Fourier Transforms and Problems on Infinite Domains

Let's consider a heat conduction problem on a *semiinfinite* space interval:

$$\frac{1}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < \infty \text{ and } 0 < t < \infty.$$

The left end of the bar is insulated, so

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad \text{for } 0 < t < \infty.$$

The initial temperature is

$$u(x, 0) = f(x) \quad \text{for } 0 < x < \infty.$$

When we try to solve this problem by separation of variables, we get as usual

$$X'' = -\lambda X, \qquad T' = -\lambda K T\,.$$

If we set $\lambda = \omega^2$, the solutions of the X equation are

$$X(x) = A\cos(\omega x) + B\sin(\omega x).$$

The boundary condition $X'(0) = 0$ forces B to be 0. We can choose $A = 1$ and write the normal mode

$$u_\omega(x,t) = \cos(\omega x)\, e^{-\omega^2 Kt}.$$

However, there is a major difference between this problem and the others we have considered: Since there is no second endpoint, there is no second boundary condition to determine the allowed values of ω. Indeed, *all* nonnegative values of ω are possible, and the complete solution $u(x,t)$ satisfying the initial data will turn out to be an *integral* over these values, not a sum.* That is why I have

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* You may wonder, then, why *complex* values of ω are not also allowed. A completely satisfying answer is not possible at the level of technicality appropriate to this course, but a standard rule of thumb is that solutions that *increase* exponentially fast at infinity ($\cosh(\kappa x)$ in this case) are not needed as eigenfunctions. We will soon see that the cosine functions by themselves are sufficient to represent all reasonable initial data.

labeled the normal mode u_ω instead of u_n ; *there is no integer variable n in this type of problem!*

Generally speaking, one has the correspondence

$$\begin{aligned} \text{Finite interval} &\Rightarrow \text{Fourier series (a sum);} \\ \text{Infinite interval} &\Rightarrow \text{Fourier transform (an integral).} \end{aligned}$$

To see that this formulation is a slight oversimplification, note that a change of variable like $y = \ln x$ can convert a finite interval into an infinite one $[(0,1)$ into $(-\infty,0)]$; obviously if a discrete sum is right in one case it will not become wrong in the other. (On the other hand, a Fourier series in x will no longer be a *Fourier* series in y, but something more general.) But this rule of thumb does apply to differential equations with *constant coefficients* and to some others. Note also that the interval referred to is one on which nonhomogeneous initial or boundary data are prescribed, not one where a homogeneous condition applies; we will see some examples of this distinction later.

It is easy to see how a Fourier series "becomes" an integral when the length of the interval goes to infinity. For this it is most convenient to use the complex-exponential form of the Fourier series. Recall that for a function on a finite interval of length $2L$, we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L},$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \, e^{-in\pi x/L} \, dx.$$

Let's write

$$k_n \equiv \frac{n\pi}{L} \,.$$

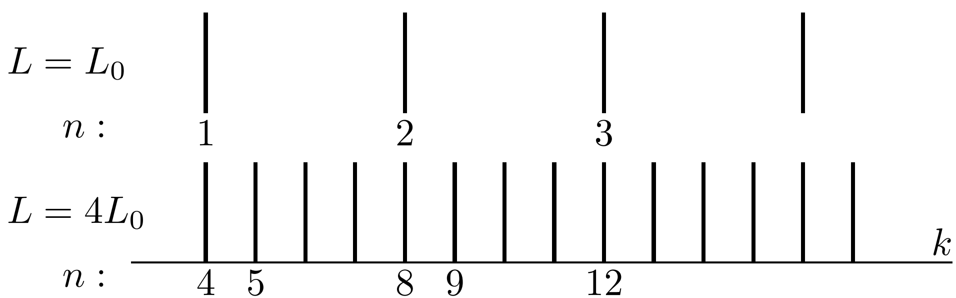
Then

$$f(x) = \sum_{k_n} c_n \, e^{ik_n x}.$$

The numbers k_n are called “frequencies” or “wave numbers”. As L increases, the frequencies become more closely spaced:

$$\Delta k_n \equiv \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}\,.$$

This suggests that for f defined on the whole real line, $-\infty < x < \infty$, *all* values of k should appear.



To make sense of the limit $L \to \infty$, we have to make a change of variable from n to k. Let

$$\hat{f}(k_n) \equiv L\sqrt{\frac{2}{\pi}}\, c_n \,.$$

Then

$$\begin{aligned} f(x) &= \sqrt{\frac{\pi}{2}} \sum_{k_n} \frac{1}{L}\, \hat{f}(k_n)\, e^{ik_n x} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k_n} \hat{f}(k_n)\, e^{ik_n x}\, \Delta k_n \,, \end{aligned}$$

$$\hat{f}(k_n) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x)\, e^{-ik_n x}\, dx.$$

As $L \to \infty$ the first formula looks like a Riemann sum. In the limit we therefore expect

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk, \\ \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx. \end{aligned}}$$

Note the surprising symmetry between these two formulas! $\hat{f}$ is called the *Fourier transform* of f, and f is the *inverse Fourier transform* of $\hat{f}$.

Sine and cosine transforms

Of course, this does not solve our example problem. There the allowed functions were $\cos(kx)$, not e^{ikx}, and we were poised to expand an initial temperature distribution, defined for *positive x only*, in terms of them: If

$$f(x) \equiv u(x,0) = \int_0^\infty A(k)\,\cos(kx)\,dk,$$

then

$$u(x,t) = \int_0^\infty A(k)\,\cos(kx)\,e^{-k^2 Kt} dk$$

is the solution.

The way to get from exponentials to sines and cosines is basically the same as in finite Fourier series. First, note that the Fourier transformation we have derived (for $-\infty < x < \infty$) can be rewritten in terms of $\sin(kx)$ and $\cos(kx)$ $(0 \leq k < \infty)$ in place of e^{ikx} $(-\infty < k < \infty)$. You can easily work out that the formulas are

$$f(x) = \int_0^\infty [A(k)\,\cos(kx) + B(k)\,\sin(kx)]\,dk,$$

$$A(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos(kx)\, f(x)\, dx,$$

$$B(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \sin(kx)\, f(x)\, dx.$$

This is seldom done in practical calculations with functions defined on $(-\infty, \infty)$, except by people with a strong hatred for complex numbers.

However, the trigonometric functions become very useful in calculations on a half-line (semiinfinite interval) with a boundary condition at the end. An arbitrary function on $0 \leq x < \infty$ can be identified with its *even extension* to the whole real line. An *even* function has a Fourier transform consisting entirely of cosines (rather than sines), and the formula for the coefficient function can be written as an integral over just the positive half of the line:

$$f(x) = \int_{0}^{\infty} A(k)\, \cos(kx)\, dk,$$

$$A(k) = \frac{2}{\pi}\int_0^\infty \cos(kx)\, f(x)\, dx.$$

An equally common normalization convention splits the constant factor symmetrically between the two formulas:

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty A(k)\, \cos(kx)\, dk,$$

$$A(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, \cos(kx)\, dx.$$

Still other people put the entire factor $\frac{2}{\pi}$ into the $A \mapsto F$ equation.* In any case, A is called the *Fourier cosine transform* of f, and it's often given a notation such as $\hat{f}_{\mathrm{c}}(k)$ or $F_{\mathrm{C}}(k)$.

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* Similar notational variations are found for the full (complex-exponential) Fourier transform.

It should now be clear how to finish the solution of the heat problem in the infinite bar with insulated left end.

Correspondingly, there is a *Fourier sine transform* related to *odd* extensions of functions. The formulas are the same except that cos is replaced by sin everywhere. The sine transform arises naturally in problems where the functions vanish at the boundary $x = 0$, and the cosine transform is appropriate when the derivative vanishes there (as we've seen).

Convergence theorems

Our derivation of the Fourier transformation formulas is not a proof that applying the two formulas in succession really will take you back to the function f from which you started; all the convergence theorems for Fourier series need to be reformulated and reproved for this new situation. In fact, since the integrals are improper, the function f needs to satisfy some technical conditions before the integral $\hat{f}$ will converge at all.

First, let's state the generalization to Fourier transforms of the pointwise convergence theorem for Fourier series. To get a true theorem, we have to make a seemingly fussy, but actually quite natural, technical condition on the function: Let's define a function with domain $(-\infty, \infty)$ to be *piecewise smooth* if its restriction to every finite interval is piecewise smooth. (Thus f is allowed to have infinitely many jumps or corners, but they must not pile up in one region of the line.) The Fourier transform is defined by

$$\hat{f}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

Pointwise convergence theorem: If $f(x)$ is piecewise smooth, and

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

(f is *absolutely integrable*, or $f \in L^1(-\infty, \infty)$), then:

a) $\hat{f}(k)$ is continuous.

b) $\hat{f}(k) \to 0$ as $|k| \to \infty$ (but $\hat{f}$ is not necessarily absolutely integrable itself). (This is a new version of the Riemann–Lebesgue theorem.)

c) The inverse Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk$$

converges pointwise to $\frac{1}{2}[f(x^+) + f(x^-)]$ (just like Fourier series).

The next theorem treats the variables x and k on a completely symmetrical basis.

Mean convergence theorem: If $f(x)$ is sufficiently smooth to be integrated, and

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

(f is *square-integrable*, or $f \in L^2(-\infty, \infty)$), then:

a) $\hat{f}(k)$ is also square-integrable. (The integral defining $\hat{f}(k)$ may not converge at every point k, but it will converge "in the mean", just like the inversion integral discussed below.)

b) A Parseval equation holds:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \, dk.$$

(If you define $\hat{f}$ so that the 2π is kept all in one place, then this formula will not be so symmetrical.)

c) The inversion formula converges in the mean:

$$\lim_{\Lambda \to \infty} \int_{-\infty}^{\infty} dx\, |f(x) - f_\Lambda(x)|^2 = 0$$

where

$$f_\Lambda(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\Lambda}^{\Lambda} \hat{f}(k)\, e^{ikx}\, dk.$$