = 0, \qquad y'(0) = 0.$$

As before, we will solve

$$\frac{\partial^2}{\partial x^2} G(x,z) + p(x)\,\frac{\partial}{\partial x} G(x,z) + q(x) G(x,z) = \delta(x-z)$$

with those initial conditions, and then expect to find y in the form

$$y(x) = \int G(x,z)\, f(z)\, dz.$$

It is not immediately obvious what the limits of integration should be, since there is no obvious "interval" in this problem.

Assume that two linearly independent solutions of the homogeneous equation

$$y'' + p(x) y' + q(x) y = 0$$

are known; call them $y_1(x)$ and $y_2(x)$. Of course, until we are told what p and q are, we can't write down exact formulas for y_1 and y_2; nevertheless, we can solve the problem in the general case — getting an expression for G in terms of y_1 and y_2, whatever they may be.

Since G satisfies the homogeneous equation for $x \neq z$, we have

$$G(x,z) = \begin{cases} A(z)y_1(x) + B(z)y_2(x) & \text{for } x < z, \\ C(z)y_1(x) + D(z)y_2(x) & \text{for } x > z. \end{cases}$$

As before we will get four equations in the four unknowns, two from initial data and two from the continuity of G and the prescribed jump in its derivative at z. Let us consider only the case $z > 0$. Then the initial conditions

$$G(0,z) = 0, \qquad \frac{\partial}{\partial x} G(0,z) = 0$$

force $A = 0 = B$. The continuity condition, therefore, says that $G(z,z) = 0$, or

$$C(z)y_1(z) + D(z)y_2(z) = 0. \tag{1}$$

The jump condition

$$\frac{\partial}{\partial x}G(z^+, z) - \frac{\partial}{\partial x}G(z^-, z) = 1$$

now becomes

$$C(z)y_1'(z) + D(z)y_2'(z) = 1. \tag{2}$$

Solve (1) and (2): The determinant is the Wronskian

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \equiv W(z).$$

Then

$$C = -\frac{y_2}{W}\,, \qquad D = \frac{y_1}{W}\,.$$

Thus our conclusion is that (for $z > 0$)

$$G(x,z) = \begin{cases} 0 & \text{for } x < z, \\ \dfrac{1}{W(z)}\big(y_1(z)y_2(x) - y_2(z)y_1(x)\big) & \text{for } x > z. \end{cases}$$

Now recall that the solution of the original ODE,

$$y'' + p(x)y' + q(x)y = f(x),$$

was supposed to be

$$y(x) = \int G(x,z)\, f(z)\, dz.$$

Assume that $f(z) \neq 0$ only for $z > 0$, where our result for G applies. Then the integrand is 0 for $z < 0$ (because $f = 0$ there) and also for $z > x$ (because $G = 0$

there). Thus

$$
\begin{aligned}
y(x) &= \int_0^x G(x,z)\, f(z)\, dz \\
&= \int_0^x \frac{y_1(z) f(z)}{W(z)}\, dz\; y_2(x) - \int_0^x \frac{y_2(z) f(z)}{W(z)}\, dz\; y_1(x).
\end{aligned}
$$

This is exactly the same solution that is found in differential equations text-books by making the ansatz

$$
y(x) = u_1(x) y_1(x) + u_2(x) y_2(x)
$$

and deriving a system of first-order differential equations for u_1 and u_2. That method is called “variation of parameters”. Writing the variation-of-parameters solution in terms of the Green function G shows in a precise and clear way how the solution y depends on the nonhomogeneous term f as f is varied. That formula is a useful starting point for many further investigations.

Example 3: *Inhomogeneous boundary data.* Consider the problem

PDE:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

BC:
$$u(x,0) = \delta(x-z).$$

Its solution is
$$G(x-z,y) \equiv \frac{1}{\pi}\,\frac{y}{(x-z)^2+y^2}\,,$$

the Green function that we constructed for Laplace's equation in the upper half plane. Therefore, the general solution of Laplace's equation in the upper half plane, with arbitrary initial data
$$u(x,0) = f(x),$$

is

$$u(x, y) = \int_{-\infty}^{\infty} dz\, G(x - z, y) f(z).$$

Similarly, the Green function

$$G(t, x - z) = \frac{1}{\sqrt{4\pi t}}\, e^{-(x-z)^2/4t}.$$

that we found for the heat equation solves the heat equation with initial data

$$u(0, x) = \delta(x - z).$$

And so on, for any linear problem with nonhomogeneous data.

Delta functions and Fourier transforms

Formally, the Fourier transform of a delta function is a complex exponential function, since

$$\int_{-\infty}^{\infty} \delta(x-z)\, e^{-ikx}\, dx = e^{-ikz}.$$

According to the Fourier inversion formula, therefore, we should have

$$\begin{aligned}
\delta(x-z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikz}\, e^{ikx}\, dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-z)}\, dk.
\end{aligned}$$

This is a very useful formula! Here is another way of seeing what it means and why it is true:

Recall that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk,$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)\, e^{-ikz}\, dz.$$

Let us substitute the second formula into the first:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(x-z)}\, f(z)\, dz\, dk.$$

Of course, this equation is useless for computing $f(x)$, since it just goes in a circle; its significance lies elsewhere. If we're willing to play fast and loose with the order of the integrations, we can write it

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{ik(x-z)}\, dk \right] f(z)\, dz,$$

which says precisely that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-z)}\,dk$$

satisfies the defining property of $\delta(x-z)$. Our punishment for playing fast and loose is that this integral does not converge (in the usual sense), and there is no function δ with the desired property. Nevertheless, both the integral and the object δ itself can be given a rigorous meaning in the modern theory of *distributions*; crudely speaking, they both make perfect sense as long as you keep them inside other integrals (multiplied by continuous functions) and do not try to evaluate them at a point to get a number.

What would happen if we tried this same trick with the Fourier *series* formulas? Let's consider the sine series,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx,$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(z) \sin nz \, dz.$$

This gives

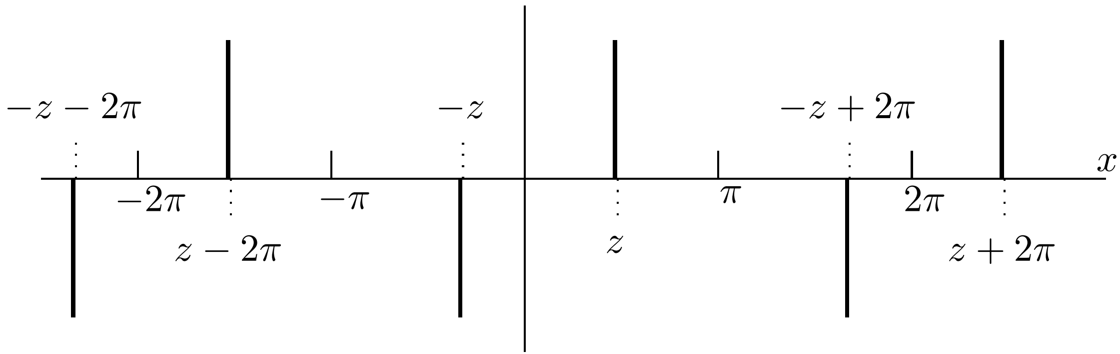
$$f(x) = \frac{2}{\pi} \int_0^{\pi} \left[\sum_{n=1}^{\infty} \sin nx \, \sin nz \right] f(z) \, dz. \tag{$\dagger$}$$

Does this entitle us to say that

$$\delta(x - z) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \, \sin nz \, ? \tag{$\ddagger$}$$

Yes and no. In (†) the variables x and z are confined to the interval $[0, \pi]$. (‡) *is a valid representation of the delta function when applied to functions whose domain is* $[0, \pi]$. If we applied it to a function on a larger domain, it would act like the odd, periodic extension of $\delta(x - z)$, as is always the case with Fourier sine series:

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \sin nx \, \sin nz = \sum_{M=-\infty}^{\infty} [\delta(x - z + 2\pi M) - \delta(x + z + 2\pi M)].$$



THE POISSON SUMMATION FORMULA

NOTE: This is *not* what Haberman calls "Poisson formula" in Exercise 2.5.4

and p. 433.

Let's repeat the foregoing discussion for the case of the full Fourier series on the interval $(-L, L)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} e^{-i\pi ny/L} f(y)\, dy$$

leads to

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} e^{-i\pi y/L} f(y)\, dy\, e^{i\pi nx/L} \\ &= \int_{-L}^{L} \left[\frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\pi n(x-y)/L}\right] f(y)\, dy. \end{aligned}$$

Therefore,

$$\frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\pi n(x-y)/L} = \delta(x-y) \quad \text{for } x \text{ and } y \text{ in } (-L, L).$$

Now consider $y = 0$ (for simplicity). For x *outside* $(-L, L)$, the sum must equal the $2L$-periodic extension of $\delta(x)$:

$$\frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\pi nx/L} = \sum_{M=-\infty}^{\infty} \delta(x - 2LM). \tag{‡}$$

Let $f(x)$ be a continuous function on $(-\infty, \infty)$ whose Fourier transform is also continuous. Multiply both sides of (‡) by $f(x)$ and integrate:

$$\frac{\sqrt{2\pi}}{2L}\sum_{n=-\infty}^{\infty} \hat{f}\left(-\tfrac{\pi n}{L}\right) = \sum_{M=-\infty}^{\infty} f(2LM).$$

Redefine n as $-n$ and simplify:

$$\sqrt{\frac{\pi}{2}}\frac{1}{L}\sum_{n=-\infty}^{\infty}\hat{f}\left(+\frac{\pi n}{L}\right)=\sum_{M=-\infty}^{\infty}f(2LM).$$

This *Poisson summation formula* says that the sum of a function over a an equally spaced grid of points equals the sum of its Fourier transform over a certain other equally spaced grid of points. The most symmetrical version comes from choosing $L=\sqrt{\frac{\pi}{2}}$

$$\boxed{\sum_{n=-\infty}^{\infty}\hat{f}(\sqrt{2\pi}\,n)=\sum_{M=-\infty}^{\infty}f(\sqrt{2\pi}\,M).}$$

However, the most frequently used version, and probably the easiest to remember, comes from taking $L = \frac{1}{2}$: Starting with a numerical sum

$$\sum_{M=-\infty}^{\infty} f(M),$$

one can replace it by

$$\sqrt{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n),$$

which is

$$\boxed{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x}\, dx}$$

(and the minus sign in the exponent is unnecessary).