Math. 412 (Fulling) 20 November 2015

Test C – Solutions

Calculators may be used for simple arithmetic operations only!

Useful information:

Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{\nu^2}{z^2}\right) Z = 0 \quad \text{has solutions } \ J_{|\nu|}(z) \ \text{ and } \ Y_{|\nu|}(z)\,.$$

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time. (In Qu. 1, don't rework the whole problem for the other sign of γ; just describe by graphs and words how the solution of the second problem would be different.)

1. *(50 pts.)* Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < t < \infty\,, \quad 0 < x < \pi),$$

$$\frac{\partial u}{\partial x}(t,0) = 0\,, \quad \frac{\partial u}{\partial x}(t,\pi) + \gamma u(t,\pi) = 0 \quad (0 < t < \infty),$$

$$u(0,x) = f(x) \quad (0 < x < \pi).$$

(a) Separate variables to reduce the problem to a Sturm–Liouville eigenvalue problem.

$$u_{\text{sep}} = T(t)X(x)\,.$$

$$\frac{T'}{T} = -\lambda = \frac{X''}{X}\,.$$

$$T(t) = e^{-\lambda t}\,.$$

$$X'' = -\lambda X\,, \quad X'(0) = 0\,, \quad X'(\pi) + \gamma X(\pi) = 0\,.$$

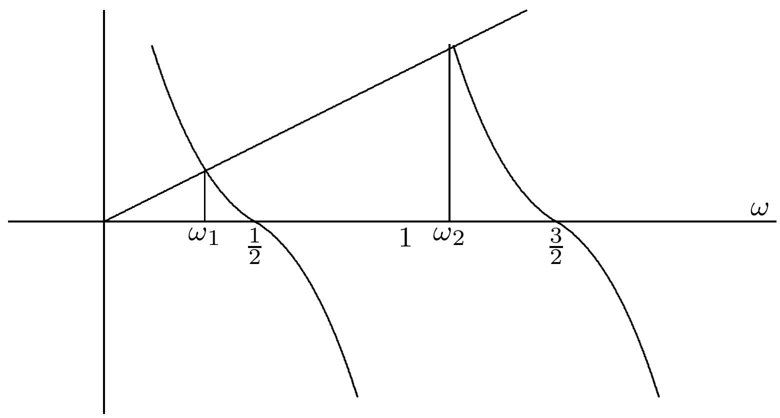
(b) Solve the Sturm–Liouville problem for

(regular) $\gamma = 2$.

Assuming for the moment that $\lambda > 0$, write $\lambda = \omega^2$. Then $X(x) = \cos(\omega x)$ because $X'(0) = 0$. The remaining condition becomes $-\omega\sin(\omega\pi) + \gamma\cos(\omega\pi) = 0$, or

$$\frac{\omega}{2} = \cot(\omega\pi)\,.$$

The allowed values of ω are the intersections of the graphs:

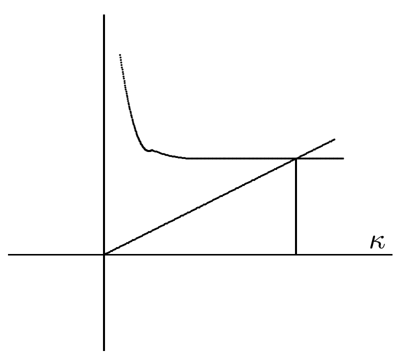


For large n , $\omega_n \approx n - 1$. Negative eigenvalues can be excluded by the general Sturm–Liouville theorem; their absence also will become clear from the analysis of the honors problem.

(honors) $\gamma = -2$.

For positive λ the solution runs just as above except that the straight graph has negative slope, so that $\omega_n \approx n$. For negative $\lambda = -\kappa^2$, $X(x) = \cosh(\kappa x)$ and $\kappa\sinh(\kappa\pi) + \gamma\cosh(\kappa\pi)$, or

$$\frac{\kappa}{2} = \coth(\kappa\pi) = \frac{1}{\tanh(\kappa\pi)}\,.$$



There is only one root, very close to $\kappa = 2$. Finally, recall that when $\gamma = 0$ we have the Neumann condition at both ends, and we know that 0 should be an eigenvalue, corresponding to constant eigenfunctions. We can see this solution emerging from the graphs for both positive and negative γ in the limit where the line becomes infinitely steep.

(c) Complete the solution of the heat problem. (Don't evaluate the normalization integrals. Instead, use some standard notation to abbreviate them.)

In the regular case,

$$u(t,x) = \sum_{n=1}^{\infty} C_n \cos(\omega_n x) e^{-\omega_n{}^2 t}\,.$$

In the honors case, u contains the same sum (with a different sequence of ω_n s) but also contains the single term

$$C_0 \cosh(\kappa x) e^{+\kappa^2 t}\,.$$

Let's write $X_n(x) \equiv \cos(\omega_n x)$ and

$$\|X_n\|^2 = \int_0^\pi \cos^2(\omega_n x)\,dx\,.$$

Similarly, $X_0(x) \equiv \cosh(\kappa x)$ and

$$\|X_0\|^2 = \int_0^\pi \cosh^2(\kappa x)\,dx\,.$$

Then

$$C_n = \frac{\int_0^\pi X_n(x) f(x)\,dx}{\|X_n\|^2}\,.$$

(d) Substitute your coefficient formula from (c) into the series solution and rearrange to get a formula for the Green function in

$$u(t,x) = \int_0^\pi G(t,x,y) f(y)\,dy\,.$$

$$\begin{aligned} u(t,x) &= \sum_n^{\infty} \int_0^\pi dy\, \frac{X_n(y)}{\|X_n\|^2}\, f(y) X_n(x) e^{-\lambda_n t} \\ &= \int_0^\pi dy\, f(y) \sum_n^{\infty} \frac{X_n(x) X_n(y)}{\|X_n\|^2}\, e^{-\lambda_n t}\,. \end{aligned}$$

Therefore,

$$G(t,x,y) = \sum_n^{\infty} \frac{X_n(x) X_n(y)}{\|X_n\|^2}\, e^{-\lambda_n t}\,.$$

Here the sums start at $n = 1$ in the regular case and $n = 0$ in the honors case. In the latter case, let's write it in more detail:

$$G(t,x,y) = \frac{\cosh(\kappa x)\cosh(\kappa y)}{\|x_0\|^2}\, e^{+\kappa^2 t} + \sum_{n=1}^{\infty} \frac{\cos(\omega_n x)\cos(\omega_n y)}{\|X_n\|^2}\, e^{-\omega_n{}^2 t}\,.$$

2. *(50 pts.)* Solve the wave equation in a sector,

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \quad (0 \le r < 1\,, \quad 0 < \theta < \tfrac{\pi}{2}\,, \quad -\infty < t < \infty),$$

$$u(0,r,\theta) = f(r,\theta)\,, \quad \frac{\partial u}{\partial t}(0,r,\theta) = 0\,,$$

$$u(t,r,0) = 0 = u(t,r,\tfrac{\pi}{2})\,, \quad u(t,1,\theta) = 0\,.$$

Start with $u_{\text{sep}} = T(t)\Phi(r,\theta)$, getting

$$\frac{T''}{T} = -\omega^2 = \nabla^2\Phi\,.$$

Then $T(t) = \cos(\omega t)$, since $T'(0) = 0$.

Press onward with $\Phi_{\text{sep}} = R(r)\Theta(\theta)$, getting

$$-\frac{\Theta''}{\Theta} = \nu^2 = \frac{r^2}{R}\,[R'' + R'/r + \omega^2 R]\,.$$

Then $\Theta(\theta) = \sin(\nu\theta)$, since $\Theta(0) = 0$. Furthermore, $\Theta(\frac{\pi}{2}) = 0$, so

$$\sin\left(\frac{\nu\pi}{2}\right) = 0 \;\Rightarrow\; |\nu| = 2n \quad \text{for positive integer } n\,.$$

In fact, we will need only positive ν in this problem, so the absolute value is unnecessary.

Now rearrange the radial equation into

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\,\frac{\partial R}{\partial r} + \omega^2 R - \frac{4n^2}{r^2}\,R = 0\,.$$

Rescaling the variable to $z = \omega r$ shows that the solutions are $J_{2n}(\omega r)$ and $Y_{2n}(\omega r)$. To avoid a singularity at the origin, we exclude the Y solutions. There is one more boundary condition, at $r = 1$, which says that $J_{2n}(\omega) = 0$. Thus $\omega = z_{2n,k}$, one of the zeros of the Bessel function.

Recapitulating, the relevant eigenfunctions of the Laplacian are

$$\Phi_{n,k}(r,\theta) = J_{2n}(z_{2n,k}r)\sin(2n\theta)$$

and the eigenfunction expansion is

$$u(t,r,\theta) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} C_{n,k}\Phi_{n,k}(r,\theta)\cos(z_{2n,k}t)\,.$$

The eigenfunctions are orthogonal, so the coefficients are

$$\begin{aligned} C_{n,k} &= \frac{\langle \Phi_{n,k}, f\rangle}{\|\Phi_{n,k}\|^2} \\ &= \frac{\int_0^{\pi/2} d\theta \int_0^1 r\,dr\,J_{2n}(z_{2n,k}r)\sin(2n\theta)f(r,\theta)}{\int_0^{\pi/2}\sin^2(2n\theta)\,d\theta\,\int_0^1 J_{2n}(z_{2n,k}r)^2 r\,dr}\,. \end{aligned}$$

The $\sin^2$ integral is $\pi/4$, as we know from the Fourier sine series formulas for an interval of length $\pi/2$.