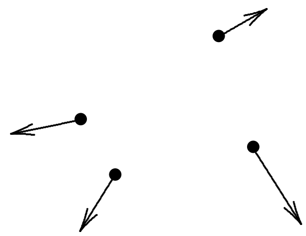
3. Equivalence classes of irrotational vector fields *(a step toward cohomology)*

Definition: A *vector field* is a (usually nonlinear) function $\vec{A}\colon \mathbf{R}^n \to \mathbf{R}^n$ — or, an assignment of an "arrow" $\vec{A}(\vec{x})$ to each point in space.



When $n = 2$, we'll use the notations $x^1 \equiv x$, $x^2 \equiv y$,

$$\vec{A}(\vec{x}) \equiv A_x(x,y)\hat{\imath} + A_y(x,y)\hat{\jmath} \equiv \sum_{i=1}^{2} A_i \hat{e}_i.$$

Let's review some well known results of vector calculus:

Theorem. *If* $A_i = \dfrac{\partial V}{\partial x^i}$ *for some* $V\colon \mathbf{R}^2 \to \mathbf{R}$, *then*

$$\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x}.$$

(We assume that second-order partial derivatives of V exist and are continuous.)

Proof: $\dfrac{\partial^2 V}{\partial x \partial y} = \dfrac{\partial^2 V}{\partial y \partial x}$.

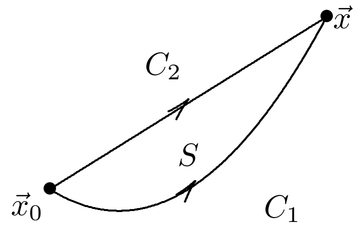
Converse theorem. *Suppose* $\dfrac{\partial A_x}{\partial y} = \dfrac{\partial A_y}{\partial x}$ **everywhere in $\mathbf{R}^2$**. *Then* $\exists V$ *such that* $A_i = \dfrac{\partial V}{\partial x^i}$ *(for short,* $\vec{A} = \nabla V$ *or* $\vec{A}\cdot d\vec{x} = dV$*). Namely,*

$$V(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} \vec{A}\cdot d\vec{x}\,' + C.$$

Here $\vec{x}_0$ *is an arbitrary point and* C *is an arbitrary constant, equal to* $V(\vec{x}_0)$. *The line integral is*

$$\int_{\vec{x}_0}^{\vec{x}} \vec{A}\cdot d\vec{x}\,' \equiv \int_{(x_0,y_0)}^{(x,y)} [A_x(x',y')\,dx' + A_y(x',y')\,dy'],$$

where one variable in each term is implicitly a function of the other.



Proof: The main point is that the line integral is *independent of path.* Green's theorem says

$$\int_{C_1} \vec{A}\cdot d\vec{x} - \int_{C_2} \vec{A}\cdot d\vec{x} = \oint_{C_1\cup C_2^{\text{reversed}}} \vec{A}\cdot d\vec{x}$$
$$= \pm \int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy = 0.$$

(The sign is + for the situation drawn here. If the curves intersect, we have to add surface integrals over several regions, with appropriate signs.)

It's then easy to calculate the gradient of the line integral $V(\vec{x})$, obtaining $\vec{A}(\vec{x})$ and completing the proof.

Condition	*Math terminology*	*Physics terminology*
$\dfrac{\partial A_x}{\partial y} = \dfrac{\partial A_y}{\partial x}$	closed	irrotational
$\vec{A} = \nabla V$	exact	conservative

So our theorems say:

(A) exact $\Rightarrow$ closed (trivial)

(B) closed *everywhere in* $\mathbf{R}^2$ $\Rightarrow$ exact

Question. *To what extent can we eliminate the condition* "everywhere in $\mathbf{R}^2$"*?*

To see why *some* condition is necessary, consider

$$A_x = \frac{-y}{x^2+y^2}, \qquad A_y = \frac{x}{x^2+y^2}.$$

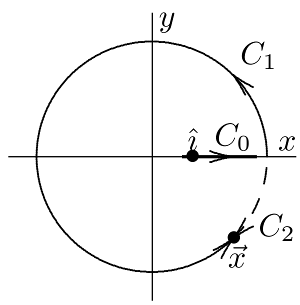
Then $\dfrac{\partial A_y}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{\partial A_x}{\partial y}$ — except at $\vec{0}$, where $\vec{A}$ and its derivatives are undefined. Nevertheless, $\vec{A}$ is not exact (on its natural domain, $\mathbf{R}^2 - \{\vec{0}\}$): Let's evaluate $\int_{\vec{x}_0}^{\vec{x}} \vec{A}\cdot d\vec{x}'$ with $\vec{x}_0 = \hat{\imath} = (1,0)$.

$$\int \vec{A}\cdot d\vec{x}' = \int \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}\,.$$

So we see that $\int_{C_0} \vec{A}\cdot d\vec{x} = 0$, since $y = 0$ and $dy = 0$ on C_0. Along the curves of constant $r \equiv \sqrt{x^2+y^2}$, we parametrize by $x \equiv r\cos\theta$, $y \equiv r\sin\theta$. Thus

$$\begin{aligned}
V(\vec{x}) &= \int \vec{A}\cdot d\vec{x}' \\
&= \int_{\theta'=0}^{\theta(x,y)} \left[\frac{-r\sin\theta'}{r^2}\, d(r\cos\theta') + \frac{r\cos\theta'}{r^2}\, d(r\sin\theta') \right] \\
&= \int_0^{\theta} (\sin^2\theta'\, d\theta' + \cos^2\theta'\, d\theta') = \int_0^{\theta} d\theta' = \theta(x,y) \\
&\big[= \tan^{-1}\frac{y}{x} + N\pi \big].
\end{aligned}$$

Path $C_1 \ \Rightarrow\ 0 \le \theta < 2\pi$. Path $C_2 \ \Rightarrow\ -2\pi < \theta \le 0$.



Path dependence! V cannot be consistently defined. Along some curve from $\vec{0}$ to ∞ it must have a jump of 2π, and $\nabla V = \vec{A}$ must be violated there.

When such a situation develops, the proper response to get a good theorem (both correct and strong) is to go back to the proof that worked for $\operatorname{dom}\vec{A} = \mathbf{R}^2$ and see what property of that domain was used which is lacking for $\operatorname{dom}\vec{A} = \mathbf{R}^2 - \{\vec{0}\}$. It's easy to see that the condition is:

> Let Ω be the domain in which $\vec{A}$ is exact. (In the complement of Ω, $\vec{A}$ may be undefined, or defined and not exact.) Every closed curve, C, in Ω is the boundary of a surface (region), S, contained in Ω.

When $\Omega = \mathbf{R}^2 - \{\vec{0}\}$, this condition is violated by closed curves encircling the origin. Similarly for any domain Ω in $\mathbf{R}^2$ with "holes". For such a domain, the *exact* (conservative) vector fields are a *proper subspace* of the *closed* (irrotational) fields. The factor space

$$H^1(\Omega) \equiv (\text{closed fields})/(\text{exact fields})$$

will be nontrivial, and its properties (in particular, its dimension) will have something to do with how many and what kind of holes Ω has.

To see what these cosets might be good for, consider an arbitrary closed field $\vec{A}$ and an arbitrary closed curve C. Note first that the integral $\oint_C \vec{A} \cdot d\vec{x}$ depends only on the coset, $\overline{A}$, of $\vec{A}$: If $\vec{A}' \in \overline{A}$, then $\vec{A}' = \vec{A} + \nabla V$ for some V and so

$$\begin{aligned} \oint_C \vec{A}' \cdot d\vec{x} &= \oint_C \vec{A} \cdot d\vec{x} + \oint_C \nabla V \cdot d\vec{x} \\ &= \oint_C \vec{A} \cdot d\vec{x} \end{aligned}$$

(since $\oint_\nabla V \cdot d\vec{x} = V(\vec{x}_1) - V(\vec{x}_1) = 0$). For a fixed C, the integral lifts to a linear function on $H^1(\Omega)$.

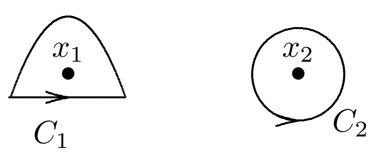
Conversely, if $\oint_C \vec{A}' \cdot d\vec{x} = \oint_C \vec{A} \cdot d\vec{x}$ for *all* closed C, then $\vec{A}'$ and $\vec{A}$ belong to the same coset:

$$\oint_C (\vec{A}' - \vec{A}) \cdot d\vec{x} = 0 \quad \forall C \;\Rightarrow$$

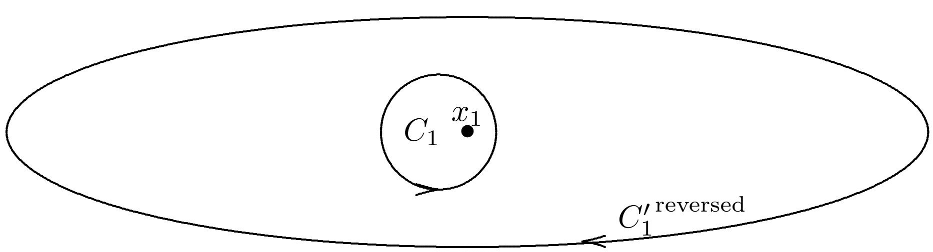
$$V(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} (\vec{A}' - \vec{A}) \cdot d\vec{x} \text{ is well-defined,}$$

so $\vec{A}' - \vec{A} = \nabla V$ is exact.

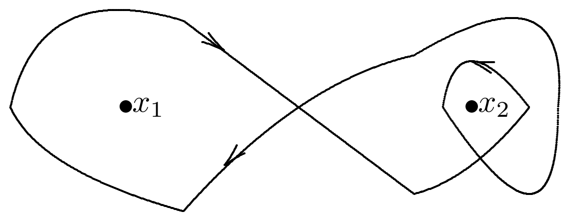
Thus, elements $\overline{A}$ of $H^1(\Omega)$ are characterized, or labelled, by the values of the integrals $\oint_C \vec{A} \cdot d\vec{x}$. Moreover, we don't need to look at *all* curves C. Consider the case of an Ω with finitely many holes, each consisting of one point, $\vec{x}_i$. (These points could be replaced by regions of finite size, as in "exterior" problems in electrostatics. The important thing is that curves can't go "through" the holes and still be in Ω; any simple closed curve has the jth hole either inside or outside.)



Note first that $\oint_C \vec{A} \cdot d\vec{x}$ is the same for any C_1 encircling $\vec{x}_1$ once counterclockwise and encircling no other $\vec{x}_j$. (Apply Green's theorem to the annulus between the curves, which lies in Ω. If the curves touch or cross, introduce a third curve to resolve the confusion, and apply the argument in two steps.)



Next, note that the integral over any closed curve is a linear combination, with integer coefficients, of the integrals over curves of the sort just described. Example:



$$\oint_C \vec{A} \cdot d\vec{x} = 2 \oint_{C_2} \vec{A} \cdot d\vec{x} - \oint_{C_1} \vec{A} \cdot d\vec{x}$$

Therefore, to label the coset $\overline{A}$ we need only to look at j basic curves, C_j. The j numbers $\oint_{C_j} \vec{A} \cdot d\vec{x}$ are called the *periods* of $\overline{A}$ (or of $\vec{A}$).

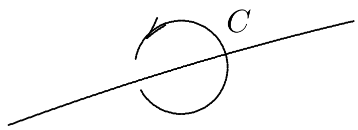
What's still missing is an existence theorem: *For every choice of periods,* $\exists\, \overline{A} \in H^1(\Omega)$ *possessing those periods.* This goes together with our previous observations to make up a special case of *DeRham's theorem* (see H. Flanders, *Differential Forms*, p. 68).

Let's finish off this topic by asking what is the analogue of the foregoing considerations for a vector field in *three* dimensions, $\vec{A}\colon \Omega \subset \mathbf{R}^3 \to \mathbf{R}^3$. It turns out that there are *two* analogues, which I shall discuss in parallel columns to point out their analogy with each other.

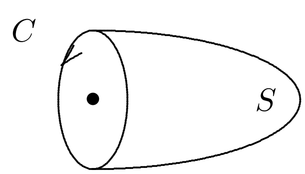
THE EASY, STARTING THEOREM:	
$\vec{A} = \nabla V$ (conservative) $\Rightarrow$ $\nabla \times \vec{A} = \vec{0}$ (irrotational)	$\vec{A} = \nabla \times \vec{W}$ *(no standard name)* $\Rightarrow$ $\nabla \cdot \vec{A} = 0$ (solenoidal)
PHYSICAL EXAMPLES:	
Electric field; velocity field of fluid in irrotational flow.	Magnetic field; velocity field of incompressible fluid.
THE NONTRIVIAL CONVERSE THEOREM:	
$\nabla \times \vec{A} = \vec{0}$ in Ω $\Rightarrow$ $\exists V : \vec{A} = \nabla V$, **provided** that every closed curve in Ω is the boundary of a surface contained in Ω. *(Proof as before.)*	$\nabla \cdot \vec{A} = 0$ in Ω $\Rightarrow$ $\exists \vec{W} : \vec{A} = \nabla \times \vec{W}$, **provided** that every closed surface in Ω is the boundary of a solid region contained in Ω. *(Proof harder.)*
DEFINITION OF THE FACTOR SPACE:	
$H^1(\Omega) \equiv$ (irrotational)/(gradients)	$H^2(\Omega) \equiv$ (solenoidal)/(curls)

“Dangerous” holes:

are those that can be lassoed (e.g., infinitely extended line singularities). (Consider the magnetic field due to an electrical current in a wire. It is not possible to define a “magnetostatic scalar potential” all around the wire.)



Note that *point* singularities are now no obstacle to showing $\oint \vec{A} \cdot d\vec{x} = 0$:



are those that can be caught in sacks (possibly nonspherical). Examples:

A) point singularity. (If a magnetic monopole exists, its magnetic field can’t be derived from a vector potential ($\vec{W}$) defined everywhere in $\mathbf{R}^3 - \{\vec{0}\}$; somewhere there must be a “Dirac string” connecting the monopole to infinity.)

B) ring singularity surrounding line singularity (can be put into a toroidal sack). (A ring of magnetic monopoles has a magnetic field not derivable from a vector potential.)

Periods:

are line integrals around elementary singularities. Irrotational $\vec{A} \Rightarrow$ period is independent of details of curve by Stokes’s theorem:

$$\oint_{C_1 - C_1'} \vec{A} \cdot d\vec{x} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = 0.$$

are surface integrals around elementary singularities. Solenoidal $\vec{A} \Rightarrow$ period is independent of details of surface by Gauss’s theorem:

$$\oint_{S_1 - S_1'} \vec{A} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{A})\, d^3\vec{x} = 0.$$