Math. 489GR (Fulling) 12 December 2005

Final Examination – Solutions

1. *(40 pts.)* Consider a three-dimensional space-time with line element (metric)

$$ds^2 = -dt^2 + [dx + V(y)\,dt]^2 + dy^2$$

where $V(y) = 0$ for $y > Y$ ("outside") and $V(y) = V_0 > 1$ for $y < -Y$ ("inside"). (Y and V_0 are large constants.)

(a) Justify this claim (paraphrased from Penrose and Floyd):

> No physical particle inside (its velocity having to be bounded by the local light velocity) can appear stationary to an observer outside.

As shown in class, the coordinate transformation from the frame that is inertial inside to the one that is inertial outside is

$$t = t', \quad x = x' - V_0 t', \quad y = y'; \qquad \frac{\partial x^\mu}{\partial x^{\nu'}} = \begin{pmatrix} 1 & 0 & 0 \\ -V_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying this matrix to a four-velocity vector U' we get

$$U = \begin{pmatrix} U_{t'} \\ U_{x'} - V_0 U_{t'} \\ U_{y'} \end{pmatrix}.$$

In particular, if $0 = U_x = U_{x'} - V_0 U_{t'}$, then $U_{x'}/U_{t'} = V_0 > 1$; this implies that U is spacelike, which is impossible.

(b) Write down the relations between the components of the four-velocity, U^α, and the components of the (covariant) four-momentum, p_α, of a particle of mass $m = 1$ moving in this space-time. Comment on why the effects of an off-diagonal metric element, $g_{0j} \neq 0$, are called "gravitomagnetic forces". (It may be helpful — though not strictly necessary — to recall the "canonical" definition of momentum as

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha},$$

where L is the Lagrangian giving rise to the particle's equation of motion.)

We have $L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, so

$$p_\alpha = g_{\alpha\nu} \dot{x}^\nu \equiv g_{\alpha\nu} U^\nu$$

(which is just a roundabout way of saying that $p^\nu = U^\nu$ when $m = 1$). In the present case

$$p_t = [-1 + V(y)^2] U^t + V(y) U^x, \quad p_x = V(y) U^t + U^x, \quad p_y = U^y.$$

The effect of the off-diagonal metric component is to mix the time and space components of the velocity in the formulas for momenta, and vice versa. This result is somewhat reminiscent of the nontrivial relation between velocity and momentum when an electromagnetic vector potential is present: $\mathbf{p} = \mathbf{U} + e\mathbf{A}$. (Like magnetism, the resulting force is proportional to a component of velocity in a transverse direction. This is most easily seen from the Lagrangian formulation:

$$L = \tfrac{1}{2}[(V^2 - 1)\dot{t}^2 + 2V\dot{t}\dot{x} + \dot{x}^2 + \dot{y}^2];$$

$$\frac{dp_y}{d\lambda} = \frac{\partial L}{\partial y} = VV'\dot{t}^2 + V'\dot{t}\dot{x},$$

which if the particle is at rest reduces to

$$\frac{dp_y}{dt} = VV' + V'\frac{dx}{dt}.$$

The last term is the one attributable to the nonzero g_{tx} .)

2. *(40 pts.)* Schutz's discussion of the dynamics of a Robertson–Walker (Friedmann) universe filled with a perfect fluid begins:

> First, consider $T^{\mu\nu}{}_{;\nu} = 0$. Since there is spatial homogeneity, only the time component of this equation is nontrivial. It is easy to show that it gives
>
> $$\frac{d}{dt}(\rho R^3) = -p\,\frac{d}{dt}(R^3)\,, \qquad (12.21)$$
>
> where $R(t)$ is the cosmological expansion factor.

Please show it. *Hints:* Recall the identity for the summed Christoffel symbols,

$$\Gamma^{\alpha}_{\mu\alpha} = (\sqrt{-g})_{,\mu}/\sqrt{-g}\,. \qquad (6.40)$$

The notation is

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\,d\Omega^2\right], \qquad (12.14)$$

$$T^{tt} = \rho\,, \qquad T^{jk} = p\,g^{jk} \quad (j,k = 1,2,3).$$

You don't need to use the explicit form of $d\Omega$ — just that it is independent of t .

By definition of covariant differentiation of a tensor,

$$\begin{aligned} 0 &= T^{t\nu}{}_{;\nu} \\ &= T^{t\nu}{}_{,\nu} + \Gamma^{\nu}_{\alpha\nu}T^{t\alpha} + \Gamma^{t}_{\alpha\nu}T^{\alpha\nu} \\ &= T^{tt}{}_{,t} + \Gamma^{\nu}_{t\nu}T_{tt} + \Gamma^{t}_{tt}T^{tt} + \sum_{j=1}^{3}\Gamma^{t}_{jj}T^{jj} \end{aligned} \qquad (\ddagger)$$

(since T is diagonal). But now

$$\begin{aligned}\Gamma^{\nu}_{t\nu} &= (\sqrt{-g})_t/\sqrt{-g} \\ &= \frac{\partial}{\partial t}\ln(\sqrt{-g}) = \frac{1}{2}\frac{\partial}{\partial t}\ln(-g) \\ &= \frac{1}{2}\frac{\partial}{\partial t}\ln[R^6(1-kr)^{-1}r^4\cdot(\text{function of angles})] \\ &= \frac{1}{2}\cdot 6R^5R'\cdot R^{-6} = 3\,\frac{R'}{R}\,;\end{aligned}$$

$$\begin{aligned}\Gamma^t_{tt} &= \frac{1}{2}\,g^{tt}(g_{tt,t}+g_{tt,t}-g_{tt,t}) \\ &= -\frac{1}{2}\,g_{tt,t} = 0\,;\end{aligned}$$

and if j is a spatial index,

$$\begin{aligned}\Gamma^t_{jj} &= \frac{1}{2}\,g^{tt}(g_{tj,j}+g_{jt,j}-g_{jj,t}) \\ &= \frac{1}{2}\,g_{jj,t} = \frac{R'}{R}\,g_{jj}\,,\end{aligned}$$

so that

$$\sum_{j=1}^{3}\Gamma^t_{jj}g^{jj} = 3\,\frac{R'}{R}\,.$$

Thus the equation (‡) becomes

$$\begin{aligned}0 &= \frac{d\rho}{dt} + 3\,\frac{R'}{R}\,\rho + 3\,\frac{R'}{R}\,p \\ &= \frac{1}{R^3}\left[\frac{d}{dt}(\rho R^3) + p\,\frac{d}{dt}(R^3)\right],\end{aligned}$$

which immediately implies (12.21).

3. *(40 pts.)* Recall the definition of a *covariant derivative* (connection) for a general type of field: The field $\phi(x)$ takes its value in a vector space (fiber) of dimension r, called F_x, and the derivative is $\nabla_\mu\phi(x) = \partial_\mu\phi(x) + w_\mu(x)\phi(x)$, where $w_\mu(x)$ (for each μ and x) is an $r\times r$ matrix. (More precisely, $w_\mu(x)$ is a linear operator, which becomes a matrix when a basis is chosen for F_x. Without such a basis choice, the term $\partial_\mu\phi(x)$ would be ill-defined.)

   (a) Let Ψ be a field taking its value (at each point x) in the *dual space* F^*_x. That is, $\Psi(\phi)$ is a scalar function, and its value at x depends linearly on the vector $\phi(x)$. Find the natural definition of the covariant derivative of Ψ by requiring that covariant derivatives of all types obey the product rule (and the covariant derivative of a scalar is the ordinary derivative).

We can regard $\Psi(\phi)$ as a matrix product $\Psi\phi$ (row vector times column vector), where Ψ is represented by its components with respect to the dual basis for each F^*_x. Then

$$\partial_\mu(\Psi\phi) = \nabla_\mu(\Psi\phi) = (\nabla_\mu\Psi)\phi + \Psi(\nabla_\mu\phi) = (\nabla_\mu\Psi)\phi + \Psi\partial_\mu\phi + \Psi w_\mu\phi\,.$$

But we must also have

$$\partial_\mu(\Psi\phi) = (\partial_\mu\Psi)\phi + \Psi(\partial_\mu\phi)\,.$$

Therefore (comparing the two equations and making some cancellations),

$$\nabla_\mu\Psi = \partial_\mu\Psi - \Psi w_\mu\,. \qquad (*)$$

(This argument roughly parallels the one for one-forms on pp. 138–139 of Schutz.)

(b) Show that when the fields ϕ are ordinary contravariant vector fields on space-time, your definition in (a) becomes the standard prescription for differentiating a one-form.

In this case the matrix indices of w are plain old space-time indices, and w_μ becomes the 3-index object $\Gamma^\beta_{\alpha\mu}$. Then $(*)$ becomes

$$\nabla_\mu p_\alpha = \partial_\mu p_\alpha - p_\beta\Gamma^\beta_{\alpha\mu}\,,$$

which is Schutz's (5.62) (up to renaming of indices).

4. *(40 pts.)* Rate each of these newspaper headlines as *implausible*, *plausible in theory* (perhaps actual in some future century), or *highly plausible*. Write a sentence or two to explain each judgment.

(a) Giant, pulsating, spherically symmetric star discovered, emitting vast amounts of gravitational radiation

Implausible. Spherically symmetric configurations do not emit gravitational waves (Birkhoff's theorem).

(b) NASA launches program to produce electrical power by extracting energy from rotating black holes

Plausible in theory. Penrose and Floyd wrote, "We now [demonstrate] the extraction of rotational energy from a black hole, [which] should, in general, be comparable with its total mass-energy."

(c) Anomalous perihelion precession of orbits around neutron stars explained by adding torsion to equation of motion (*Hint:* The geodesic equation is $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$. How does torsion enter into it?)

Implausible. Torsion is, by definition, the antisymmetric part of the Christoffel symbols. Since $\dot{x}^\beta\dot{x}^\gamma$ is symmetric, the torsion cancels out of the geodesic equation. Adding torsion to the connection does not change the equation of motion at all! (Disclaimer: If a connection with torsion is metric-compatible, its symmetric part is not equal to the usual Levi-Civita expression. In that sense introducing torsion would indirectly change the geodesic equation.)

(d) Cosmological constant measured by observing accelerated expansion of universe

Highly plausible. Accelerated expansion is already observed, and a cosmological constant is one proposed explanation. (We are still some way from convincing "measurement", however.)

5. *(40 pts.)*

(a) List all the basic index symmetries of the Riemann tensor, $R_{\alpha\beta\gamma\delta}$ (*not* including those that involve derivatives, such as the "second Bianchi identity").

Antisymmetry: $R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta}$
Cyclic identity: $R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$
Pair symmetry: $R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta}$
The other antisymmetry: $R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$

There are many other identities obtainable from the cyclic identity together with the others, but they are not "basic".

(b) Use those symmetries to show that there are exactly *three* independent quantities that can be formed by contracting in pairs all the indices in a product of two Riemann tensors, $R_{\alpha\beta\gamma\delta}R_{\mu\nu\rho\tau}$. (One of these is $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, called "the square of the Riemann tensor". Another is the square of the Ricci scalar, $R^2 = R^{\alpha\beta}{}_{\alpha\beta}R^{\gamma\delta}{}_{\gamma\delta}$. Your task is to find a third one and to show that all the other possibilities, such as $R^{\alpha\beta\gamma\delta}R_{\delta\beta\alpha\gamma}$, can be written in terms of those three. The hardest part is getting from four down to three.)

First suppose that two indices on the first factor are contracted (with each other). We know that the only independent object that can be obtained in this way is the Ricci tensor, $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$. Now there are only two indices left on that tensor, so a pair on the other factor must be contracted also. So the only possibilities are the contractions of $R_{\alpha\beta}R_{\gamma\delta}$. Here we can contract indices on the same factor, getting R^2, or on different factors, getting "the square of the Ricci tensor", $R^{\alpha\beta}R_{\alpha\beta}$.

Second consider the other possibility, that $\mu\nu\rho\tau$ is some permutation of $\alpha\beta\gamma\delta$. A priori there are $4! = 24$ cases to consider. However, because of antisymmetry and pair symmetry, we can assume that $\mu = \alpha$. Using antisymmetry again, we can assume that $\rho\tau$ are in alphabetical order. So there are three cases left: $R^{\alpha\beta\gamma\delta}$ contracted with $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\gamma\beta\delta}$, or $R_{\alpha\delta\beta\gamma}$. Using cyclic symmetry and antisymmetry, we see that these three objects are linearly dependent:

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta}R_{\alpha\gamma\beta\delta} - R^{\alpha\beta\gamma\delta}R_{\alpha\delta\beta\gamma}\,. \tag{\dagger}$$

Now comes the tricky part — in the last term of (†) rename γ as δ and vice versa:

$$-R^{\alpha\beta\gamma\delta}R_{\alpha\delta\beta\gamma} = -R^{\alpha\beta\delta\gamma}R_{\alpha\gamma\beta\delta} = R^{\alpha\beta\gamma\delta}R_{\alpha\gamma\beta\delta}\,.$$

Thus two seemingly independent cases with the indices in scrambled order are actually equal to each other and (because of (†)) equal to half the (unscrambled) square of the Riemann tensor.