Math. 489GR (Fulling) 15 Dec 2009

Final Examination – Solutions

1. *(Multiple choice – each 5 pts.)*

   (a) What is the curvature scalar of the metric $ds^2 = -dt^2 + x^2\,dx^2$?

   (A) A positive constant, because this is one of the many known forms of the de Sitter metric.

   (B) Zero, by an obvious coordinate transformation.

   (C) $\Gamma^\alpha_{\beta\nu,\mu} + \Gamma^\alpha_{\beta\mu,\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$, which is too complicated to calculate by hand in this case.

   (D) [none of these]

B (Let $dy = x\,dx$ ($y = \frac{1}{2}x^2$); then $ds^2 = -dt^2 + dy^2$, so the space-time is (at least a piece of) 2-dimensional Minkowski space.)

   (b) Wolfgang Rindler (of U.T. Dallas) is well known for

   (A) discovering the acceleration of the universe attributed to “dark energy”.

   (B) defining covariant derivatives for sections of an arbitrary field bundle.

   (C) studying hyperbolic coordinates in flat space and their analogy with Schwarzschild coordinates around a black hole.

   (D) solving the Einstein equations for a static star with an arbitrary equation of state.

C

   (c) Gauge symmetries and the resulting conservation laws reduce the number of degrees of freedom

   (A) from 4 to 2 for all massless fields.

   (B) from 4 to 3 for electromagnetism, from 16 to 2 for gravity.

   (C) from 3 to 2 for electromagnetism, from 10 to 6 for gravity.

   (D) from 4 to 2 for electromagnetism, from 10 to 2 for gravity.

D

   (d) The equation

$$R_{\alpha\beta\mu\nu} = \tfrac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$$

   (A) is a useful formula for the Riemann tensor of any metric.

   (B) is valid for spherically symmetric metrics only.

   (C) is valid in the weak-field limit only.

   (D) is valid for cosmological (Robertson–Walker) metrics only.

C (The complete Riemann tensor is quadratic in the Christoffel symbols and hence nonlinear in the metric.)

   (e) An *equation of state* is essential for obtaining a definite solution for

   (A) cosmological expansion.

   (B) a dense star in equilibrium.

   (C) [both]

   (D) [neither]

C

(f) Analogies between electromagnetism and gravity (general relativity) involve all of these EXCEPT
   (A) linearity of the field equations.
   (B) gauge transformations.
   (C) conservation laws for the sources.
   (D) long-distance forces.

A

(g) A *uniformly accelerated* observer experiences all of these EXCEPT
   (A) three-acceleration a that is time-independent in an inertial observer's rest frame.
   (B) time dilation relative to an inertial observer.
   (C) a hyperbolic trajectory in space-time.
   (D) a time-translation group that is a "boost" subgroup of the Lorentz group of an inertial observer.

A (Acceleration is constant in the *accelerated* frame.)

(h) The *contracted Bianchi identity* for the Riemann tensor is equivalent to
   (A) the antisymmetry of the Christoffel symbols in their lower indices.
   (B) the relation between distance and redshift in a Friedmann (Robertson–Walker) universe.
   (C) the impossibility of gravitational radiation from a spherical source.
   (D) the energy-momentum conservation law for the matter source.

D

(i) *(Bonus question)* A recent conference at Texas A&M commemorated a research achievement with an Aggie connection, which won the Dannie Heineman Prize. It was
   (A) the correct handling of gauge (coordinate) freedom in the dynamics of general relativity (Arnowitt, Deser, and Misner).
   (B) discovering the acceleration of the universe attributed to "dark energy" (Suntzeff, Riess, and Permutter).
   (C) finding black hole solutions in dimensions greater than 4 (Pope, Gibbons, and Lü).
   (D) superb pedagogy in teaching general relativity to undergraduates (Fulling, Schutz, and Yasskin).

A

2. *(50 pts.)*

(a) List all the independent index symmetries of the Riemann tensor. (For example, $R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$ is an index symmetry, albeit one that is not strictly independent of the others you should list. As this example indicates, you should consider the curvature tensor for the standard Levi-Civita connection and write it in fully covariant form (all indices *down*).)

(b) Show that the Ricci tensor, $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$, is the only independent rank-2 tensor that can be formed from the Riemann tensor; that is, contracting on any other pair of indices (raising and lowering indices with the metric tensor when necessary) does not lead to anything new.

(c) Show that the only independent scalars that can be formed from the product of two Riemann tensors are

$$R^2, \quad R^{\alpha\beta}R_{\alpha\beta}\,, \quad \text{and} \quad R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}.$$

(In particular, index permutations such as $R^{\alpha\beta\gamma\delta}R_{\delta\gamma\alpha\beta}$ do not give anything new.)

[See Fall 2005 final, Qu. 5.]

3. *(Essay – 40 pts.)* The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2.$$

It has mathematical singularities at $r = 2M$ and $r = 0$. Describe (physically and geometrically) what actually happens at those places.

[The main point is to distinguish between the true curvature singularity at $r = 0$ and the *horizon*, marked by a coordinate singularity, at $r = 2M$.]

4. *(30 pts.)* Recall that a non-Abelian gauge theory (in flat space) involves a covariant derivative

$$\nabla_\mu\phi = \partial_\mu\phi + w_\mu\phi,$$

where each component of $w(x)_\mu$ is a matrix, and the value $\phi(x)$ at each point x is a vector belonging to a vector space F_x .

(a) Let $\Psi(x)$ be a section of the *dual bundle*, which simply means that its value $\Psi(x)$ at a point x is a linear functional mapping F_x into the real numbers. ("One-forms" are a special case.) Explain why the only sensible definition of the covariant derivative of Ψ is

$$\nabla_\mu\Psi = \partial_\mu\Psi - \Psi w_\mu.$$

Explain clearly what the last term *means*. (How do I multiply something with a matrix on the *right*?)

[I'm following the solution of Sean Grant. I did essentially the same thing twice in class, once for one-forms and once in general.]

Require the Leibniz rule,

$$\nabla_\mu(\Psi\phi) = \nabla_\mu\Psi\,\phi + \Psi\nabla_\mu\phi = \nabla_\mu\Psi\,\phi + \Psi\partial_\mu\phi + \Psi w_\mu\phi.$$

Because $\Psi\phi$ is a scalar, this must match

$$\nabla_\mu(\Psi\phi) = \partial_\mu\Psi\,\phi + \Psi\partial_\mu\phi.$$

That is possible if and only if

$$\nabla_\mu\Psi + \Psi w_\mu = \partial_\mu\Psi.$$

Multiplication by a matrix on the right is meaningful and natural when the object on the left is a row vector, representing a linear functional. (More abstractly, Ψw_μ is a linear functional calculated by applying the operator w_μ to the input vector (from F_x) and then applying Ψ to the result.)

(b) Let $M(x)$ be a field whose value at x is the matrix of a linear operator mapping F_x into itself. Show (by the same sort of argument as in (a)) that the covariant derivative of M is

$$\nabla_\mu M = \partial_\mu M + [w_\mu, M].$$

(The last term is a matrix commutator, $[A, B] \equiv AB - BA$.)

Method 1: Argue just as above, except use the fact that $M\phi$ is a vector (so we know how to write ∇ of it) instead of the fact that $\Psi\phi$ is a scalar.

Method 2: "Saturate" M with an arbitrary vector on the right and an arbitrary functional on the left. The result is a scalar, and you know how to apply ∇ to the vector and the functional. So the same sort of calculation yields ∇M.

5. *(40 pts.)* When solving in lecture the Friedmann–RW cosmological equations, we assumed that the derivative of the cosmological scale factor was never zero: $\dot{a}(t) \neq 0$. If we assume to the contrary that a is a constant, then the two equations become

$$\frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho, \tag{1}$$

$$\frac{k}{3a^2} - \frac{\Lambda}{3} = -\frac{8\pi G}{3}\,p. \tag{2}$$

Investigate the existence of solutions under various assumptions about k, Λ, ρ, and p (in particular, their signs — zero, positive, or negative). One of the solutions you should find is the one called "Einstein's greatest blunder".

[I'm following in part the solutions of Sean Grant (points 1–4) and Siying Peng (point 5).]

0. First suppose $k = 0$ and $\Lambda = 0$. Then necessarily $\rho = 0$ and $p = 0$. This is a valid solution: it is empty, flat space-time.
1. If $k = 0$, then $p = -\rho$ and Λ has the same sign as p. This is the "pure dark energy" solution; the matter stress tensor is indistinguishable from a cosmological-constant term and precisely compensates the one that is there.
2. If $\Lambda = 0$, then $p = -\frac{1}{3}\rho$ and k has the same sign as ρ (presumably positive). This solution has no physical interpretation that I know of. (It is not "radiation-dominated", which would be $p = +\frac{1}{3}\rho$.)
3. If $p = 0$, then $\Lambda = 4\pi G\rho$ and k also has the same sign as ρ (presumably positive). This is "Einstein's blunder" — see Ex. 12.20 and p. 358.
4. In order to get a solution with both ρ and p positive, we need Λ and k of the same sign and

$$\frac{|\Lambda|}{3} < \frac{|k|}{a^2} < |\Lambda|.$$

5. In the cases where $k \neq 0$, you can solve for a. For example, in the radiation-dominated case, $p = \frac{1}{3}\rho > 0$, we have

$$a^2 = \frac{3k}{2\Lambda}\,, \qquad \rho = \frac{\Lambda}{8\pi G}\,.$$