Fourier Series

Now we need to take a theoretical excursion to build up the mathematics that makes separation of variables possible.

Periodic functions

Definition: A function f is *periodic* with *period* p if

$$f(x+p) = f(x) \quad \text{for all } x.$$

Examples and remarks: (1) $\sin(2x)$ is periodic with period 2π — and also with period π or 4π. (If p is a period for f, then an integer multiple of p is also a period. In this example the *fundamental* period — the smallest positive period

— is π.) (2) The smallest common period of $\{\sin(2x), \sin(3x), \sin(4x), \ldots\}$ is 2π. (Note that the fundamental periods of the first two functions in the list are π and $2\pi/3$, which are smaller than this common period.) (3) A constant function has every number as period.

The strategy of separation of variables raises this question:

Is *every* function with period 2π of the form*

$$(*) \qquad \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \bigl[a_n \cos(nx) + b_n \sin(nx)\bigr]} \; ?$$

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* Where did the cosines come from? In the previous example we had only sines, because we were dealing with Dirichlet boundary conditions. Neumann conditions would lead to cosines, and periodic boundary conditions (for instance, heat conduction in a ring) would lead to both sines and cosines, as we'll see.

(Note that we could also write (*) as

$$f(x) = \sum_{n=0}^{\infty} \big[a_n \cos(nx) + b_n \sin(nx)\big],$$

since $\cos(0x) = 1$ and $\sin(0x) = 0$.)

More precisely, there are three questions:

1. What, exactly, does the infinite sum mean?

2. Given a periodic f, are there numbers a_n and b_n that make (*) true?

3. If so, how do we calculate a_n and b_n ?

It is convenient to answer the last question first. That is, let's *assume* (*) and then find formulas for a_n and b_n in terms of f. Here we make use of the ...

Orthogonality relations: If n and m are nonnegative integers, then

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0;$$

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \begin{cases} 0 & \text{if } n \neq 0, \\ 2\pi & \text{if } n = 0; \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0;$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n = m \neq 0; \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n = m \neq 0. \end{cases}$$

Proof: These integrals are elementary, given such identities as

$$2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi).$$

Now multiply $(*)$ by $\cos(mx)$ and integrate from $-\pi$ to π. Assume temporarily that the integral of the series is the sum of the integrals of the terms. (To justify this we must answer questions 1 and 2.) If $m \neq 0$ we get

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\, f(x)\, dx &= a_0 \int_{-\pi}^{\pi} \cos(mx)\, dx \\ &\quad + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\, dx \\ &= \pi a_m\, .\end{aligned}$$

We do similar calculations for $m = 0$ and for $\sin(mx)$. The conclusion is: *If f*

has a Fourier series representation at all, then the coefficients must be

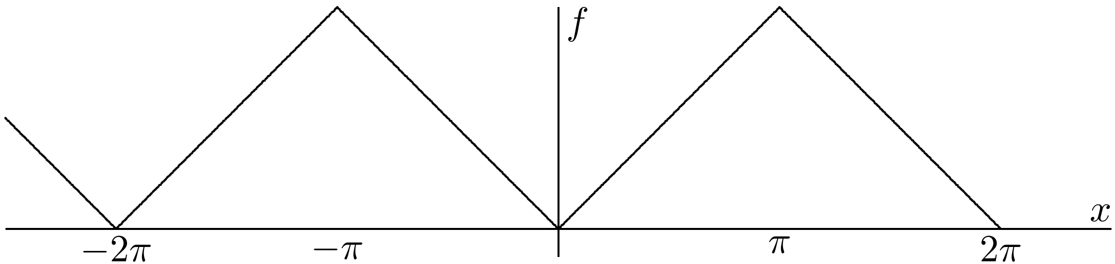
$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\,f(x)\,dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\,f(x)\,dx. \end{aligned}}$$

Note that the first two equations can't be combined, because of an annoying factor of 2. (Some authors get rid of the factor of 2 by defining the coefficient a_0 differently:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big[a_n\cos(nx) + b_n\sin(nx)\big]. \qquad \textbf{(* NO *)}$$

In my opinion this is worse.)

Example: Find the Fourier coefficients of the function ("triangle wave") which is periodic with period 2π and is given for $-\pi < x < \pi$ by $f(x) = |x|$.



$$
\begin{aligned}
\pi a_n &= \int_{-\pi}^{\pi} |x| \cos(nx)\, dx \\
&= \int_{-\pi}^{0} (-x) \cos(nx)\, dx + \int_{0}^{\pi} x \cos(nx)\, dx.
\end{aligned}
$$

In the first term, let $y = -x$:

$$
\begin{aligned}
\pi a_n &= 2 \int_{0}^{\pi} x \cos(nx)\, dx \\
&= \frac{2}{n} \left[x \sin(nx) \Big|_0^{\pi} - \int_0^{\pi} \sin(nx)\, dx \right] \\
&= 0 - \frac{2}{n} \frac{(-1)}{n} \cos(nx) \Big|_0^{\pi} \\
&= \frac{2}{n^2} \big(\cos(n\pi) - 1 \big).
\end{aligned}
$$

Thus

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even (and not 0),} \\ -\dfrac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, one finds that $a_0 = \dfrac{\pi}{2}$. Finally,

$$\pi b_n = \int_{-\pi}^{0} (-x)\, \sin(nx)\, dx + \int_0^\pi x\, \sin(nx)\, dx.$$

Here the first term equals $\int_0^\pi y\, \sin(-ny)\, dy$, but this is just the negative of the second term. So $b_n = 0$. (This will always happen when an *odd* integrand is integrated over an interval centered at 0.)

Putting the results together, we get

$$
\begin{aligned}
f(x) &\sim \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2} \cos[(2k+1)x] \\
&= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \cdots\right].
\end{aligned}
$$

(The symbol "$\sim$" is a reminder that we have calculated the coefficients, but haven't proved convergence yet. The important idea is that this "formal Fourier series" must have *something* to do with f even if it doesn't converge, or converges to something other than f.)

It's fun and informative to graph the first few *partial sums* of this series with suitable software, such as *Maple*. By taking enough terms of the series we really do get a good fit to the original function. Of course, with a *finite* number of terms we can never *completely* get rid of the wiggles in the graph, nor reproduce the sharp points of the true graph at $x = n\pi$.

Fourier series on a finite interval

If $f(x)$ is defined for $-\pi < x \le \pi$, then it has a *periodic extension* to all x: just reproduce the graph in blocks of length 2π all along the axis. That is,

$$f(x \pm 2\pi n) \equiv f(x) \quad \text{for any integer } n.$$

If f is continuous on $-\pi < x \le \pi$, then the periodic extension is continuous if and only if

$$\lim_{x\downarrow -\pi} f(x) \equiv f(-\pi) = f(\pi) = \lim_{x\uparrow\pi} f(x).$$

(Here the operative equality (the target of "if and only if") is the middle one. The left one is a definition, and the right one is a consequence of our continuity assumption. The notation $\lim_{x\uparrow\pi}$ means the same as $\lim_{x\to\pi^-}$, etc.) This issue of continuity is important, because it influences how well the infinite Fourier series converges to f, as we'll soon see.

The Fourier coefficients of the periodically extended f,

$$\int_{-\pi}^{\pi} \cos(nx)\, f(x)\, dx \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\, f(x)\, dx,$$

are completely determined by the values of $f(x)$ in the original interval $(-\pi, \pi]$ (or, for that matter, any other interval of length 2π — all of which will give the same values for the integrals). Thus we think of a Fourier series as being associated with

(1) an arbitrary function on a finite interval

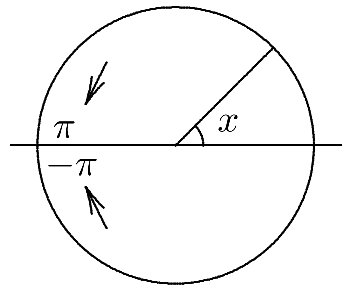
as well as

(2) a periodic function on the whole real line.

Still another approach, perhaps the best of all, is to think of f as

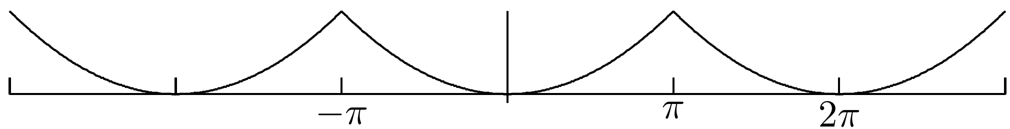
(3) an arbitrary function defined on a circle

with x as the *angle* that serves as coordinate on the circle. The angles x and $x + 2\pi n$ represent the same point on the circle.



In particular, π and $-\pi$ are the same point, no different in principle from any other point on the circle. Again, f (given for $x \in (-\pi, \pi]$) qualifies as a *continuous* *function on the circle* only if $f(-\pi) = f(\pi)$. The behavior $f(-\pi) \neq f(\pi)$ counts as a *jump discontinuity* in the theory of Fourier series.

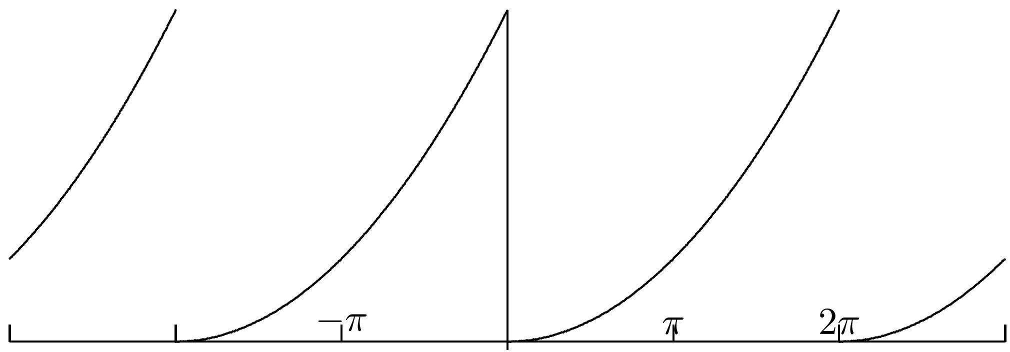
Caution: The periodic extension of a function originally given on a finite interval is not usually the natural extension of the algebraic expression that defines the function on the original interval. The Fourier series belongs to the periodic extension, not the algebraic extension. For example, if $f(x) = x^2$ on $(-\pi, \pi]$, its Fourier series is that of



(axes not to scale!) and has nothing to do with the full parabola,

$$f(x) = x^2 \quad \text{for all } x.$$

The coefficients of this scalloped periodic function are given by integrals such as $\int_{-\pi}^{\pi} \cos(mx)\, x^2\, dx$. If we were to calculate the integrals over some other interval of length 2π, say $\int_0^{2\pi} \cos(mx)\, x^2\, dx$, then we would get the Fourier series of a very different function:



This does not contradict the earlier statement that the integration interval is irrelevant when you start with a function that is already periodic.

An *even* function satisfies

$$f(-x) = f(x).$$

Examples: cos, cosh, x^{2n}.

An *odd* function satisfies

$$f(-x) = -f(x).$$

Examples: sin, sinh, x^{2n+1}.

In either case, the values $f(x)$ for $x < 0$ are determined by those for $x > 0$ (or vice versa).

Properties of even and odd functions (schematically stated)**:**

(1) even + even = even; odd + odd = odd; even + odd = neither.

In fact, anything = even + odd:

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)].$$

In the language of linear algebra, the even functions and the odd functions each form *subspaces*, and the vector space of all functions is their *direct sum.*

(2) even × even = even; odd × odd = even; even × odd = odd.

(3) (even)$'$ = odd; (odd)$'$ = even.

(4) $\int$ odd = even; $\int$ even = odd $+\, C$.

Theorem: If f is even, its Fourier series contains only cosines. If f is odd, its Fourier series contains only sines.

Proof: We saw this previously for an even example function. Let's work it out in general for the odd case:

$$\begin{aligned}
\pi a_n &\equiv \int_{-\pi}^{\pi} f(x)\, \cos(nx)\, dx \\
&= \int_{-\pi}^{0} f(x)\, \cos(nx)\, dx + \int_{0}^{\pi} f(x)\, \cos(nx)\, dx \\
&= \int_{0}^{\pi} f(-y)\, \cos(-ny)\, dy + \int_{0}^{\pi} f(x)\, \cos(nx)\, dx \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\pi b_n &\equiv \int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx \\
&= \int_{-\pi}^{0} f(x)\,\sin(nx)\,dx + \int_{0}^{\pi} f(x)\,\sin(nx)\,dx \\
&= \int_{0}^{\pi} f(-y)\,\sin(-ny)\,dy + \int_{0}^{\pi} f(x)\,\sin(nx)\,dx \\
&= 2\int_{0}^{\pi} f(x)\,\sin(nx)\,dx.
\end{aligned}$$

This was for an odd f defined on $(-\pi, \pi)$. Given *any* f defined on $(0, \pi)$, we can extend it to an odd function on $(-\pi, \pi)$. Thus it has an Fourier series consisting entirely of sines:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$\text{for odd } f \text{ on } -\pi < x < \pi$$

$$\text{or any } f \text{ on } 0 < x < \pi.$$

Similarly, the *even* extension gives a series of *cosines* for any f on $0 < x < \pi$. This series includes the constant term, $n = 0$, for which the coefficient formula has an extra factor $\frac{1}{2}$. The formulas are

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos(nx)$$

$$\text{where} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \quad \text{for } n > 0,$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

for even f on $-\pi < x < \pi$

or any f on $0 < x < \pi$.

For an interval of arbitrary length, L, we let $x = \pi y/L$ and obtain

$$\boxed{\begin{gathered}\overline{f}(y) \equiv f\left(\frac{\pi y}{L}\right) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{L} \\ \text{where}\quad b_n = \frac{2}{L}\int_0^L \overline{f}(y)\sin\frac{n\pi y}{L}\,dy \\ \text{for odd } \overline{f} \text{ on } -L < y < L \\ \text{or any } \overline{f} \text{ on } 0 < y < L.\end{gathered}}$$

To keep the formulas simple, theoretical discussions of Fourier series are conducted for the case $L = \pi$; the results for the general case then follow trivially.

Summary: Given an arbitrary function on an interval of length K, we can expand it in

(1) sines *or* cosines of period $2K$ (taking $K = L$, interval $= (0, L)$),

or

(2) sines *and* cosines of period K (taking $K = 2L$, interval $= (-L, L)$).

In each case, the arguments of the trig functions in the series and the coefficient formulas are

$$\frac{m\pi x}{L}, \quad m = \text{integer}.$$

Which series to choose (equivalently, which extension of the original function) depends on the context of the problem; usually this means the *type of boundary conditions.*

Complex Fourier series

A quick review of complex numbers:

$$i \equiv \sqrt{-1}.$$

Every complex number has the form $z = x + iy$ with x and y real. To manipulate these, assume that $i^2 = -1$ and all rules of ordinary algebra hold. Thus

$$(a + ib) + (c + id) = (a + c) + i(b + d);$$

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad).$$

We write $x \equiv \operatorname{Re} z$, $y \equiv \operatorname{Im} z$;

$$|z| \equiv \sqrt{x^2 + y^2} = \text{modulus of } z;$$

$$z^* \equiv x - iy = \text{complex conjugate of } z.$$

Note that

$$(z_1 + z_2)^* = {z_1}^* + {z_2}^*, \qquad (z_1 z_2)^* = {z_1}^* {z_2}^*.$$

Define

$$e^{i\theta} \equiv \cos\theta + i \sin\theta \qquad (\theta \text{ real});$$

then

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y); \end{aligned}$$

$$\left|e^{i\theta}\right| = 1 \quad \text{if } \theta \text{ is real}; \quad e^{z+2\pi i} = e^z;$$

$$e^{i\pi} = -1, \quad e^{i\pi/2} = i, \quad e^{-i\pi/2} = e^{3\pi i/2} = -i = \frac{1}{i}, \quad e^{2\pi i} = e^0 = 1;$$

$$\left(e^{i\theta}\right)^* = e^{-i\theta} = \frac{1}{e^{i\theta}}; \quad e^{-i\theta} = \cos\theta - i \sin\theta;$$

$$\boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).}$$

Remark: Trig identities become trivial when expressed in terms of $e^{i\theta}$, hence easy to rederive. For example,

$$\begin{aligned}\cos^2\theta &= \tfrac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)^2\\ &= \tfrac{1}{4}\left(e^{2i\theta}+2+e^{-2i\theta}\right)\\ &= \tfrac{1}{2}\big(\cos(2\theta)+1\big).\end{aligned}$$

In the Fourier formulas $(*)$ for periodic functions on the interval $(-\pi,\pi)$, set

$$c_0 = a_0\,, \quad c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \tfrac{1}{2}(a_n + ib_n).$$

The result is

$$\boxed{\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n\, e^{inx},\\ \text{where } c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx.\end{aligned}}$$

(Note that we are now letting n range through negative integers as well as non-negative ones.) Notice that now there is *only one* coefficient formula. This is a major simplification!

Alternatively, the complex form of the Fourier series can be derived from *one* orthogonality relation,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\, e^{-imx}\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

As usual, we can scale these formulas to the interval $(-L, L)$ by the variable change $x = \pi y/L$.