On the Relation between Inversion and Index Swapping

In special relativity, Schutz writes $\{\Lambda^{\beta}{}_{\bar\alpha}\}$ for the matrix of the coordinate transformation inverse to the coordinate transformation

$$x^{\bar\alpha} = \Lambda^{\bar\alpha}{}_{\beta}\, x^{\beta}. \qquad (*)$$

However, one might want to use that same notation for the transpose of the matrix obtained by raising and lowering the indices of the matrix in $(*)$:

$$\Lambda_{\bar\alpha}{}^{\beta} = g_{\bar\alpha\bar\mu}\Lambda^{\bar\mu}{}_{\nu}g^{\nu\beta}.$$

Here $\{g_{\alpha\beta}\}$ and $\{g_{\bar\alpha\bar\beta}\}$ are the matrices of the metric of Minkowski space with respect to the unbarred and barred coordinate system, respectively. (The coordinate transformation $(*)$ is linear, but not necessarily a Lorentz transformation.) Let us investigate whether these two interpretations of the symbol $\Lambda^{\beta}{}_{\bar\alpha}$ are consistent.

If the answer is *yes*, then (according to the first definition) $\delta^{\bar\alpha}_{\bar\gamma}$ must equal

$$\begin{aligned}\Lambda^{\bar\alpha}{}_{\beta}\Lambda_{\bar\gamma}{}^{\beta} &\equiv \Lambda^{\bar\alpha}{}_{\beta}\left(g_{\bar\gamma\bar\mu}\Lambda^{\bar\mu}{}_{\nu}g^{\nu\beta}\right)\\ &= g_{\bar\gamma\bar\mu}\left(\Lambda^{\bar\mu}{}_{\nu}g^{\nu\beta}\Lambda^{\bar\alpha}{}_{\beta}\right)\\ &= g_{\bar\gamma\bar\mu}g^{\bar\mu\bar\alpha}\\ &= \delta^{\bar\alpha}_{\bar\gamma}\,, \qquad \text{Q.E.D.}\end{aligned}$$

(The first step uses the second definition, and the next-to-last step uses the transformation law of a $\binom{2}{0}$ tensor.)

In less ambiguous notation, what we have proved is that

$$\left(\Lambda^{-1}\right)^{\beta}{}_{\bar\alpha} = g_{\bar\alpha\bar\mu}\Lambda^{\bar\mu}{}_{\nu}g^{\nu\beta}. \qquad (\dagger)$$

Note that if Λ is not a Lorentz transformation, then the barred and unbarred g matrices are not numerically equal; at most one of them in that case has the form

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

If Λ *is* Lorentz (so that the g matrices are the same) and the coordinates are with respect to an orthogonal basis (so that indeed $g = \eta$), then $(\dagger)$ is the indefinite-metric counterpart of the "inverse = transpose" characterization of an orthogonal matrix in Euclidean space: *The inverse of a Lorentz transformation equals the transpose with the indices raised and lowered (by η).* (In the Euclidean case, η is replaced by δ and hence $(\dagger)$ reduces to

$$\left(\Lambda^{-1}\right)^{\beta}{}_{\bar\alpha} = \Lambda^{\bar\alpha}{}_{\beta}\,,$$

in which the up-down index position has no significance.) For a general linear transformation, (†) may appear to offer a free lunch: How can we calculate an inverse matrix without the hard work of evaluating Cramer's rule, or performing a Gaussian elimination? The answer is that in the general case at least one of the matrices $\left\{g_{\bar{\alpha}\bar{\mu}}\right\}$ and $\left\{g^{\nu\beta}\right\}$ is nontrivial and somehow contains the information about the inverse matrix.

Alternative argument: We can use the metric to map between vectors and covectors. Since

$$v^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} v^{\beta}$$

is the transformation law for vectors, that for covectors must be

$$\begin{aligned}
\tilde{v}_{\bar{\mu}} &= g_{\bar{\mu}\bar{\alpha}} v^{\bar{\alpha}} \\
&= g_{\bar{\mu}\bar{\alpha}} \Lambda^{\bar{\alpha}}{}_{\beta} v^{\beta} \\
&= g_{\bar{\mu}\bar{\alpha}} \Lambda^{\bar{\alpha}}{}_{\beta} g^{\beta\nu} \tilde{v}_{\nu} \\
&\equiv \Lambda_{\bar{\mu}}{}^{\nu} \tilde{v}_{\nu}
\end{aligned}$$

according to the second definition. But the transformation matrix for covectors is the transpose of the inverse of that for vectors — i.e.,

$$\tilde{v}_{\bar{\mu}} = \Lambda^{\nu}{}_{\bar{\mu}} \tilde{v}_{\nu}$$

according to the first definition. Therefore, the definitions are consistent.