Inner products *(Sec. 12)*

So far our vector spaces carry no concepts of

length
distance
angle
orthogonality (perpendicularity).

An inner product (an *extra* operation) provides these, and also provides *one* possible source of

limits
infinite sums or linear combinations
open and closed sets
continuous functions, differentiable functions, etc.

— in short, a topology.

An inner product is also called a "scalar product" (especially by physicists) and generalizes the "dot product" in $\mathbf{R}^3$. For the moment, following Bowen & Wang, I use the dot notation and state the definition for $\mathcal{F} = \mathbf{C}$; if $\mathcal{F} = \mathbf{R}$, ignore the conjugations.

Definition: An *inner product* is a function

$$\mathcal{V} \times \mathcal{V} \to \mathbf{C} \quad [\mathbf{R}] \qquad\qquad (\vec{u}, \vec{v}) \mapsto \vec{u} \cdot \vec{v}$$

with the properties

1. $\vec{u} \cdot \vec{v} = \overline{\vec{v} \cdot \vec{u}}$ (*[Hermitian] symmetry*)

2. $(\lambda \vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v})$ — hence $\vec{u} \cdot (\lambda \vec{v}) = \overline{\lambda}(\vec{u} \cdot \vec{v})$.

3. $(\vec{u} + \vec{w}) \cdot \vec{v} = \vec{u} \cdot \vec{v} + \vec{w} \cdot \vec{v}$ — hence additive also in the right variable.
((2) and (3) together define *sesquilinearity* ($\mathbf{C}$) or *bilinearity* ($\mathbf{R}$).)

4. $\vec{u} \cdot \vec{u} \equiv \|\vec{u}\|^2 \geq 0$, with equality only when $\vec{u} = \vec{0}$. (*positive definiteness*)

$\|\vec{u}\|$ is called the *norm* (or "length") of $\vec{u}$.

NOTATIONAL VARIATIONS:

(1) $\vec{u} \cdot \vec{v} \equiv (\vec{u}, \vec{v}) \equiv \langle \vec{u}, \vec{v} \rangle \equiv \langle \vec{u} | \vec{v} \rangle$. Such "bracket" notations are standard when the vectors belong to a function space (and may even be necessary then to avoid ambiguity).

(2) Physicists insist the inner product is linear on the *right*:

$$\langle \vec{u}, \lambda\vec{v} \rangle = \lambda\langle \vec{u}, \vec{v} \rangle = \langle \overline{\lambda}\vec{u}, \vec{v} \rangle.$$

This convention has genuine advantages when inner products are related to linear functionals (Chap. 7). (Of course, when the scalars are real, there's no difference.)

For these reasons, in the future we may occasionally adopt a bracket notation with the complex conjugation in the physicists' place.

Examples:

1. $\mathbf{R}^n$ or $\mathbf{C}^n$; $\qquad \vec{u} \cdot \vec{v} \equiv \sum_{j=1}^{n} u^j\, \overline{v^j}$

2.A) Space of terminating sequences of scalars: e.g., $\vec{u} = (u^1, 0, u^3, 0, 0, \dots)$.

$$\|\vec{u}\|^2 \equiv \sum_{j=1}^{\infty} |u^j|^2 < \infty.$$

Remark: Given a formula for $\|\vec{u}\|^2$ for some inner-product space, one can usually guess the formula for the inner product, $\vec{u} \cdot \vec{v}$, itself. Here,

$$\vec{u} \cdot \vec{v} \equiv \sum_{j=1}^{\infty} u^j\, \overline{v^j}.$$

Indeed, the norm determines the inner product via a *polarization identity*,

$$\vec{u} \cdot \vec{v} = \frac{1}{2}[\|\vec{u} + \vec{v}\|^2 + i\|\vec{u} + i\vec{v}\|^2 - (1+i)\|\vec{u}\|^2 - (1+i)\|\vec{v}\|^2]$$

or

$$\vec{u} \cdot \vec{v} = \frac{1}{4}[\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 + i\|\vec{v} - i\vec{u}\|^2 - i\|\vec{v} + i\vec{u}\|^2].$$

2.B) This inner product can't be defined on the space of *all* sequences, since the series would usually diverge. The natural vector space where it lives is

$$\ell^2 \equiv \{\, \vec{u} : \mathbf{Z}_+ \to \mathcal{F}, \quad \sum_{j=1}^{\infty} |u^j|^2 < \infty \}.$$

3. Similarly, we can make inner products of functions, say with a domain $(a, b) \subset \mathbf{R}$, such that

$$\|f\|^2 \equiv \int_a^b |f(x)|^2\, dx < \infty.$$

Here there's a new complication: $\int_a^b |f(x)|^2\,dx$ may be 0 even though $\exists x : f(x) \neq 0$. To satisfy the positive definiteness condition, we must consider two functions "the same" if $\int_a^b |f_1 - f_2|^2\,dx = 0$. I.e., the elements of this space, $\mathcal{L}^2(a,b)$, are *equivalence classes* of functions. More about this when we study *factor spaces*.

REMARK: *Complete* inner product spaces (*Hilbert spaces*) have such nice properties that they are often the best arena for doing analysis — e.g., studying solutions of differential or integral equations. For the moment, think of "complete" as meaning that all sequences or functions for which the inner product converges are included in the space, as we demanded for ℓ^2 and $\mathcal{L}^2$.

4. *Sobolev spaces* have more complicated formulas for their inner products. These spaces are valuable technical tools in analysis.

   A) $\|\vec{u}\|_s^2 = \displaystyle\sum_{j=1}^{\infty} j^{2s}|u^j|^2 < \infty$

   B) $\|f\|_s^2 = \displaystyle\int_a^b [|f^{(s)}(x)|^2 + |f(x)|^2]\,dx < \infty$

   Fourier series coefficients of functions of type B are sequences of type A, roughly speaking.

NORM and METRIC: See Bowen & Wang.

Theorem 12.1 (Schwarz Inequality). [also attached to names of Cauchy and Bunyakovsky]

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$

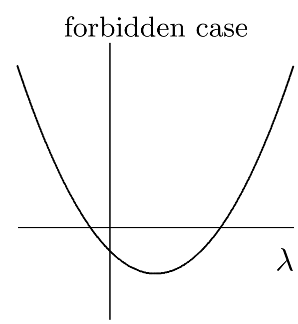
with equality if and only if $\vec{u}$ and $\vec{v}$ are proportional (parallel, dependent, collinear).

PROOF of the equality is in the homework. The proof of the inequality in the book is of the "black magic" type. I'll try to make it more instructive (if verbose):

PROOF FOR REAL CASE: $\forall\lambda$, $\vec{u}$, $\vec{v}$,

$$\begin{aligned} 0 &\leq (\vec{u}+\lambda\vec{v})\cdot(\vec{u}+\lambda\vec{v}) \\ &= \|\vec{u}\|^2 + 2\lambda\vec{u}\cdot\vec{v} + \lambda^2\|\vec{v}\|^2. \end{aligned}$$

$$\text{Therefore } 0 \geq \frac{b^2-4ac}{4} \equiv B^2 - AC = (\vec{u}\cdot\vec{v})^2 - \|\vec{u}\|\|\vec{v}\|^2.$$



Hence QED.

Try complex case by same method:

$$0 \le (\vec{u} + \lambda\vec{v}) \cdot (\vec{u} + \lambda\vec{v}) = \|\vec{u}\|^2 + \lambda\vec{v}\cdot\vec{u} + \overline{\lambda}\vec{u}\cdot\vec{v} + |\lambda|^2\|\vec{v}\|^2.$$

Let $\lambda = \mu + i\nu$, $\quad \vec{u}\cdot\vec{v} = x + iy$. Then

$$\begin{aligned} \lambda\overline{\vec{u}\cdot\vec{v}} &= (\mu + i\nu)(x - iy) = \mu x + \nu y + i(\nu x - \mu y) \\ \Rightarrow \qquad \lambda\vec{v}\cdot\vec{u} + \overline{\lambda}\vec{u}\cdot\vec{v} &= 2\,\mathrm{Re}\,[\lambda\vec{v}\cdot\vec{u}] = 2(\mu x + \nu y) \\ \Rightarrow \qquad 0 &\le \|\vec{u}\|^2 + 2\mu x + 2\nu y + (\mu^2 + \nu^2)\|\vec{v}\|^2 \end{aligned}$$

Take $\nu = 0$: $\qquad 0 \le \|\vec{u}\|^2 + 2\mu x + \mu^2\|\vec{v}\|^2$

$$\Rightarrow \qquad x^2 \le \|\vec{u}\|^2\|\vec{v}\|^2 \quad \text{as in real case}$$
$$\Rightarrow \qquad |\mathrm{Re}\,(\vec{u}\cdot\vec{v})| \le \|\vec{u}\|\|\vec{v}\|.$$

Similarly, $\mu = 0 \quad \Rightarrow \quad |\mathrm{Im}\,(\vec{u}\cdot\vec{v})| \le \|\vec{u}\|\|\vec{v}\|$.
Thus $|\vec{u}\cdot\vec{v}| = \sqrt{x^2 + y^2} \le \sqrt{2}\|\vec{u}\|\|\vec{v}\|$.

This is nice, but not good enough. We haven't yet explored the best direction in the complex plane — the one that will allow us to pick up *all* of $|\vec{u}\cdot\vec{v}|$, not just Re or Im of it, in the quadratic inequality. We must get λ "in phase" with $\vec{u}\cdot\vec{v}$ in some sense.

Try $\lambda = re^{i\theta}$, $\qquad \vec{u}\cdot\vec{v} = se^{i\phi} \qquad (s = |\vec{u}\cdot\vec{v}|)$.

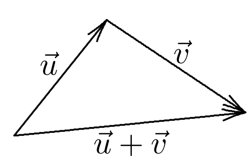
$$0 \le \|\vec{u}\|^2 + re^{i\theta}se^{-i\phi} + re^{-i\theta}se^{i\phi} + r^2\|\vec{v}\|^2.$$

So, choose $\theta = \phi$: $\qquad 0 \le \|\vec{u}\|^2 + 2r|\vec{u}\cdot\vec{v}| + r^2\|\vec{v}\|^2$

$$\Rightarrow \qquad 0 \ge B^2 - AC = |\vec{u}\cdot\vec{v}|^2 - \|\vec{u}\|^2\|\vec{v}\|^2, \qquad \text{QED.}$$

Corollary 12.2 (Triangle Inequality).

$$\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$



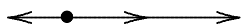
This is a key property of a norm or distance (metric).

Remark: One may study *indefinite* inner products, such that $\vec{u}\cdot\vec{u}$ may be negative (as for space-time vectors), or $\vec{u}\cdot\vec{u}$ may be 0 although $\vec{u} \ne 0$ (as for function spaces before passing to equivalence classes). But these do not satisfy the basic properties of a metric, and hence do not define useful topologies.

Remark: $\dfrac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\|\vec{v}\|}$ measures the difference in direction of $\vec{u}$ and $\vec{v}$. Its extreme values:

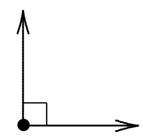
1 if vectors parallel

(from Schwarz)



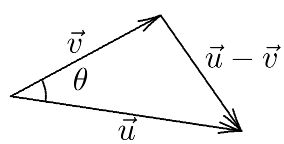
0 if vectors *orthogonal*

(**definition**)



For a *real* inner product space, we define the *angle* θ between vectors by

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \qquad (-1 \le \cos\theta \le 1, \qquad 0 \le \theta \le \pi).$$



Proof that this coincides with the elementary notion of angle:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \quad \text{from def. of norm}$$
$$\text{cf. } \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta \quad \text{from trig}$$

[Book says (p. 69), "... derive the law of cosines," but that's a fraud. You need the independent high-school proof of the law in order to justify the definition of θ.]