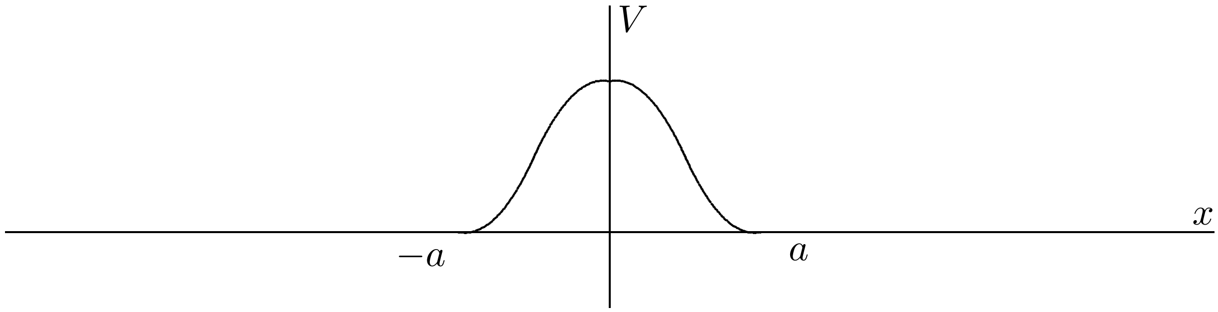
Scattering and the Delta Potential

Consider

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - V(x)u,$$

where $V(x) = 0$ for $|x| > a$.



Separation of variables leads to

$$\begin{aligned} \frac{d^2X}{dx^2} &= [V(x) - \lambda]X \\ &= -\lambda X \quad \text{for } |x| > a. \end{aligned}$$

Suppose (for the moment) that λ is positive, and set $\lambda = \omega^2$ with $\omega > 0$.

There are constants such that two independent solutions satisfy

$$X_\omega(x) = \begin{cases} e^{i\omega x} + R_\omega e^{-i\omega x} & \text{if } x < -a, \\ T_\omega e^{i\omega x} & \text{if } x > a; \end{cases}$$

$$X_{-\omega}(x) = \begin{cases} T_{-\omega} e^{-i\omega x} & \text{if } x < -a, \\ e^{-i\omega x} + R_{-\omega} e^{i\omega x} & \text{if } x > a. \end{cases}$$

Example: Suppose $V(x) = \alpha\delta(x)$. Integrate

$$\frac{d^2X}{dx^2} = [\alpha\delta(x) - \lambda]X \ :$$

you get

$$X'(0_+) - X'(0_-) = \alpha X(0), \qquad X(0_+) = X(0_-).$$

Let's apply this to $X_{+\omega}$. We get

$$1 + R_\omega = T_\omega\,, \qquad i\omega(1 - R_\omega) = i\omega T_\omega + \alpha T_\omega\,.$$

Solving this system we get

$$R_\omega = \frac{-\alpha}{2i\omega + \alpha}\,, \qquad T_\omega = \frac{2i\omega}{2i\omega + \alpha}\,.$$

$X_{-\omega}$ can be treated similarly.

Now let's put together the general solution. Let $-\infty < k < \infty$ and $|k| = \omega$. We expect

$$u(x,t) = \int_{-\infty}^{\infty} C(k) X_k(x) e^{-i\omega t}\, dk + \int_{-\infty}^{\infty} D(k) X_k(x) e^{i\omega t}\, dk.$$

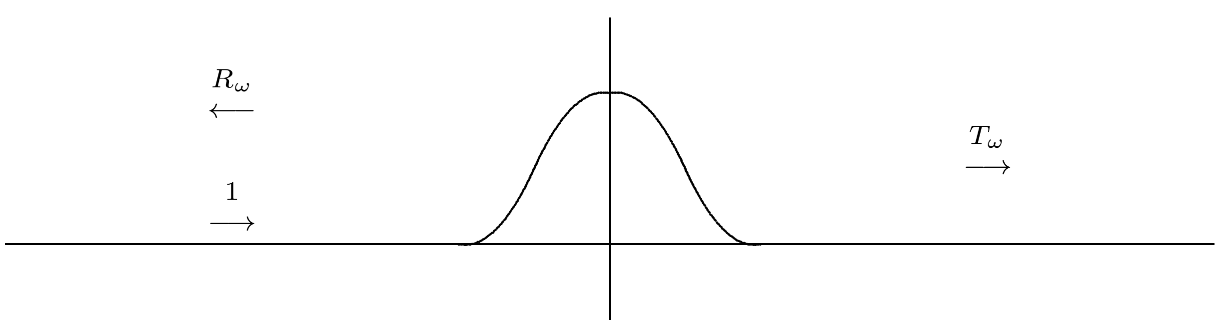
(Note that the sign of k and the sign of the $\pm\omega$ in the exponent are independent.) I'll concentrate on the first term and call it just u. (In an application where we know the solutions are real-valued, the second term must be the complex conjugate of the first. the decomposition of the solution into sines and cosines of ωt is then determined by the phases of the complex numbers $C(k)$.) Notice that for $x > a$,

$$u(x,t) = \int_0^\infty C(\omega) T_\omega e^{i\omega(x-t)}\, d\omega + \int_0^\infty C(-\omega)[e^{-i\omega(x+t)} + R_{-\omega} e^{i\omega(x-t)}]\, d\omega.$$

And for $x < -a$,

$$u(x,t) = \int_0^\infty C(\omega)[e^{i\omega(x-t)} + R_\omega e^{-i\omega(x+t)}\, d\omega + \int_0^\infty C(-\omega) T_{-\omega} e^{-i\omega(x+t)}\, d\omega.$$

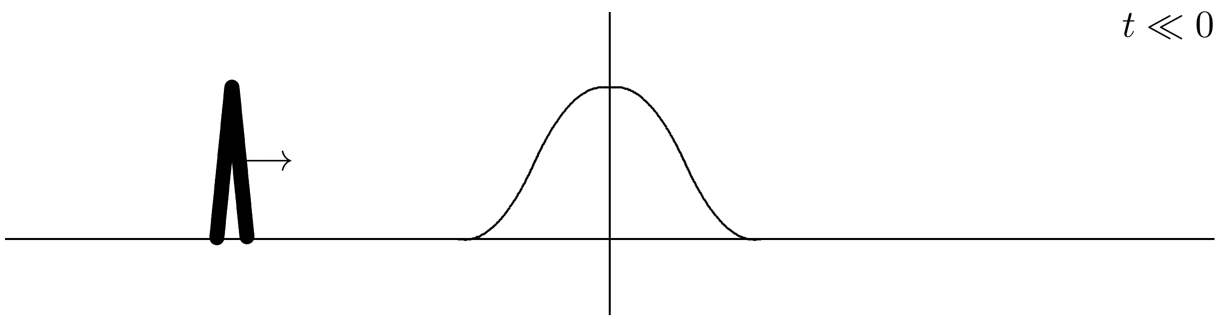
I claim that for each sign of $k \equiv \pm\omega$, each of these solution consists of three (out of four possible) beams. The $C(\omega)$ terms describe a beam coming in from the left with amplitude unity, which is partially reflected back to the left and partially transmitted into the region to the right of the potential hill:

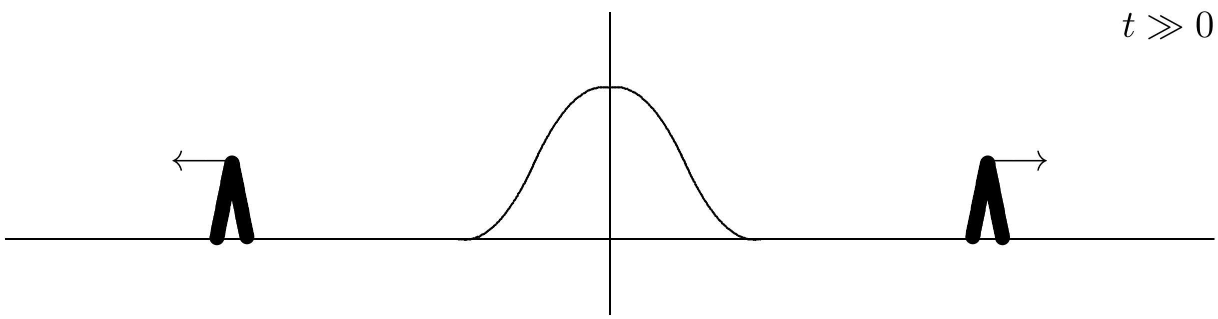


To see that, recall that in the regions $|x| > a$ our PDE reduces to the ordinary wave equation, so the solution must have a form

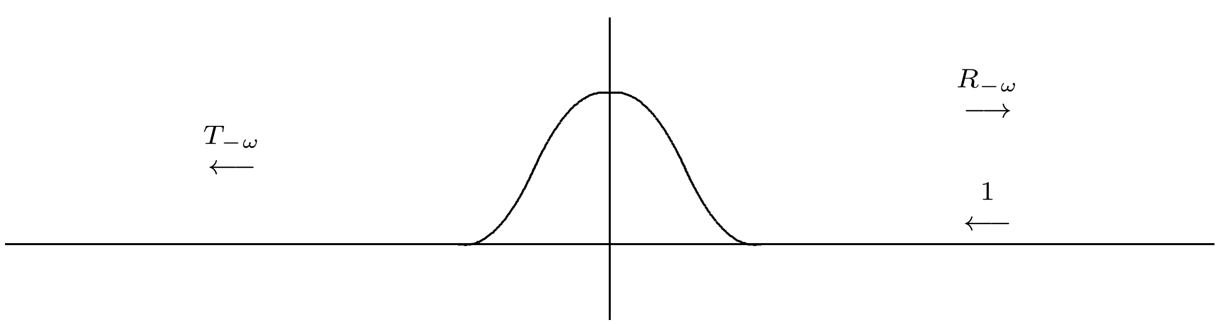
$$u(x,t) = B(x-t) + C(x+t)$$

in each of those regions. In fact, we can see terms of these types in the solutions we've constructed. For sufficiently smooth functions C, the terms will be localized "bumps", which are inverse Fourier transforms. Still considering a solution with $C(-\omega) = 0$, think of running the clock back to large negative t; the R and T bumps will eventually back into the regions where their terms don't apply, so they will disappear. There is only an initial bump far to the left of the potential. As time moves forward, this incident wave packet moves toward the potential at the origin; hits it; and scatters into two outgoing packets.





Similarly, a solution with only $C(-\omega)$ nonzero consists of a bump incident from the right, which scatters into two outgoing packets.



Thus these basic solutions are the right ones for posing a realistic initial-value problem where a localized disturbance propagates and scatters in both directions where the physics tells it to go, with definite *reflection and transmission amplitudes.* A bit of thought shows that $C(k)$ can be computed from initial data exactly like a Fourier transform:

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_k(x)^* f(x)\, dx, \qquad \text{etc.}$$

If you accept that the eigenfunctions must be complete, the only uncertainty is whether their normalization is correct, and that can be checked by demanding that you get the right answer back in the early-time region, where the integrals reduce to Fourier transforms (for sufficiently localized wave pulses).

Returning to the delta example, note that

$$|R_\omega|^2 = \frac{\alpha^2}{4\omega^2 + \alpha^2}\,, \qquad |T_\omega|^2 = \frac{4\omega^2}{4\omega^2 + \alpha^2}\,,$$

so $|R_\omega|^2 + |T_\omega|^2 = 1$. This equation expresses the conservation of the energy or flux in the wave (in quantum mechanics, the probability). It can be proved in the general case, using the constancy of the Wronskian of X_ω and X_ω^*.

Remark: Eigenfunctions with $\lambda < 0$ can occur; they are localized "bound states". If $\lambda = -\kappa^2$ with $\kappa > 0$, then

$$X(x) = \begin{cases} Ae^{\kappa x} & \text{if } x < -a, \\ Be^{-\kappa x} & \text{if } x > a. \end{cases}$$

In the delta case, we find

$$A = B, \qquad -\kappa(A + B) = \alpha A,$$

hence

$$\kappa = -\frac{\alpha}{2}\,.$$

So there are no bound states if α is positive, and exactly one bound state if α is negative. The bound state must be included (in addition to the generalized Fourier transform) in expanding a generic data function in eigenfunctions; it corresponds to solutions that grow or decay exponentially in time but are localized in space.