Lecture for Week 3 (Secs. 2.5 and 2.6)

Infinity and Continuity

(including vertical asymptotes from Sec. 2.2)

Infinity is not a number. It is a figure of speech.

Stewart pp. 86–88 and 109 shows graphical examples of functions with *vertical asymptotes*, or, equivalently, *infinite limits*. The asymptote is a vertical line which is approached by the graph. Typically, it appears at a spot where the denominator of the function vanishes (equals 0) but the numerator is *not* zero.

$$\lim_{x \longrightarrow a} f(x) = +\infty$$

means that the values of $f(x)$ can be forced to be arbitrarily large (and positive) by considering only x sufficiently close to a.

This definition doesn't allow x to be actually *equal* to a. Usually $f(a)$ is not even defined.

Note that a function like

$$\left| \frac{\sin\left(\frac{1}{x}\right)}{x} \right|$$

has arbitrarily large values near $a = 0$, but it does *not* approach $+\infty$, because it also has *small* values arbitrarily close to a. (For instance, it's 0 when $x = \frac{1}{N\pi}$.)

The limit is $-\infty$ if the function becomes arbitrarily large and *negative* around the asymptote.

Often a function will approach $+\infty$ from one side and $-\infty$ from the other:

Exercise 2.2.18 (p. 90), extended

Discuss the behavior of $f(x) = \dfrac{6}{x-5}$ around $x = 5$.

As $x \to 5$ from the right, the denominator becomes very small but remains positive:

$$\lim_{x \to 5^+} \frac{6}{x-5} = +\infty.$$

As $x \to 5$ from the left, the denominator becomes very small and negative:

$$\lim_{x \to 5^-} \frac{6}{x-5} = -\infty.$$

The line $x = 5$ is a vertical asymptote. ■

In Sec. 2.6 we have graphs with *horizontal asymptotes*, representing functions with definite *limits at infinity*. The asymptote is a horizontal line that the graph approaches at either the extreme right of the graph, or the extreme left, or both. Notice (last picture on p. 123) that the graph does not need to stay on one side of the asymptote — it can wiggle around it.

For example,

$$\lim_{x \to -\infty} f(x) = L$$

means that the values $f(x)$ can be forced to be arbitrary close to L by considering only values of x that are sufficiently large and negative. (More precisely, I should say "negative and sufficiently large in absolute value.")

Exercise 2.6.33 (p. 133)

$$\lim_{x \to \infty} \frac{x+4}{x^2 - 2x + 5}$$

Exercise 2.6.17

$$\lim_{x \to \infty} (\sqrt{1+x} - \sqrt{x})$$

$$\lim_{x \to \infty} \frac{x+4}{x^2 - 2x + 5}$$

is a *rational function* (ratio of two polynomials). The basis trick for finding limits of such things at infinity is:

Divide numerator and denominator by the highest power appearing in the denominator.

$$\frac{x+4}{x^2-2x+5} = \frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{2}{x}+\frac{5}{x^2}}.$$

The point of this maneuver is that now the denominator approaches 1 as $x \to \infty$, so all we need to do is to take the limit of the numerator, which is 0 in this case.

$$\lim_{x\to\infty} \frac{x+4}{x^2-2x+5} = 0. \qquad \blacksquare$$

It's easy to see what will happen in all problems of this type.

1. If the denominator has higher degree than the numerator, the limit is 0.

2. If the numerator and denominator have the same degree, the limit is some nonzero number (the coefficient of the leading term in the numerator).

3. If the numerator has the higher degree, the

two limits at infinity are infinite (possibly of opposite signs). Example:

$$\frac{x^3+3}{x^2-1} = \frac{x+\frac{3}{x^2}}{1-\frac{1}{x^2}}\,,$$

$$\lim_{x\to+\infty} \frac{x^3+3}{x^2-1} = +\infty, \qquad \lim_{x\to-\infty} \frac{x^3+3}{x^2-1} = -\infty.$$

With rational functions we can't construct an example for which $\lim_{x\to+\infty}$ and $\lim_{x\to-\infty}$ are finite and different. But the inverse trig function $\tan^{-1} x$ has that property (see graph p. 279).

In a problem like

$$\lim_{x\to\infty} (\sqrt{1+x} - \sqrt{x})$$

it helps to "rationalize the numerator" by multiplying numerator and denominator by the "conjugate" expression, in this case $\sqrt{1+x}+\sqrt{x}$. You get

$$\frac{(1+x)-x}{\sqrt{1+x}+\sqrt{x}}\,.$$

The numerator now goes to 1 while the denominator goes to infinity, so the limit is 0. ■

In both of our exercise examples, the horizontal axis $y = 0$ was a horizontal asymptote. Here is an example with a different result:

Exercise 33, extended

Find any asymptotes of $y = f(x) = \frac{x}{x+4}$, and state the corresponding limits involving infinity.

There is a vertical asymptote at $x = -4$.

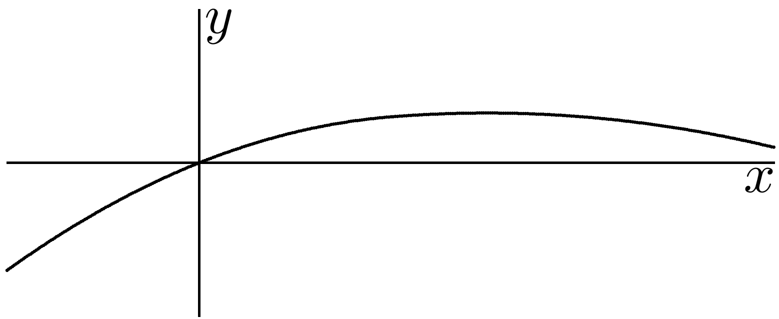
$$\lim_{x \to -4^+} f(x) = -\infty, \qquad \lim_{x \to -4^-} f(x) = +\infty.$$

$$\lim_{x \to \pm\infty} f(x) = \lim \frac{1}{1 + \frac{4}{x}} = 1.$$

There is a horizontal asymptote at $y = 1$. $\blacksquare$

Now, what is continuity?

The intuitive idea is that a function is continuous if its graph can be drawn in one stroke, never lifting the pencil from the paper.



The practical meaning of continuity for doing calculations is related to something we talked about last week. Remember that the crucial practical question about limits is, when do we know that

$$\lim_{x \to a} f(x) = f(a)\,?$$

More generally, in evaluating limits we often wanted to "push the limit through" a function:

$$\lim_{u \to b} f(g(u)) = f\left(\lim_{u \to b} g(u)\right) ?$$

If $\lim_{u \to b} g(u)$ exists and equals a, then that maneuver is correct, provided that

$$\lim_{x \to a} f(x) = f(a). \qquad (*)$$

If $(*)$ is true, we say that "f is continuous at a" — a very convenient property for a function to have!

If a function *fails* to be continuous at a, then one of three things has happened:

1. $f(a)$ is not defined.

2. $\lim_{x \to a} f(x)$ does not exist.

3. Those two numbers exist but are not equal.

(It is possible for both 1 and 2 to happen at once.)

The limits that define derivatives, of the type

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a},$$

are the classic example of discontinuous functions of type 1. They are the main reason for studying limits at the beginning of calculus.

We have seen various examples of type 2, including (a) vertical asymptotes; (b) points where left and right limits exist but are not equal.

It is easy to draw graphs of functions of type 3: take a continuous function and move one point on it vertically to a strange place. (For instance, see the point $(-2, -3)$ in the graph for Exercise 2.2.1. Also do Exercise 2.5.17.)

It is harder to find such functions in "real life", but here's an attempt. Suppose you shoot a gun at a target containing a hole exactly the size of a bullet. If your aim is perfect, the bullet lands on the far side. Otherwise, it ricochets off the target and lands on your side.

Another way to summarize continuity is that **nearby inputs yield nearby outputs**: *If x is close to a, then $f(x)$ is close to $f(a)$.*

As it stands, this is rather vague and says nothing about $f(x)$ for any one particular x. The precise definition of a limit (and hence of continuity) is a statement about the "collective" behavior of *all* points $(x, f(x))$ for x sufficiently near a.

Exercise 2.5.23

Explain why $h(x) = \sqrt[5]{x-1}(x^2-2)$ is continuous on its domain. State the domain.

Remember that common functions (given by formulas) are continuous at most points. Just look for points where something strange might be happening.

The only problem with the domain might be that the argument of the root function might be negative. But that is a problem only for *even* roots; 5 is odd. So the domain is the entire real line.

Polynomials are always continuous, and so are root functions. So this function is continuous everywhere. (We've used Theorems 5, 6, 8, and 4 (part 4).) ■

More examples:

$\dfrac{1}{\sqrt{x-1}}$ has domain $x > 1$. It is continuous there (but not at $x = 1$, which is a vertical asymptote).

$\sqrt{x-1}$ has domain $x \geq 1$. Unlike the first example, it *is* defined when $x = 1$. It is even continuous there, because (according to the definition and example on p. 114) the points $x < 1$ that are outside the domain don't count in deciding whether the limit exists at a point inside the domain.

$(x+2)^{-1/3}$ is defined and continuous for all x except -2, which is a vertical asymptote. (This counts as a "zero in the denominator", although the denominator is hidden in the negative exponent.)

$\cot x$ also has a hidden denominator. It equals $\frac{\cos x}{\sin x}$, so it is undefined whenever $x = N\pi$ (N an integer). It is defined and continuous everywhere else.

Exercise 2.5.39

Find the values of c and d that make h continuous on $\mathbf{R}$.

$$h(x) = \begin{cases} 2x & \text{if } x < 1, \\ cx^2 + d & \text{if } 1 \leq x \leq 2, \\ 4x & \text{if } x > 2. \end{cases}$$

Since polynomials are continuous, the only places of possible discontinuity are the "joints" at $x = 1$ and 2. We must insist that the polynomials match there.

$$2 \times 1 = c \times 1^2 + d \quad \text{and} \quad c \times 2^2 + d = 4 \times 2.$$

$$c + d = 2, \qquad 4c + d = 8.$$

$$c = 2, \qquad d = 0$$

is the only solution. ■