Math. 489GR (Fulling) 20 February 2008

Test A – Solutions

Extra Credit: The test is worth 120 points, but 100 counts as a perfect score.

1. *(30 pts.)*

   (a) Sam is moving in the positive x direction at speed v relative to me. Write down the Lorentz transformation from my coordinate system to Sam's. (Take $c = 1$.)

The transformation has matrix

$$\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(I chose to type the version that involves the fewest square roots.)

   (b) Karen is moving in the positive x direction at speed u relative to Sam. Suppressing the irrelevant y and z directions, find the Lorentz transformation from my coordinate system to Karen's. (Multiply two 2×2 matrices.)

In analogy to (a), the transformation from Sam's coordinates to Karen's is $\frac{1}{\sqrt{1-u^2}}\begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix}$. So to get from my coordinates to Karen's we need

$$\frac{1}{\sqrt{1-u^2}}\begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix}\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} = \frac{1}{\sqrt{1-u^2}\sqrt{1-v^2}}\begin{pmatrix} 1+uv & -u-v \\ -u-v & 1+uv \end{pmatrix}.$$

   (c) From your answer to (b), deduce the (one-dimensional) relativistic velocity composition law (the formula for Karen's speed relative to me).

We need to simplify (b) to the form $\frac{1}{\sqrt{1-w^2}}\begin{pmatrix} 1 & -w \\ -w & 1 \end{pmatrix}$. The quickest way to do the algebra is to note that $-w$ must be the ratio of the off-diagonal entries to the diagonal ones:

$$w \equiv -\frac{\Lambda^1{}_0}{\Lambda^0{}_0} = \frac{u+v}{1+uv}.$$

This is the well known correct answer. To be completely careful, we now check that the "γ" factor comes out right:

$$\begin{aligned}
\gamma^{-2} &\equiv 1 - w^2 \\
&= 1 - \frac{(u+v)^2}{(1+uv)^2} \\
&= \frac{(1+2uv+u^2v^2) - (u^2+2uv+v^2)}{(1+uv)^2} \\
&= \frac{1-(u^2+v^2)+u^2v^2}{(1+uv)^2} \\
&= \frac{(1-u^2)(1-v^2)}{(1+uv)^2} \\
&= (\Lambda^0{}_0)^{-2}.
\end{aligned}$$

2. *(40 pts.)* Consider the coordinate transformation

$$t = b\bar t, \qquad x = \bar x - v\bar t \qquad (b \text{ and } v \text{ constant})$$

in a two-dimensional space-time whose metric tensor in the unbarred coordinates is the usual one,

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Note that this is *not* a Lorentz transformation. It is a linear transformation, however.)

(a) Calculate the tangent vectors to the coordinate curves, $\vec e_{\bar t}$ and $\vec e_{\bar x}$ (also called $\vec e_{\bar 0}$ and $\vec e_{\bar 1}$, or $\frac{\partial}{\partial \bar t}$ and $\frac{\partial}{\partial \bar x}$.)

$\vec e_{\bar t}$ is the tangent vector to the curve $\binom{t}{x}$ regarded as a function of $\bar t$ with $\bar x$ fixed:

$$\vec e_{\bar t} = \begin{pmatrix} b \\ -v \end{pmatrix}.$$

By the same reasoning,

$$\vec e_{\bar x} = \frac{\partial}{\partial \bar x}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For later use note that these vectors go together to make up the Jacobian matrix of the transformation,

$$J = \begin{pmatrix} b & 0 \\ -v & 1 \end{pmatrix} = \Lambda^\alpha{}_{\bar\beta}\,.$$

(In the present case J is the same as Λ, the matrix of the coordinate transformation itself, because the transformation is linear.)

(b) Calculate the normal one-forms to the coordinate "surfaces", $\tilde d\bar t$ and $\tilde d\bar x$ (also called $dx^{\bar 0}$ and $dx^{\bar 1}$, or $\tilde E^{\bar 0}$ and $\tilde E^{\bar 1}$. (Check that the reciprocal-basis condition, $\tilde E^\mu(\vec e_\nu) = \delta^\mu_\nu$, is satisfied.)

$$\tilde d\bar t = \frac{\partial \bar t}{\partial t}\,\tilde dt + \frac{\partial \bar t}{\partial x}\,\tilde dx\,, \qquad \text{etc.}$$

The easiest way to find the coefficients in these equations is to note that they are the *rows* of the inverse of J, the matrix whose *columns* are the tangent vectors. By Cramer's rule,

$$J^{-1} = \frac{1}{b}\begin{pmatrix} 1 & 0 \\ v & b \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ vb^{-1} & 1 \end{pmatrix}.$$

$$\tilde d\bar t = \frac{1}{b}\,\tilde dt\,, \qquad \tilde d\bar x = \frac{v}{b}\,\tilde dt + \tilde dx\,.$$

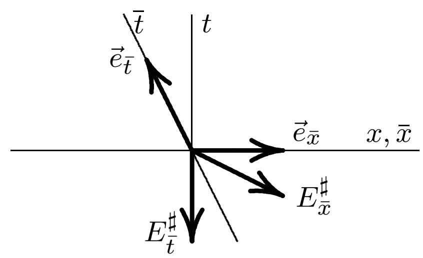
In another notation,

$$\tilde E^{\bar t} = (1/b, 0)\,, \qquad \tilde E^{\bar x} = (v/b, 1)\,,$$

and it's easy to check that these are reciprocal to the tangent-vector basis.

(c) Take $b = 1$ and $v = \frac{1}{2}$. At the origin of the (t, x) Cartesian coordinate grid, sketch the two tangent vectors, $\vec{e}_{\bar{t}}$ and $\vec{e}_{\bar{x}}$, and the two normal vectors, $E^{\sharp}_{\bar{t}}$ and $E^{\sharp}_{\bar{x}}$, related to the normal one-forms via the metric ("index-raising"). (Recall that the normal vectors may not look normal to the surfaces, but they are normal with respect to the Lorentz inner product.)

$$\vec{e}_{\bar{t}} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}, \qquad \vec{e}_{\bar{x}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \qquad E^{\sharp}_{\bar{t}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad E^{\sharp}_{\bar{x}} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}.$$



(d) Calculate the metric tensor in the barred coordinates, $g_{\bar{\alpha}\bar{\beta}}$.

Method 1: $-dt^2 + dx^2 = -(b\,d\bar{t})^2 + (d\bar{x} - v\,d\bar{t})^2 = (-b^2 + v^2)\,d\bar{t}^2 + d\bar{x}^2 - 2v\,d\bar{x}\,d\bar{t}$. Therefore, the matrix of g is $\begin{pmatrix} -b^2 + v^2 & -v \\ -v & 1 \end{pmatrix}$.

Method 2: $g_{\bar{\alpha}\bar{\beta}} = \Lambda_{\bar{\alpha}}{}^{\mu} \Lambda_{\bar{\beta}}{}^{\nu} \eta_{\mu\nu}$ with an appropriate matrix Λ. Namely, the matrix that maps covector components from unbarred to barred is the contragredient of the one that maps vector components from unbarred to barred — that is, the transpose of the one that maps vector components from barred to unbarred, which is J. In matrix product terms,

$$g = J^{\mathrm{t}} \eta J = \begin{pmatrix} b & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ -v & 1 \end{pmatrix} = \begin{pmatrix} -b^2 + v^2 & -v \\ -v & 1 \end{pmatrix}.$$

Method 3: Evaluate $g_{\bar{t}\bar{t}} = g(\vec{e}_{\bar{t}}, \vec{e}_{\bar{t}})$ in the unbarred system as $\eta_{00} b^2 + \eta_{11}(-v)^2 = -b^2 + v^2$. The other three components work out similarly.

3. *(50 pts.)* Let T be a $\binom{0}{2}$ tensor, and let $\{\vec{e}_\alpha\}$ be a basis (not necessarily orthonormal) for the space of contravariant vectors, $\mathcal{V}$ (alias $\binom{1}{0}$ tensors).

(a) State the modern definition of a $\binom{0}{2}$ tensor as a function of some kind acting on inputs from $\mathcal{V}$. Give the formula for the tensor components $T_{\alpha\beta}$ with respect to the given basis.

T is a bilinear functional on $\mathcal{V}$. That is, it takes two vectorial arguments and yields a (real) number, and it depends linearly on each argument when the other is fixed. For example, $T(\vec{u} + \lambda\vec{v}, \vec{w}) = T(\vec{u}, \vec{w}) + \lambda T(\vec{v}, \vec{w})$. We have $T_{\alpha\beta} = T(\vec{e}_\alpha, \vec{e}_\beta)$ in terms of the basis vectors.

(b) T is called *symmetric* if $T_{\beta\alpha} = T_{\alpha\beta}$ (for all indices). Explain why this condition is independent of the basis chosen.

Using the bilinearity it is easy to show that the condition is equivalent to $T(\vec{u},\vec{v}) = T(\vec{v},\vec{u})$ for all vectors $\vec{u}$ and $\vec{v}$. This condition makes no reference to a particular coordinate system.

Alternative argument: For some transformation matrix Λ,

$$T_{\bar{\mu}\bar{\nu}} = \Lambda_{\bar{\mu}}{}^{\alpha}\Lambda_{\bar{\nu}}{}^{\beta}T_{\alpha\beta} = \Lambda_{\bar{\mu}}{}^{\alpha}\Lambda_{\bar{\nu}}{}^{\beta}T_{\beta\alpha} = T_{\bar{\nu}\bar{\mu}}\,.$$

(c) T is called *antisymmetric* if $T_{\beta\alpha} = -T_{\alpha\beta}$. Prove that every $\binom{0}{2}$ tensor is a sum of a symmetric and an antisymmetric tensor.

Given a T, define (in terms of its transpose, T^{t})

$$T_{\mathrm{s}} = \frac{1}{2}(T + T^{\mathrm{t}}), \qquad T_{\mathrm{a}} = \frac{1}{2}(T - T^{\mathrm{t}}).$$

Then T_{s} is symmetric (since taking the transpose just means swapping the index positions), T_{a} is antisymmetric, and $T = T_{\mathrm{s}} + T_{\mathrm{a}}$. (This is the same as the proof that every function is the sum of an even and an odd function. It is an example of the simplest decomposition of a group representation into a sum of irreducible representations. Note that nothing but the zero tensor is both symmetric and antisymmetric, so the decomposition is unique.)

(d) For a $\binom{1}{1}$ tensor, show that the component condition $T^{\beta}{}_{\alpha} = T^{\alpha}{}_{\beta}$ is *not* independent of basis. (Suggestion: Construct a counterexample assuming that the dimension of $\mathcal{V}$ is 2.)

For a mixed tensor, the basis transformation law is the familiar *similarity transformation* of matrices, $\overline{T} = MTM^{-1}$. Suppose that

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Now if T is a symmetric but nondiagonal matrix, say $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$\overline{T} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 1 \end{pmatrix},$$

which is not symmetric.

(e) Show that the condition $T^{\beta}{}_{\alpha} = T^{\alpha}{}_{\beta}$ *is* preserved by Lorentz transformations (for which $(\Lambda^{-1})^{\nu}{}_{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu}$.)

In analogy to the alternative argument for (b), calculate

$$T^{\bar{\alpha}}{}_{\bar{\beta}} = \Lambda^{\bar{\alpha}}{}_{\mu}(\Lambda^{-1})^{\nu}{}_{\bar{\beta}}T^{\mu}{}_{\nu} = \Lambda^{\bar{\alpha}}{}_{\mu}\Lambda^{\bar{\beta}}{}_{\nu}T^{\mu}{}_{\nu} = \Lambda^{\bar{\alpha}}{}_{\mu}\Lambda^{\bar{\beta}}{}_{\nu}T^{\nu}{}_{\mu} = \Lambda^{\bar{\beta}}{}_{\nu}(\Lambda^{-1})^{\mu}{}_{\bar{\alpha}}T^{\nu}{}_{\mu} = T^{\bar{\beta}}{}_{\bar{\alpha}}\,.$$