Math. 152, WebCalc Sections 513–514 9 May 2001

Final Examination – Solutions

Name: ____________________ Section: __________

Part I: Multiple Choice (3 points each)

There is no partial credit. You may not use a calculator.

1. Which of these integrals converges?

(A) $\displaystyle\int_1^\infty \frac{dx}{x}$

(B) $\displaystyle\int_0^1 \frac{dx}{x}$

(C) $\displaystyle\int_1^\infty \frac{dx}{x^2}$ $\Leftarrow$ correct

(D) $\displaystyle\int_0^1 \frac{dx}{x^2}$

(E) none of them

Integrals of x^{-p} out to ∞ converge if and only if $p > 1$. Integrals of x^{-p} down to 0 converge if and only if $p < 1$.

2. Which of these series converges?

(A) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ $\Leftarrow$ correct

(B) $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$

(C) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{1/2}}$

(D) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{-1/2}}$

(E) all of them

A p-series converges if and only if $p > 1$, just like the corresponding integral to ∞.

3. Calculate the vector cross product, $\vec{v} \times \vec{w}$, of the vectors $\vec{v} = \langle 1, 2, 3 \rangle$ and $\vec{w} = \langle 1, 1, 1 \rangle$.

(A) $\langle 0, 2, -2 \rangle$

(B) $\langle 2, -1, 1 \rangle$

(C) $\langle -2, -1, 2 \rangle$

(D) $\langle 1, -2, -1 \rangle$

(E) $\langle -1, 2, -1 \rangle$ $\Leftarrow$ correct

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -\hat{\imath} + 2\hat{\jmath} - \hat{k}.$$

4. Calculate the projection of the vector $\vec{v} = \langle 1, 2, 3 \rangle$ onto (i.e., in the direction of) the vector $\vec{w} = \langle 1, 1, 1 \rangle$.

(A) $\frac{1}{\sqrt{3}} \langle -1, 2, -1 \rangle$

(B) $\langle 2, 2, 2 \rangle$ $\Leftarrow$ correct

(C) $\frac{1}{\sqrt{3}} \langle 1, 2, 3 \rangle$

(D) $\frac{1}{\sqrt{14}} \langle 1, 1, 1 \rangle$

(E) $\frac{1}{3} \langle 0, 1, 2 \rangle$

Let $\hat{u}$ be the unit vector along $\vec{w}$.

$$\hat{u} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle.$$

The projection is

$$(\hat{u} \cdot \vec{v})\hat{u} = \frac{1}{3}(\vec{w} \cdot \vec{v})\vec{w} = \frac{1}{3}(1 + 2 + 3)\vec{w} = 2\vec{w}.$$

5. The series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$ represents

(A) $x^2 e^{x^2}$

(B) $\ln(x+2)$

(C) $\dfrac{1}{x-2}$

(D) e^{x+2} $\Leftarrow$ correct

(E) $\dfrac{e^x - 2}{x}$

Set $z = x+2$ in the well known Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,.$$

6. A bar 2 meters long has a linear mass density of $\ln x$ kilograms per meter, where $x = 3$ at one end of the bar and $x = 5$ at the other end. Find the TOTAL MASS (NOT the center of mass) of the bar. (*Hint:* Integrate by parts.)

(A) $\ln \dfrac{5}{3}$

(B) $2 \ln 2$

(C) $5 \ln 5 - 3 \ln 3 - 2$ $\Leftarrow$ correct

(D) $\dfrac{1}{5} - \dfrac{1}{3}$

(E) $25 \ln 5 - 9 \ln 3$

$$m = \int_3^5 \ln x \, dx = \big[x \ln x - x\big]_3^5$$
$$= 5 \ln 5 - 5 - 3 \ln 3 + 3 = 5 \ln 5 - 3 \ln 3 - 2.$$

Details of the integration: Let $u = \ln x$, $dv = dx$. Then $du = \frac{dx}{x}$, $v = x$. Thus

$$\int \ln x \, dx = x \ln x - \int x \, \frac{dx}{x}$$
$$= x \ln x - \int dx = x \ln x - x + C.$$

7. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$ equals

(A) $\dfrac{1}{4!}$

(B) $-\dfrac{1}{6}$ $\Leftarrow$ correct

(C) $+\infty$

(D) $-\infty$

(E) 0

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

So the quotient is

$$\frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots - x}{x^3}$$
$$= \frac{-\dfrac{x^3}{6} + \cdots}{x^3} = -\frac{1}{6} + \cdots.$$

8. The series $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$ converges in the interval

(A) $-1 < x < 1$

(B) $0 < x < 2$ $\Leftarrow$ correct

(C) $-2 < x < 0$

(D) $-\infty < x < \infty$

(E) the point $x = 1$ only

By the ratio test (*OR* the basic theorem on geometric series (which is proved by the ratio test)), the largest open interval of convergence is

$$|x-1| < 1.$$

9. The partial fraction decomposition of $\dfrac{x^2-1}{(x^2+1)(x+5)^2}$ has the form

(A) $\dfrac{A}{x^2+1}+\dfrac{B}{x+5}+\dfrac{C}{(x+5)^2}$

(B) $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+5}+\dfrac{D}{x+5}$

(C) $\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x+5)^2}+\dfrac{E}{x+5}$

(D) $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+5}+\dfrac{D}{(x+5)^2}$ $\Leftarrow$ correct

(E) $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{(x+5)^2}$

10. Suppose that the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$ is given by $s_n = 1-\dfrac{\ln n}{n}$. Then the series $\sum_{n=1}^{\infty} a_n$

(A) diverges, because $\lim_{n\to\infty}\left[1-\dfrac{\ln n}{n}\right]\neq 0$.

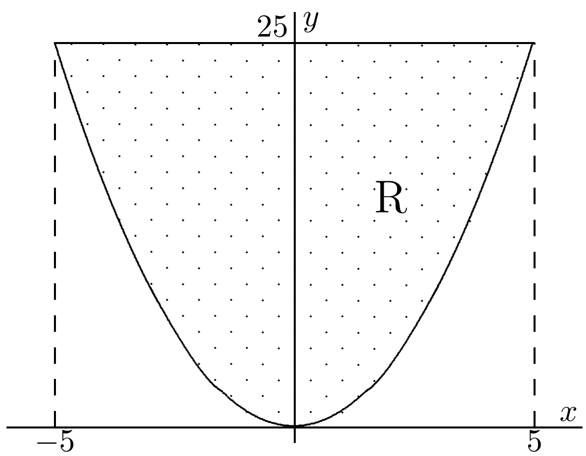
(B) converges, because $1-\dfrac{\ln n}{n}\to 1$. $\Leftarrow$ correct

(C) converges, by the alternating series test.

(D) diverges, by the integral test.

(E) diverges, because $\sum_{n=1}^{\infty}\dfrac{\ln n}{n}$ diverges by comparison with the harmonic series.

(Essentially the same as a classic problem from the third exam.)



Let's see how many things we can do with a parabola. More precisely, let R be the plane region bounded above by the line $y = 25$ and below by the parabolic segment

$$y = x^2, \qquad -5 < x < 5.$$

The next seven problems will be based on this geometry. (In the drawing the horizontal and vertical scales are different.)

11. The area of R is

(A) 10

(B) $\frac{25}{4}$

(C) 250

(D) 100

(E) $\frac{500}{3}$ ⇐ correct

$$\begin{aligned} A &= \int_{-5}^{5} (y_{\max} - y_{\min})\, dx = \int_{-5}^{5} (25 - x^2)\, dx \\ &= 2\int_0^5 (25 - x^2)\, dx = 2\left[25x - \frac{1}{3}x^3\right]_0^5 \\ &= 2(5^3 - \frac{1}{3} \times 5^3) = \frac{4}{3} \times 125 = \frac{500}{3}\,. \end{aligned}$$

12. The area of R can be calculated EXACTLY by

(A) Simpson's rule ⇐ correct

(B) the trapezoidal rule

(C) both

(D) neither

Simpson's rule is constructed so as to integrate every quadratic polynomial exactly.

We can check this in the present case, with $\Delta x = 5$ and $n + 1 = 3$ function evaluations:

$$S_3 = \frac{\Delta x}{3}(1 \times 0 + 4 \times 25 + 1 \times 0) = \frac{500}{3}\,,$$

which is correct.

$$T_3 = \Delta x(\tfrac{1}{2} \times 0 + 1 \times 25 + \tfrac{1}{2} \times 0) = 125,$$

which is wrong.

13. The volume of the solid region obtained by revolving R around the line $x = 5$ is

(A) $\displaystyle\int_0^{10} 2\pi r[25-(5-r)^2]\,dr$, where $r = 5 - x$ ⇐ correct

(B) $\displaystyle\int_{-5}^{5} \pi(25^2 - x^4)\,dx$

(C) $\displaystyle\int_0^{25} 2\pi y\,dy$

(D) $\displaystyle\int_0^{25} \pi\sqrt{y}\,dy$

(E) $\displaystyle\int_{-5}^{5} \pi[r^2-(5-r)^4]\,dr$, where $r = 5 - x$

Use cylindrical shells:

$$V = (\text{circumference})(\text{height})\,dr,$$

where the radius $r = 5-x$ varies from 0 to 10. Since $x = 5 - r$ and $y_{\text{min}} = x^2$,

$$V = \int_0^{10} 2\pi r(25 - x^2)\,dr = \text{(A)}.$$

14. A water tank is 10 feet long and has our parabolic region R as cross section. The pressure force (in ft-lb) on the parabolic end of the tank (when it's full) is

(A) $\displaystyle 10\int_{-5}^{5} 9.8y\sqrt{y}\,dy$

(B) $\displaystyle 2\int_0^{25} 62.5(25-y)\sqrt{y}\,dy$ ⇐ correct

(C) $\displaystyle\int_{-5}^{5} 62.5(25-x^2)dx$

(D) $\displaystyle 9.8\int_0^{25} 62.5(25-y)y^2\,dy$

(E) $\displaystyle\int_0^{25} 9.8(25-y)y\,dy$

$$\begin{aligned} F &= \int_{y_{\text{min}}}^{y_{\text{max}}} (\text{depth})(\text{weight density})(\text{width})\,dh \\ &= \int_0^{25} (25-y)(62.5)(\sqrt{y} - (-\sqrt{y}))\,dy \\ &= 2\int_0^{25} 62.5(25-y)\sqrt{y}\,dy. \end{aligned}$$

15. The surface area of the paraboloid swept out when our parabolic segment is revolved around the vertical axis is

(A) $\int_{-5}^{5} (25 - x^2)(2x)\, dx$

(B) $\int_{0}^{5} 2\pi x\sqrt{1+4x^2}\, dx$ $\Leftarrow$ correct

(C) $\int_{0}^{25} 2\pi\sqrt{y}\, dy$

(D) $\int_{-5}^{5} \frac{2\pi x\, dx}{\sqrt{1+4x^2}}$

(E) $2\pi \int_{0}^{5} (1+4x^2)\, dx$

$$\int (\text{circumference})\, d(\text{arc length}) = \int_0^5 2\pi x\sqrt{1+(y')^2}\, dx = \int_0^5 2\pi x\sqrt{1+(2x)^2}\, dx.$$

Part II: Write Out (10 points each)

Show all your work. Appropriate partial credit will be given. You may not use a calculator.

16. (a) Make up (and clearly state) one more problem concerning the parabola (page 6), WHOSE ANSWER IS THE INTEGRAL $\int_{-5}^{5} \sqrt{1+4x^2}\, dx$. (*Hint:* Concentrate on the parabolic segment itself, not the region R above it.)

What is the arc length of the parabolic segment [from the point $(-5, 25)$ to the point $(5, 25)$]?

(b) DO NOT try to evaluate the integral in part (a). Instead, do this one, which you'll like better: $\int_0^2 \sqrt{4-x^2}\, dx$. *Hint:* $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Let $x = 2\sin\theta$, so that $dx = 2\cos\theta\, d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. Thus the indefinite integral becomes $\int 4\cos^2\theta\, d\theta$. From the easily-memorized hint and $\cos^2\theta + \sin^2\theta = 1$ we recover the needed identity

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}.$$

So the integral is (without " $+C$ ")

$$2\int (\cos 2\theta + 1)\, d\theta = \sin 2\theta + 2\theta = 2\sin\theta\cos\theta + 2\theta = x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}.$$

Thus the definite integral is

$$\left[x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = 0 + 2\sin^{-1}1 - 0 - 0 = \pi.$$

Alternative method: When $x = 0$, θ equals 0; when $x = 2$, θ equals $\pi/2$. Thus the definite integral is

$$[\sin 2\theta + 2\theta]_0^{\pi/2} = \pi.$$

Third method: Recognize the integral as the area of one quadrant of a circle of radius 2: $\frac{1}{4}\pi(2)^2 = \pi$.

17. Finally, calculate the volume of the solid region obtained by revolving the parabolic region R (see page 6) around the horizontal line $y = 25$. *Hint:* $5^4 = 625$.

$$\begin{aligned} V &= \int_{-5}^{5} \pi r^2\,dx = \int_{-5}^{5} \pi(25-x^2)^2\,dx = 2\pi\int_0^5 (625 - 50x^2 + x^4)\,dx \\ &= 2\pi\left[625x - \frac{50}{3}x^3 + \frac{x^5}{5}\right]_0^5 = 2\pi\left(5^5 - \frac{2}{3}5^5 + \frac{5^5}{5}\right) = 2\pi 5^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) \\ &= 2\pi 5^5\frac{8}{15} = \frac{16\pi}{3}\times 625. \end{aligned}$$

18. Recall that an "ideal" spring is one that obeys Hooke's law, also known as a linear restoring force.

 (a) The spring constant of an ideal spring was determined by observing that 6 Newtons of force are necessary to stretch it 3 meters from its natural length. How much work was done in that stretching operation?

The force law is $F(x) = -kx$. (The minus sign is there because the force exerted *by* the spring is in the opposite direction from the displacement. The force that needs to be applied to stretch the spring is the negative of that, which causes the minus sign to disappear in the work calculation when one does it carefully. Of course, we know that the work must come out positive.)

$$6 = 3k \;\Rightarrow\; k = 2.$$

$$W = -\int_0^3 F(x)\,dx = +2\int_0^3 x\,dx = x^3\Big|_0^3 = 9 \text{ Joules}.$$

(b) The spring was replaced by a wad of bubble gum that exerts a restoring force $F(x) = -xe^{-2x}$. How much work is required now to stretch the spring from $x = 0$ to $x = 3$?

$$W = -\int_0^3 F(x)\,dx = \int_0^3 x\,e^{-2x}\,dx.$$

Let $u = x$, $dv = e^{-2x}\,dx$, so that $du = dx$, $v = -\frac{1}{2}e^{-2x}$.

$$\begin{aligned} W &= -\left.\frac{x}{2}e^{-2x}\right|_0^3 + \int_0^3 \frac{1}{2}e^{-2x}\,dx = -\frac{3}{2}e^{-6} - 0 - \left.\frac{1}{4}\,e^{-2x}\right|_0^3 \\ &= -\frac{3}{2}\,e^{-6} - \frac{1}{4}\,e^{-6} + \frac{1}{4} = \frac{1}{4} - \frac{7}{4}\,e^{-6}. \end{aligned}$$

19. Determine whether each of these series converges or diverges. Say why: Be sure to name or quote the test(s) you use and check out the requirements of the test.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ CIRCLE ONE: Converges <u>Diverges</u>

Explain:

This is a geometric series with $r \geq 1$. (*OR*, note that the terms do not go to 0 as $n \to \infty$.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ CIRCLE ONE: <u>Converges</u> Diverges

Explain:

This satisfies the alternating series test:

- The terms alternate in sign.
- $\dfrac{1}{n^{1/2}}$ is decreasing.
- $\dfrac{1}{n^{1/2}}$ approaches 0 as $n \to \infty$.

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{n-2}{n^3+3n+6}$ CIRCLE ONE: <u>Converges</u> Diverges

Explain:

Comparison test:

$$\frac{n-2}{n^3+3n+6} < \frac{n}{n^3} = \frac{1}{n^2},$$

and the p-series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges.

20. On the night before commencement, a 6-pound box of soap was dumped into Rudder fountain, which contained 500 gallons of water. To clean up the mess, the physical plant personnel ran fresh water into the basin at the rate of 10 gallons per minute, allowing the well-mixed solution to drain off at the same rate.

(a) Write the DIFFERENTIAL equation ($\frac{dS}{dt} = \ldots$) satisfied by the amount of soap in the basin.

$$\frac{dS}{dt} = -(\text{outflow})(\text{concentration}) = -10\,\frac{S}{500} = -\,\frac{S}{50} \quad (\text{or } -0.02S).$$

(b) Solve for $S(t)$, the amount of soap left after t minutes.

This is a separable equation:

$$\int \frac{dS}{S} = -\int \frac{dt}{50} \Rightarrow \ln S = -\frac{t}{50} + c \Rightarrow S = Ce^{-t/50}.$$

We must have $6 = S(0) = C$. Therefore,

$$S(t) = 6e^{-t/50} = 6e^{-0.02t}.$$

21. A plane passes through the points $P\ (1,0,2)$, $Q\ (-1,1,0)$, and $R\ (2,1,1)$.

(a) Find an equation for this plane of the form $Ax + By + Cz = D$.

Let us find two vectors parallel to the plane by subtracting coordinates of two of the vectors from the other one (P):

$$\overrightarrow{PQ} = \langle -2, 1, -2 \rangle, \qquad \overrightarrow{PR} = \langle 1, 1, -1 \rangle.$$

Their cross product is normal to the plane:

$$\begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ -2 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = \hat\imath \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} - \hat\jmath \begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix} + \hat k \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = \hat\imath - 4\hat\jmath - 3\hat k.$$

Take $\langle A, B, C\rangle = \vec n = \langle 1, -4, -3\rangle$. The equation of the plane is $\vec n \cdot (X - X_0) = 0$, where $X = (x, y, z)$ and X_0 comprises the coordinates of any point on the plane; for instance, let's take $X_0 = P$.

$$\vec n \cdot (1, 0, 2) = 1 - 6 = -5.$$

Thus the plane is

$$x - 4y - 3z = -5.$$

(The equation of the plane and the formula for $\vec n$ could be multiplied by any nonzero number.)

(b) Find a parametric equation for this plane.

Let's use the same vectors as in part (a). The plane is

$$(x, y, z) = (1, 0, 2) + t\langle -2, 1, -2\rangle + s\langle 1, 1, -1\rangle,$$

or

$$x = 1 - 2t + s, \qquad y = t + s, \qquad z = 2 - 2t - s.$$

(Many other equally correct answers are possible.)