Lecture for Week 12 (Secs. 5.5 and 5.7)

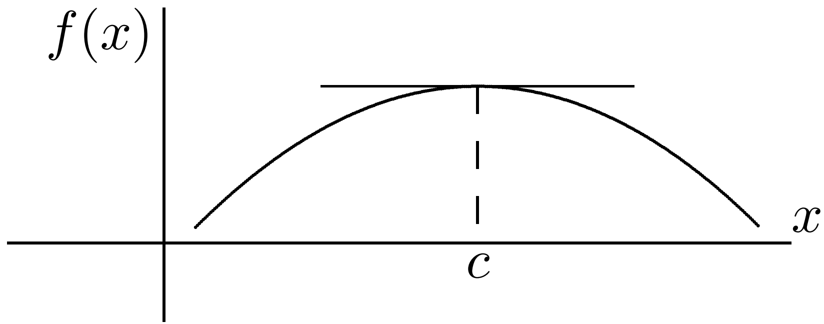
Optimization Problems

and Antiderivatives

We are concerned this week with finding the maximum and minimum values of a function in practical problems.

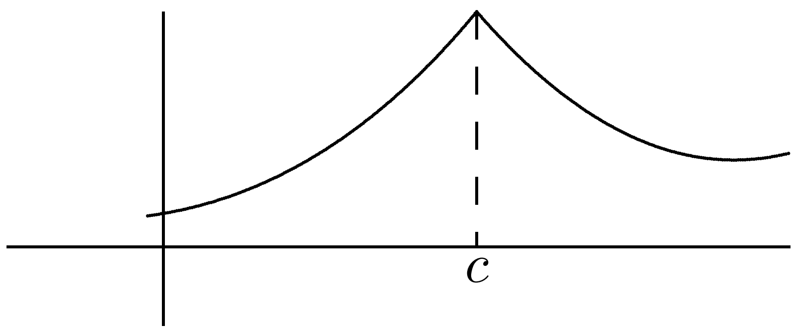
Extremum is the generic term for either type of extreme value. *Optimum* is the generic term for whichever type is considered desirable. All these words form plurals by changing “-um” to “-a”.

The main principle: Maximum and minimum values of a function f occur at places (say $x = c$) where $f'(x) = 0$ (i.e., the tangent line is horizontal).

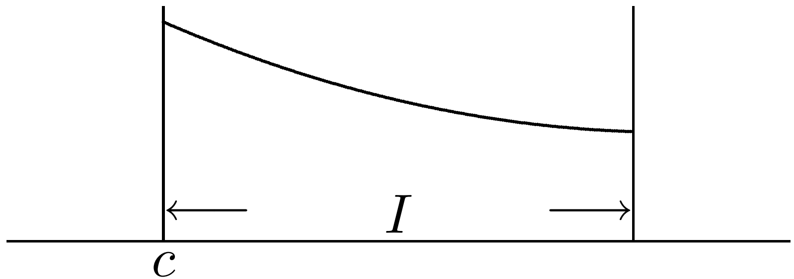


But there are **Complications**

1. A local extremum of f may occur at c even if the tangent line does not become horizontal there. This happens if $f'(c)$ doesn't exist.



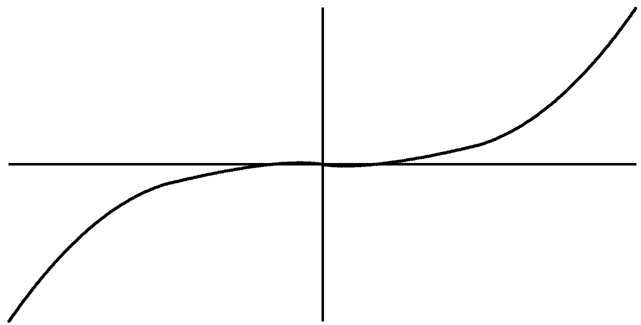
2. If c is an endpoint of the interval I, then $f(c)$ may be the absolute extremum of f on I even if $f'(c)$ exists and $f'(c) \neq 0$.



Remark: According to what I consider the standard terminology, an absolute maximum is automatically a local maximum. However, in Exercise 12(a) (which I assigned last week) Stewart seems to be reserving the term "local maximum" for an interior maximum, occurring where $f'(c)$ is either 0 or undefined.

By the way, "relative" and "global" can be used instead of "local" and "absolute".

3. $f'(c)$ may be 0 even if $f(c)$ is *not* an extremum (even a local one). This happens for $f(x) = x^3$, $c = 0$, for example.



Procedure for finding the absolute extrema of a function on an interval

1. Find all *critical numbers* in the interval (i.e., places where $f'(c) = 0$ **or** $f'(c)$ is undefined).

2. Calculate $f(c)$ at the critical numbers **and** at the endpoints of the interval.

3. Compare the results; pick out the largest and smallest.

Example

Your dream house will be built on a rectangular lot. Along the side facing the street you will have a stone wall that will cost \$10 per foot (measured horizontally along the street). The other three sides will be enclosed by a steel fence costing \$5 per foot. You have \$2500 to spend on wall and fence together. Find the dimensions of the lot with the maximum area consistent with your plans.

I will work out this example, then go back and point out the general principles and strategies that it illustrates.

Let L be the length of the lot (the dimension parallel to the street) and W be the width (from front to back). The area is $A = LW$. The cost of the fence is

$$2500 = C = 10L + 5(2W + L) = 15L + 10W.$$

Let's solve this equation for W:

$$W = \frac{2500 - 15L}{10} = 250 - \frac{3}{2}\,L.$$

Then

$$A = LW = 250L - \frac{3}{2}\,L^2.$$

Now find critical numbers:

$$0 = A'(L) = 250 - 3L \;\Rightarrow\; L = \frac{250}{3}\,.$$

We should also consider "the endpoints of the interval" — but what is the interval? Well, it would make no sense for L to be negative, so the interval should start at $L = 0$. Also, if L is too big, then the constraint (cost) equation will force W to be negative. So the interval ends at the value of L that makes $W = 0$: $L = 2500/15 = 500/3$. At these two endpoints, $A = 0$, whereas in the interior A is obviously positive. Therefore, the endpoints are absolute minima of A, and the absolute maximum must occur at the critical point, $L = 250/3$. Calculate the other quantities:

$$W = 250 - \frac{3}{2}\,\frac{250}{3} = 125, \qquad A = \frac{250}{3}\,125 = \frac{31250}{3}\,.$$

Alternative argument that the critical point is the minimum: $A''(L) = -3 < 0$ everywhere, hence in particular at the critical point. So the function is concave downward there, and that extremum must be a maximum. (So there was no real need to study the endpoints in detail.)

■

Strategy for optimization problems

1. Read the problem carefully. Understand –
   - What quantity is to be extremized? (Let's call it Q.)
   - What other quantities can vary? What quantities are fixed?
2. Introduce notation. **Draw a diagram** if appropriate.

3. Write down the relations among the variables. (They may be given in the problem, or deducible from general knowledge.) You need

   1) an **objective function** expressing Q in terms of other variables;

   2) **constraint equations** relating those other variables so that you can write Q as a function of *just one* independent variable (let's call it x).

4. Solve the constraints and substitute the results into the objective function.

   - *Don't* differentiate the constraints.

   - *Don't* differentiate Q *until* you have eliminated all variables but x.

5. Differentiate $Q(x)$ to find its critical points.

6. Verify that your favorite critical point is the correct extremum.

    - Is it a max or a min? or neither?
    - Is it in the physically allowed interval?
    - Remember to check endpoints as possible candidates.

Exercise

Redo the dream house example, but this time suppose that the budget is flexible and you want a lot of exactly 10,000 ft^2. Find the dimensions of the lot with the minimal wall-fence cost (and find that cost).

We still have the area and cost formulas

$$A = LW, \qquad C = 15L + 10W.$$

But now the roles of objective and constraint formulas are interchanged. The constraint (solved) is $W = 10000/L$, so the objective is

$$C(L) = 15L + \frac{100000}{L}\,.$$

(In either problem I could have eliminated L in favor of W instead of the reverse. That would change the intermediate algebra but not the answers.)

$$0 = C'(L) = 15 - \frac{100000}{L^2}\,.$$

$$L = \sqrt{\tfrac{100000}{15}} = 100\sqrt{\tfrac{2}{3}}\,.$$

$$W = 100\sqrt{\tfrac{3}{2}}\,.$$

$$C = 1500\sqrt{\tfrac{2}{3}} + 1000\sqrt{\tfrac{3}{2}}\,. \qquad \blacksquare$$

And now something completely different: *Antiderivatives*

There is not a whole lot to say here, because (1) we've been talking all along about the problem of finding a function whose derivative is a given function, and (2) such problems are either very easy (if you know all the differentiation formulas) or hard, and the hard part is postponed to Sec. 6.5 and next semester.

"Given f, find a function (or all functions) F such that $F'(x) = f(x)$."

Let's go ahead and write this problem in the notation you will be using for the rest of your life, even though it won't be explained till next week:

$$\text{Find} \quad F(x) = \int f(x)\,dx.$$

There is one crucial theoretical point: Any two antiderivatives of a function on an interval differ only by a constant.

Example: We know that one antiderivative of $(1-x^2)^{-1/2}$ is $\sin^{-1} x$. Therefore, a formula for *all possible* antiderivatives of $(1-x^2)^{-1/2}$ is

$$\int (1-x^2)^{-1/2}\,dx = \sin^{-1} x + C,$$

where "C" stands for an arbitrary constant.

There is one exception to the rule that only a constant is needed to get all antiderivatives. It is handled by the disclaimer “on an interval” in the theorem on the previous slide. If f is defined on two or more disjoint intervals, we could choose a different antiderivative on each interval.

(example on next slide)

Example: The domain of

$$f(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

is $(-\infty, -1) \cup (1, \infty)$. According to pp. 280–281, an antiderivative of this function is $F(x) = \sec^{-1} x$. Another antiderivative is (for instance)

$$\mathcal{F}(x) = \begin{cases} \sec^{-1} x - 4\pi & \text{if } x < -1, \\ \sec^{-1} x + 20 & \text{if } x > 1. \end{cases}$$

Exercise 5.5.7

Find the most general antiderivative of

$$g(t) = (t^3 + 2t^2)/\sqrt{t}.$$

Exercise 5.5.27

Find $f(x)$ if $f'(x) = 3\cos x + 5\sin x$ and $f(0) = 4$.

Converting roots to fractional powers and multiplying out factors is usually a good strategy:

$$\begin{aligned}\int g(t)\,dt &= \int (t^{5/2} + 2t^{3/2})\,dt \\ &= \frac{2}{7}t^{7/2} + \frac{4}{5}t^{5/2} + C \\ &= \sqrt{t}\left(\frac{2}{7}t^{3} + \frac{4}{5}t^{2}\right) + C\end{aligned}$$

(where the last step is nonmandatory). ■

The other problem is an example of a *differential equation*. It exemplifies the general principle that to get a unique answer, we need to have as many *initial conditions* in the problem as there are derivatives. The initial condition, $f(0) = 4$, will determine the unknown constant C.

$$\begin{aligned}\int f'(x)\,dx &= \int (3\cos x + 5\sin x)\,dx \\ &= 3\sin x - 5\cos x + C.\end{aligned}$$

Choose C so that

$$4 = f(0) = 3\sin 0 - 5\cos 0 + C = C - 5.$$

$$f(x) = 3\sin x - 5\cos x + 9. \qquad \blacksquare$$

Exercise 5.7.77

Find the velocity and position of a particle if

$$\mathbf{a}(t) = t^2\,\mathbf{i} + \cos(2t)\,\mathbf{j},$$

$$\mathbf{v}(0) = \mathbf{j}, \qquad \mathbf{r}(0) = \mathbf{i}.$$

$$
\begin{aligned}
\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt = \int \left(t^2\,\mathbf{i} + \cos(2t)\,\mathbf{j}\right) dt \\
&= \frac{1}{3}t^3\,\mathbf{i} + \frac{1}{2}\sin(2t)\,\mathbf{j} + \mathbf{v}_0\,,
\end{aligned}
$$

where $\mathbf{v}_0$ is some constant vector. We can find $\mathbf{v}_0$ now from the given initial velocity; or we can wait until after the next step. I'll do the latter, since it shows better the *general* structure of the solution of such a mechanics problem.

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = \int \left(\tfrac{1}{3}t^3\,\mathbf{i} + \tfrac{1}{2}\sin(2t)\,\mathbf{j} + \mathbf{v}_0\right)\,dt \\ &= \frac{1}{12}t^4\,\mathbf{i} - \frac{1}{4}\cos(2t)\,\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0\,.\end{aligned}$$

Now we have

$$\mathbf{j} = \mathbf{v}(0) = 0 + 0 + \mathbf{v}_0\,, \qquad \mathbf{i} = \mathbf{r}(0) = 0 - \frac{1}{4}\,\mathbf{j} + \mathbf{r}_0\,.$$

So $\mathbf{v}_0 = \mathbf{j}$ and $\mathbf{r}_0 = \mathbf{i} + \frac{1}{4}\,\mathbf{j}$. ■

Points worth noting:

1. The initial data are not always given at 0. (In the first example, we might have had $f(1) = 4$ instead of $f(0) = 4$.)

2. The simplest or "most natural" antiderivative is not always equal to 0 when the independent variable is 0. (In both examples we encountered $\cos 0 = 1$. So the constants and the initial data were not the same numbers.)