Kernel and range

Definition: The *kernel* (or *null-space*) of $\underline{A}$ is

$$\ker\,\underline{A} \equiv \{\vec{v} \in \mathcal{V} :\ \underline{A}\vec{v} = \vec{0}\ (\in \mathcal{U})\}.$$

THEOREM 15.3. *$\ker\,\underline{A}$ is a subspace of $\mathcal{V}$. (In particular, it always contains $\vec{0}_{\mathcal{V}}$.)*

Definition: $\underline{A}$ is *one-to-one* (or *injective*, or *regular*) if

$$\underline{A}\vec{v}_1 = \underline{A}\vec{v}_2\ \Rightarrow\ \vec{v}_1 = \vec{v}_2.$$

It suffices to check the case $\underline{A}\vec{v}_1 = \vec{0}$:

Theorem 15.4. *A* **linear** *operator $\underline{A}$ is injective iff $\ker\,\underline{A} = \{\vec{0}\}$.*

PROOF:

$\Rightarrow$: trivial (special case).

$\Leftarrow$: $\underline{A}\vec{v}_1 = \underline{A}\vec{v}_2\ \Rightarrow\ \underline{A}(\vec{v}_1 - \vec{v}_2) = \vec{0}\ \Rightarrow\ \vec{v}_1 - \vec{v}_2 \in \ker\,\underline{A}\ \Rightarrow\ \vec{v}_1 = \vec{v}_2$.

[Pulling everything back to $\vec{0}$ is a standard trick in dealing with linear operators; recall solution of inhomogeneous ODE via solution of homogeneous ODE.]

Definition: A *homogeneous linear equation* is an equation of the form $\underline{A}\vec{v} = \vec{0}$ ($\underline{A}$ linear).

Thus its solutions are precisely the elements of the kernel of $\underline{A}$.

Definition: An *inhomogeneous linear equation* is one of the form $\underline{A}\vec{v} = \vec{b}$ ($\vec{b} \in \mathcal{U}$ given).

Thus we can reformulate and sharpen Theorem 15.4: *If the corresponding homogeneous equation has nontrivial ($\neq \vec{0}$) solutions, then the solution of an inhomogeneous equation is nonunique (if it exists), and conversely.*

The existence question is related to another concept:

Definition: The *range* of $\underline{A}$ is

$$\operatorname{ran}\,\underline{A} \equiv \{\vec{u} \in \mathcal{U} :\ \exists \vec{v} \in \mathcal{V} \text{ such that } \underline{A}\vec{v} = \vec{u}\}.$$

THEOREM 15.6. *ran $\underline{A}$ is a subspace of $\mathcal{U}$.*

Thus the range of $\underline{A}$ is precisely those elements $\vec{b}$ of $\mathcal{U}$ for which the inhomogeneous equation $\underline{A}\vec{v} = \vec{b}$ has solutions.

Definition: $\underline{A}$ is *onto* $\mathcal{U}$ (or *surjective*) if ran $\underline{A} = \mathcal{U}$.

Theorem 15.8. *dim dom $\underline{A}$ = dim ker $\underline{A}$ + dim ran $\underline{A}$.*

PROOF: For the moment assume that $\mathcal{V}$ ($\equiv$ dom $\underline{A}$) is finite-dimensional. Pick a basis for ker $\underline{A}$ and extend it to a basis for $\mathcal{V}$:

$$\{\vec{w}_1, \dots, \vec{w}_p, \vec{v}_{p+1}, \dots, \vec{v}_n\} \qquad (p \equiv \text{dim ker } \underline{A}).$$

Consider the images,

$$\{\underbrace{\underline{A}\vec{w}_1, \dots, \underline{A}\vec{w}_p}_{\text{all}=\vec{0}}, \underline{A}\vec{v}_{p+1}, \dots, \underline{A}\vec{v}_n\}.$$

They span ran $\underline{A}$; hence $\{\underline{A}\vec{v}_{p+1}, \dots, \underline{A}\vec{v}_n\}$ spans ran $\underline{A}$. In fact, this is a basis for ran $\underline{A}$:

$$\vec{0} = \sum_{p+1}^{n} \lambda^j \underline{A}\vec{v}_j = \underline{A}\left(\sum \lambda^j \vec{v}_j\right) \Rightarrow \sum \lambda^j \vec{v}_j \in \text{ker } \underline{A} \Rightarrow \lambda^j = 0$$

(since $\vec{v}_j$ (for $j > p$) is independent of ker $\underline{A}$). Therefore, dim ran $\underline{A} = n - p =$ dim dom $-$ dim ker. QED.

If dim $\mathcal{V} = \infty$, we need only to show that dim ker $< \infty \Rightarrow$ dim ran $= \infty$. Assume to the contrary that $\{\vec{w}_1, \dots, \vec{w}_p\}$ is a basis for ker $\underline{A}$ and $\{\underline{A}\vec{v}_{p+1}, \dots, \underline{A}\vec{v}_q\}$ is a basis for ran $\underline{A}$. Let $\vec{v}_{q+1}$ be a vector independent of $\{\vec{w}_1, \dots, \vec{v}_q\}$. Then $\underline{A}\vec{v}_{q+1} = \sum_{p+1}^{q} \lambda^j (\underline{A}\vec{v}_j) \Rightarrow \vec{v}_{q+1} - \sum \lambda^j \vec{v}_j \in \text{ker } \underline{A}$, contradicting linear independence of $\{\vec{w}_1, \dots, \dots, \vec{v}_{q+1}\}$.

Finally, the converse — that infinite-dimensional range implies infinite-dimensional domain — is left as an exercise.

COROLLARY 15.7. *dim ran $\underline{A} \leq \min(\dim \mathcal{V}, \dim \mathcal{U})$.*

DEFINITION: dim ran $\underline{A}$ is called the *rank* of $\underline{A}$.

REMARK: rank $\underline{A}$ = dimension of the subspace of $\mathbf{R}^n$ spanned by the columns of the matrix A. By a later theorem, the *column rank* equals the *row rank* of the matrix.

Corollary 15.10. *If* $\dim \mathcal{V} = \dim \mathcal{U} < \infty$, *then* $\underline{A}$ *is injective if and only if it is surjective.*

Cf. the theorem that a set of $\dim \mathcal{V}$ vectors is linearly independent iff it spans.

PROOF: injective $\iff$ dim ker = 0 $\iff$ dim ran = dim $\mathcal{V}$ $\iff$ surjective (since ran $\underline{A} \subseteq \mathcal{U}$).

COUNTEREXAMPLE to Cor. 15.10 if $\dim \mathcal{V} = \dim \mathcal{U} = \infty$: $\mathcal{V} = \mathcal{U}$ = space of sequences $(u^1, u^2, \dots)$; $\mathcal{U}^1$ = space of sequences with $u^1 = 0$; $\underline{A}$ = *right shift operator*:

$$\underline{A}(u^1, u^2, \dots) = (0, u^1, u^2, \dots).$$

Then ran $\underline{A} = \mathcal{U}_1$, and $\underline{A}$ is injective. It is obviously not surjective, despite the fact that its range has the same dimension as $\mathcal{V}$ (even after the distinctions among transfinite cardinal numbers are taken into account).

RANK: A CLOSER LOOK

dim ran $\underline{A}$ + dim ker $\underline{A}$ = dim dom $\underline{A} = n$ (fixed).

Therefore, kernel increases $\Rightarrow$ range decreases. The rank of $\underline{A}$ is thus a doubly important characteristic of $\underline{A}$.

Note: rank $\underline{A}$ = dim ran $\underline{A}$ = codim ker $\underline{A}$. If $m = n$, then dim ker $\underline{A}$ = codim ran $\underline{A}$.

Let's look at this more concretely. Let $m = n = 3$. We find the kernel by reducing matrix A to row echelon form, A_{red}.

Case I: $A_{\text{red}} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ (A nonsingular).

$\underline{A}\vec{v} = \vec{0}$ has unique solution $\vec{v} = \vec{0}$. Thus dim ker = 0. $\underline{A}\vec{v} = \vec{b}$ translates into an augmented matrix $\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{pmatrix}$, hence uniquely solvable equations for v^3, v^2, v^1. Thus dim ran = 3, consistent with dim ker = 0.

Case II: $A_{\text{red}} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}$. Thus dim ker = 1 ($v^3$ arbitrary).

$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & \xi \end{pmatrix} \Rightarrow \underline{A}\vec{v} = \vec{b}$ solvable iff $\xi = 0$. Thus ran $\underline{A}$ has codim 1; dim ran = 2.

What are the other cases?

III. $\begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ IV. $\begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(II–IV are rank 2.)

V. $\begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ VI. $\begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ VII. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(V–VII are rank 1.)

VIII. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

VIII is rank 0. (It was the $\underline{0}$ matrix all along.)

General observation:

$$A_{\text{red}} = \begin{pmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1) Row rank of A = number of nonzero rows of A_{red}.

2) dim ker $\underline{A} = n -$ number of nontrivial homogeneous equations

$$= n - \text{row rank}.$$

But Theorem 15.8 says $\quad \dim\ker = n - \overbrace{\text{column rank}}^{\text{dim ran}}.$

Therefore, row rank = column rank.

(This argument can be made into a complete proof, but we'll find a slicker proof later — Sec. 18.)

3) To see directly what dim ran is, consider solving the inhomogeneous equation, $\underline{A}\vec{v} = \vec{b}$, by the augmented matrix, encountering $\left(A_{\text{red}} \mid \vec{\xi}\right)$. Each zero row of A_{red} gives a constraint on $\vec{\xi}$, hence on $\vec{b}$. Each remaining row gives a solvable equation. Thus dim ran = number of nontrivial rows = row rank.

What happens for an $m \times n$ matrix with $m \neq n$ (i.e., number of equations $\neq$ number of unknowns)? As a crude rule of thumb, we expect

$m > n \Rightarrow$ no solution.

$m < n \Rightarrow$ solution not unique.

But we know there are exceptions. Let's see why:

(1) $m > n \Rightarrow A_{\text{red}}$ has zero-rows: $\begin{pmatrix} 1 & 0 & | & \xi^1 \\ 0 & 1 & | & \xi^2 \\ 0 & 0 & | & \xi^3 \\ 0 & 0 & | & \xi^4 \end{pmatrix}$ If the ξ's next to the zeros are nonzero (the generic case), there is no solution — in accord with the rule of thumb. If all of those ξ's are 0, then some of the original equations were *redundant*. Therefore, the system was "really" $m' \times n$, where $m' \leq n$. We know from the case $m' = n$ that now there can be one solution, or many, or none.

(2) $m < n$: The typical case is nonuniqueness: $\begin{pmatrix} 1 & 0 & * & | & \xi^1 \\ 0 & 1 & * & | & \xi^2 \end{pmatrix}$. (Here v^3 is arbitrary.) Thus $\dim \ker > 0$, $\dim \operatorname{ran} \leq m < n$. But there may be no solution: $\begin{pmatrix} 1 & * & * & | & \xi^1 \\ 0 & 0 & 0 & | & \xi^2 \end{pmatrix}$. Here $\dim \operatorname{ran} < m$, $\dim \ker > 1$. Is it ever possible to have a unique solution? [Hint: Consider the homogeneous case. Recall a homework exercise.]

Linear transformations as vectors

$$\mathcal{L}(\mathcal{V}; \mathcal{U}) \equiv \text{set of linear maps } \underline{A} : \mathcal{V} \to \mathcal{U}.$$

Sums and scalar multiples of such are defined in the usual way for functions. This makes $\mathcal{L}(\mathcal{V}; \mathcal{U})$ a vector space.

THEOREM 16.1. $\dim \mathcal{L}(\mathcal{V}; \mathcal{U}) = mn$ ($= \infty$ if m or $n = \infty$).

PROOF: The infinite-dimensional case is left as an exercise. In the finite-dimensional case, we have seen that $\mathcal{L}$ is isomorphic to the $m \times n$ matrices. A basis for the latter space is obviously $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ etc., which has mn elements. (The proof on pp. 84–85 of Bowen & Wang is the same — it just looks different.)

We turn next to a precise definition of "isomorphic", which we have been using (as here) informally.