The number of degrees of freedom of a gauge theory
(Addendum to the discussion on pp. 59–62 of notes)

Let us work in the Fourier picture (Remark, p. 61 of notes). In a general gauge, the Maxwell equation for the vector potential is

$$-\partial^\mu \partial_\mu A^\alpha + \partial^\alpha \partial_\mu A^\mu = J^\alpha. \tag{1}$$

Upon taking Fourier transforms, this becomes

$$k^\mu k_\mu A^\alpha - k^\alpha k_\mu A^\mu = J^\alpha, \tag{1$'$}$$

where $\vec{A}$ and $\vec{J}$ are now functions of the 4-vector $\vec{k}$. (One would normally denote the transforms by a caret ($\hat{A}^\alpha$, etc.), but for convenience I won't.) The field strength tensor is

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \tag{2}$$

or

$$F^{\alpha\beta} = ik^\alpha A^\beta - ik^\beta A^\alpha. \tag{2$'$}$$

The relation between field and current is (factor 4π suppressed)

$$J^\alpha = \partial_\beta F^{\alpha\beta}, \tag{3}$$

or

$$J^\alpha = ik_\mu F^{\alpha\mu}. \tag{3$'$}$$

Of course, (2′) and (3′) imply (1′).

(1′) can be written in matrix form as

$$\vec{J} = M\vec{A}, \tag{4}$$

$$M(\vec{k}) = k^\mu k_\mu \, I - \vec{k} \otimes \tilde{k} = \begin{pmatrix} \vec{k}^2 - k^0 k_0 & -k^0 k_1 & -k^0 k_2 & -k^0 k_3 \\ -k^1 k_0 & \vec{k}^2 - k^1 k_1 & -k^1 k_2 & -k^1 k_3 \\ -k^2 k_0 & -k^2 k_1 & \vec{k}^2 - k^2 k_2 & -k^2 k_3 \\ -k^3 k_0 & -k^3 k_1 & -k^3 k_2 & \vec{k}^2 - k^3 k_3 \end{pmatrix} \tag{5}$$

Consider a generic $\vec{k}$ (not a null vector). Suppose that $\vec{A}$ is a multiple of $\vec{k}$:

$$A^a(\vec{k}) = k^\alpha \chi(\vec{k}). \tag{6$'$}$$

Then it is easy to see that $\vec{A}$ is in the kernel (null space) of $M(\vec{k})$; that is, it yields $\vec{J}(\vec{k}) = 0$. (In fact, by (2′) it even yields a vanishing F.) Conversely, every vector in the kernel is of that form, so the kernel is a one-dimensional subspace. Back in space-time, these observations correspond to the fact that a vector potential of the form

$$\vec{A} = \nabla \chi \tag{6}$$

is "pure gauge". This part of the vector potential obviously cannot be determined from $\vec{J}$ and any initial data by the field equation, since it is entirely at our whim. (Even if the Lorenz gauge condition is imposed, we can still perform a gauge transformation with χ a solution of the scalar wave equation.)

Now recall a fundamental theorem of finite-dimensional linear algebra: *For any linear function, the dimension of the kernel plus the dimension of the range equals the dimension of the domain.* In particular, if the dimension of the domain equals the dimension of the codomain (so that the linear function is represented by a square matrix), then the dimension of the kernel equals the codimension of the range (the number of vectors that must be added to a basis for the range to get a basis for the whole codomain). Thus, in our situation, there must be a one-dimensional set of vectors $\vec{J}$ that are left out of the range of $M(\vec{k})$. Taking the scalar product of $\vec{k}$ with (1′), we see that

$$k_\alpha J^\alpha = 0 \tag{7'}$$

is the necessary (and sufficient) condition for (4) to have a solution, $\vec{A}$. In space-time, this condition is the conservation law,

$$\partial_\alpha J^\alpha = 0. \tag{7}$$

(7′) can be solved to yield

$$\rho = -\frac{\mathbf{k}\cdot\mathbf{J}}{k_0}\,. \tag{8''}$$

In terms of $\vec{A}$, the right-hand side of (8″) cannot contain ${k_0}^2$ (since (1′) is quadratic in $\vec{k}$); that is, the Fourier transform of (8″) is a linear combination of components of the field equation that does not contain second-order time derivatives. In fact, a few more manipulations show that

$$\rho = i\mathbf{k}\cdot\mathbf{E}, \tag{8'}$$

whose transform is

$$\rho = \nabla\cdot\mathbf{E}. \tag{8}$$

That is, the conservation law is essentially equivalent (up to the "swindle" mentioned in the Remark) to the Gauss law, which is a *constraint* on the allowed initial data (including first-order time derivatives) for $\vec{A}$.

Conclusion: At each $\vec{k}$ (hence at each space-time point) there are only *two* independent physical degrees of freedom, not four or even three. One degree of freedom is lost to the gauge ambiguity; another is cut out of the space of candidate solutions by the constraint related to the conservation law. But by Noether's theorem, the conservation law is itself a consequence of the gauge invariance. In the Fourier picture the fact that degrees of freedom are lost in pairs is consequence of the dimension theorem for linear functions.