Direct sums *(Sec. 10)*

Recall that a *subspace* of $\mathcal{V}$ is a subset closed under addition and scalar multiplication.

$\mathcal{V}$ and $\{\vec{0}\}$ are subspaces of $\mathcal{V}$. A *proper* subspace is any subspace other than $\mathcal{V}$ itself. A *nontrivial* subspace is any subspace other than $\{\vec{0}\}$. [*Warning:* Some authors use "proper" to mean "proper and nontrivial".]

THEOREM 10.1. *Every subspace contains* $\vec{0}$.

EXAMPLE: Planes through the origin are subspaces of $\mathbf{R}^3$. Planes that don't pass through the origin are something else (*affine subspaces* or *cosets* (Sec. 11)).

THEOREM (10.2, 10.3). $\mathcal{U}$ *a subspace of* $\mathcal{V} \Rightarrow \dim \mathcal{U} \le \dim \mathcal{V}$. *If* $\dim \mathcal{V} < \infty$, *then equality holds only if* $\mathcal{U} = \mathcal{V}$.

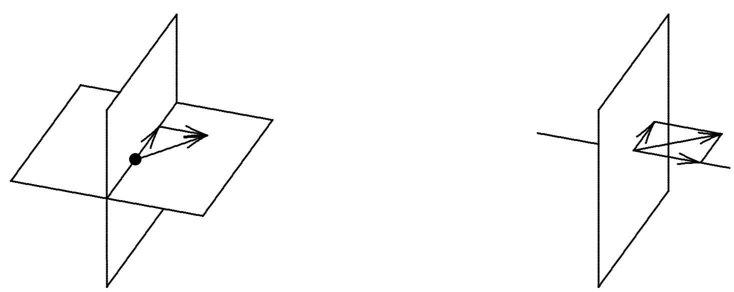
Operations on subspaces: (see diagrams below)

$\mathcal{U} \cap \mathcal{W}$ is a subspace.

$\mathcal{U} \cup \mathcal{W}$ generally isn't one. However —

$\mathcal{U} + \mathcal{W} \equiv \{\vec{u} + \vec{w} : \vec{u} \in \mathcal{U},\ \vec{w} \in \mathcal{W}\}$ is the subspace generated by $\mathcal{U} \cup \mathcal{W}$ ($\equiv$ span $(\mathcal{U} \cup \mathcal{W})$).

Definition: $\mathcal{U} + \mathcal{W}$ is called $\mathcal{U} \oplus \mathcal{W}$ (a *direct sum*) if $\mathcal{U} \cap \mathcal{W} = \{\vec{0}\}$.



The sum of two planes shown on the left is *not* direct. But we can have a direct sum of a plane and a line: $\mathbf{R}^3 = \mathbf{R}^2 \oplus \mathbf{R}^1$, as shown on the right.

Theorem 10.8. *Any* $\vec{v} \in \mathcal{U} \oplus \mathcal{W}$ *has a* **unique** *decomposition* $\vec{v} = \vec{u} + \vec{w}$ ($\vec{u} \in \mathcal{U}$, $\vec{w} \in \mathcal{W}$). (If a sum is not direct, the decomposition is not unique.)

EXAMPLES: In the right sketch, the vector in space decomposes into its projections on the plane and the line. But in the left sketch, a vector in one plane is also the sum of a different vector in that plane with a nonzero vector in the other plane.

Remark: Such a decomposition is analogous to expansion of a vector with respect to a basis, except that here the "pieces" of the vector space have dimensions > 1, possibly. Directness of a sum $\approx$ linear independence of a spanning set.

Theorem 10.9′. $\dim \mathcal{U} \oplus \mathcal{W} = \dim \mathcal{U} + \dim \mathcal{W}$. *More generally,* $\dim \mathcal{U} + \mathcal{W} = \dim \mathcal{U} + \dim \mathcal{W} - \dim \mathcal{U} \cap \mathcal{W}$ *if* $\mathcal{U} \cap \mathcal{W}$ *is finite-dimensional.*

Proof: Let $\mathcal{B}$ be a basis for $\mathcal{U} \cap \mathcal{W}$. By Thm. 9.8′, $\mathcal{B}$ can be extended to a basis $\mathcal{B}_{\mathcal{U}}$ for $\mathcal{U}$, and likewise $\mathcal{B}$ can be extended to a basis $\mathcal{B}_{\mathcal{W}}$ for $\mathcal{W}$. Let $r = \dim \mathcal{U} \cap \mathcal{W}$, $s = \dim \mathcal{U}$, $t = \dim \mathcal{W}$. If s or $t = \infty$, the theorem is trivial. Otherwise,

$$\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_r\}, \quad \mathcal{B}_{\mathcal{U}} = \{\vec{v}_1, \ldots, \vec{v}_r, \vec{u}_1, \ldots, \vec{u}_{s-r}\},$$

$$\mathcal{B}_{\mathcal{W}} = \{\vec{v}_1, \ldots, \vec{v}_r, \vec{w}_1, \ldots, \vec{w}_{t-r}\}.$$

Let $\bar{\mathcal{B}} = \mathcal{B}_{\mathcal{U}} \cup \mathcal{B}_{\mathcal{W}}$ — as a *set*, not a *sequence*. It contains $r + (s-r) + (t-r) = s + t - r$ vectors. Therefore, we will have proved the theorem if we show that $\bar{\mathcal{B}}$ is a basis for $\mathcal{U} + \mathcal{W}$. Clearly $\bar{\mathcal{B}}$ spans $\mathcal{U} + \mathcal{W}$. We show it is independent: Suppose

$$\vec{0} = \sum \lambda^j \vec{u}_j + \sum \mu^j \vec{v}_j + \sum \nu^j \vec{w}_j.$$

Notice that no nontrival linear combination of the $\vec{u}_j$ can lie in $\mathcal{W}$, since then $\mathcal{B}_{\mathcal{U}}$ would be dependent. Therefore, the first (λ) sum is not in $\mathcal{W}$ (unless it is zero), but the remaining terms are in $\mathcal{W}$. Thus we must have

$$\sum \lambda^j \vec{u}_j = \vec{0} = \sum \mu^j \vec{v}_j + \sum \nu^j \vec{w}_j.$$

Then $\mathcal{B}_{\mathcal{U}}$, $\mathcal{B}_{\mathcal{W}}$ each independent $\Rightarrow$ all coefficients $= 0$.

Cartesian product "=" direct sum

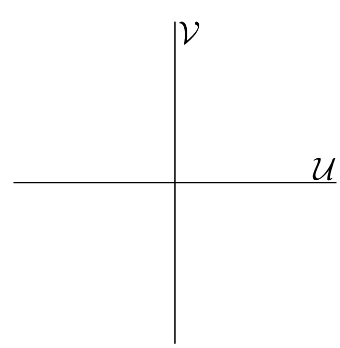
$$\mathcal{U} \times \mathcal{V} \equiv \{(\vec{u}, \vec{v}) : \vec{u} \in \mathcal{U},\ \vec{v} \in \mathcal{V}\}$$

Addition and scalar multiplication are defined componentwise:

$$(\vec{u}_1, \vec{v}_1) + \lambda(\vec{u}_2, \vec{v}_2) \equiv (\vec{u}_1 + \lambda\vec{u}_2,\ \vec{v}_1 + \lambda\vec{v}_2).$$

We blur distinctions: identify

$$\mathcal{U} \times \mathcal{V} \cong \mathcal{U} \oplus \mathcal{V}$$
$$\mathcal{U} \cong \mathcal{U} \times \{\vec{0}\} \subset \mathcal{U} \times \mathcal{V}$$
$$(\vec{u}, \vec{v}) \cong \vec{u} + \vec{v}$$
$$(\vec{u}, \vec{0}_{\mathcal{V}}) \cong \vec{u}$$
$$(\vec{0}_{\mathcal{U}}, \vec{v}) \cong \vec{v}$$



We already did this when we wrote $\mathbf{R}^3 = \mathbf{R}^2 \oplus \mathbf{R}^1$.

In this construction, $\mathcal{U}$ and $\mathcal{V}$ are not, *a priori*, subspaces of a larger space, but we create by brute force a larger space for them to sit inside.

Direct sums of more than 2 spaces: $\mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_3 \oplus \cdots$

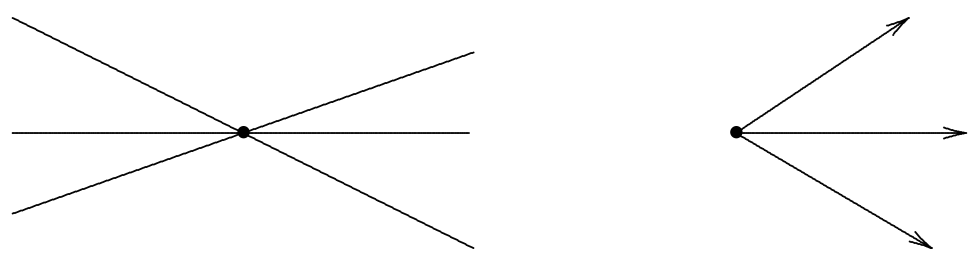
[One motivation: Diagonalization of a matrix with multiple eigenvalues. A definite choice of basis in each eigenspace is excess baggage.]

In the Cartesian-product picture there is no problem in generalizing to more than two factors. If there are infinitely many factors (say countable), we have a choice of considering arbitrary sequences or terminating sequences: $(\vec{v}_1, \vec{v}_2, \vec{0}, \vec{v}_4, \vec{0}, \vec{0}, \vec{0}, \ldots)$. So there are two different definitions.

If the $\mathcal{U}_j$ start as subspaces of $\mathcal{V}$, what replaces the condition $\mathcal{U} \cap \mathcal{W} = \{\vec{0}\}$? This time we'd better stick to finite sums, corresponding to terminating sequences.

First guess: $\mathcal{U}_j \cap \mathcal{U}_k = \{\vec{0}\}, \quad \forall j,\, k.$

This is wrong: Consider three coplanar but noncollinear lines in $\mathbf{R}^3$. It's wrong for the same reason that we can't define linear dependence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$ by: "Some $\vec{v}_j$ is a scalar multiple of some other $\vec{v}_k$."



An analogue of the definition of linear independence that seems suitable for our purpose is:

$$(*) \qquad \text{If } \vec{v}_j \in \mathcal{U}_j \text{ and } \vec{0} = \sum \vec{v}_j, \text{ then } \forall \vec{v}_j = \vec{0}.$$

But this corresponds more nearly to the *conclusion* of Thm. 10.8 than to its hypothesis ($\mathcal{U} \cap \mathcal{V} = \{\vec{0}\}$). Let us conjecture

Theorem. *The following are equivalent:*

1. (†) $\quad \mathcal{U}_j \cap \sum_{k=1}^{j-1} \mathcal{U}_k = \{\vec{0}\} \quad$ *for* $j = 2, 3, \ldots.$

2. (book) [book's similar but stronger condition, p. 57]

$$\mathcal{U}_j \cap \sum_{k=1}^{j-1} \mathcal{U}_k + \mathcal{U}_j \cap \sum_{k=j+1}^{\infty} \mathcal{U}_k = \{\vec{0}\}$$

3. (uniqueness) $\vec{v} \in \sum \mathcal{U}_j \Rightarrow \vec{v}$ *has unique decomposition* $\vec{v} = \sum \vec{u}_j,\ \vec{u}_j \in \mathcal{U}_j$.

4. $(*)$

Any of these defines directness of the sum: $\sum \mathcal{U}_j = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots$.

PROOF: Clearly (book) $\Rightarrow$ (†). Let's show (†) $\Rightarrow$ $(*)$ $\Rightarrow$ (uniqueness) $\Rightarrow$ (book).

(†) $\Rightarrow$ $(*)$: $\displaystyle \vec{0} = \sum_{j=1}^{j_{\max}} \vec{v}_j \Rightarrow -\vec{v}_{j_{\max}} = \sum_{j=1}^{j_{\max}-1} \vec{v}_j.$

Therefore both sides $= 0$ by (†). By induction, all terms $= 0$.

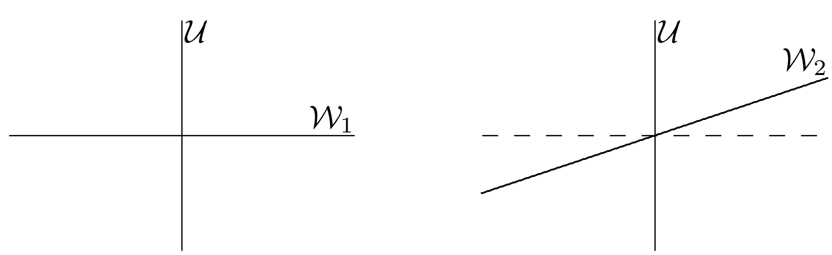
$(*)$ $\Rightarrow$ (uniq): Subtract two candidate sums for $\vec{v}$ and conclude that the terms are individually $\vec{0}$. [Cf. two earlier proofs.]

not(book) $\Rightarrow$ not(uniq): Suppose $\vec{0} \neq \vec{v} \in \mathcal{U}_j \cap \sum_{k=1}^{j-1} \mathcal{U}_k + \mathcal{U}_j \cap \sum_{k=j+1}^{\infty} \mathcal{U}_k$. Then $\exists \vec{w} \in \mathcal{U}_j \cap \sum_{k=1}^{j-1} \mathcal{U}_k$ or $\exists \vec{w} \in \mathcal{U}_j \cap \sum_{k=j+1}^{\infty} \mathcal{U}_k$ $(\vec{w} \neq \vec{0})$. Consider the second case (first is similar): $\vec{w} = \vec{w} + \vec{0}_{\Sigma \mathcal{U}_k} = \vec{0}_{\mathcal{U}_j} + \vec{w}$.

COMPLEMENTS

Definition: If $\mathcal{U} \oplus \mathcal{W} = \mathcal{V}$, then $\mathcal{W}$ is called a *direct complement* of $\mathcal{U}$ [in $\mathcal{V}$].

Complements are not unique:



(So far we have no notion of perpendicularity.)

DEFINITION: $\dim \mathcal{W} \equiv$ *codimension* of $\mathcal{U}$ [relative to $\mathcal{V}$].

The codimension may be finite even when $\dim \mathcal{U}$ and $\dim \mathcal{V}$ are ∞. Although the direct complement is not unique, the codimension is. [Can you prove this?]

EXAMPLE: [Milne p. 32] $\mathcal{V}$ = all continuous functions on $\mathbf{R}$. $\mathcal{U} \equiv$ all such functions satisfying $f(0) = 0$. A direct complement of $\mathcal{U}$ is the span of the constant function 1 (the space of constant functions):

$$f \in \mathcal{V} \Rightarrow f(x) = \alpha + g(x), \text{ where}$$

$$g(x) \equiv f(x) - f(0) \in \mathcal{U}, \text{ and}$$

$$\alpha = \alpha(x) \equiv f(0), \forall x.$$

Thus $\operatorname{codim} \mathcal{U} = 1$.

[Discussion of Sec. 11 (factor spaces) postponed.]