Math. 489GR November 2009

Test B – Solutions

1. *(35 pts.)* A two-dimensional space-time has the metric tensor given by the line element

$$ds^2 = -dt^2 + \cosh^2 t\, dx^2$$

(i.e., $g_{tt} = -1$, $g_{xx} = \cosh^2 t$, $g_{xt} = 0$).

(a) Find the geodesic equation (the equation of motion of a free particle in this gravitational background).

A Lagrangian that yields the geodesic equation is

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = -\frac{1}{2}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\cosh^2 t \left(\frac{dx}{ds}\right)^2 .$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -\dot{t}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = (\cosh^2 t)\dot{x}.$$

$$\frac{\partial \mathcal{L}}{\partial t} = (\cosh t \sinh t)\dot{x}^2, \qquad \frac{\partial \mathcal{L}}{\partial x} = 0.$$

Thus the equation $\dfrac{\partial \mathcal{L}}{\partial t} - \dfrac{d}{ds}\dfrac{\partial \mathcal{L}}{\partial \dot{t}} = 0$ is

$$\ddot{t} + (\cosh t \sinh t)\dot{x}^2 = 0,$$

and the other component of the geodesic equation is

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{ds}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= -\frac{d}{ds}(\dot{x}\cosh^2 t) \\ &= -\ddot{x}\cosh^2 t - 2(\cosh t \sinh t)\dot{x}\dot{t}. \end{aligned}$$

This last equation is better rewritten

$$\ddot{x} + 2(\tanh t)\dot{x}\dot{t} = 0.$$

(b) Find the curvature (Riemann) tensor for this space-time. *Avoid unnecessary work:* Note that the tensor has very few independent components in dimension 2.

The basic formula is

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}\,,$$

and from (a) we read off the Christoffel symbols

$$\Gamma^t_{xx} = \cosh t \sinh t, \qquad \Gamma^t_{tt} = \Gamma^t_{tx} = \Gamma^t_{xt} = 0,$$

$$\Gamma^x_{tt} = \Gamma^x_{xx} = 0, \qquad \Gamma^x_{xt} = \Gamma^x_{tx} = \tanh t.$$

(Alternatively, you could calculate the Christoffel symbols from the formula

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\mu}(g_{\mu\gamma,\beta} + g_{\beta\mu,\gamma} - g_{\beta\gamma,\mu}). \tag{1}$$

The results can then be used to answer (a) as well as (b).)

By the antisymmetry of the Riemann tensor in each index pair and the diagonality of the metric tensor, we may assume without loss of generality that $\alpha \neq \beta$, $\mu \neq \nu$, and the indices in each pair are in a certain order, say t before x. Thus there is only *one* independent component, which we may take as

$$\begin{aligned} R^t{}_{xtx} &= \Gamma^t_{xx,t} - \Gamma^t_{xt,x} + \Gamma^t_{\sigma t}\Gamma^\sigma_{xx} - \Gamma^t_{\sigma x}\Gamma^\sigma_{xt} \\ &= \frac{d}{dt}(\cosh t \sinh t) - 0 + 0 - (\cosh t \sinh t)\tanh t \\ &= \sinh^2 t + \cosh^2 t - \sinh^2 t; \end{aligned}$$

so

$$R^t{}_{xtx} = \cosh^2 t = -R^t{}_{xxt}\,.$$

To get the other nonzero components, lower the index and use antisymmetry:

$$R_{txtx} = -\cosh^2 t = -R_{xttx}\,,$$

so

$$R^x{}_{ttx} = 1 = -R^x{}_{txt}\,.$$

(c) Find the Ricci tensor and its covariant trace, the Ricci curvature scalar.

The Ricci tensor is

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}\,.$$

In our case we have

$$R_{tt} = R^x{}_{txt} = -1,$$

$$R_{xx} = R^t{}_{xtx} = \cosh^2 t,$$

$$R_{xt} = R_{tx} = R^\mu{}_{x\mu t} = 0.$$

The covariant trace, $R = g^{\alpha\beta} R_{\alpha\beta}$, is

$$-R_{tt} + \frac{1}{\cosh^2 t} R_{xx} = 2.$$

Interestingly, this is a *constant*.

2. *(15 pts.)*

(a) Show that if a vector field ξ^α satisfies Killing's equation,

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0,$$

then

$$p_\alpha \xi^\alpha \,(= p^\alpha \xi_\alpha) = \text{constant}$$

along the geodesic with tangent vector $\dfrac{\vec{p}}{m}$. [Such a field is called a *Killing vector*. The theorem is true in any space-time, but no Killing vectors will exist unless the space-time has some degree of symmetry.]

If s is the affine parameter along the geodesic, then we are supposed to prove that

$$\frac{d}{ds}\big(p^\alpha \xi_\alpha\big) = 0.$$

We can apply the product rule to this derivative in a geometrically covariant way by employing the absolute derivative, which may be written as the contraction of the covariant derivative with the tangent vector to the curve:

$$\frac{D}{ds} = \frac{1}{m}\, p^\beta \nabla_\beta\,;$$

this equation is literally true for $\tilde{\xi}$ and symbolic for $\vec{p}$. But in any event, the absolute derivative of $\vec{p}$ along its own curve vanishes, by definition of a geodesic. So we find

$$\begin{aligned} \frac{d}{ds}\big(p^\alpha \xi_\alpha\big) &= \frac{Dp^\alpha}{ds}\,\xi_\alpha + p^\alpha\,\frac{D\xi_\alpha}{ds} \\ &= 0 + \frac{1}{m}\, p^\alpha p^\beta \nabla_\beta \xi_\alpha \\ &= \frac{1}{2m}\, p^\alpha p^\beta \big(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\big) \\ &= 0. \end{aligned}$$

(The next-to-last step uses the symmetry of $p^\alpha p^\beta$ to relabel dummy indices in half of the expression.)

(b) Show (without using (7.29)) that the "important result" in italics on p. 179 of Schutz is a corollary of the theorem in (a). (In the old edition it's on p. 189.)

By definition of covariant derivative,

$$\nabla_\alpha \xi^\beta = \partial_\alpha \xi^\beta + \Gamma^\beta_{\nu\alpha} \xi^\nu,$$

so

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = g_{\beta\mu} \partial_\alpha \xi^\mu + g_{\beta\mu} \Gamma^\mu_{\nu\alpha} \xi^\nu + g_{\alpha\mu} \partial_\beta \xi^\mu + g_{\alpha\mu} \Gamma^\mu_{\nu\beta} \xi^\nu. \tag{2}$$

If

$$\xi^\beta = 1, \quad \xi^\alpha = 0 \text{ for } \alpha \neq \beta \qquad (\text{i.e., } \xi^a = \delta^{\alpha\beta}), \tag{3}$$

then $p_\beta = p_\alpha \xi^a$ and we have a chance of proving what we want from part (a).

Assume (3) and also that $g_{\mu\nu}$ is independent of x^β. Then (2) becomes (after renaming an index to avoid ambiguity)

$$\nabla_\alpha \xi_\gamma + \nabla_\gamma \xi_\alpha = 0 + g_{\gamma\mu}\Gamma^\mu_{\beta\alpha} + 0 + g_{\alpha\mu}\Gamma^\mu_{\beta\gamma}\,. \tag{3}$$

The metric factor in front merely undoes the index raising in (1):

$$g_{\alpha\mu}\Gamma^\mu_{\beta\gamma} = \tfrac{1}{2}(g_{\alpha\gamma,\beta} + g_{\beta\alpha,\gamma} - g_{\beta\gamma,\alpha}).$$

In our case the first term is zero (like all β derivatives), and the other two terms are antisymmetric in α and γ. So (3) collapses to 0, as claimed.

3. *(25 pts.)* In three-dimensional Minkowski space-time with the line element

$$ds^2 = -dt^2 + dx^2 + dy^2$$

consider the surface $\mathcal{M}$ defined by

$$-t^2 + x^2 + y^2 = 1.$$

Introduce a new coordinate system (r, τ, θ) by

$$\begin{aligned} t &= r \sinh \tau, \\ x &= r \cosh \tau \cos \theta, \\ y &= r \cosh \tau \sin \theta. \end{aligned}$$

(a) Show that $\mathcal{M}$ is the surface $r = 1$. What must be the ranges of the variables τ and θ so that we get *every* point of $\mathcal{M}$, exactly *once*?

Using

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{and} \qquad \cosh^2\tau - \sinh^2\tau = 1$$

we find that

$$-t^2 + x^2 + y^2 = r^2.$$

(This is a hyperboloid of revolution.) To trace out the entire hyperboloid exactly once, let

$$-\infty < \tau < \infty \qquad \text{and} \qquad 0 \le \theta < 2\pi$$

(or any other interval of length 2π.)

(b) Find the metric tensor on $\mathcal{M}$ with respect to the coordinates (τ, θ). ($\mathcal{M}$ "inherits" its geometry from the three-dimensional Minkowski space just as a sphere or other surface gets its geometry from three-dimensional Euclidean space.)

Set $r = 1$ in the coordinate transformation equations and take differentials:

$$\begin{aligned} dt &= \cosh\tau\, d\tau, \\ dx &= \sinh\tau\cos\theta\, d\tau - \cosh\tau\sin\theta\, d\theta, \\ dy &= \sinh\tau\sin\theta\, d\tau + \cosh\tau\cos\theta\, d\theta. \end{aligned}$$

Then you find (intermediate algebra omitted)

$$-dt^2 + dx^2 + dy^2 = -d\tau^2 + \cosh^2\tau\, d\theta^2.$$

(I.e., $g_{\tau\tau} = -1$, $g_{\theta\theta} = \cosh^2\tau$, $g_{\theta\tau} = 0$).

An equivalent way of stating this calculation is: We construct the basis vectors $\vec{e}_\tau$ and $\vec{e}_\theta$, where, for example,

$$\vec{e}_\tau \to \left(\frac{\partial t}{\partial\tau}, \frac{\partial x}{\partial\tau}, \frac{\partial y}{\partial\tau}\right),$$

and take their dot products to get the components of the metric tensor. (These vectors are in three-space, but they are tangent to $\mathcal{M}$ and form a basis for the two-dimensional space of tangent vectors to $\mathcal{M}$ at each point.) The foregoing manipulation with differentials just mechanizes this calculation.

A conceptually simpler but calculationally longer way to reach the same result is to find the metric tensor for the entire three-space in terms of the coordinates (r, τ, θ), then specialize to $\mathcal{M}$ by setting $r = 1$, $dr = 0$.

(c) Show that the Ricci curvature scalar of $\mathcal{M}$ is a constant (independent of τ and θ).

Identify the coordinates t and x of Question 1 with the τ and θ of Question 3. The space-time described in Question 1 is our $\mathcal{M}$, provided that the ranges of the variables there are those we established in part (a) here. (Note that the periodicity of x, or lack thereof, is irrelevant to the *local* differential geometry studied in Question 1.)

The constancy of R indicates that the hyperboloid, as a submanifold of Minkowski space, has constant curvature, just as a sphere has constant curvature as a submanifold of Euclidean space. This should not be surprising, since hyperboloids are the Lorentzian analogues of spheres. [As a space-time, $\mathcal{M}$ is called *two-dimensional de Sitter space*.]

4. *(25 pts.)*

(a) Write out the three components of the Killing equation $\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0$ for the case of the two-dimensional universe $\mathcal{M}$ with (τ, θ) coordinates.

Using the Christoffel symbols from Question 1 and the covariant derivative definition

$$\xi_{\alpha;\beta} = \xi_{\alpha,\beta} - \Gamma^\mu_{\alpha\beta}\xi_\mu$$

you get

$$\frac{\partial\xi_\tau}{\partial\tau} = 0\,, \qquad \frac{\partial\xi_\theta}{\partial\theta} = \cosh\tau\sinh\tau\,\xi_\tau\,, \qquad \frac{\partial\xi_\theta}{\partial\tau} + \frac{\partial\xi_\tau}{\partial\theta} = 2\tanh\tau\,\xi_\theta\,.$$

(b) Find three (independent) conserved quantities for particle motion in $\mathcal{M}$. **Hint:** One Killing vector should be immediately obvious. Hunt for two more by finding a solution of the form

$$\xi_\tau = f(\tau)e^{i\theta}, \quad \xi_\theta = g(\tau)e^{i\theta}$$

(i.e., periodic in θ), and taking real and imaginary parts at the end.

According to Question 2, one conserved quantity is

$$p_\theta = \cosh^2\tau\, p^\theta.$$

It is the linear momentum associated with invariance of the geometry under translation in the spatial coordinate θ. [The corresponding Killing vector in contravariant components is $\xi^\tau = 0$, $\xi^\theta = 1$. Its covariant components are $\xi_\tau = 0$, $\xi_\theta = \cosh^2\tau$, and one can easily check that it satisfies the three Killing equations in (a).]

Substituting the $e^{i\theta}$ ansatz into the three Killing equations, you get

$$f'(\tau) = 0, \qquad ig(\tau) = \cosh\tau\sinh\tau\, f(\tau), \qquad g'(\tau) + if(\tau) = 2\tanh\tau\, g(\tau).$$

Therefore, from the first two equations $f(\tau) = \text{constant}$ and $g(\tau) = -i\cosh\tau\sinh\tau\, f$. (The third equation is then satisfied, too.) It follows that two linearly independent real-valued Killing vectors are

$$\xi_\tau = \cos\theta\,, \quad \xi_\theta = \cosh\tau\sinh\tau\sin\theta$$

and

$$\xi_\tau = \sin\theta\,, \quad \xi_\theta = -\cosh\tau\sinh\tau\cos\theta\,.$$

[To see the geometical significance of these vector fields, look at their contravariant components near the origin (i.e., to first order in (τ,θ)): the first one is

$$\xi^\tau \approx -1\,, \quad \xi^\theta \approx 0,$$

which looks like a time translation; the second one is

$$\xi^\tau \approx -x\,, \quad \xi^\theta \approx -\tau,$$

which is an infinitesimal Lorentz boost. If you were to look one quarter of the way around the world ($\theta = \frac{\pi}{2}$), these two vector fields would exchange roles! Like Minkowski space, the de Sitter space has a very large symmetry group, and consequently has three independent Killing vectors and constants of motion, the maximum number allowed in dimension 2.]

The two resulting constants of motion, according to the formula in Question 2, are

$$p^\tau\cos\theta + p^\theta\cosh\tau\sinh\tau\sin\theta$$

and

$$p^\tau\sin\theta - p^\theta\cosh\tau\sinh\tau\cos\theta.$$

[Again, these become more intuitive if you expand to first order near the origin. The first one becomes p^τ, the energy in a local Lorentz frame, and the second one becomes $p^\tau\theta - p^\theta\tau$, which is a kind of "angular momentum" associated with local Lorentz transformations.]