Math. 489GR (Fulling) 9 April 2008

Test B – Solutions to nonessay questions

1. *(40 pts.)* Consider the two-dimensional space-time with metric

$$ds^2 = -dt^2 + t^2\,dx^2 \qquad (-\infty < x < \infty\,, \quad 0 < t < \infty)\,.$$

Note that parts (a) and (b) can be done in either order.

(a) Find the differential equation(s) defining geodesics in this space-time.

(b) Find the Christoffel symbols.

(a), then (b): Consider the Lagrangian for the geodesics,

$$L = \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha x^\beta = \tfrac{1}{2}(-\dot{t}^2 + t^2\dot{x}^2)\,.$$

The Euler–Lagrange equations are

$$0 = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = -\ddot{t} - t\dot{x}^2\,,$$
$$0 = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{d\lambda}[t^2\dot{x}] = t^2\ddot{x} + 2t\dot{t}\dot{x}\,.$$

Compare with the geodesic equations,

$$0 = \ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,.$$

Conclusion: $\Gamma^t_{xx} = t$, $\Gamma^x_{xt} = t^{-1}$, and all others are 0 (except Γ^x_{tx} , of course).

(b), then (a): Use

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}[g_{\beta\nu,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}]\,.$$

Any nonvanishing term must contain $g_{xx,t}$, and $g^{\mu\nu}$ is diagonal. So it is easy to see that

$$\Gamma^t_{tt} = 0\,, \qquad \Gamma^t_{xx} = -\tfrac{1}{2}[0 + 0 - 2t] = t\,, \qquad \Gamma^t_{tx} = 0\,,$$

$$\Gamma^x_{tt} = 0, \qquad \Gamma^x_{xx} = 0\,, \qquad \Gamma^x_{xt} = \tfrac{1}{2}t^{-2}[0 + 2t - 0] = t^{-1}.$$

Now construct the geodesic equations as

$$0 = \ddot{t} + t\dot{x}^2\,, \qquad 0 = \ddot{x} + \frac{2}{t}\,\dot{t}\dot{x}\,.$$

(c) Introduce new coordinates by

$$\overline{t} = t \cosh x\,,$$
$$\overline{x} = t \sinh x\,.$$

Show that that metric in the barred coordinates is $ds^2 = -d\overline{t}^2 + d\overline{x}^2$.

Easy way: Start from the barred form.

$$d\overline{t} = \cosh x\, dt + t\, \sinh x\, dx, \qquad d\overline{x} = \sinh x\, dt + t\, \cosh x\, dx\,.$$

Therefore,

$$\begin{aligned} ds^2 &= -(\cosh x\, dt + t\, \sinh x\, dx)^2 + (\sinh x\, dt + t\, \cosh x\, dx)^2 \\ &= (-\cosh^2 x + \sinh^2 x)\, dt^2 + \cdots \\ &- dt^2 + t^2\, dx^2\,. \end{aligned}$$

Hard way: Solve for

$$t = \sqrt{\overline{t}^2 - \overline{x}^2}\,, \qquad x = \tanh^{-1} \frac{\overline{x}}{\overline{t}}\,.$$

Crank through — it works!

(d) Find the Riemann curvature tensor for this space-time. (Don't waste too much time on this part!)

Easy way: Part (c) shows that this universe is (a portion of) two-dimensional flat Minkowski space-time, so the Riemann tensor is identically zero.

Hard way: Because of antisymmetry, in dimension 2 there is only one independent component, R_{txtx}. The general formula for $R^\alpha{}_{\beta\alpha\beta}$, or the formula for the commutator of covariant derivatives, can be used to calculate it (as 0).

2. *(20 pts.)* Recall that the main dynamical equation for a homogeneous, isotropic universe is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho\,.$$

(Here $a(t)$ is the Robertson–Walker scale factor, loosely called "radius of the universe", which is sometimes denoted $R(t)$.) G, k, and Λ are constants (G positive).

(a) Explain what k is.

k represents the *spatial* curvature of the universe (as a three-dimensional manifold at each fixed time). It is positive if the space is a three-dimensional sphere, zero if the space is flat, and negative if the space is hyperboloidal (like a saddle). When k is not zero, a can be rescaled to make $|k| = 1$.

(b) Suppose that $\rho = Ca^{-3}$. (C is a positive constant.) What sort of matter predominates in this universe, and what is the corresponding pressure?

massive particles (dust), for which $p = 0$.

(c) Suppose that ρ is as in (b) and that $k > 0$. Repeat Einstein's famous "blunder" by showing that there exists a Λ capable of making a independent of time.

If $\dot{a} = 0$, the Einstein equation becomes

$$\frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi GC}{3} a^{-3} .$$

Solve:

$$\Lambda = \frac{3k}{a^2} - \frac{8\pi GC}{a^3} .$$

So, for any a_0 there is a Λ that makes $a(t) = a_0$ a solution.

(d) Show that the other Einstein equation,

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = -\,8\pi G p ,$$

provides (unexpectedly) an additional constraint among k, C, Λ, and a in this static situation.

In our case this equation becomes just

$$\Lambda = \frac{k}{a^2} .$$

Combined with the result of (c), this implies

$$\frac{2k}{a^2} = \frac{8\pi GC}{a^3} ,$$

or

$$a = \frac{4\pi GC}{k} .$$

Thus any two of the parameters determine the other two; there is less freedom than we thought in (c). (Naturally, these static solutions are unstable, and they are not taken seriously nowadays.)

Remarks: The proof that the second Einstein equation adds no information to the first involves cancelling a factor $\dot{a}/a$, so it is not valid for a static solution. Closer examination of the second equation without prior assumptions on k and the equation of state shows that k must indeed be positive and that pure radiation ($\rho = 3p$, $\rho = Ca^{-4}$) is not consistent with a static solution. These restrictions do not enter in part (c), but leaving them (and part (d)) out, as I originally intended, would require stating the question as "Compound Einstein's famous blunder ... ".

3. *(Essay – 40 pts.)* Tell me what you know about **TWO** of these topics. (Extra credit for **THREE**.)

(A) The independent degrees of freedom of the electromagnetic and gravitational fields.

(B) Parallel transport and curvature. (Emphasize geometrical concepts rather than trying to reconstruct all the equations.)

(C) Lagrangians and variational principles for geodesics.

(D) The *equation of geodesic deviation*,

$$\frac{D^2 w^\alpha}{d\lambda^2} = R^\alpha{}_{\mu\nu\beta}\, u^\mu u^\nu w^\beta .$$

(Explain what the symbols mean and what the equation has to do with tides.)