Math. 412 (Fulling) 14 December 2015

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

Some possibly useful information

Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

Laplacian operator in spherical coordinates ("physicists' notation"):

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\,.$$

Spherical harmonics satisfy

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_l^m(\theta,\phi) = -l(l+1) Y_l^m(\theta,\phi).$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z)\,.$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{l(l+1)}{z^2}\right) Z = 0 \quad \text{has solutions } j_l(z) \text{ and } y_l(z)\,.$$

Legendre's equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } P_l(\cos\theta)\,.$$

Airy's equation:

$$\frac{\partial^2 y}{\partial z^2} - zy = 0 \quad \text{has solutions } \mathrm{Ai}(z) \text{ and } \mathrm{Bi}(z)\,.$$

Famous Green function integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-|k|y}\, dk = \frac{1}{\pi}\frac{y}{x^2+y^2}\,.$$

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1. *(30 pts.)* Classify each equation as

   (i) elliptic, hyperbolic, or parabolic,

and

   (ii) linear homogeneous, linear nonhomogeneous, or nonlinear.

   (a) $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + t^2 - x^2 = 0$$

nonlinear, hyperbolic

   (b) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + e^{-(x-y)^2} = 0$$

linear nonhomogeneous, elliptic

   (c) $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + x^2 u = 0$$

linear homogeneous, parabolic

In any remaining problems that involve spherical coordinates, their range is always the whole sphere,

$$0 < \theta < \pi, \qquad -\pi < \phi \ \leq \pi \quad \textbf{(or equivalent)}.$$

I reiterate that the meanings of θ and ϕ are reversed compared to Haberman's book.

2. *(40 pts.)* Solve the heat equation in a ball,

$$\nabla^2 u = \frac{\partial u}{\partial t} \quad \text{for} \quad 0 \leq r < R, \quad 0 < t < \infty,$$
$$u(R,\theta,\phi,t) = 0, \qquad u(r,\theta,\phi,0) = f(r,\theta,\phi).$$

   (The spherical harmonic notation is strongly advised.)

After separating the time variable we have to solve $\nabla^2 u_\lambda = -\lambda u_\lambda$ with homogeneous Dirichlet condition on the sphere. The solutions are well known to be $Y_l^m(\theta,\phi) j_l(\omega_{ln} r)$, where j_l is called a spherical Bessel function (related to an ordinary Bessel function of order $l + \frac{1}{2}$), the ω_{ln} are the numbers for which $j_l(\omega_{ln} R) = 0$, and $\lambda_{ln} = \omega_{ln}^2$. (Here ln is a pair of indices, not their product.) Then we can write the general solution

$$u(r,\theta,\phi,t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} C_{lmn} Y_l^m(\theta,\phi) j_l(\omega_{ln} r) e^{-\lambda_{ln} t}.$$

Finally

$$f(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} C_{lmn} Y_l^m(\theta,\phi) j_l(\omega_{ln} r),$$

so

$$C_{lmn} = \frac{\int_{-\pi}^{\pi} d\phi \int_0^{\pi} d\theta \int_0^R dr\, r^2 \sin\theta\, j_l(\omega_{ln} r) Y_l^m(\theta,\phi)^* f(r,\theta,\phi)}{\int_0^R j_l(\omega_{ln} r)^2\, r^2\, dr}.$$

3. *(50 pts.)*

(a) Solve by separation of variables or an equivalent transform technique:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < \infty, \quad 0 < y < \infty),$$

$$\frac{\partial u}{\partial x}(0, y) = 0 \quad (0 < y < \infty), \qquad u(x, 0) = f(x) \quad (0 < x < \infty).$$

We see that we will need a transform on the half-line of x. The boundary condition indicates a Fourier cosine transform, since the cosines are the eigenfunctions that satisfy the Neumann condition. The infinite interval in y calls for decaying exponentials. So the expected form of the solution is

$$u(x, y) = \int_0^\infty A(k) \cos(kx) e^{-ky}\, dk.$$

Thus

$$f(x) = \int_0^\infty A(k) \cos(kx)\, dk,$$

so

$$A(k) = \frac{2}{\pi} \int_0^\infty f(x) \cos(kx)\, dx.$$

(b) Find the Green function that gives the solution to (a) in the form

$$u(x, y) = \int_0^\infty G(x, z, y) f(z)\, dz\,.$$

(There are two methods. **Do** evaluate the integral if your method leads to one.)

Substituting the formula for A into the one for u, we get

$$\begin{aligned} u(x, y) &= \int_0^\infty \frac{2}{\pi} \int_0^\infty f(z) \cos(kz)\, dz \cos(kx) e^{-ky}\, dk \\ &= \int_0^\infty dz\, f(z) \left[\frac{2}{\pi} \int_0^\infty dk\, \cos(kz) \cos(kx) e^{-ky} \right]. \end{aligned}$$

So the object in the brackets is the Green function. Using $\cos(kz) = \frac{1}{2}(e^{ikz} + e^{-ikz})$ etc., you can reduce this to integrals that can be evaluated by the second "famous Green function" formula. The result will be the same as the one I shall now get by the image method.

Alternative method: Remember that the second "famous Green function integral", with x replaced by $x - z$, is the Green function for the similar problem with $f(z)$ given on the entire real line. For the half-line with the homogeneous Neumann condition at the end, the Green function is obtained by adding the effect of an image source at $-z$ instead of z:

$$G(x, z, y) = \frac{1}{\pi} \left[\frac{y}{(x - z)^2 + y^2} + \frac{y}{(x + z)^2 + y^2} \right].$$

4. *(40 pts.)* Solve Laplace's equation in the region between two concentric spheres:

$$\nabla^2 u = 0 \quad \text{for} \quad 1 < r < 3\,,$$

$$\frac{\partial u}{\partial r}(1,\theta,\phi) = 0, \qquad u(3,\theta,\phi) = f(\theta) \quad (\text{independent of } \phi\,).$$

(Note that ϕ is the **azimuthal** angle, not the polar one.)

Again we have spherical harmonics, but this time the radial solution involves r^l and $r^{-(l+1)}$, not Bessel functions. Which linear combination satisfies the homogeneous boundary condition?

$$R(r) = Ar^l + Br^{-(l+1)},$$

$$0 = R'(1) = lA - (l+1)B.$$

Of course we can only determine the ratio at this point, so take $A = 1$, $B = \frac{l}{l+1}$.

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_l^m(\theta,\phi) \left[r^l + \frac{l}{l+1} r^{-(l+1)} \right].$$

But because there is no ϕ dependence, only $m = 0$ contributes:

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} C_l Y_l^0(\theta,\phi) \left[r^l + \frac{l}{l+1} r^{-(l+1)} \right].$$

(This could also be written in terms of $P_l(\cos\theta)$, but the spherical harmonic notation automatically takes care of the normalization factor.) Then

$$f(\theta) = \sum_{l=0}^{\infty} C_l Y_l^0(\theta,\phi) \left[3^l + \frac{l}{l+1} 3^{-(l+1)} \right].$$

(The spherical harmonic here is actually independent of ϕ and is some multiple of $P_l(\cos\theta)$. Also, it is real.) So

$$C_l = \frac{2\pi \int_0^\pi d\theta\, \sin\theta\, Y_l^0(\theta,\phi) f(\theta)}{3^l + \frac{l}{l+1} 3^{-(l+1)}}.$$

5. *(40 pts.)* Using Fourier series, solve this modified wave equation on a circle (with periodic boundary conditions):

$$\frac{\partial^2 u}{\partial t^2} + tu = \frac{\partial^2 u}{\partial x^2} \quad (1 < t < \infty, \quad -\pi < x \leq \pi),$$

$$u(x,1) = f(x), \qquad \frac{\partial u}{\partial t}(x,1) = g(x).$$

(If you don't know the solutions for the time dependence, give them convenient names and proceed.)

Since this one involves a nonstandard operator, it is prudent to go through the whole separation of variables from the beginning. Let $u_{\text{sep}} = X(x)T(t)$. Then $XT'' + tXT = X''T$, so

$$\frac{T'' + tT}{T} = \frac{X''}{X} = -\lambda.$$

The spatial (angular!) equation is $X'' = -\lambda X$, and because of the periodicity we must have $\lambda = n^2$ (for integer n) and

$$X_n = e^{inx} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

or, alternatively,

$$X_n = \sin(nx) \quad (n = 1, 2, \ldots) \quad \text{or} \quad \cos(nx) \quad (n = 0, 1, \ldots).$$

I will use the exponential form because it calls for less typing.

The time equation is $T'' + tT = -n^2 T$. It is second-order, linear, homogeneous, so it has a two-dimensional vector space of solutions. In other words, any solution is a linear combination of two basis solutions. Furthermore, the equation is nonsingular, so each solution can be characterized by its initial data at $t = 1$. (In this problem the initial time was chosen to be 1 instead of 0 just to keep the coefficient t from changing sign in the region of interest, but that turns out not to be very important.) Let's introduce the notation p and q for the solutions with the data

$$p(1) = 1, \quad p'(1) = 0, \qquad q(1) = 0, \quad q'(1) = 1.$$

These functions also depend on $\lambda = n^2$, so they should carry a subscript n ; but p_n and p_{-n} are the same thing, and the same for q .

Now we write the general solution. For each λ there are four independent solutions, each of which needs a yet-to-be-determined coefficient:

$$u(x,t) = \sum_{n=-\infty}^{\infty} \left[A_n p_n(t) e^{inx} + B_n q_n(t) e^{inx} \right].$$

(If you use the trigonometric basis, you must literally write four terms:

$$u(x,t) = \sum_{n=0}^{\infty} \left[a_n p_n(t) \cos(nx) + b_n q_n(t) \cos(nx) + c_n p_n(t) \sin(nx) + d_n q_n(t) \sin(nx) \right].$$

Recall that writing the most general solutions of the X equation and the T equation and multiplying them does *not* give the correct (linear) construction.)

Determine the coefficients by imposing the initial conditions:

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}, \qquad g(x) = \sum_{n=-\infty}^{\infty} B_n e^{inx}.$$

Therefore,

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx, \qquad B_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} g(x)\, dx.$$

Finally, can we say anything about what a_n and b_n are? Try introducing a new variable $\tau = -(t + n^2)$. This transforms the T equation to $T'' - \tau T = 0$ (where the primes now indicate derivatives with respect to τ — but the second derivative is the same for either variable). This is Airy's equation. So we know that $p_n(t)$ and $q_n(t)$ are certain linear combinations of the Airy fuctions $\mathrm{Ai}(-t - n^2)$ and $\mathrm{Bi}(-t - n^2)$. To find the coefficients in these linear combinations we would need to use a table or a computer program to find the values of the Airy functions and their derivatives at all the points $z = -(n^2 + 1)$.