Lecture for Week 4 (Secs. 3.1–3)

Derivatives

(Finally, we get to the point.)

There are two equivalent forms of the definition of a derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\,.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\,.$$

One uses whichever seems most natural or useful in a given context. Usually the second is more convenient for calculating derivatives from first principles.

I assume that you have read all the explanations in the book, so I will mostly do examples.

Exercise 3.1.27

Find the derivative of $f(x) = x^4$ using the definition of derivative.

Actually, we already did this, for the special case $x = 1$, remember?

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} .$$

I showed you how to do this in Lecture 2, slides 25–27.

$$(x+h)^4 = x^4 + 4x^3 h + O(h^2)$$

where $O(h^2)$ is shorthand for "terms containing

h^2 or even higher powers". So

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{4x^3h + O(h^2)}{h} \\ &= \lim_{h\to 0} \left(4x^3 + O(h)\right) \\ &= 4x^3. \end{aligned} \qquad \blacksquare$$

Now I hear some of you saying, "Why are we doing this? Don't we all know that the derivative of x^n is nx^{n-1}? And that easy formula is in the next section."

The immediately practical reason is that you'll have to do something like this on the first exam, and maybe also on the final, so you'd better get used to it.

The deeper reason is that formulas are use-less unless you know what they mean. In science and engineering courses you'll study how equations involving derivatives arise out of the physics of a problem, by considering how some variables change as other variables change. There *the derivative is the answer, not the question.* In those situations you don't know beforehand what the functions involved are; you just have to understand that one is the derivative of another. Applying the definition to particular functions is simple, *less* abstract, practice for that kind of thinking.

Also, of course, we will need the definition, and the skills of working with it, to find derivatives of nonpolynomial functions such as $\sin x$ later.

So, let's do one more, pretending that none of you have ever heard of the chain rule and the power rule.

Exercise 3.1.23

Find the derivative of $g(x) = \sqrt{1 + 2x}$ using the definition of derivative.

$$g'(x) = \lim_{h\to 0} \frac{\sqrt{1+2x+2h} - \sqrt{1+2x}}{h}\,.$$

Somehow we have to squeeze an h out of the numerator so we can cancel it. Experience shows that multiplying top and bottom by

$$\sqrt{1+2x+2h} + \sqrt{1+2x}$$

might help. (See Lecture 3, slide 14.)

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{(1+2x+2h)-(1+2x)}{h(\sqrt{1+2x+2h}+\sqrt{1+2x})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}} \\
&= \frac{1}{\sqrt{1+2x}}\,. \qquad \blacksquare
\end{aligned}
$$

Ok, let's go on to the next section and use the power rule and other theorems!

Exercise 3.2.7

Differentiate $Y(t) = 6t^{-9}$.

This is a simple instance of the power rule:

$$Y'(t) = (-9)6t^{-10} = -54t^{-10}$$

(alias $-\dfrac{54}{t^{10}}$). $\blacksquare$

Exercise 3.2.15

Differentiate

$$y = \frac{x^2 + 4x + 3}{\sqrt{x}}.$$

Hard way: Use the quotient rule.

Easy way: $\sqrt{x} = x^{1/2}$, so

$$y = x^{3/2} + 4x^{1/2} + 3x^{-1/2}.$$

$$y' = \tfrac{3}{2}x^{1/2} + 2x^{-1/2} - \tfrac{3}{2}x^{-3/2}.$$

If you like, you can write this back in terms of square roots, as in the book's answer. ■

Exercise 3.2.37

Find the equation of the tangent line to the curve

$$y = x + \frac{4}{x}$$

at the point $(2, 4)$.

The slope of the line is the derivative of the function.

$$y' = 1 - \frac{4}{x^2}.$$

$$y'(2) = 1 - \frac{1}{1} = 0.$$

So the tangent line is the horizontal line through the point:

$$y = 4. \qquad \blacksquare$$

Usually the tangent will *not* be horizontal, so you have to use the point-slope form of a line:

$$y = f'(a)(x - a) + f(a)$$

is the tangent line to the graph of f at the point $(a, f(a))$.

If g is the derivative of f, then f is called an *antiderivative* of g.

Later we will see that any two antiderivatives of g differ only by a constant (provided that the domain of the functions is the whole real line, or a single interval).

You need to find antiderivatives in first-semester physics, so we will treat them briefly now.

Exercise 5.7.1 (p. 353)

Find the most general antiderivative of

$$f(x) = 12x^2 + 6x - 5.$$

Exercise 5.7.9

Find the most general antiderivative of

$$g(t) = \frac{t^3 + 2t^2}{\sqrt{t}}.$$

$$f(x) = 12x^2 + 6x - 5.$$

Use the power rule in reverse, remembering to divide each term by the *new* exponent:

$$F(x) = 4x^3 + 3x^2 - 5x + C$$

(C = arbitrary constant) is the most general antiderivative of f. ■

As you may know already, later we'll write

$$\int (12x^2 + 6x - 5)\, dx = 4x^3 + 3x^2 - 5x + C.$$

$$
\begin{aligned}
g(t) &= \frac{t^3 + 2t^2}{\sqrt{t}} \\
&= t^{5/2} + 2t^{3/2}.
\end{aligned}
$$

So

$$G(t) = \tfrac{2}{7}t^{7/2} + \tfrac{4}{5}t^{5/2} + C$$

is the most general antiderivative.

Replacing the square root factor by a power, which was convenient in an earlier example (slide 15), is almost essential now. ■

Exercise 3.3.7

The position function of a particle is

$$s = t^3 - 4.5t^2 - 7t \quad (\text{for } t \geq 0).$$

When does the particle reach a velocity of 5 m/s?

$$s = t^3 - 4.5t^2 - 7t \quad (\text{for } t \geq 0).$$

The velocity is the derivative of the position:

$$v = 3t^2 - 9t - 7.$$

$$5 = 3t^2 - 9t - 7 \ \Rightarrow\ 3t^2 - 9t - 12 = 0.$$

$$0 = t^2 - 3t - 4 = (t - 4)(t + 1).$$

The root $t = -1$ is outside the stated domain, so $t = 4$ s. ■

Exercise 3.3.11

A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the circle's area is increasing after

(a) 1 s

(b) 3 s

(c) 5 s

We need the formula for the area of a circle of radius r: $A = \pi r^2$. We also need to know what the radius is at each time; from the problem statement, that is clearly $r = 60t$. Therefore,

$$A = (60)^2 \pi t^2 = 3600\pi t^2.$$

$$\frac{dA}{dt} = 7200\pi t.$$

To get the numerical answers, insert $t = 1, 3, 5$. (Units $= \mathrm{cm}^2/\mathrm{s}$.) $\blacksquare$