etry, which are called the parabolic, the hyperbolic and the elliptic geometries. They are also called the Geometries of Euclid, of Lobachevski, and of Riemann." Now Saccheri in his demonstration of Prop. XI. makes, almost in the words of Archimedes, an assumption, introduced by the words 'it is manifest,' which we now call, for convenience, Archimedes' Axiom. In his futile attempts at demonstrating the parallel-postulate, Legendre set forth two theorems, called Legendre's theorems on the angle-sum in a triangle. They are:

1. In a triangle the sum of the three angles can never be greater than two right angles.

2. If in any triangle the sum of the three angles is equal to two right angles, so is it in every triangle.

In addition to assuming the infinity or two-sidedness of the straight, in his proofs of these theorems Legendre uses essentially the Archimedes Axiom. Thence he gets that the angle-sum in a triangle equaling two right angles is equivalent to the parallel-postulate, all of which is really what Saccheri gave a century before him, now just reproduced by Barbarin and Manning, as before by De Tilly. Even Hilbert in his 'Vorlesung ueber Euklidische Geometrie' (winter semester, 1898–99), for a chance to see Dr. von Schafer's Authographie of which I am deeply grateful to Professor Bosworth, gives the following five theorems and then says: "Finally we remark, that it seems as if each of these five theorems could serve precisely as *equivalent of the Parallel Axiom.*" They are

1. The sum of the angles of a triangle is always equal to two right angles.

2. If two parallels are cut by a third straight, then the opposite (corresponding) angles are equal.

3. Two straights, which are parallel to a third, are parallel to one another.

4. Through every point within an angle less than a straight angle, I can always draw straights which cut both sides [not perhaps their prolongations].

5. All points of a straight have from a parallel the same distance.

But since then a wonderful discovery has been made by M. Dehn.

It was known that Euclid's geometry could be built up without the Archimedes axiom. So arises the weighty question: *In such a geometry do the Legendre theorems necessarily hold good?*

In other words: Can we prove the Legendre theorems without making use of the Archimedes axiom?

This is the question which, at the instigation of Hilbert, was taken up by Dehn.

His article was published July 10, 1900 (*Mathematische Annalen*, 53 Band, pp. 404–439).

Dehn was able to demonstrate Legendre's second theorem without using any postulate of continuity, a remarkable advance over Saccheri, Legendre, De Tilly.

But his second result is far more remarkable, namely, that Legendre's first theorem is indemonstrable without the Archimedes axiom.

To prove this startling position, Dehn constructs a new non-Euclidean geometry, which he calls a 'non-Legendrean' geometry, in which through every point an infinity of parallels to any straight can be drawn, yet in which nevertheless the angle sum in every triangle is greater than two right angles.

Thereby is the undemonstrability of the first Legendre theorem without the help of the Archimedes axiom proven.

Dehn then discusses the connection between the three different hypotheses relative to the sum of the angles [the three hypotheses of Saccheri, Barbarin, Manning] and the three different hypotheses relative to the number and existence of parallels.

He reaches the following remarkable propositions:

From the hypothesis that through a given point we can draw an infinity of parallels to a given straight it follows, if we exclude the Archimedes axiom, *not* that the sum of the angles of a triangle is less than two right angles, but on the contrary that this sum may be (*a*) greater than two right angles, (*b*) equal to two right angles.

The first case (*a*) is proven by the existence of the non-Legendrean geometry.

To demonstrate the second case (*b*), Dehn constructs a geometry wherein the parallel-axiom does not hold good, and wherein nevertheless are verified all the theorems of Euclidean geometry; the sum of the angles of a triangle is equal to two right angles, similar triangles exist, the extremities of equal perpendiculars to a straight are all situated on the same straight, etc.

As Dehn states this result: There are non-Archimedian geometries, in which the parallel-axiom is not valid and yet the angle-sum in every triangle is equal to two right angles.

Such a geometry he calls '*semi-Euclidean.*'

Therefore, it follows that none of the theorems, the angle-sum in the triangle is two right angles, the equidistantial is a straight, etc., can be considered as equivalent to the parallel-postulate, and that Euclid in setting up the parallel-postulate hit just the right assumption.

This is a marvelous triumph for Euclid.

Finally Dehn arrives at this surprising theorem:

From the hypothesis that there are no parallels, it follows that the sum of the angles of a triangle is greater than two right angles.

Thus the two non-Euclidean hypotheses about parallels act in a manner absolutely different with regard to the Archimedes Axiom.

The different results obtained may now be tabulated thus:

The angle-sum in the triangle is:	Through a given point we can draw to a straight:		
	No parallel.	One parallel.	An infinity of parallels.
$> 2R$	Elliptic geometry	(Impossible)	Non-Legendrean geometry
$= 2R$	(Impossible)	Euclidean geometry	Semi-Euclidean geometry
$< 2R$	(Impossible)	(Impossible)	Hyperbolic geometry

Riemann, Helmholtz and Sophus Lie founded geometry on an analytical basis in contradistinction to Euclid's pure synthetic method.

They elected to conceive of space as a manifold of numbers. In the Columbus report is an account of the Helmholtz-Lie investigation of the essential characteristics of space by a consideration of the movements possible therein.

This is notably simplified if we suppose given *à priori* the graphic concepts of straight and plane, and admit that movement transforms a straight or a plane into a straight or respectively a plane. Killing determines analytically the three types of projective groups, but the same results are reached in a way geometric and purely elementary by Roberto Bonola in a beautiful little article entitled, 'Determinazione, per via geometrica, dei tre tipi di spazio: Iperbolico, Ellittico, Parabolico (*Rendiconti del Circolo Matematico di Palermo*, Tomo XV., pp. 56–65, April, 1901).

In 1833 was published in London the fourth edition of an extraordinary book (3d Ed., 1830) by T. Perronet Thompson of Queen's College, Cambridge, with the following title:

'Geometry without Axioms.'

" Being an attempt to get rid of Axioms and Postulates; and particularly to establish the theory of parallel lines without recourse to any principle not grounded on previous demonstration.