Fundamental concepts
about vector spaces *(Secs. 8–10)*

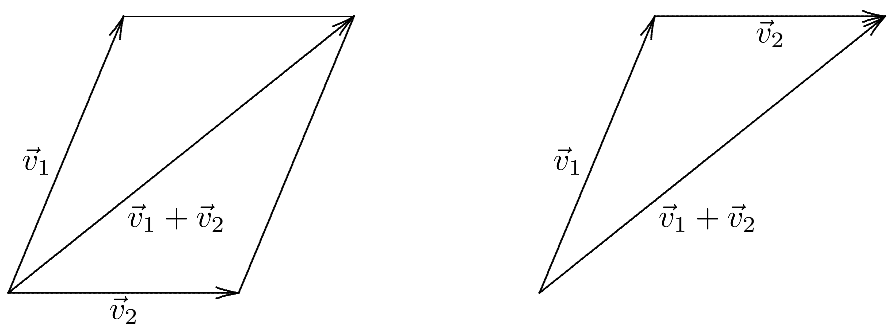
In a nutshell, *vectors* are things which can be

- *added* to each other
- *multiplied* by numbers (*scalars*).

To be worthy of the names, the operations of addition and scalar multiplication must satisfy certain conditions, which make up the famous "eight axioms" in the definition of a vector space. (The definition will be stated in due course.)

Examples — showing how vector spaces arise naturally in concrete contexts

1. Vectorial physical quantities, such as velocities and forces; "quantities with both magnitude and direction." Addition is defined by the parallelogram or triangle construction:



Remark: In examples, the definition of scalar multiplication is usually obvious once addition is described. For integer n, the axioms imply

$$n\vec{v} = \text{sum of } n \text{ copies of } \vec{v} : \qquad \to\to\to\to\to$$

In practice, there is usually a "natural" extension to nonintegral n.

2. $\mathbf{R}^n$, $\mathbf{C}^n$ — spaces of n-tuples of scalars. Addition is defined componentwise:

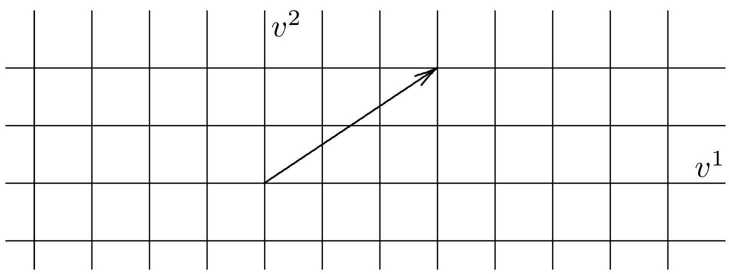
$$\vec{u} = (u^1, u^2, u^3), \quad \vec{v} = (v^1, v^2, v^3);$$

$$\vec{u} + \vec{v} \equiv (u^1 + v^1, u^2 + v^2, u^3 + v^3).$$

I.e., $(\vec{v}_1 + \vec{v}_2)^j = v_1^j + v_2^j \qquad (j = 1, 2, \ldots, n).$

REMARK: The superscript notation for components (which is not universal) must not be confused with exponents. For the moment, its advantage is that it distinguishes a component index from an index labelling members of a set of vectors (see last equation above). A deeper significance to upper and lower indices will be developed in Chapter 7.

CONNECTION BETWEEN EXAMPLES 1 AND 2: Introducing a coordinate system (alias "choice of basis") identifies geometrical vectors with n-tuples. (This is an *isomorphism* — the sums and scalar multiples of vectors can be computed either way, with the same results.)



3. The set of all functions, $f(x)$, on some fixed domain, $\mathcal{D}$:

   f takes values in the field of scalars (**R** or **C**);

   x is a variable running over $\mathcal{D}$, which may be *any set whatsoever.*

   Addition is defined **pointwise** (as usual):

$$(f_1 + f_2)(x) \equiv f_1(x) + f_2(x).$$

REMARKS: (1) If $\mathcal{D}$ is an infinite set (e.g., **R**), then the vector space of functions is infinite-dimensional. (A precise definition of "dimension" will come later.)

(2) Example 2 is a special case of example 3: Take domain $= \{1, 2, \ldots, n\}, \quad x \equiv j$.

(3) In these spaces *multiplication of two vectors* is defined: $(fg)(x) \equiv f(x)\, g(x)$ $[\mathcal{V} \times \mathcal{V} \to \mathcal{V}]$ — but this is a special situation! Let us clarify the distinction among various "products" of vectors:

- Scalar multiplication is a function of type $\mathcal{F} \times \mathcal{V} \to \mathcal{V}$.
- An inner product (treated later) is of type $\mathcal{V} \times \mathcal{V} \to \mathcal{F}$.
- The cross product is of type $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$, where one of the three vectors involved is an *antisymmetric tensor* in disguise (remark to be explained toward the end of the course).

- Multiplication of functions is of type $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

4. Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0.$$

Every complex-valued solution is of the form

$$y = f(x) = A\,e^{i\omega x} + B\,e^{-i\omega x} \qquad (A, B \in \mathbf{C}).$$

The space of all these is a vector space under addition defined as in ex. 3. (It's a *subspace* of the space in ex. 3.) The formula sets up an isomorphism with $\mathbf{C}^2$: $f \leftrightarrow (A, B)$. A different isomorphism is given by

$$f(x) = C \cos \omega x + D \sin \omega x$$

— cf. rotation of axes in ex. 1.

Recall that even if we are interested only in *real-valued* solutions, the complex numbers are useful in finding them. At root, the reason is that $y = e^{rx}$ (r in $\mathbf{C}$, possibly) is an *eigenvector* [Chap. 6] of the differentiation operation:

$$\frac{d}{dx} y = ry.$$

This converts the calculus problem $d^2y/dx^2 + \omega^2 y = 0$ into the algebraic problem $r^2 + \omega^2 = 0$ (hence $r = \pm i\omega$).

REMARK: The solution space of a linear homogeneous *partial* differential equation is an infinite-dimensional vector space. Hence functional analysis is a central tool of PDE theory, as elementary linear algebra is of ODE theory.

5. A complex vector space which arises in classical physics is the space of possible pure polarization states of a light wave. (Details upon request.)

Now we get down to business:

Let $\mathcal{F}$ denote the real numbers, the complex numbers, or some other field. [Read Chap. 2 for definition of "field".] The field of integers modulo a power of 2 is of some interest in applied mathematics, because of computers. Perhaps some future philosopher of mathematics will make the pronouncement,

> *"Intel™ makes the integers from* 0 *through* $2^{32} - 1$*; all the rest is the work of man."*

Definition: A *vector space* is a set $\mathcal{V}$ endowed with two operations,

$$\textit{addition} \quad \mathcal{V} \times \mathcal{V} \to \mathcal{V} : \qquad (\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v}$$

and

$$\textit{scalar multiplication} \quad \mathcal{F} \times \mathcal{V} \to \mathcal{V} : \qquad (\lambda, \vec{u}) \mapsto \lambda\vec{u},$$

satisfying the following conditions:

1. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ $\qquad (\forall \vec{u}, \vec{v}, \vec{w} \in \mathcal{V})$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\exists^{(}\vec{0}^{)} \in \mathcal{V}\ \forall \vec{u} \in \mathcal{V}:\ \vec{u} + \vec{0} = \vec{u}$
4. $\forall \vec{u} \in \mathcal{V}\ \exists \vec{v} \equiv -\vec{u} \in \mathcal{V} : \vec{u} + (-\vec{u}) = \vec{0}.$

[So far we have said $\mathcal{V}$ is a commutative group.]

5. $\lambda(\mu\vec{u}) = (\lambda\mu)\vec{u}$ $\qquad (\forall \lambda, \mu \in \mathcal{F},\ \forall \vec{u} \in \mathcal{V})$
6. $1\vec{u} = \vec{u}$ $\qquad$ (1 = multiplicative identity element of $\mathcal{F}$ (= the number 1))
7. $(\lambda + \mu)\vec{u} = \lambda\vec{u} + \mu\vec{u}$
8. $\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$

Subtraction and *scalar division* are defined in the obvious ways. All the obvious and familiar calculational procedures are valid (if one avoids nonsensical things like dividing by a vector!) and will be used without comment.

Exercise: Verify that the space $\mathcal{H}$ of all complex-valued functions defined on a (fixed but arbitrary) set $\mathcal{A}$ satisfies all the axioms in the definition of a vector space.

Subspaces

Consider this question: Let $\mathcal{U}$ be the set of all continuous functions on $[0, \infty)$ satisfying $f(1) = 0$. Is $\mathcal{U}$ a vector space?

It is *not* necessary to verify the 8 axioms, since we just did that, for addition and scalar multiplication of functions, in the exercise above! What's at issue here is two unnumbered assertions in the definition:

$$\forall \vec{u}, \vec{v} \in \mathcal{V},\ \vec{u} + \vec{v} \text{ is } \textit{defined} \text{ and is } \textit{in the set } \mathcal{V};$$

and similarly for scalar multiplication. In a situation like this, we need only show that $\mathcal{U}$ is *closed* under the vector operations. ($\mathcal{U}$ must be assumed nonempty. Also, one must and can prove, once and for all, that closure $\Rightarrow \mathcal{U}$ contains $\vec{0}$ and $-\vec{v}$. Cf. Thms. 8.1, 8.2.)

Definition: A (nonempty) subset $\mathcal{U}$ of a vector space $\mathcal{V}$ is a *subspace* if one of the following equivalent conditions holds:

(1)
$$\forall \vec{u}, \vec{v} \in \mathcal{U}: \ \vec{u} + \vec{v} \in \mathcal{U}, \quad \text{and}$$
$$\forall \vec{u} \in \mathcal{U}, \forall \lambda \in \mathcal{F}: \ \lambda \vec{u} \in \mathcal{U}.$$

(2)
$$\forall \vec{u}, \vec{v} \in \mathcal{U}, \forall \lambda \in \mathcal{F} \ : \ \lambda \vec{u} + \vec{v} \in \mathcal{U}.$$

(3)
$$\forall n, \forall \vec{u}_1, \ldots, \vec{u}_n \in \mathcal{U}, \forall \lambda_1, \ldots, \lambda_n \in \mathcal{F}:$$
$$\sum_{j=1}^{n} \lambda_j \vec{u}_j \in \mathcal{U}.$$

[(3) follows from (1) or (2) by induction. (1) is weakest, (3) strongest. Which is more useful depends on whether "$\mathcal{U}$ is a subspace" is the conclusion or the hypothesis of your argument!]

Examples [cf. Milne p. 20].

1. $\mathcal{V}$ = space of sequences of scalars, $(x^1, x^2, \ldots, x^j, \ldots)$ [= function space based on domain $\mathcal{A} = \mathbf{Z}^+$].

   In analysis, subspaces of "well-behaved" sequences are more useful than $\mathcal{V}$; e.g.,

   a) $\mathcal{U}$ = space of bounded sequences ($\sup |x^j| < \infty$).

   b) $\mathcal{U}$ = space of finite sequences ($x^j = 0$ for $j >$ some J).

   c) $\mathcal{U}$ = space of absolutely summable sequences ($\sum_{j=1}^{\infty} |x^j| < \infty$).

2. For functions of a real (rather than integral) variable there are similar conditions at infinity, and also local conditions — e.g.,
   measurable
   locally integrable
   continuous
   differentiable
   analytic

It's easy to verify that these are algebraically closed.

3. $\mathcal{V}$ = space of *polynomials*: $P(t) = a_0 + a_1 t + \cdots + a_n t^n$. [This time a superscript *does* mean exponentiation!] $\mathcal{V}$ is isomorphic to the space of finite sequences (ex. 1(b)):

$$P \leftrightarrow (a_0, a_1, \ldots, a_n, 0, 0, \ldots).$$

   P is of *degree* n if $a_n \neq 0$ and $a_m = 0$ for $\forall m > n$. (Nonzero constants have degree 0; the zero polynomial has degree $-\infty$, or undefined.)

   Which of these subsets are subspaces?

   a) $P(t)$ of degree exactly n

   b) $P(t)$ of degree $\leq n$ [cf. $\mathbf{C}^{n+1}$]

   c) homogeneous $P(t)$ of degree n. [This one is more interesting in several variables. E.g., $x^2 - 2xy + 5y^2$ has degree 2.]