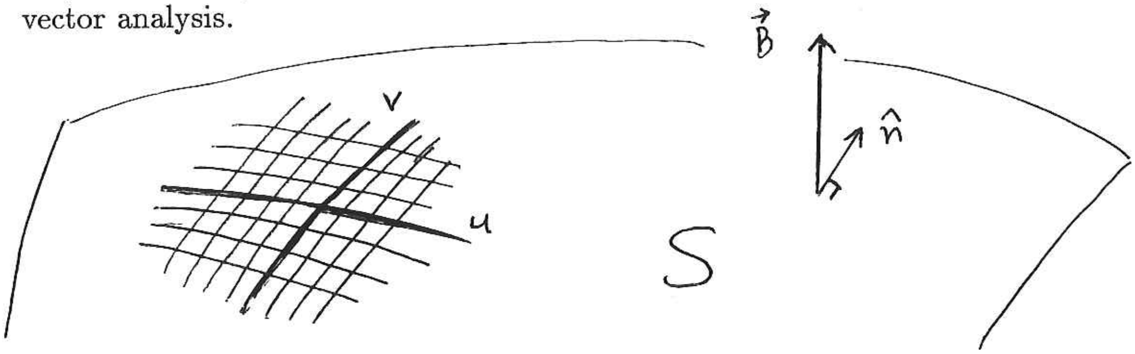
Totally antisymmetric tensors *(Chap. 8)*

We have run out of time, but I want to indicate briefly the relationships among antisymmetric tensors, determinants, volume, and integration. ("Totally antisymmetric" means antisymmetric under interchange of any pair of indices (or argument vectors). The indices or arguments must therefore be either all covariant or all contravariant.)

Certain integrals over hypersurfaces of dimension p are naturally described in terms of rank-p antisymmetric tensors, *without reference to a metric* (inner product). Such integrals generalize the notion of "flux through a surface" in classical vector analysis.



The classical formulation of the magnetic flux through a surface is

$$\Phi_S = \iint_S (\vec{B}\cdot\hat{n})\,dS,$$

where dS is the element of surface area. In terms of two coordinates, u and v, parametrizing S, the unit normal vector is

$$\hat{n} = \frac{\frac{\partial \vec{x}}{\partial u}\times\frac{\partial \vec{x}}{\partial v}}{\left\|\frac{\partial \vec{x}}{\partial u}\times\frac{\partial \vec{x}}{\partial v}\right\|};$$

but the denominator just cancels a factor in the definition of dS, so that the integral in fact doesn't depend on the inner product used to define length and orthogonality. The upshot is that

$$\Phi_S = \iint_S \left[B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy\right],$$

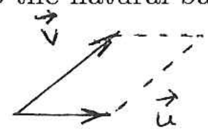
which can be written as

$$\iint_S \frac{1}{2}\sum_{j,k}\Omega_{jk}\,dx^j\,dx^k$$

whre $\underline{\Omega}$ is the antisymmetric tensor associated to $\vec{B}$ via the cross product.

To see how antisymmetric tensors relate to determinants, consider the elementary antisymmetrized monomial ($\{\hat{E}^j\} \equiv$ dual basis to the natural basis of $\mathbf{R}^N$)

$$\hat{E}^1 \wedge \hat{E}^2 \equiv \hat{E}^1 \otimes \hat{E}^2 - \hat{E}^2 \otimes \hat{E}^1,$$
$$[\hat{E}^1 \wedge \hat{E}^2](\vec{v}, \vec{u}) \equiv v^1 u^2 - v^2 u^1.$$



If $\dim \mathcal{V} = 2$, this is the determinant of the matrix whose columns are $\vec{v}$ and $\vec{u}$; furthermore, this number is, up to sign, the area of the parallelogram spanned by those two vectors. In $\mathbf{R}^3$, the area of the parallelogram determined by two vectors is $\|\vec{v} \times \vec{u}\|$, which is still the norm of an antisymmetric combination of the vectors. For three vectors in $\mathbf{R}^3$, the volume of the parallelepiped they span is

$$|\vec{v} \cdot (\vec{u} \times \vec{w})| = \pm \begin{vmatrix} v^1 & v^2 & v^3 \\ u^1 & u^2 & u^3 \\ w^1 & w^2 & w^3 \end{vmatrix} \equiv [\hat{E}^1 \wedge \hat{E}^2 \wedge \hat{E}^3](\vec{v}, \vec{u}, \vec{w}).$$

This pattern continues to higher dimensions and provides the key to defining integration on hypersurfaces (p-dimensional curved sets in N-dimensional space) or manifolds (abstract p-dimensional spaces). The "wedge" operation, $\wedge$, incidentally, is defined for any list of covectors as arguments (and also for any list of vectors), although here we've used it only on the elements of an ON basis for $\mathcal{V}^*$. The definition of the wedge often differs by a factor of $p!$ from the one I've given here.

REMARK: A general tensor of rank greater than 2 is *not* a sum of a totally symmetric and a totally antisymmetric part. More complicated intermediate symmetry types exist, symbolized by *Young diagrams.*

DUALITY

The connection between vectors and antisymmetric 2-tensors in $\mathbf{R}^3$ extends in $\mathbf{R}^N$ to an isomorphism

$$(\text{antisymmetric } p\text{-tensors}) \leftrightarrow (\text{antisymmetric } (N-p)\text{-tensors}).$$

The cross product corresponds to the case $N = 3$, $p = 2$, $N - p = 1$. (An object with only one index counts as antisymmetric by default.) The classical notation for this duality isomorphism is

$$B^{j_1 \ldots j_{N-p}} = \epsilon^{j_1 \ldots j_{N-p} k_1 \ldots k_p} A_{k_1 \ldots k_p},$$

where the tensor components are those with respect to an ON basis, and the object ϵ is defined by

$$\epsilon^{12 \ldots N} \equiv 1,$$
$$\epsilon^{j_1 \ldots j_N} \equiv 0 \text{ if any two indices are equal,}$$
$$\epsilon \text{ is antisymmetric under permutations.}$$