Tensor Calculus in Hyperbolic Coordinates

We shall do for hyperbolic coordinates in two-dimensional space-time all the things that Schutz does for polar coodinates in two-dimensional Euclidean space.

The coordinate transformation

[Put drawing here.]

Introduce the coordinates (τ, σ) by

$$t = \sigma \sinh \tau,$$
$$x = \sigma \cosh \tau.$$

Then

$$\frac{t}{x} = \tanh \tau, \qquad -t^2 + x^2 = \sigma^2. \tag{1}$$

The curve $\tau =$ const. is a straight line through the origin. The curve $\sigma =$ const. is a hyperbola. As σ varies from 0 to ∞ and τ varies from $-\infty$ to ∞ (endpoints not included), the region

$$x > 0, \qquad -x < t < x$$

is covered one-to-one. In some ways σ is analogous to r and τ is analogous to θ, but geometrically there are some important differences.

From Exercises 2.21 and 2.19 we recognize that the hyperbola $\sigma =$ const. is the path of a uniformly accelerated body with acceleration $1/\sigma$. (The parameter τ is not the proper time but is proportional to it with a scaling that depends on σ.)

From Exercises 1.18 and 1.19 we see that translation in τ (moving the points (τ, σ) to the points $(\tau + \tau_0, \sigma)$) is a Lorentz transformation (with velocity parameter τ_0).

Let unprimed indices refer to the inertial coordinates (t, x) and primed indices refer to the hyperbolic coordinates. The equations of small increments are

$$\begin{aligned} \Delta t &= \frac{\partial t}{\partial \tau}\,\Delta\tau + \frac{\partial t}{\partial \sigma}\,\Delta\sigma = \sigma \cosh \tau\,\Delta\tau + \sinh \tau\,\Delta\sigma, \\ \Delta x &= \sigma \sinh \tau\,\Delta\tau + \cosh \tau\,\Delta\sigma. \end{aligned} \tag{2}$$

Therefore, the matrix of transformation of (tangent or contravariant) vectors is

$$V^\beta = \Lambda^\beta{}_{\alpha'} V^{\alpha'}, \qquad \Lambda^\beta{}_{\alpha'} = \begin{pmatrix} \sigma \cosh \tau & \sinh \tau \\ \sigma \sinh \tau & \cosh \tau \end{pmatrix}. \tag{3}$$

Inverting this matrix, we have

$$V^{\alpha'} = \Lambda^{\alpha'}{}_\beta V^\beta, \qquad \Lambda^{\alpha'}{}_\beta = \begin{pmatrix} \frac{1}{\sigma}\cosh \tau & -\frac{1}{\sigma}\sinh \tau \\ -\sinh \tau & \cosh \tau \end{pmatrix}. \tag{4}$$

(Alternatively, you could find from (1) the formula for the increments $(\Delta\tau, \Delta\sigma)$ in terms of $(\Delta t, \Delta x)$. But in that case the coefficients would initially come out in terms of the inertial coordinates, not the hyperbolic ones. These formulas would be analogous to (5.4), while (4) is an instance of (5.8–9).)

If you have the old edition of Schutz, be warned that the material on p. 128 has been greatly improved in the new edition, where it appears on pp. 119–120.

Basis vectors and basis one-forms

Following p. 122 (new edition) we write the transformation of basis vectors

$$\vec{e}_{\alpha'} = \Lambda^{\beta}{}_{\alpha'}\vec{e}_{\beta}\,,$$

$$\begin{aligned}\vec{e}_{\tau} &= \sigma\cosh\tau\,\vec{e}_t + \sigma\sinh\tau\,\vec{e}_x\,,\\ \vec{e}_{\sigma} &= \sinh\tau\,\vec{e}_t + \cosh\tau\,\vec{e}_x\,;\end{aligned} \tag{5}$$

and the transformation of basis covectors

$$\tilde{E}^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\tilde{E}^{\beta}\,,$$

which is now written in a new way convenient for coordinate systems,

$$\begin{aligned}\tilde{d}\tau &= \frac{1}{\sigma}\cosh\tau\,\tilde{d}t - \frac{1}{\sigma}\sinh\tau\,\tilde{d}x\,,\\ \tilde{d}\sigma &= -\sinh\tau\,\tilde{d}t + \cosh\tau\,\tilde{d}x\,.\end{aligned} \tag{6}$$

To check that the notaion is consistent, note that (because our two Λ matrices are inverses of each other)

$$\tilde{d}\xi^{\alpha'}(\vec{e}_{\beta'}) = \delta^{\alpha'}_{\beta'} \equiv \tilde{E}^{\alpha'}(\vec{e}_{\beta'}).$$

Note that equations (6) agree with the "classical" formulas for the differentials of the curvilinear coordinates as scalar functions on the plane; it follows that, for example, $\tilde{d}\tau(\vec{v})$ is (to first order) the change in τ under a displacement from $\vec{x}$ to $\vec{x}+\vec{v}$. Note also that the analog of (6) in the reverse direction is simply (2) with Δ replaced by $\tilde{d}$.

The metric tensor

Method 1: By definitions (see (5.30))

$$g_{\alpha'\beta'} = \mathsf{g}(\vec{e}_{\alpha'},\vec{e}_{\beta'}) = \vec{e}_{\alpha'}\cdot\vec{e}_{\beta'}\,.$$

So

$$g_{\tau\tau} = -\sigma^2, \qquad g_{\sigma\sigma} = 1, \qquad g_{\tau\sigma} = g_{\sigma\tau} = 0.$$

These facts are written together as

$$ds^2 = -\sigma^2\,d\tau^2 + d\sigma^2,$$

or

$$\mathsf{g} \overset{\mathcal{O}'}{\to} \begin{pmatrix} -\sigma^2 & 0\\ 0 & 1\end{pmatrix}.$$

The inverse matrix, $\{g^{\alpha'\beta'}\}$, is

$$\begin{pmatrix} -\frac{1}{\sigma^2} & 0\\ 0 & 1\end{pmatrix}.$$

Method 2: In inertial coordinates

$$\mathsf{g} \overset{\mathcal{O}}{\to} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now use the $\binom{0}{2}$ tensor transformation law

$$g_{\alpha'\beta'} = \Lambda^{\gamma}{}_{\alpha'} \Lambda^{\delta}{}_{\beta'} g_{\gamma\delta} \,,$$

which in matrix notation is

$$\begin{pmatrix} g_{\tau\tau} & g_{\tau\sigma} \\ g_{\sigma\tau} & g_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} \Lambda^{t}{}_{\tau} & \Lambda^{t}{}_{\sigma} \\ \Lambda^{x}{}_{\tau} & \Lambda^{x}{}_{\sigma} \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda^{t}{}_{\tau} & \Lambda^{t}{}_{\sigma} \\ \Lambda^{x}{}_{\tau} & \Lambda^{x}{}_{\sigma}, \end{pmatrix}$$

which, with (3), gives the result.

This calculation, while conceptually simple, is cumbersome and subject to error in the index conventions. Fortunately, there is a streamlined, almost automatic, version of it:

Method 3: In the equation $ds^2 = -dt^2 + dx^2$, write out the terms via (2) and simplify, treating the differentials as if they were numbers:

$$\begin{aligned} dx^2 &= -(\sigma \cosh\tau \, d\tau + \sinh\tau \, d\sigma)^2 + (\sigma \sinh\tau \, d\tau + \cosh\tau \, d\sigma.)^2 \\ &= -\sigma^2 \, d\tau^2 + d\sigma^2. \end{aligned}$$

Christoffel symbols

A generic vector field can be written

$$\vec{v} = v^{\alpha'} \vec{e}_{\alpha'} \,.$$

If we want to calculate the derivative of $\vec{v}$ with respect to τ, say, we must take into account that the basis vectors $\{\vec{e}_{\alpha'}\}$ depend on τ. Therefore, the formula for such a derivative in terms of components and coordinates contains extra terms, with coefficients called *Christoffel symbols*. [See ($*$) and the next equation in the old notes, or (5.43,46,48,50) in the book.]

The following argument shows the most elementary and instructive way of calculating Christoffel symbols for curvilinear coordinates in flat space. Once we get into curved space we won't have inertial coordinates to fall back upon, so other methods of getting Christoffel symbols will need to be developed.

Differentiate (5) to get

$$\begin{aligned} \frac{\partial \vec{e}_\tau}{\partial \tau} &= \sigma \sinh\tau \, \vec{e}_t + \sigma \cosh\tau \, \vec{e}_x = \sigma \vec{e}_\sigma \,, \\ \frac{\partial \vec{e}_\tau}{\partial \sigma} &= \cosh\tau \, \vec{e}_t + \sinh\tau \, \vec{e}_x = \frac{1}{\sigma} \, \vec{e}_\tau \,, \\ \frac{\partial \vec{e}_\sigma}{\partial \tau} &= \cosh\tau \, \vec{e}_t + \sinh\tau \, \vec{e}_x = \frac{1}{\sigma} \, \vec{e}_\tau \,, \\ \frac{\partial \vec{e}_\sigma}{\partial \sigma} &= 0. \end{aligned}$$

Since by definition

$$\frac{\partial \vec{e}_{\alpha'}}{\partial x^{\beta'}} = \Gamma^{\mu'}_{\alpha'\beta'} \vec{e}_{\mu'} \,,$$

we can read off the Christoffel symbols for the coordinate system (τ, σ):

$$\Gamma^{\tau}_{\sigma\sigma} = 0, \quad \Gamma^{\sigma}_{\sigma\sigma} = 0,$$

$$\Gamma^{\tau}_{\tau\sigma} = \Gamma^{\tau}_{\sigma\tau} = \frac{1}{\sigma} \,,$$

$$\Gamma^{\sigma}_{\tau\sigma} = \Gamma^{\sigma}_{\sigma\tau} = 0,$$

$$\Gamma^{\sigma}_{\tau\tau} = \sigma, \quad \Gamma^{\tau}_{\tau\tau} = 0.$$

Later we will see that the Christoffel symbol is necessarily *symmetric in its subscripts*, so in dimension d the number of independent Christoffel symbols is

$$d \text{ (superscripts)} \times \frac{d(d+1)}{2} \text{ (symmetric subscript pairs)} = \tfrac{1}{2} d^2 (d+1).$$

For $d = 2, 3, 4$ we get $6, 18, 40$ respectively. In particular cases there will be geometrical symmetries that make other coefficients equal, make some of them zero, etc.

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