Lecture for Week 7 (Secs. 3.10–12)

Derivative Miscellany II

Related rates

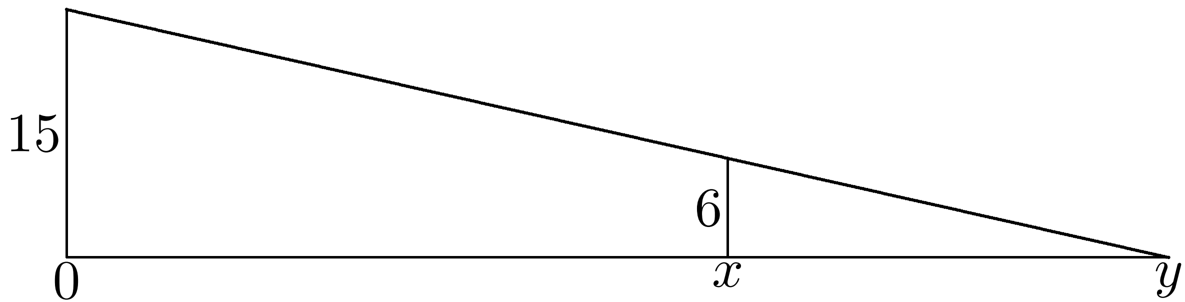
"Related rates" means "applications of the chain rule and implicit differentiation."

"Applications" in this context means "practical applications" (i.e., word problems).

Exercise 3.10.7

A lamp is at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s. How fast is the tip of his shadow moving when he is 40 ft from the pole? How fast is the shadow lengthening at that point?

Every nontrivial word problem's solution starts with some hard work specific to that problem, having nothing to do with the mathematical concept being demonstrated. Often the necessary reasoning is geometrical. For example, in Exercise 3.10.5 we need to know the formula for the surface area of a sphere; I chose not to do that problem, because it is so similar to the sphere problem that I gave so much attention to while we were studying the chain rule. In the present problem we need *similar triangles*.



Here x and y are coordinates, not lengths. The key equation is

$$\frac{y}{15} = \frac{y - x}{6},$$

which simplifies to $y = \frac{5}{3}x$. So

$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{25}{3} = 8.33 \text{ ft/s.}$$

That's how fast the tip moves. The rate of change of the length is

$$\frac{d}{dt}(y - x) = \frac{dy}{dt} - \frac{dx}{dt} = \frac{10}{3} = 3.33 \text{ ft/s.}$$

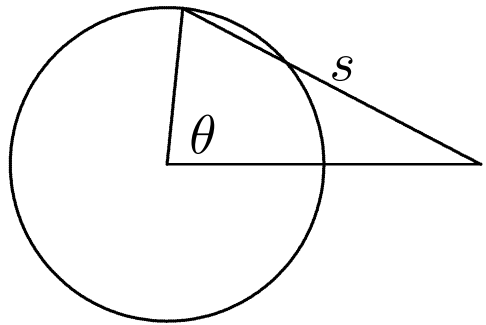
■

Notice that we never had to use the number 40 ft, because the rates turned out to be constants. That is an accident of this problem. However, the point of general importance is that it was necessary to consider a general x, not just $x = 40$, to derive the relation between the rates (derivatives). **Do not plug in the instantaneous values of changing quantities until you have finished differentiating!**

Exercise 3.10.33

A runner runs around a circular track of radius 100 m at speed 7 m/s. Her friend stands a distance 200 m from the center. How fast is the distance between them changing when the distance is 200 m?

Start by drawing a figure, of course!



This time we need the *law of cosines*, the generalization of the Pythagorean theorem to a non-right triangle. Recall that the known sides are 100 and 200.

$$s^2 = 100^2 + 200^2 - 2 \times 100 \times 200 \cos \theta.$$

So $s = 200$ when $100 = 400\cos\theta$, or $\cos\theta = \frac{1}{4}$. Don't reach for the calculator's arccos key yet — we may not need it.

Differentiate the formula for s^2 with respect to t:

$$2s\,\frac{ds}{dt} = 40{,}000\sin\theta\,\frac{d\theta}{dt}\,.$$

Two things to note: (1) **Don't** plug in numbers for s and θ first. (2) Taking the square root first would make the calculus harder, not easier.

Now

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{\frac{15}{16}}\,,$$

so after dividing by $2s = 400$ we get

$$\frac{ds}{dt} = 100\frac{\sqrt{15}}{4}\,\frac{d\theta}{dt}\,.$$

And the angular velocity is $\dfrac{d\theta}{dt} = \dfrac{7}{100}$, so finally

$$\frac{ds}{dt} = \frac{7}{4}\sqrt{15}\text{ m/s.} \qquad \blacksquare$$

The language of differentials

The key idea: $f'(a)$ tells us how to build a *linear approximation* to $f(x)$ that is good for x near a.

$$f(x) \approx f_{\text{approx}}(x) \equiv f(a) + f'(a)(x - a).$$

(f_{approx} is the function whose graph is the tangent line at a.)

Recall that $f'(a) = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$, where Δx (often called h) is a change in x and $\Delta y = f(a+h) - f(a)$ is the resulting change in y. Now let's turn that equation around:

$$\Delta y \approx f'(a)\Delta x \quad \text{if } \Delta x \text{ is small.}$$

You have been taught to interpret $\frac{dy}{dx}$ as $\frac{d}{dx}\, y$, an operation on the function $y = f(x)$. Now we see that, at least intuitively, $\frac{dy}{dx}$ can be broken up like a fraction, $\frac{(dy)}{(dx)}$.

"dx" is what Δx is called when you are announcing your intention to ignore powers of Δx — or to ignore anything that vanishes *faster* than Δx as $\Delta x \to 0$.

(More precisely, one ignores things that vanish even after being divided by Δx; for example,

$$\frac{(\Delta x)^3}{\Delta x} = (\Delta x)^2 \to 0.)$$

In that approximation, Δy equals dy.

Thus $\frac{dy}{dx}$ is not just a notation for f', but can be interpreted as a ratio of two numbers: dx is any change in x, and dy is the corresponding change in y **when you adopt the tangent-line approximation**.

Δx and Δy are called *increments*. dx and dy are called *differentials*.

Leibniz and other early mathematicians thought of dx and dy as *infinitesimal* numbers — so small that the approximate equation

$$\Delta y \approx \frac{dy}{dx}\,\Delta x$$

became *exact*:

$$dy = \frac{dy}{dx}\,dx.$$

Today we avoid that kind of talk in discussing fundamentals; we use *limits* instead. But thinking of $f' = \frac{dy}{dx}$ as a ratio of *small changes* is still very helpful in *applying* calculus.

Example: Find a formula for the rate of change of the area of a circle with respect to its radius. (A plays the role of y, r the role of x.)

1. *Derivation in the language of differentials.* As the radius goes from r to $r + dr$, the area goes from πr^2 to $\pi(r + dr)^2$. (I leave it to you to draw the obvious diagram.)

$$\begin{aligned} dA &= \pi(r + dr)^2 - \pi r^2 \\ &= \pi[r^2 + 2r\,dr + (dr)^2] - \pi r^2. \end{aligned}$$

The terms without any dr factors cancel. We neglect the *second-order* term dr^2 because it is *very* small. (Whereas dr is just “small”.)

$$\begin{aligned} dA &= \pi[r^2 + 2r\,dr + (dr)^2] - \pi r^2 \\ &= 2\pi r\,dr. \end{aligned}$$

Conclusion:
$$\frac{dA}{dr} = 2\pi r.$$

This kind of argument appears often in science and engineering textbooks. *Note that the distinction between dA and ΔA has been blurred.*

2. *Careful restatement in the language of difference quotients and limits.* When the radius changes from r to $r + \Delta r$, the area changes from πr^2 to $\pi(r + \Delta r)^2$.

$$\begin{aligned}\Delta A &= \pi[r^2 + 2r\,\Delta r + (\Delta r)^2] - \pi r^2 \\ &= \pi(2r + \Delta r)\,\Delta r.\end{aligned}$$

Thus $\frac{\Delta A}{\Delta r} = \pi(2r + \Delta r)$, and therefore

$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \pi(2r + \Delta r) = 2\pi r.$$

The differential argument is just shorthand for this.

Better than the linear approximation is the *quadratic approximation*, which fits the graph, not with the line that matches it best, but with the parabola that matches it best:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

You can go on to even higher-degree polynomials, called *Taylor approximations*. These are covered in Chapter 10.

Exercise 19, p. 228, extended

Use differentials to approximate $\sqrt{36.1}$ and $\sqrt{40}$.

We know that $\sqrt{36} = 6$, so that is a good starting point, a.

$$f(x) = \sqrt{x}\,, \qquad f'(x) = \frac{1}{2\sqrt{x}}\,.$$

$$\begin{aligned} \sqrt{x} &\approx f(a) + f'(a)(x-a) \\ &= 6 + \frac{1}{12}(x-36). \end{aligned}$$

$$\sqrt{36.1} \approx 6 + \frac{1}{120} \approx 6.008333.$$

Check: $6.00833^2 = 36.10007$. Pretty good.

$$\sqrt{40} \approx 6 + \frac{4}{12} \approx 6.333333.$$

Check: $6.333333^2 = 40.11111$. Not so good. The approximation is better when $x - a$ is smaller. This method is not very useful for calculating $\sqrt{x}$ numerically unless x is close to a perfect square.

There are two different kinds of approximations ("≈") in the calculations. Besides the differential approximation there is *roundoff error*. (I used *Maple* to do the arithmetic decimally.) ■

Newton's method

One needs a method for finding approximate solutions of equations such as

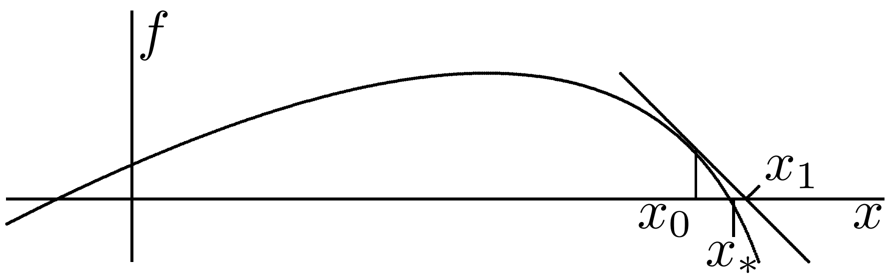
$$\tan x = x \qquad \text{or} \qquad x^5 + 4x^4 + 3x = 10,$$

for which ordinary algebra fails us. We can assume that the equation has the form

$$f(x) = 0$$

(otherwise transpose all the terms on the right side to the left).

First, sketch the graph of f so that you can guess roughly where the roots of $f(x) = 0$ are. Newton's method enables you to improve your guesses. Suppose you guessed x_0 but the correct root is x_*. Slide down (or up) the tangent line at x_0 to its intersection with the horizontal axis. That point, x_1, should be closer to x_*.



We can derive the formula for x_1 by differentials. We don't know what x_* is, but we do know that it is an exact root of f:

$$f(x_*) \equiv y_* = 0.$$

Let $y_0 = f(x_0)$ and calculate the differential $dy = f'(x_0)dx$. That means

$$y_* - y_0 \approx f'(x_0)(x_* - x_0)$$

for any point (x_*, y_*) on the graph near (x_0, y_0).

Solve for x_* , using $y_* = 0$:

$$
\begin{aligned}
y_* - y_0 &\approx f'(x_0)(x_* - x_0) \\
\Rightarrow x_* - x_0 &\approx \frac{-y_0}{f'(x_0)} \\
\Rightarrow x_* \approx x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \,.
\end{aligned}
$$

Yes, there will be trouble if $f'(x_0) = 0$!

If x_0 was an OK guess and x_1 was a good approximation, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

should even better. Keep going (*iterate* the process),

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\,,$$

until the answer is changing only in the highest decimal places.

When it works at all, Newton's method works very well; typically the number of decimal places of accuracy will double at each step. But if the graph is either very wiggly or very flat (f' is small) near x_*, Newton may have trouble finding x_*. The tangent lines then are almost horizontal, so the algorithm may shoot an x_n off to a distant part of the graph, where its successors x_{n+m} may converge to a different root or not converge at all. The extreme case is where one of the $f'(x_n)$ turns out to be 0 and the process breaks down completely.

The *secant method* is slower than Newton's but more reliable in those troublesome cases. Calculus books don't usually mention it, because its theory involves no calculus, just geometry. The method is to find x_1 and x_2 so that $f(x_1)$ and $f(x_2)$ have opposite signs. The intermediate value theorem guarantees that there is a root x_* between them (if f is continuous). Construct the secant line between those two points on the graph. Let its intersection with the axis be x_3, a better guess for x_*. For the next step, use x_3 in place of x_1 or

x_2 (whichever had the same sign for $f(x)$). Iterate. The formula is

$$x_{n+1} = x_n \frac{f(x_{n-1})}{f(x_{n-1}) - f(x_n)} + x_{n-1} \frac{f(x_n)}{f(x_n) - f(x_{n-1})}.$$

Reference: F. S. Acton, *Numerical Methods that Work.*

Exercise (similar to 3.12.7)

Use Newton's method to calculate $\sqrt{36.1}$ and $\sqrt{40}$.

I use the same numbers as in the previous example to stress that differential approximation and Newton's method are two different things, even though we used the former to derive the latter.

When we're computing square roots, in the master formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

the function $f(x)$ is **not** $\sqrt{x}$! To find $\sqrt{a}$ we need to solve the equation

$$0 = x^2 - a \equiv f(x).$$

So $f'(x) = 2x$,

$$x_{n+1} = x_n - \frac{(x_n)^2 - a}{2x_n}.$$

I will implement this in a *Maple* worksheet. The process can be carried to any desired accuracy (up to the limit set by roundoff error) regardless of whether a is close to a perfect square. ■