Lecture for Week 15 (Sec. 6.5)

Integration by Substitution

(also called "by change of variable")

I will start with an example that I'll do several different ways, to stress that they are all equivalent. Consider the *indefinite* integral

$$I(x) = \int \sqrt{3x + 4}\, dx.$$

(That is, the problem is to find the most general antiderivative of the integrand, $\sqrt{3x + 4}$.)

Let $u = 3x + 4$. Then $\frac{du}{dx} = 3$, which we write as $du = 3\,dx$, and hence conclude that $dx = \frac{1}{3}\,du$.

(This goes against the grain of what you were taught, which is that the derivative is not really a ratio of tiny numbers. At the end I'll sketch a justification of pretending that du and dx really are numbers.)

Now

$$I \equiv \int \sqrt{3x + 4}\,dx = \int \sqrt{u}\,\frac{1}{3}\,du,$$

which we further process as

$$\frac{1}{3} \int u^{1/2} \, du = \frac{1}{3} \frac{2}{3} u^{3/2} + C = \frac{2}{9} (3x + 4)^{3/2} + C.$$

Here is a slightly different way of doing the same problem. Again write $u = 3x + 4$. Then $x = \frac{1}{3}(u - 4)$, so $dx = \frac{1}{3} \, du$. Thus

$$I \equiv \int \sqrt{3x + 4} \, dx = \int \sqrt{u} \, \frac{1}{3} \, du = \cdots$$

and we can continue exactly as before.

In more complicated problems you may need to substitute the formula for x in terms of u back into the integrand, too (example later). Whether it's neater to work with the old variable as a function of the new one or the new one as a function of the old one depends on the problem.

(Apologies if you think this is a fuss over nothing. Some students see the two methods as very different and get upset in Math. 152 when the first method needs to be replaced by the second one in certain problems.)

Now let's consider a *definite* integral, $I \equiv \int_0^1 \sqrt{3x + 4}\,dx$.

Method 1: By our previous result for the indefinite integral,

$$I = \frac{2}{9}(3x + 4)^{3/2}\Big|_0^1 = \frac{2}{9} \times 7^{3/2} - \frac{2}{9} \times 4^{3/2}.$$

Method 2: Suppose we had not already found the antiderivative. It would be just as easy to argue this way:

Let $u = 3x + 4$. I see that

$$u = 4 \quad \text{when } x = 0, \qquad u = 7 \quad \text{when } x = 1.$$

Therefore (using $dx = \frac{1}{3} \, du$ as before),

$$I \equiv \int_0^1 \sqrt{3x + 4} \, dx = \int_4^7 \sqrt{u} \, \frac{1}{3} \, du$$

$$= \frac{1}{3} \frac{2}{3} \, u^{3/2} \Big|_4^7 = \frac{2}{9} \left(7^{3/2} - 4^{3/2} \right).$$

Another example: $I(x) = \int \dfrac{2x\,dx}{\sqrt{x^2 + 1}}$.

Let $u = x^2 + 1$, so $du = 2\,dx$. (You should use this kind of substitution whenever you see $x\,dx$ times a function of $x^2 + (\text{constant})$.)

$$I = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$$

$$= 2u^{+1/2} + C = 2\sqrt{x^2 + 1} + C.$$

Let's do it the other way: $u = x^2 + 1$ implies $x = \sqrt{u-1}$ (if x is positive), so

$$dx = \frac{1}{2\sqrt{u-1}}\, du.$$

$$I = \int \frac{1}{\sqrt{u}} \times 2\sqrt{u-1} \times \frac{1}{2\sqrt{u-1}}\, du = \cdots$$

as before.

And finally, let's make it a definite integral:

$$I \equiv \int_0^1 \frac{2x\,dx}{\sqrt{x^2 + 1}}\,.$$

I'll do this by "Method 2":

$$u = x^2 + 1, \quad du = 2\,dx.$$

$$x = 0 \;\Rightarrow\; u = 1\,; \quad x = 1 \;\Rightarrow\; u = 2.$$

$$I = \int_1^2 u^{-1/2}\,dx = 2u^{+1/2}\Big|_1^2 = 2\sqrt{2} - 2.$$

The big point here is, **don't write** $I = \int_0^1 u^{-1/2} \, du$. That is false! This issue is similar to those that arose in connection with the chain rule and the implicit function theorem, where we need to keep in mind the difference between the numerical values of the independent and dependent variables. The notation

$$I = \int_{x=0}^{x=1} u^{-1/2} \, du$$

is unambiguous and acceptable, however.

I promised a problem that required an additional back-substitution.

$$I(x) \equiv \int \frac{x^3 \, dx}{\sqrt{x^2 + 1}} \, .$$

$$u = x^2 + 1, \quad du = 2x \, dx.$$

$$I = \int \frac{x^2 \times \frac{1}{2} \, du}{\sqrt{u}} \, .$$

Now you somehow need to recognize that $x^2 = u - 1$; that's fairly obvious, but it might be more

obvious if you had solved for $x = \sqrt{u-1}$ originally. Now

$$I = \frac{1}{2} \int \left(u^{1/2} - u^{-1/2} \right) du$$
$$= \frac{1}{2}(x^2 + 1)^{3/2} - (x^2 + 1)^{1/2} + C.$$

(You can fill in the details.)

Of course, there are lots of integrals by substitution that don't involve square roots. Here are some examples:

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du,$$

$$\int (4x + 9)^{50} \, dx = \frac{1}{4} \int u^{50} \, du,$$

$$\int \frac{(\ln x)^{2/3}}{x} \, dx = \int u^{2/3} \, du.$$

Finally, let's take a brief theoretical look at integration by substitution. Suppose that the formula introducing the new variable has the form $u = g(x)$. Then the substitution procedure for an indefinite integral (antiderivative) gives us

$$\int f\big(g(x)\big)\, g'(x)\, dx = \int f(u)\, du = F\big(g(x)\big) + C,$$

where F is any function satisfying $F' = f$. From this general pattern it is easy to see that *integration by substitution is the integration rule that*

reverses the chain rule:

$$\frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big)g'(x).$$

Applied to a definite integral, the rule yields

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$
$$= F\big(g(b)\big) - F\big(g(a)\big).$$

Of course, this follows from the indefinite case by the fundamental theorem; but there is an alternative way of looking at it, directly from the definition of an integral. We have $f(u) = (f \circ g)(x)$. Let's construct a Riemann sum for approximating $\int_{g(a)}^{g(b)} f(u)\, du$ with strip widths $\Delta u_i \approx g'(x_i^*)\, \Delta x$ with Δx constant (see graph on next slide). Because of the stretching of the horizontal scale, this Riemann sum also approximates $\int_a^b (f \circ g)(x)\, g'(x)\, dx$, as the formula says.

$f(u) = (f \circ g)(x)$

a 1   2   3 b x

$g(a)$ 1   2   3   4   5 $g(b)$ u
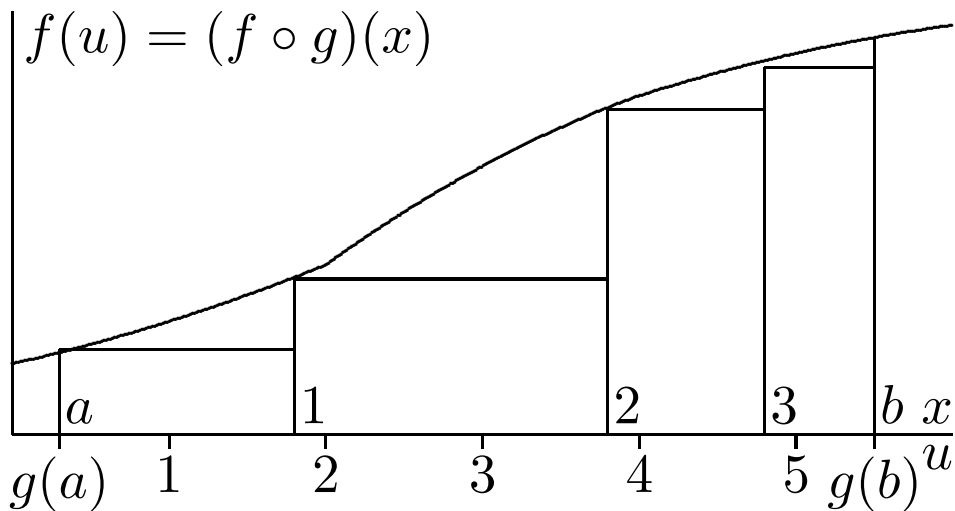
Note that this graph is that of $(f \circ g)(x)$, not of $(f \circ g)g'$. $\Delta x = 1$ but the true strip size is $g'(x_i^*)$.

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