The Ergosphere

A rotating (Kerr) black hole in the most common (Boyer–Lindquist) coordinate system has a metric of the form

$$ds^2 = -(\text{mess})dt^2 - (\text{mess})dt\,d\phi + (\text{mess})d\phi^2 + (\text{mess})dr^2 + (\text{mess})d\theta^2$$

(see Schutz (11.71) and Penrose–Floyd p. 2). The (mess)s are independent of t and ϕ, so the energy, $-p_t$, and angular momentum, p_ϕ, of a particle are conserved. (Recall that p_0 is normally a *negative* number in our metric signature, since U^0 is positive.)

Suppose the hole were replaced by a spinning flywheel. A particle could hit it and be batted away with more energy than it had coming in. This does not contradict conservation of energy, because there is a nontrivial interaction with the flywheel and the wheel will slow down slightly by recoil. The Penrose process is an analog that allows energy to be extracted from the Kerr black hole.

As in the Brans–Stewart cylinder universe, there is no *global* rotating Lorentz frame. (This is true of any rotational situation in relativity — it has nothing to do with horizons or even with gravity.) The best one can do is to construct a rotating frame that is related to local Lorentz frames by *Galilean* transformations (i.e., leaving the hypersurfaces of constant time fixed).

The model

Here I present a simple model related to the Kerr black hole in somewhat the same way that the uniformly accelerated (Rindler) frame is related to the Schwarzschild black hole. Consider the line element

$$ds^2 = -dt^2 + (dx + V(y)\,dt)^2 + dy^2.$$

(We could add a third spatial dimension, dz, but it adds nothing conceptually so I'll omit it.) That is, the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -1 + V(y)^2 & V(y) & 0 \\ V(y) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the order of the coordinates is t, x, y. Since $g_{\mu\nu}$ is independent of t and x, $-p_t$ and p_x are conserved. Notice that something strange is going to happen when $|V(y)| \geq 1$, because then g_{tt} changes sign.

Consider now the Galilean transformation

$$t = t', \qquad x = x' - V_0 t', \qquad y = y',$$

with inverse

$$t' = t, \qquad x' = x + V_0 t, \qquad y' = y.$$

Then $dx = dx' - V_0\,dt'$ implies

$$ds^2 = -dt'^2 + [dx' + (V(y) - V_0)dt']^2 + dy'^2.$$

In particular, in a region where $V(y) = \text{constant}$, choose $V_0 = V(y)$; then

$$ds^2 = -dt'^2 + dx'^2 + dy'^2$$

— space is flat!

Suppose that $V(y) = 0$ for $y \gg 0$ ("outside"), so the space is flat and the unprimed coordinates are inertial there; and that $V(y) = V_0$ for $y \ll 0$ ("inside"), so the space is flat and the primed coordinates are inertial there. In the Kerr–Boyer–Lindquist situation, r is analogous to y and ϕ is analogous to x. Like the Schwarzschild black hole, the Kerr black hole has a horizon at some small $r \equiv r_+$ (and a singularity inside that), but that does not concern us today. We are interested in a region $r_+ < r < r_0$ called the *ergosphere.* (See Schutz p. 312 for formulas for r_+ and r_0 ; r_0 is where $g_{tt} = 0$, and r_+ is where $g_{rr} = \infty$.) In our model, the ergosphere is the inside region, $-\infty < y \ll 0$.

Basis vectors and basis change matrices

Let us look at the unprimed basis vectors in primed terms; in other words, look at the (natural interior extension of the) inertial frame of an observer in the exterior region from the point of view of an observer "going with the flow" in the interior region. The change-of-basis matrices are

$$\Lambda^\mu{}_{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}} = \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda^{\nu'}{}_\mu = \frac{\partial x^{\nu'}}{\partial x^\mu} = \begin{pmatrix} 1 & 0 & 0 \\ V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Recall that the columns of the second matrix are the basis tangent vectors $\vec{e}_t$, etc., and the rows of the first matrix are the basis one-forms dual to them. The important thing to note is that if $V_0 > 1$, then $\vec{e}_t$, the time-translation vector, is spacelike in the ergosphere! (On the other hand, ∇t, the normal vector to the surfaces of constant t, is still timelike.)

Similarly, in Kerr, $\vec{e}_t$ in the ergosphere leans over and points primarily in the ϕ direction. (In any rotating system in GR, it will lean slightly; this is called the *Lense–Thirring frame-dragging effect*, or *gravitomagnetism*; see Schutz pp. 310–311. But usually it remains timelike. An ergosphere is a region where it leans so far it becomes spacelike.)

Velocity

Let's use Λ to transform the 4-velocity vector of a particle:

$$\vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{v}' = \begin{pmatrix} v_{t'} \\ v_{x'} - V_0 v_{t'} \\ v_{y'} \end{pmatrix}.$$

Now suppose that the spatial velocity is 0 in the unprimed frame; then

$$v_{x'} = V_0 v_{t'}\,.$$

But if $|V_0| > 1$, this equation would say that $\vec{v}$ is spacelike, which is impossible for a physical particle. Conclusion: *A particle inside the ergosphere cannot be motionless as viewed by an observer outside.*

Momentum and geodesic equations

Because the metric is nondiagonal, the canonical momentum is not proportional to the velocity. The Lagrangian for particle motion is

$$L = \tfrac{1}{2}[(V(y)^2 - 1)\dot{t}^2 + 2V(y)\dot{t}\dot{x} + \dot{x}^2 + \dot{y}^2].$$

Therefore,

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y}, \qquad \frac{dp_y}{dt} = \frac{\partial L}{\partial y} = VV'\dot{t}^2 + V'\dot{t}\dot{x} = V'\dot{t}p_x\,,$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = V\dot{t} + \dot{x} = \gamma(V + v), \qquad \frac{dp_x}{dt} = \frac{\partial L}{\partial x} = 0,$$

$$p_t = \frac{\partial L}{\partial \dot{t}} = (V^2 - 1)\dot{t} + V\dot{x}, \qquad \frac{dp_t}{dt} = \frac{\partial L}{\partial t} = 0.$$

We can further reduce

$$p_t = -\dot{t} + V(V\dot{t} + \dot{x}) = -\dot{t} + Vp_x\,.$$

Thus $\dot{t} = -p_t + Vp_x$, and we can write

$$\ddot{y} = \frac{dp_y}{dt} = V'(y)p_x[V(y)p_x - p_t],$$

which is the only nontrivial equation of motion. (Recall that p_x and p_t are constants.) Note that $p_y =$ constant whenever the particle is in either of the asymptotic regions.

Energy extraction

Consider a particle originating outside with

$$p_t = p_0 < 0, \qquad p_x = 0, \qquad p_y = -k < 0.$$

Since the inertial frame outside is the unprimed one, $p_t < 0$ is required for a physical particle. The condition $p_y < 0$ assures that the particle will fall in. In the primed frame these momentum components are the same:

$$\vec{p}\,' = (p_0, 0, p_y) \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (p_0, 0, p_y) = \vec{p}.$$

In general, p_y will change with time, but p_x and p_t are conserved. Let's say that $p_y = -K < 0$ when the particle is inside.

Now suppose that after it enters the ergosphere, the particle decays:

$$\vec{p} = \vec{p}_1 + \vec{p}_2\,.$$

(This is a vectorial equation, hence valid in either frame.) Suppose also that

$$p'_{2y} = +K > 0, \qquad \text{so} \quad p'_{1y} = -2K < 0.$$

Thus particle 1 gets swallowed by the "black hole", but particle 2 reemerges. In exterior coordinates

$$\vec{p}_1 = (p'_{1t}, p'_{1x}, p'_{1y}) \begin{pmatrix} 1 & 0 & 0 \\ +V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (p'_{1t} + V_0 p'_{1x}, p'_{1x}, p'_{1y}).$$

Note that $p_{1t} = p'_{1t} + V_0 p'_{1x}$ can be positive if (and only if) $|V_0| > 1$ (since $|p'_{1x}| < |p'_{1t}|$). (This is not a physical contradiction, since the unprimed frame is not inertial at points inside.) Now do the same calculation for the escaping particle:

$$\vec{p}_2 = (p'_{2t}, p'_{2x}, p'_{2y}) \begin{pmatrix} 1 & 0 & 0 \\ +V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (p'_{2t} + V_0 p'_{2x}, p'_{2x}, p'_{2y}).$$

Here $p_{2t} = p'_{2t} + V_0 p'_{2x}$ can be less than p_0 (i.e., $|p_{2t}| > |p_0|$) if and only if $|V_0| > 1$. But p_{2t} is conserved, so it is the physical momentum of particle 2 after it emerges from the ergosphere.

Conclusion: *Mechanical energy has been extracted from the "black hole". Total energy is conserved, because the energy of the hole has been reduced by the amount $|p_{1t}| = |p_0|$, the negative energy carried in by particle 1.* In the true rotating-black-hole case, the angular momentum is reduced similarly (corresponding to the conserved quantity p_x in the model).

There is an analogue of the Penrose process for waves, called *superradiance*: For waves of certain values of angular momentum (angular quantum number or separation constant), the scattered wave amplitude exceeds the incident amplitude. In quantum field theory this effect leads to production of particle-antiparticle pairs by the rotating black hole, in analogy to something called the *Klein paradox* for quantum particles in a strong electric field. (This is different from *Dicke superradiance* in atomic physics.)