Math. 412 (Fulling) 23 October 2015

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

1. *(30 pts.)* Use Fourier transforms (or an equivalent separation of variables) to solve

(regular)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty, \quad 0 < t < \infty),$$

$$u(x,0) = f(x) \quad (0 < x < \infty),$$

$$u(0,t) = 0 \quad (0 < t < \infty).$$

Here I will take the transform approach. To minimize writing constants, I define the Fourier sine transform by

$$F(k) = \int_0^\infty f(x) \sin kx \, dx \,, \qquad f(x) = \frac{2}{\pi} \int_0^\infty F(k) \sin kx \, dk \,,$$

with the corresponding formulas for $U(k,t)$. Then the transforms of the equations are

$$\frac{\partial U}{\partial t} = -k^2 U \,, \qquad U(k,0) = F(k) \,.$$

Therefore,

$$U(k,t) = F(k) e^{-k^2 t} \,.$$

Hence

$$\begin{aligned} u(x,t) &= \frac{2}{\pi} \int_0^\infty U(k,t) \sin kx \, dk \\ &= \frac{2}{\pi} \int_0^\infty F(k) \sin kx \, e^{-k^2 t} \, dk \,, \end{aligned}$$

where $F(k)$ is given by a formula above.

(honors)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty, \quad -\infty < t < \infty),$$

$$u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad (-\infty < x < \infty).$$

Here I will take the variable-separation approach. (Either problem can be done either way.) Write $u_{\text{sep}}(x,t) = X(x)T(t)$ and conclude

$$\frac{T'}{T} = \frac{X''}{X} = -k^2\,.$$

(We know from experience that imaginary k will not appear, and $k = 0$ only as the lower limit of the integral.) Thus $X_k(x) = e^{ikx}$, where k can be either positive or negative for a given k^2, and

$$T_k(t) = C(k)\cos kt + D(k)\sin kt\,.$$

Therefore, the full solution has the form

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}[C(k)\cos kt + D(k)\sin kt]\,dk\,.$$

The initial conditions become

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} C(k)\,dk\,,$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} kD(k)\,dk\,.$$

Therefore,

$$C(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx\,,$$

$$D(k) = \frac{1}{k}\int_{-\infty}^{\infty} e^{-ikx} g(x)\,dx\,.$$

Remark: It is equally valid to write

$$X_k(x) = A\cos kx + B\sin kx\,,$$

where now $0 < k < \infty$. However, it is important to remember that the full solution must be a *linear combination* of normal modes. Therefore, it is *wrong* to write

$$u(x,t) = \int_0^{\infty} (A\cos kx + B\sin kx)(C\cos kt + D\sin kt)\,dk\,.$$

Not every solution factors in that way, and for those that do, the solution for A, B, C, D will not be unique. The correct form is

$$u(x,t) = \int_0^{\infty} [a(k)\cos kx\cos kt + b(k)\cos kx\sin kt + c(k)\sin kx\cos kt + d(k)\sin kx\sin kt]$$

(possibly with a different normalization convention, of course.)

2. *(30 pts.)* Construct the Green function that solves

$$y'' - 4y = f(x) \quad (0 < x < 1),$$

$$y(0) = 0 = y(1)\,.$$

Clearly state the formula for calculating y from G and f.
Hint: $\sinh a \cosh b - \cosh a \sinh b = \sinh(a-b)$.

This is the same as Qu. 5 of Fall 2012 with a different number. The formula will be

$$y(x) = \int_0^1 G(x,z) f(z)\,dz\,.$$

The Green function must satisfy

$$\frac{\partial^2 G}{\partial x^2} - 4G = \delta(x-z)\,, \quad G(0,z) = 0 = G(1,z)\,.$$

The differential equation is interpreted as

$$\frac{\partial^2 G}{\partial x^2} - 4G = 0 \quad \text{for } x < z \text{ and } x > z\ .$$

$$G(z^+, z) = G(z^-, z)\,, \quad \frac{\partial G}{\partial x}(z^+, z) - \frac{\partial G}{\partial x}(z^-, z) = 1\,.$$

Therefore, in view of the boundary conditions,

$$G(x,z) = \begin{cases} A(z)\sinh(2x) & \text{for } 0 < x < z, \\ B(z)\sinh(2(x-1)) & \text{for } z < x < 1. \end{cases}$$

The jump conditions give

$$A(z)\sinh(2z) = B(z)\sinh(2(z-1))\,, \quad 2A(z)\cosh(2z) - 2B(z)\cosh(2(z-1)) = -1\,.$$

After some algebra using the hint, you get

$$A(z) = \frac{\sinh(2(z-1))}{2\sinh 2}\,, \quad B(z) = \frac{\sinh(2z)}{2\sinh 2}\,.$$

3. *(Essay – 40 pts.)* Outline a strategy to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (0 < x < \pi\,, \quad 0 < y < 2\,, \quad 0 < t < \infty)\,,$$

$$u(x,0,t) = 0\,, \quad u(x,2,t) = g(x)\,,$$

$$u(0,y,t) = T = u(\pi,y,t) \qquad (T = \text{nonzero constant})\,,$$

$$u(x,y,0) = f(x,y)\,.$$

Then carry out as many of the steps as you have time for.

This is the same as Qu. 3 of Fall 2006 with a few changes.

Because there are three different kinds of nonhomogeneous data, we expect to have to break the solution into three terms. Because the boundary data are independent of t, we expect that two of those terms can be steady-state solutions, independent of t. Suppose that we have found a steady-state solution, $v(x,y)$, that satisfies all the spatial boundary conditions. (Later we will break v into two parts.) Then $w = u - v$ satisfies the heat equation with completely homogeneous boundary conditions and the initial condition $w(x,y,0) = f(x,y) - v(x,y)$. The problem for w can be solved by separation of variables, and the eigenfunctions will be products of sine functions in the x and y directions.

Now consider the problem of finding v. It will satisfy Laplace's equation. Since T does not depend on y, the best strategy is to look for a solution of Laplace's equation that does not depend on y and satisfies the T boundary conditions. It is fairly easy to see that the constant function T will work, so I won't bother to give that function another name. Let $s(x,y) = v(x,y) - T$. Then s must satisfy Laplace, vanish on the sides $x = 0,\ \pi$, and satisfy

$$s(x,0) = -T\,, \quad s(x,2) = g(x) - T\,.$$

This can be solved by a routine separation of variables. (There will be two kinds of terms corresponding to the two nonhomogeneous boundary data functions, but since both those conditions refer to the same variable (i.e., parallel sides) it is not necessary to split s into two parts before separating variables.)

Finally, the solution is $u = w(x,y,t) + s(x,y) + T$. [An alternative, but suboptimal, solution is to write $v = v_1 + v_2$ with g as boundary data for v_1 and T as boundary data as v_2 (and no other nonhomogeneous data). In that approach one will need to expand the constant function T as a sine series in x. (But we'll end up doing that anyway, it turns out.)]

Sketch of details of w: It satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\,,$$

$$w(x,0,t) = 0 = w(x,2,t)\,, \qquad w(0,y,t) = 0 = w(\pi,y,t)\,,$$

$$w(x,y,0) = f(x,y) - v(x,y) \equiv h(x,y)\,.$$

Find eigenfunctions $X_n(x)Y_m(y) = \sin(nx)\sin\left(\frac{m\pi y}{2}\right)$. Then

$$w(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(nx)\sin\left(\frac{m\pi y}{2}\right)e^{-(n^2+m^2\pi^2/4)t}$$

and

$$a_{nm} = \frac{2}{\pi} \int_0^2 \int_0^\pi \sin(nx) \sin\left(\frac{m\pi y}{2}\right) h(x,y)\, dx\, dy\,.$$

Sketch of details of T *:* We need a function of x alone that satisfies Laplace's equation (i.e., has zero second derivative) and equals T at two points. It is of the form $Ax + B$, and you quickly find $A = 0$, $B = T$.

Sketch of details of s *:* It satisfies

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = 0\,,$$

$$s(x,0) = -T\,, \quad s(x,2) = g(x) - T \equiv k(x)\,,$$

$$s(0,y) = 0 = s(\pi, y)\,.$$

Separation of variables leads to eigenfunctions $X_n(x) = \sin(nx)$ and complementary solutions $Y(y) = \sinh(ny)$ and $\sinh[n(2-y)]$ (each chosen to vanish on one of the relevant boundaries). Summing up,

$$s(x,y) = \sum_{n=1}^\infty \sin(nx) \big[a_n \sinh(ny) + b_n \sinh[n(\pi - y)]\big]\,.$$

Then we find a_n from the Fourier sine coefficients of k and b_n from those of $-T$ (divided by $\sinh(2n)$).