Math. 151–540 (Fulling) 2 May 2008

Final Examination – Solutions

Name: ____

Part I: Multiple Choice (4 points each)

1. Which of these limits must **NOT** be evaluated by l'Hospital's rule?

(A) $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x^2}$

(B) $\displaystyle\lim_{x\to 0} \frac{x^3}{\sin x^2}$

(C) $\displaystyle\lim_{x\to 0} \frac{\cos x^2}{x^2}$ $\Leftarrow$ correct (numerator doesn't $\to 0$)

(D) $\displaystyle\lim_{x\to 0} \frac{\cos t - 1}{t^2}$

(E) $\displaystyle\lim_{x\to \pi} \frac{\sin x}{x-\pi}$

2. Kepler's Third Law states that the distance of a planet from the sun is related to the period of its orbit by $D^3 = CP^2$, where C is a constant. It follows that $\dfrac{dP}{dD} =$

(A) $\dfrac{CP}{D}$

(B) $\dfrac{CP^2}{D}$

(C) $\dfrac{3D^2}{2CP}$ $\Leftarrow$ correct (implicit differentiation)

(D) $\dfrac{D^3}{CP^2}$

(E) $\dfrac{2CP}{D^2}$

3. When you use Newton's formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ to approximate $\sqrt[3]{63}$, the function used is

(A) $f(x) = x - \sqrt[3]{63}$

(B) $f(x) = x^3$

(C) $f(x) = \sqrt[3]{x}$

(D) $f(x) = \sqrt[3]{x} - 63$

(E) $f(x) = x^3 - 63 \Leftarrow$ correct

4. This semester we have learned how to find all of the following **EXCEPT**

(A) an antiderivative of $\dfrac{1}{x}$.

(B) an antiderivative of $\dfrac{1}{\sqrt{1-x^2}}$.

(C) an antiderivative of $\sqrt{x^2+1}$. $\Leftarrow$ correct

(D) an antiderivative of $(x^2+1)^{-1}$.

(E) solutions of the differential equation $\dfrac{dy}{dx} = y$.

5. $\dfrac{d}{dx}\displaystyle\int_1^x \cos(t^4)\,dt =$

(A) $\cos(x^4) - \cos 1$

(B) $\sin(x^4)$

(C) $4\sin(x^4)x^3$

(D) $\cos(x^4) \Leftarrow$ correct (fundamental theorem part 1)

(E) $\sin(x^4) - \sin 1$

6. Evaluate the one-sided limit $\displaystyle\lim_{x \to 5^+} \frac{x-5}{|x-5|}$.

(A) 5

(B) 1 $\Leftarrow$ correct

(C) ± 1

(D) $-\infty$

(E) Limit doesn't exist.

7. Which of these maneuvers is valid?

(A) $\dfrac{d}{dx}\dfrac{1}{x^3+x^4} = \dfrac{d}{dx}\dfrac{1}{x^3} + \dfrac{d}{dx}\dfrac{1}{x^4}$

(B) $\dfrac{d}{dx}\dfrac{1}{x^3+x^4} = \dfrac{1}{3x^2+4x^3}$

(C) $\dfrac{d}{dx}\ln(x^3+x^4) = \dfrac{d\ln(x^3)}{dx}\,\dfrac{d\ln(x^4)}{dx}$

(D) $\dfrac{d}{dx}\ln(x^3+x^4) = \ln(x^4)\,\dfrac{d\ln(x^3)}{dx} + \ln(x^3)\,\dfrac{d\ln(x^4)}{dx}$

(E) None of these. $\Leftarrow$ correct

8. $\displaystyle\lim_{x \to +\infty} x^2 e^{-x} =$

(A) $+\infty$

(B) $-\infty$

(C) 1

(D) 0 $\Leftarrow$ correct (l'Hospital or general knowledge)

(E) Limit doesn't exist.

9. The tangent line to the curve $\mathbf{r}(t) = \langle t, 1-t^2 \rangle$ at the point where $t = 1$ is

(A) $\mathbf{r}(t) = \langle 0, 1 \rangle + (t-1)\langle -1, 2 \rangle$

(B) $\mathbf{r}(t) = \langle 0, 0 \rangle + (t-1)\langle 1, 2 \rangle$

(C) $\mathbf{r}(t) = \langle 1, 0 \rangle + (t-1)\langle -2, 1 \rangle$

(D) $\mathbf{r}(t) = \langle 0, 1 \rangle + (t-1)\langle 1, 0 \rangle$

(E) $\mathbf{r}(t) = \langle 1, 0 \rangle + (t-1)\langle 1, -2 \rangle$ $\Leftarrow$ correct

10. A cannon ball is fired vertically; eventually it falls back to earth. Which of these statements is **FALSE**? (The position (height) is measured upwards from the earth's surface.)
    (A) The acceleration is always negative.
    (B) The acceleration is constant.
    (C) The velocity is zero at the top of the trajectory.
    (D) The velocity changes from negative to positive somewhere. $\Leftarrow$ correct (vice versa)
    (E) The graph of the height as a function of time has no inflection points.

Questions 11–13 concern the function $f(x) = \dfrac{x-2}{x+1}$.

11. The graph of f has
    (A) a horizontal asymptote at $y = -1$ and a vertical asymptote at $x = 1$.
    (B) a horizontal asymptote at $y = 2$ and a vertical asymptote at $x = 0$.
    (C) a horizontal asymptote at $y = 0$ and no vertical asymptote.
    (D) a horizontal asymptote at $y = 1$ and a vertical asymptote at $x = -1$. $\Leftarrow$ correct
    (E) a vertical asymptote at $x = 2$ and no horizontal asymptote.

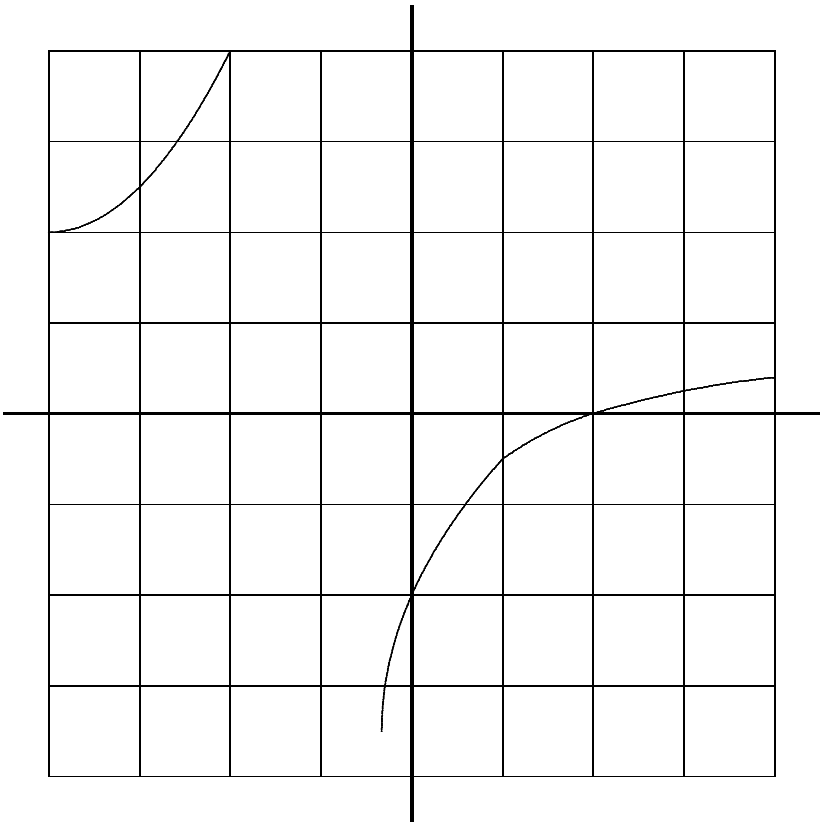
12. The graph of f is
    (A) concave up everywhere in its domain.
    (B) concave up for $x > -1$ only.
    (C) concave down everywhere in its domain.
    (D) increasing everywhere in its domain. $\Leftarrow$ correct
    (E) increasing for $x > -1$ only.

$$f'(x) = \frac{3}{(x+1)^2} > 0\,. \qquad f''(x) = \frac{-6}{(x+1)^3} \text{ changes from } + \text{ to } - \text{ at } x = -1\,.$$

Part II: Write Out (point values as indicated)

Give complete solutions ("show work"). Appropriate partial credit will be given.

13. *(4 pts.)* Sketch the graph of f (using the information in Questions 11–12).



14. *(4 pts.)* A function f is called *odd* if $f(-x) = -f(x)$. Show that if f is odd, then

$$\int_{-2}^{2} f(x)\,dx = 0.$$

Hint: Split the integral into two terms and apply integration by substitution to one of them.

It's $\int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$. In the first term let $x = -u$, so that integral is

$$\int_{2}^{0} f(-u)\,(-1)du = +\int_{0}^{2} (-1) f(u)\,du\,.$$

This precisely cancels the second integral. (Note: This theorem is proved at the end of Stewart Sec. 6.5.)

15. *(each 5 pts.)* Evaluate these integrals. (Note that some are definite and some are indefinite.)

(a) $\displaystyle\int_0^{\pi/3} \frac{dx}{1+x^2}$

$$\tan^{-1} x\Big|_0^{\pi/3} = \tan^{-1}(\pi/3)$$

(which can't be simplified further).

(b) $\displaystyle\int e^{-x^2} x\,dx$

Let $u = x^2$. Then $du = 2x\,dx$ and hence the integral is

$$\frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2} + C\,.$$

(c) $\displaystyle\int \left(t^{3/2} + t^{1/2} + t^{-1/2} + t^{-3/2}\right)\,dt$

$$\frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2} + 2t^{1/2} - 2t^{-1/2} + C\,.$$

(d) $\displaystyle\int_0^{\pi/6} \cos(3x - 5\pi)\,dx$

Let $u = 3x - 5\pi$. Then $du = 3\,dx$, and the integral is

$$\frac{1}{3}\int_{-5\pi}^{-9\pi/2} \cos u\,du = \frac{1}{3}\,\sin u\Big|_{-5\pi}^{-9\pi/2} = -\frac{1}{3}\sin(9\pi/2) = -\frac{1}{3}[\sin(4\pi) + \sin(\pi/2)] = -\frac{1}{3}\,.$$

16. *(5 pts.)* Find the first-order (also called "differential") approximation to $\sin(0.790)$.

Useful information: $\dfrac{\pi}{4} \approx 0.785$, $\dfrac{\sqrt{2}}{2} \approx 0.707$.

$$\sin(x + dx) \approx \sin x + (\cos x)\,dx \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0.005) \approx (0.707)(1.005) = 0.7105\,.$$

17. *(6 pts.)* Let $\mathbf{a} = \langle 3, -1 \rangle$, $\mathbf{b} = \langle 2, 1 \rangle$. Calculate and sketch these vectors:

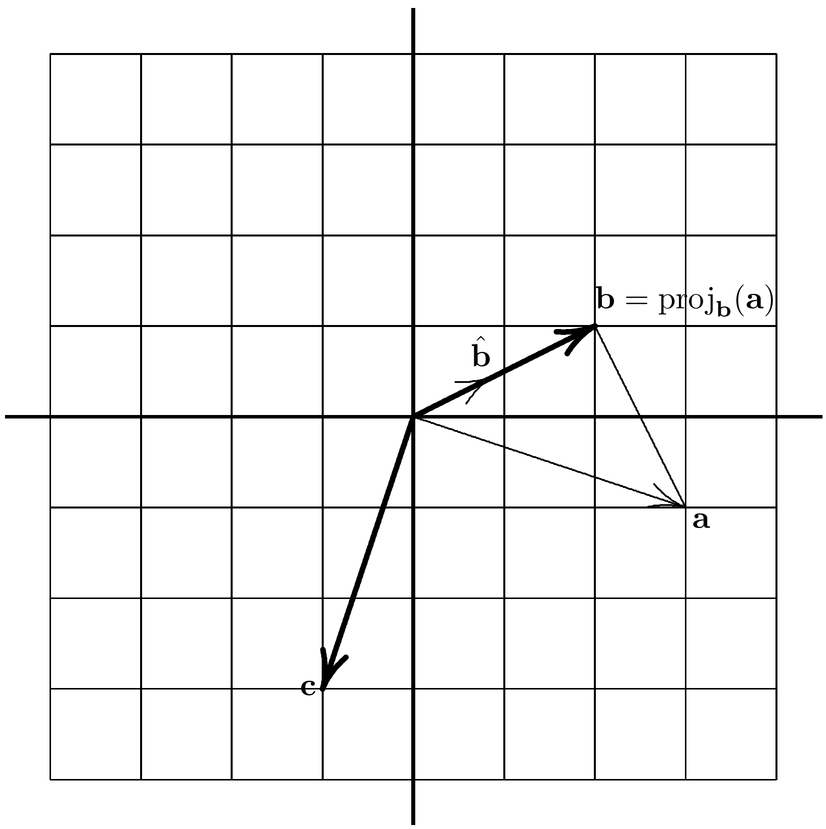
(a) $\mathbf{c} = \mathbf{a} - 2\mathbf{b}$

$$\langle -1, -3 \rangle$$

(b) The vector projection $\mathrm{proj}_{\mathbf{b}}(\mathbf{a})$ (also known as "the component of $\mathbf{a}$ along $\mathbf{b}$")

$$|\mathbf{b}| = \sqrt{5}\,; \qquad \hat{\mathbf{b}} = \frac{1}{\sqrt{5}}\langle 2, 1 \rangle\,; \qquad \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{1}{\sqrt{5}}(6-1) = \sqrt{5}\,;$$

so $\mathrm{proj}_{\mathbf{b}}(\mathbf{a}) = \sqrt{5}\,\hat{\mathbf{b}} = \langle 2, 1 \rangle = \mathbf{b}$. This result may be a bit surprising, so let's check it: It says that the perpendicular from the head of the arrow representing $\mathbf{a}$ to the line parallel to $\mathbf{b}$ must hit the head of the arrow representing $\mathbf{b}$. Well, $\mathbf{a} - \mathbf{b} = \langle 1, -2 \rangle$, and this is indeed perpendicular to $\mathbf{b}$.



18. *(Essay – 6 pts.)* What is the "fundamental theorem of calculus"? Why do you think it is called "fundamental"?

[A good answer will state *both* parts of the theorem, will remark that the second part is the primary tool for evaluating definite integrals in practice, and will comment that the two parts of the theorem show how differentiation and integration are related.]

19. *(7 pts.)* Answer **ONE** of these. (Extra credit for **TWO**.)

(A) The Zircon Auto Company can sell x cars per week at a price of p dollars each, where $p = 3000 - x$. The cost of production is $500 + 800x$ dollars. Find the production level that maximizes the firm's profit (revenue minus cost).

The revenue is $R(x) = xp = 3000x - x^2$. Therefore, the profit is $P(x) = -500 + 2200x - x^2$. Calculate $P'(x) = 2200 - 2x$; it equals 0 when $x = 1100$ (which is the answer). Since $P'' = -2 < 0$, it is indeed a maximum.

(B) A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the whole window is 30 feet, find the dimensions for the window that maximize its area.

There are several ways to choose variables for the dimensions. I will take r to be the radius of the semicircle, which is also half the width of the rectangle. Let h be the height of the rectangle. Then the area of the window is

$$A = 2rh + \frac{1}{2}\,\pi r^2$$

and the perimeter is

$$30 = 2h + 2r + \pi r.$$

Solve the constraint: $2h = 30 - (2 + \pi)r$. Therefore,

$$A(r) = 30r - (2 + \pi)r^2 + \frac{1}{2}\,\pi r^2 = 30r - \left(2 + \frac{\pi}{2}\right) r^2 .$$

So $0 = A'(r) = 30 - (4 + \pi)r$, so

$$r = \frac{30}{4 + \pi}\,, \qquad h = 15 - \left(1 + \frac{\pi}{2}\right)\left(\frac{30}{4 + \pi}\right) .$$