Math. 489GR (Fulling) 5 May 2008

Open-Book Final Examination – Solutions

1. *(70 pts.)* A two-dimensional space-time is called *conformally flat* if there exist coordinates in which the line element (metric) takes the form $ds^2 = \mathcal{C}(t,x)(-dt^2 + dx^2)$. We shall assume that $\mathcal{C}$ is smooth, bounded, and strictly positive.

   (a) Show that in such a case there exist coordinates (u,v) (called *null coordinates*) in which the line element is $ds^2 = -C(u,v)\,du\,dv$. Here C is simply $\mathcal{C}$ regarded as a function of the new coordinates: $C(u,v) = \mathcal{C}\big(t(u,v), x(u,v)\big)$.

Let $u = t - x$, $v = t + x$. Then $t = \frac{1}{2}(v+u)$, $x = \frac{1}{2}(v-u)$, so

$$-dt^2 + dx^2 = -\tfrac{1}{4}(dv^2 + du^2 + 2\,du\,dv) + \tfrac{1}{4}(dv^2 + du^2 - 2\,du\,dv) = -du\,dv\,.$$

The assertion follows immediately. (My definitions of u and v are those standard in the relativity literature. Yours may differ by signs or interchange of u with v .)

   (b) Show that in the null coordinate system the only nonzero Christoffel symbols are

$$\Gamma^u_{uu} = \frac{1}{C}\frac{\partial C}{\partial u}\,, \qquad \Gamma^v_{vv} = \frac{1}{C}\frac{\partial C}{\partial v}\,.$$

   Suggestion: Throughout this problem write C_u for $\partial C/\partial u$, etc.

I'll use the variational method with $\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -C\dot{u}\dot{v}$.

$$\frac{\partial \mathcal{L}}{\partial \dot{u}} = -C\dot{v}, \quad \frac{\partial \mathcal{L}}{\partial u} = -C_u\dot{u}\dot{v} \;\Rightarrow\; 0 = -C\ddot{v} - C_u\dot{u}\dot{v} - C_v\dot{v}^2 + C_u\dot{u}\dot{v} = -C\ddot{v} - C_v\dot{v}^2\,,$$

or $\ddot{v} + C^{-1}C_v\dot{v}^2 = 0$. Similarly, $\ddot{u} + C^{-1}C_u\dot{u}^2 = 0$. Comparing with the general geodesic equation, we see that the assertion is true.

   (c) Show that the scalar wave equation, $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0$, reduces in the null coordinates to something almost trivial, and thereby find the general solution of the wave equation in the (t,x) coordinates ("d'Alembert's method" from Math. 412).

We have $g_{\mu\nu} = \begin{pmatrix} 0 & -\frac{C}{2} \\ -\frac{C}{2} & 0 \end{pmatrix}$ and hence $g^{\mu\nu} = \begin{pmatrix} 0 & -\frac{2}{C} \\ -\frac{2}{C} & 0 \end{pmatrix}$. So the wave equation reduces to

$$\begin{aligned} 0 = 2g^{uv}\nabla_u\nabla_v\phi &= -\frac{4}{C}\nabla_u\partial_v\phi \\ &= -\frac{4}{C}(\partial_u\partial_v\phi - \Gamma^\alpha_{uv}\partial_\alpha\phi) = -\frac{4}{C}\partial_u\partial_v\phi. \end{aligned}$$

The factor $4/C$ can be discarded. So the wave equation is exactly the same as in flat space and can be solved in the same way:

$$\partial_v\phi = f(v) \;\Rightarrow\; \phi = A(v) + B(u) \quad (f = A')\,,$$

so in terms of t and x the general solution is

$$\phi(t,x) = A(t+x) + B(t-x)$$

where A and B are arbitrary functions of one variable.

(d) Note that the null coordinates are not unique: a coordinate transformation $u = f(U)$, $v = g(V)$ leaves the metric in the null form. Verify that, nevertheless, the quantity

$$R = 4C^{-3}\left(C\,\frac{\partial^2 C}{\partial u\,\partial v} - \frac{\partial C}{\partial u}\,\frac{\partial C}{\partial v}\right)$$

is independent of which null coordinate system is used.

Remark: R is, in fact, the Ricci curvature scalar for this space-time. *(free information)*

$$ds^4 = -C\,du\,dv = -cf'g'\,dU\,dV\,.$$

So in the R formula we must replace C by $Cf'g'$ and $\frac{\partial}{\partial u}$ by $\frac{\partial}{\partial U}$, etc., and check that the result doesn't change. Keep in mind that f' depends only on u and g' only on v. I construct the ingredients:

$$\frac{\partial}{\partial U}(Cf'g') = C_u(f')^2g' + Cf''g', \qquad \frac{\partial}{\partial V}(Cf'g') = C_vf'(g')^2 + Cf'g''\,.$$

$$\frac{\partial^2}{\partial U\,\partial V}(Cf'g') = C_{uv}(f'g')^2 + C_u(f')^2g'' + C_vf''(g')^2 + Cf''g''\,.$$

Therefore,

$$Cf'g'\,\frac{\partial^2}{\partial U\,\partial V}(Cf'g') - \frac{\partial}{\partial U}(Cf'g')\,\frac{\partial}{\partial V}(Cf'g')$$
$$= CC_{uv}(f'g')^3 - C_uC_v(f'g')^3 + (6\text{ terms that cancel in pairs})\,.$$

Multiplying by $4(Cf'g')^{-3}$, we get exactly the original formula for R!

An alternative approach is to use the Christoffel symbols in (b) to calculate the curvature scalar. (The off-diagonal nature of C makes this slightly tricky. For instance, $R^u{}_{vuv} = 0$ but $R^v{}_{vuv}$ is nonzero!) Since R is a true scalar, its formula must be the same in all coordinate systems of the null form. It is easy to see that the calculation must proceed in the same way in both systems, since the formulas for the metric and Christoffel symbols are precisely analogous.

(e) Show that all the curves $u =$ const. and $v =$ const. are null (lightlike) geodesics (not necessarily affinely parametrized!), and, conversely, all null geodesics are of this form. (*Hint* for the converse: How many null geodesics pass through a given point in 2D space-time?)

Consider a curve on which u is constant. It satisfies the geodesic equation $\ddot{u} + C^{-1}C_u\dot{u}^2 = 0$ found in our solution to (b). (If you calculated the Christoffel symbols directly, the geodesic equation follows easily from the information in (b).) Since we are not demanding that the geodesic parameter λ is affine, v can be an arbitrary monotonic function of λ and the other geodesic equation is irrelevant. Since $\dot{u} = 0$ and $g_{vv} = 0$, we have $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, which proves the geodesic is null. Similarly, the curves $v =$ const. are null geodesics. In 2D space-time each point has exactly two null geodesics through it, one in the v direction and one in the u direction, so we have them all.

2. *(40 pts.)* Consider a two-dimensional space-time with line element

$$ds^2 = -d\tau^2 + e^{2H\tau}\,dx^2 \qquad (H = \text{constant}).$$

Its independent nonzero Christoffel symbols are *(free information)*

$$\Gamma^{\tau}_{xx} = He^{2H\tau}, \qquad \Gamma^{x}_{\tau x} = H.$$

Remark: This is one form of the 2D *de Sitter* metric. Its four-dimensional analogue is believed to be a good model of the actual expansion of the universe during both the early *inflationary* stage and the late (contemporary) *accelerating* or *dark-matter* stage.

(a) Show (by an explicit coordinate transformation) that this space-time is conformally flat.

We need to find t such that $d\tau^2 = e^{2H\tau}\,dt^2$. That is, $dt = e^{-H\tau}\,d\tau$ (up to sign, see below).

$$t = \int e^{-H\tau}\,d\tau = -\frac{1}{H}\,e^{-H\tau} + C.$$

We can set $C = 0$. Note that t now comes out negative; that's not a problem. (Note that $-t$ would equally well satisfy our original condition, but then t and τ would run in opposite directions ($\partial t/\partial\tau < 0$), which would be even more confusing than having the world end at $t = 0$.)

$$t \equiv -\frac{1}{H}\,e^{-H\tau}, \qquad \tau = -\frac{1}{H}\,\ln(-Ht).$$

$$ds^2 = e^{2H\tau}(-dt^2 + dx^2) = (Ht)^{-2}(-dt^2 + dx^2).$$

(b) Show that the Ricci curvature scalar of this metric is a constant.

(c) Check your answer to (b) by an independent argument.

Method 1: Given the Christoffel symbols, which are mostly zero, we can easily calculate the curvature scalar. The only independent component of the Riemann tensor is

$$\begin{aligned} R^{\tau}{}_{x\tau x} &= -\Gamma^{\tau}{}_{x\tau,x} + \Gamma^{\tau}{}_{xx,\tau} + \Gamma^{\tau}_{\gamma\tau}\Gamma^{\gamma}_{xx} - \Gamma^{\tau}_{\gamma x}\Gamma^{\gamma}_{x\tau} \\ &= 0 + 2H^2e^{2H\tau} + 0 - H^2e^{2H\tau} = H^2e^{2H\tau}. \end{aligned}$$

Thus

$$R_{\tau x\tau x} = -H^2e^{2H\tau} = R_{x\tau x\tau} \;\Rightarrow\; R^{x}{}_{\tau x\tau} = -H^2.$$

The Ricci tensor is

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = R^{\tau}{}_{\mu\tau\nu} + R^{x}{}_{\mu x\nu}.$$

The first term is 0 unless $\mu = \nu = x$, in which case it's $H^2e^{2H\tau}$. The second term is 0 unless $\mu = \nu = \tau$, in which case it's $-H^2$. Thus $R^x_x = H^2 = R^\tau_\tau$, so the trace of this tensor is the constant

$$R = 2H^2.$$

Method 2: Part (a) places this problem into the framework of Question 1, so we can apply the formula in 1(d).

$$C = (Ht)^{-2} = \frac{4}{H^2}\,(v+u)^{-2}\,.$$

$$C_u = -\frac{8}{H^2}\,(v+u)^{-3} = C_v\,, \qquad C_{uv} = \frac{24}{H^2}(v+u)^{-4}\,.$$

$$CC_{uv} - C_uC_v = \frac{96}{H^4}\,(v+u)^{-6} - \frac{64}{H^4}\,(v+u)^{-6} = \frac{32}{H^4}\,(v+u)^{-6}\,.$$

$$R = 4\,\frac{H^6}{64}\,(v+u)^6 \times \frac{32}{H^4}\,(v+u)^{-6} = 2H^2\,.$$

3. *(Essay – 30 pts.)* Discuss the *horizon* of a black hole. In particular, explain in what sense the horizon is, and in what sense it is not, a "singularity". Like a good colloquium talk, your essay should start with a qualitative explanation understandable by a general audience, then continue with a slightly more technical discussion. In the technical part you can assume that the black hole is nonrotating (Schwarzschild). **If you copy sentences (not just equations), put them in quotation marks and state the sources.**

4. *(30 pts.)* Show that the curvature tensor of a non-Abelian gauge theory does indeed transform as a gauge tensor under gauge transformations $U(x)$:

$$\tilde{Y}_{\mu\nu}(x) = U(x)Y_{\mu\nu}(x)U(x)^{-1}.$$

(This is Exercise 32 of Chapter 8 of *Aspects* ... , which was also left as an exercise in lecture. $\tilde{Y}$ is defined in terms of $\tilde{w}$, the transformed connection form, by the same formula that gives Y in terms of w.)

Exercise 32 points to the equation (8.10) as the formula for y in terms of w. The analogous formula in the other gauge is, therefore,

$$\tilde{Y}_{\mu\nu} = \tilde{w}_{\nu,\mu} - \tilde{w}_{\mu,\nu} + [\tilde{w}_\mu, \tilde{w}_\nu]\,. \qquad (*)$$

The other crucial relation needed is in Exercise 29:

$$\tilde{w}_\mu = U[w_\mu - U^{-1}\partial_\mu U]U^{-1}\,.$$

Substitute it into $(*)$:

$$\begin{aligned}
\tilde{Y}_{\mu\nu} &= \partial_\mu\big[U[w_\nu - U^{-1}\partial_\nu U]U^{-1}\big] - \partial_\nu\big[U[w_\mu - U^{-1}\partial_\mu U]U^{-1}\big] \\
&\quad + U[w_\mu - U^{-1}\partial_\mu U][w_\nu - U^{-1}\partial_\nu U]U^{-1} - U[w_\nu - U^{-1}\partial_\nu U][w_\mu - U^{-1}\partial_\mu U]U^{-1} \\
&= (\text{lots of cancelling terms involving derivatives of } U) \\
&\quad + U(w_{\nu,\mu} - w_{\mu,\nu} + [w_\mu, w_\nu])U^{-1} \\
&= UY_{\mu\nu}U^{-1}\,.
\end{aligned}$$

(To show the cancellations in detail, you need to use the fact that

$$\partial_\alpha(U^{-1}) = -U^{-1}\partial_\alpha U\,U^{-1}\,,$$

which is true for any matrix-valued function (by implicit differentiation of $UU^{-1} = 1$).)

5. *(30 pts.)* Larry, Moe, and Curly are identical triplets. Moe travels clockwise in a perfect circle at a very high, constant speed. Curly travels counterclockwise around the same circle at the same speed. Larry remains motionless at the center of the circle.

   (a) What can you say about the relative ages of the three when this process ends?

By symmetry, Moe and Curly must have the same age. Since Larry is stationary, he can use special relativity in a fixed inertial frame to describe all his observations; thus he is correct to conclude from the time dilation effect that the others are younger than he by the factor $\gamma^{-1} = \sqrt{1 - v^2}$.

   (b) Each triplet "sees" the clock of each of the others running *slow* relative to his own clock. Yet it is not possible that each is younger relative to every other at their final rendezvous. Resolve this paradox. (Note that because of the symmetry between Moe and Curly, you can't just say, "One of them is accelerated.")

Moe (for example) is correct to say that Curly's (or Larry's) clock is running slower than his, in Moe's instantaneous inertial frame. However, since Moe is accelerating, his natural notion of "hypersurfaces of constant time" is continually changing. Thus the clock rates in his inertial frame are not the same thing as the histories of the clocks with respect to his instantaneous hypersurfaces of constant time, parametrized by his proper time. In Schutz's parable of Diana and Artemis (the conventional "away and back in a straight line" version of the twin paradox) the accelerating twin experiences a sudden jump in her constant-time hypersurface at the instant of acceleration. In Moe's case, this adjustment of constant-time hypersurfaces takes place continually throughout the journey.

6. *(Extra Credit – 20 pts.)* Use parts (a) and (e) of Question 1 to prove that all two-dimensional space-times are conformally flat. (Label every null geodesic in your space-time by a number in a smooth, monotonic manner, and denote each point by the labels (u, v) of the right-moving and left-moving null geodesics through it.)

Construct the coordinate system (u, v) as described in the hint. What is the metric tensor in that system? All that needs to be proved is that g_{uu} and g_{vv} equal 0 , because you can then define $C = -2g_{uv}$. (If C comes out with the wrong sign, change the sign of u .) Well, just reverse the argument in the solution to 1(e): Let $\dot{x}^\mu = (\dot{u}, \dot{v})$ be the tangent vector to one of our geodesics. For definiteness, let it be a curve with $u = \text{const.}$; then $\dot{u} = 0$, so

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = g_{vv}\dot{v}^2.$$

But $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$ by definition of a null geodesic, so $g_{vv} = 0$. Similarly, $g_{uu} = 0$.