Lecture for Week 14 (Secs. 6.3–4)

Definite Integrals

and the Fundamental Theorem

Integral calculus — the second half of the subject, after differential calculus — has two aspects:

1. "Undoing" differentiation. This is the problem of finding *antiderivatives*, which we've already discussed.

2. The study of "adding things up" or "accumulating" something. This is very closely related to the *area* topic of last week.

The *fundamental theorem of calculus* shows that these two things are *essentially the same*.

To reveal the basic idea, consider a **speed-distance problem**: We know that if an object moves at a *constant* speed for a certain period of time, then the total distance traveled is

$$\text{distance} = \text{speed} \times \text{time}.$$

Suppose instead that the speed is a varying function of time. If we consider a very *short* time interval, $[t_{i-1}, t_i]$, then the speed is approximately constant (at least if f is continuous) and hence we can approximate the distance by

$$\text{distance} = \text{speed in that interval} \times (t_i - t_{i-1}).$$

As the speed we may choose the maximum speed in the short interval, or the minimum, or anything in between — say $f(w_i)$ for some $w_i \in [t_{i-1}, t_i]$. The choice won't make any difference in the end.

(There should be a picture here, but I don't have time to draw it now.)

So the *total* distance traveled between $t = a$ and $t = b$ is approximately

$$\sum_{i=1}^{n} f(w_i)\Delta t_i\,, \qquad \text{where } \Delta t_i - t_{i-1}\,.$$

As we let $n \to \infty$, the approximation should become exact.

In effect, we have concluded that:

(A) The distance traveled between times a and b is the area under the graph of the speed function between the vertical lines $t = a$ and $t = b$ (if area is defined in appropriate units, and if the speed is always nonnegative).

On the other hand,

(B) The distance function is an antiderivative of the speed function.

Conclusion: *Areas* and *antiderivatives* are very closely related. In some sense, they are the same thing!

Before continuing we need a more precise definition of the definite integral as a limit of a sum,

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i\,.$$

That occupies the first half of Sec. 6.3, and it involves all the issues about varying the sizes Δx_i of the strips, etc., that I discussed last week about area.

One important difference between areas and generic definite integrals: The *integrand* function is allowed to be negative in some (or all) places). *Areas must always be positive (or zero), but integrals can be negative.* In general,

$$\int_a^b f(x)\,dx = A_+ - A_-\,,$$

where A_+ is the area below the graph and above the x axis, and A_- is the area above the graph and below the x axis. (In Fig. 3 on p. 379, A_+ is yellow and A_- is blue.)

Note also that the "dummy variable" in an integral can be any letter that doesn't cause confusion:

$$\int_2^6 x^3\,dx = \int_2^6 t^3\,dt.$$

Also, this thing is not a function of the variable of integration, it is just a number.

Here is an example where a bad choice of letter *would* cause confusion:

$$x^2 - x \int_2^{x^2} t^3\, dt \tag{$*$}$$

must not be written as

$$x^2 - x \int_2^{x^2} x^3\, dx.$$

(The integral in $(*)$ *is* a function of x, though not of t.)

Algebraic properties of integrals

1. $$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

2. $$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$$

for a constant c.

3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

4. $$\int_a^b dx \quad \text{means} \quad \int_a^b 1\,dx$$

and equals $\quad b - a.$

Remarks: Formula (4) is just the area of a rectangle. The others are also rather obvious for areas. Recall that making (3) obvious was one of the reasons for allowing strips of varying widths Δx_i .

5. $$\int_b^a f(x)\,dx \quad \text{means} \quad -\int_a^b f(x)\,dx$$

if $b > a$.

It follows that in (3), c does not need to lie between a and b.

Perhaps more important (from the point of view of avoiding mistakes) is an identity that is *not* in the list:

$$\int f(x)g(x)\,dx = \int f(x)\,dx \times \int g(x)\,dx$$

FALSE!

The integral of a product is not the product of the integrals, just as (and because) the derivative of a product is not the product of the derivatives. Roughly speaking, every derivative formula turns around to give an integral formula. The integral formula corresponding to the product rule for derivatives is **integration by parts** (Sec. 8.1).

Order properties of the integral

1. If $f(x) \geq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

(Here we assume $a < b$, of course.) In particular, if f is nonnegative, then

$$\int_a^b f(x)\,dx \geq 0 \quad \text{also.}$$

2. If f is **continuous** and $f(x) \geq 0$, then $\displaystyle\int_a^b f(x)\,dx$ (with $a < b$) is *strictly* greater than 0 unless $f(x) = 0$ for all x in $[a, b]$.

3. $$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f((x)|\,dx.$$

4. **Mean value theorem for integrals:** If f is **continuous**, then there is a z in (a, b) such that $\int_a^b f(x)\,dx = f(z)\,(b-a)$.

5. If $m \le f(x) \le M$ in $[a, b]$, then

$$m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a).$$

Now back to the "fundamental theorem". Roughly speaking, it says that differentiation "undoes" integration, and vice versa. They are *inverse* operations (almost), like squaring and taking the square root, except that they operate on functions instead of numbers.

The two functions involved are related as are the two readings on a car's speedometer-odometer panel. The odometer reading is the integral of the speedometer reading. The speedometer reading is the derivative of the odometer reading. That is the essence of calculus!

The theorem has two parts, one for each order of the operations. And I state each part in two versions, depending on which function (the integral or the derivative) takes center stage.

In stating the theorem, we assume for simplicity that f (the derivative) is continuous.

Fundamental Theorem, Part 1:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

That is,

$$G(x) \equiv \int_a^x f(x)\,dt$$

is an antiderivative of f.

Exercise

Evaluate

$$\frac{d}{dx}\int_{10}^{x} 2u\,du,$$

$$\frac{d}{dy}\int_{3}^{y} \frac{1}{t}\,dt,$$

$$\frac{d}{dv}\int_{10}^{v^2} 2u\,du.$$

For the first two, just apply the theorem:

$$\frac{d}{dx}\int_{10}^{x} 2u\,du = 2x,$$

$$\frac{d}{dy}\int_{3}^{y} \frac{1}{t}\,dt = \frac{1}{y}\,.$$

We know these facts without necessarily knowing what the integrals themselves are. You may know that the first one is

$$\int_{10}^{x} 2u\,du = x^2 - 100,$$

either (the hard way) in analogy to Exercise 6.3.16 or (the easy way) peeking ahead to Part 2 of the theorem. The second integral requires a logarithm function (see Sec. 6.6).

For the third one, use the chain rule:

$$\frac{d}{dv}\int_{10}^{v^2} 2u\,du = 2v^2 \times 2v = 4v^3.$$

(Make sure you understand this. The integral is a *function of* v^2 and therefore of v.) ■

Fundamental Theorem, Part 2:

$$\int_a^b H'(x)\,dx = H(b) - H(a).$$

That is,

$$\int_a^b f(x)\,dx = H(b) - H(a),$$

where H is *any* antiderivative of f.

Now we can fill in the remarks I made on the previous example:

Exercise

Find

$$\int_{10}^{x} 2u\,du,$$

$$\int_{3}^{y} \frac{1}{t}\,dt.$$

Since $\dfrac{d}{dx}x^2 = 2x$,

$$\begin{aligned} \int_{10}^{x} 2u\,du &= u^2\Big|_{u=x} - u^2\Big|_{u=10} \\ &\equiv u^2\Big|_{10}^{x} \\ &= x^2 - 100. \end{aligned}$$

Certainly much easier than “Use Theorem 5” on p. 387!

From Sec. 4.4 or 5.7, we know that an antiderivative of $\frac{1}{y}$ is $\ln y$. (This is under the assumption that $y > 0$ in the interval concerned. If it's negative, we should write $\ln|y|$.) So

$$\begin{aligned}\int_3^y \frac{1}{t}\,dt &= \ln t\big|_3^y \\ &= \ln y - \ln 3.\end{aligned}$$

(Since 3 is positive, so is y. The formula does not apply to negative y, because the logarithm is discontinuous at $t = 0$.) ■

Why did I say that integration and differentiation are "almost" inverse operations? Look back at slide 23. It says that, if we think of H as a function of b, then differentiating it and then integrating it *almost* gives back $H(b)$, but not quite: There is a "constant of integration", $-H(a)$, stuck on at the end. This complication is inevitable, because a given function has many antiderivatives, differing by constants.

In fact, Part 2 is easy to prove, if you know Part 1 and the theorem that two antiderivatives (on an interval) differ only by a constant. Because (Part 1)

$$\frac{d}{db}\int_a^b f(x)\,dx = f(b),$$

it must be that

$$\int_a^b f(x)\,dx \equiv G(b)$$

is an antiderivative of $f(b)$; the only question is which one. If H is *any* antiderivative (see Part 2), then

$$\int_a^b f(x)\,dx = H(b) + C$$

(Theorem 2, p. 345). To find C, apply the **initial condition** that

$$\int_a^a f(x)\,dx = 0.$$

It tells us that $0 = H(a) + C$. So

$$\int_a^b f(x)\,dx = H(b) - H(a). \qquad \blacksquare$$

Proving Part 1 is harder, and you should read the details in the book. The basic idea is the same as in the earlier argument that the distance traveled is the area under the graph of the speed function, but run in reverse: we will pick the sum apart instead of building it up.

By definition of a derivative,

$$\begin{aligned}
\frac{d}{dx}\int_a^x f(t)\,dt &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]\\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt\\
&\approx \lim_{h\to 0}\frac{1}{h}\times(\text{area of ``last strip''})\\
&= \text{height of ``last strip''}\\
&= f(x). \qquad \blacksquare
\end{aligned}$$

Practical aspects of the fundamental theorem

Integrals (defined by Riemann sums) are equal to *antiderivatives* according to the theorem. So,

1. If you are trying to "integrate" (in either sense) a function *analytically* (i.e., in terms of exact formulas), the antiderivative is almost always easier to evaluate than the limit of Riemann sums. So you use Part 2:

$$\int_a^b f(t)\,dt = H(b) - H(a) \qquad \bigl(H'(t) = f(t)\bigr).$$

To find the left side (which might arise in an application as an area, for example) you calculate the right side.

Any antiderivative H will do for this purpose, so it is permissible to leave out the "$+\,C$" in this context.

2. If you are integrating *numerically*, sums are usually easier than antiderivatives. So the computer programs for such work use some refinement of the definition

$$\int_a^x f(t)\,dt = \lim_{\|P\|\to 0} \sum_i f(t_i^*)\Delta t_i\,.$$

To find the left side (which might arise in an application as a distance traveled at speed f, for example) you compute one of the sums on the right side with a small $\|P\|$ and trust that that is a good approximation to the answer.