Toward Longer Investment:
Authority Versus Inclusive Governance

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Abstract

This paper investigates how a governance arrangement shapes the sustainability of private capital investment via affecting capital-income tax rate, savings motive, and information availability to investors. We propose inclusive governance that admits a cooperative equilibrium. It is shown that both the degree of government transparency and the degree of capital mobility matter in sustaining private capital investment. In providing incentives for longer investments: for authority to dominate inclusive governance, a lower degree of government transparency must be accompanied by a lower degree of capital mobility, while inclusive governance dominates authority whenever capital is sufficiently mobile.

Keywords: Governance design; Delayed information; Exit cost; Optimal exit time; Sustainable investment; Stochastic differential game.

JEL Codes: D72; H11; H30; P26.

1 Introduction

In market economies, tax authorities face the constraint that capitalists can vote with their feet by means of capital flight.[1] The issue of capital flight is especially worse for developing countries. In

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[1] It is a realization of the exit choice emphasized by Hirschman (1970).

China, for example, just in 2011, 2.8 trillion RMB was transferred overseas, and emerging markets in 2015 saw an estimated \$735 billion in net capital outflows with all but \$59 billion of that coming from China.[2] Likewise, Russia warns of capital flight. According to the Central Bank of Russia, capital outflow hit \$151.5 billion in 2014, 2.5 times greater than 2013 numbers.[3] A 2012 report for Global Financial Integrity estimated that, from 2001 to 2010, capital flight from developing countries increased from \$477.1 billion to \$1,138 billion, registering a trend rate of growth of 12.6% per annum.[4]

Such scales of outward capital flight would be detrimental to investment and, thereby, to sustainable growth.[5] Other things equal, the capability of sustaining private capital investment is desirable for implementing investment-based economic development as well as enlarging tax base along time dimension. The question addressed is thus: what kind of governance arrangement can provide more incentives for capitalists to invest for a longer time? More precisely, what kind of relationship connecting government and capitalists is more desirable for sustaining private investment?

Although it is consistent with our intuition that governance arrangement should be relevant in affecting the choice of capital investment horizon, we find by reviewing existing literature that such a connection is left unexplored in theory. For example, related papers (e.g., Lensink et al., 2000; Collier et al., 2001; Hermes and Lensink, 2001; Le and Zak, 2006) just focus on empirically estimating how political-economic risks and policy uncertainties affect capital flight. More importantly, they are silent on how the endogenous investment horizon changes under alternative governance arrangements. In terms of discouraging capital flight and incentivizing sustainable investments, one should design an incentive-compatible governance arrangement rather than suggesting direct punishments as in Segal and Vincent (1998). In a word, we are motivated to offer a theory helping us understand how governance arrangements shape endogenous investment endurance.

In addition to the authoritarian governance characterized as a top-down hierarchy of authority, we design an inclusive governance arrangement. Absolute political authority supports the government to unilaterally determine a tax rate, whereas inclusive governance allows for a bargaining table on which capitalists and government may reach a mutually-beneficial tax rate. We derive equilibrium capital-income tax rate as a component of a subgame perfect equilibrium under authority, while it is established as a component of a cooperative equilibrium under bargaining. In particular, we

[2] See *Financial Times*, January 20, 2016.

[3] See *Forbes*/Investing, March 2, 2015.

[4] For more details: http://www.gfintegrity.org/the-adverse-economic-consequences-of-capital-flight-and-illicit-flows-from-developing-countries/

[5] See, e.g., Cuddington (1986) and Pastor (1990).

demonstrate that the cooperative equilibrium simultaneously satisfies individual rationality, group rationality, subgame consistency and Pareto efficiency, and no one unilaterally deviates from cooperation, no matter Nash bargaining solution, Shapley value or proportional distribution is adopted as an allocation principle. The inclusive governance is therefore justified by these desired properties.

Among many policy variables, capital income tax is the one we choose to compare these two types of governance arrangements. Everything else equal, a linear capital-income tax rate distorts capitalists' inter-temporal savings motive and hence the path of capital accumulation, thereby being a relevant policy affecting the sustainability of private capital investment. Alternative governance arrangements may induce different levels of distortion and hence affect the endurance of investment differently. To characterize the difference, we formalize the two governance arrangements as two different game forms between the government and a representative capitalist. To make our theory more complete, we not only show the equilibrium effect resulted from different game forms under the same information structure but also show the equilibrium effect resulted from different information structures within the same game form.

In the current stochastic environment, government transparency is embedded by assuming that capitalist exhibits delayed information availability relative to government, and we normalize[6] the size of delayed information to zero under inclusive governance so that the degree of government transparency under authoritarian governance is equal to the discrepancy of government transparency between the two governance arrangements. This specification captures in part the information constraint appearing in reality and enables us to focus on the primary concern of this paper. In addition, we use exit cost to capture market failures or institutional frictions. For example, we can interpret it as a kind of transaction cost originated from the incompleteness or imperfectness of capital market, or as an exogenous "exit tax"[7] imposed by the government. Intuitively, a higher exit cost facing the capitalist implies a lower degree of capital mobility.

We obtain the following main results. Firstly, the higher the degree of government transparency, the longer the expected investment horizon under uncertainty, implying that, ceteris paribus,

[6] As shall be further explained in the model, imposing this normalization is for simplicity as it is not essential for deriving our formal results.

[7] For example, Hillary Clinton planed to impose an exit tax on businesses that relocate outside the U.S. (see *The Wall Street Journal*, Aug.21, 2016); Japan's government targets wealthy individuals with an exit tax in hope of preventing them moving to a location where taxes are low (see *The Wall Street Journal*, Dec.18, 2014); in China, an article published in the state-run *People's Daily* (Nov., 2011), entitled "We Should Make it Harder for the Wealthy to Emigrate", proposes an exit tax on wealthy Chinese leaving the country (see *The Atlantic*, Apr.11, 2013).

strengthening government transparency is desirable even under authoritarian governance.

Secondly, there is an endogenous threshold of the degree of capital mobility such that: below the threshold (namely capital is relatively immobile), authoritarian governance dominates inclusive governance when the discrepancy of government transparency between them is smaller than a critical value, otherwise inclusive governance dominates authoritarian governance when the discrepancy is greater than this critical value; above the threshold (namely capital is relatively mobile), inclusive governance dominates authoritarian governance even if there is no discrepancy of government transparency between them, implying that inclusive governance dominates authoritarian governance whenever capital is sufficiently mobile.

Therefore, to identify their relative advantage in sustaining private capital investment, both the degree of government transparency and the degree of capital mobility are relevant factors. By calibrating our model to match some characteristics of the US economy, we find via using numerical experiments that: the lower the degree of government transparency under authoritarian governance, the higher the threshold of the degree of capital immobility is required so that authoritarian governance dominates inclusive governance above this threshold.

Our work is related to the following literature. Concerning the mechanism of voting by feet, our paper is related to Tiebout (1956), Qian and Roland (1998), Cai and Treisman (2005), and Bai et al. (2016), to name just a few. Departing from them who focus on static models, we solve for the optimal exit strategy in a dynamic stochastic environment. Technically, we rely on using optimal stopping theory under uncertainty. We use a dynamic model due to the following two considerations. First, the activity of private capital investment is dynamic in nature, and hence a dynamic model represents a better approach. Second, analyzing the effect of capital taxation on inter-temporal savings decisions in general calls for a dynamic model other than a static model. More importantly, they analyze how the threat of voting by feet may constrain governmental behavior, while we study how governance arrangement in turn shapes the equilibrium choice of voting by feet, so our paper complements the literature in exploring the interaction between governance and foot-voting mechanism.

Alesina and Tabellini (1989) develop a model to argue that it is the uncertainty over the fiscal policies of future governments that generates capital flight, whereas we show that capital flight can be rationalized as an equilibrium choice of sustainable capital accumulation in a stochastic environment. Based on a cross-country data of 40 countries in 7 years, Zhao et al. (2003) present the empirical evidence that a low government transparency is likely to significantly reduce the magnitude of capital inflows to host countries. As a theoretical complement, our results show that, ceteris paribus, a low government transparency is also likely to hurt the sustainability of private capital investment.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 derives equilibria. Section 4 establishes the major result. Section 5 offers further discussions. Section 6 concludes. All proofs appear in Appendix.

2 The Model

To avoid inessential complications, we consider an economy populated by a representative capitalist and a government consisting of heterogenous politicians.

2.1 Capitalist

The capitalist owns initial capital $k(0) \equiv k_0 > 0$, a deterministic constant, and accumulates it by[8]

$$dk(t) = [(1-\tau_k)(r-\delta)k(t) - c(t)]dt + \sigma k(t)dB(t), \tag{1}$$

where $\sigma > 0$ is a constant percentage volatility measuring a set of unpredictable events occurring during this motion, and $B(t)$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\le t\le\tau}, P)$ with $0 < \tau \le \infty$, $B(0) = 0$ a.s.-P and usual conditions fulfilled. Also, τ_k is the capital-income tax rate, $r > 0$ is a constant capital return rate, $0 < \delta < 1$ is a constant depreciation rate, and c is consumption. The economy is characterized by three parameters: r, δ and σ. The capitalist is assumed to initially invest k_0 amounts of capital in the economy (through either production sectors or financial sectors).

By imposing a log preference[9], the intertemporal objective reads as

$$\mathbb{E}_{t_0}\left[\int_0^\tau e^{-\rho(t_0+t)} \ln c(t)dt\right],$$

where $\mathbb{E}_{t_0}$ is the expectation operator depending on information flow up to time $t_0 \ge 0$, $0 < \rho < 1$ is a subjective discount factor, and τ is an exit time which determines the investment horizon. We define two sets of admissible exit times by $\mathcal{T}_0 \equiv \{t \ge 0; \mathfrak{F}_t\text{-adapted exit times}, P\text{-almost surely finite}\}$ and $\mathcal{T}_\Delta \equiv \{t \ge 0; \mathfrak{F}_{t-\Delta}\text{-adapted exit times for a constant } \Delta > 0, P\text{-almost surely finite}\}$, meaning,

[8]Stochastic differential equation is often used to characterize capital accumulation under uncertainty (see, e.g., Merton, 1975; Leong and Huang, 2010).

[9]This assumption simplifies greatly the tractability of the model and enables us to derive formal results transparently. As a caveat, we admit that our results might not necessarily easily carry over to general utility functions.

respectively, the capitalist has **overall information** denoted by filtration $\mathfrak{F}_t$ and **Δ-delayed information** denoted by filtration $\mathfrak{F}_{t-\Delta}$ at time t. To focus on the key issue, we assume that Δ is a commonly-known constant. We can relax this assumption, for example, via incorporating uncertainty into the size of delayed information. However, this assumption is not essential for establishing the following formal results, as we can use an expected value of Δ to replace the current Δ and our major predictions still hold true.

The capitalist first chooses an exit time τ, namely **the timing of terminating (or withdrawing) investment** written in a formal contract, based on a sustainability consideration, then he chooses an optimal consumption plan during $[0, \tau]$. In particular, if $\mathbb{E}_{t_0}(\tau) = 0$ in equilibrium, then the capitalist will not initially invest in the current economy. This specification of decision-making procedure is consistent with our intuition, and can be interpreted as a natural extension of the classic intertemporal optimization through endogenizing the planning horizon. Here sustainability[10] of capital accumulation requires that the τ be a solution of

$$\Phi_{\cdot}(t_0, k_0) \equiv \sup_{\tau \in \mathcal{T}} \mathbb{E}^{t_0,k_0} \left[e^{-\rho(t_0+\tau)}(k(\tau) - \varpi) \right] \tag{2}$$

subject to (1). The subscript $\cdot = 0$ or Δ for $\mathcal{T} = \mathcal{T}_0$ or $\mathcal{T}_\Delta$, $\mathbb{E}^{t_0,k_0}$ is the expectation with respect to probability law P^{t_0,k_0} of time-space process $dZ(t) \equiv (dt, dk(t))'$ with initial state $Z(0) \equiv (t_0, k_0)'$ and transpose $'$, and $\varpi > 0$ is a constant **exit cost** which measures the barriers to inter-jurisdictional or inter-national capital mobility as well as the associated transaction cost. For any given τ, optimal consumption plan solves the problem:

$$\max_{c(t)>0} \mathbb{E}_{t_0} \left[\int_0^\tau e^{-\rho(t_0+t)} \ln c(t) dt \right] \tag{3}$$

subject to (1).

2.2 Government

To be as realistic as possible, the government is assumed to consist of both types of benevolent and selfish politicians. The measure of politicians is normalized to one with a constant fraction ε of the benevolent, who share the same utility as the capitalist, and the remaining $1 - \varepsilon$ of the selfish who maximize the utility generated by tax revenue.

[10] In the literature (e.g., Radner, 1961; Kurz, 1965; McKenzie, 1963, 1976) regarding optimal capital accumulation, sustainability is usually defined by maximizing terminal stocks (or final states). Departing from these studies, we use optimal stopping theory which enables us to make both optimal terminal stock and optimal exit time be simultaneously determined.

We focus on two governance arrangements implying two alternative tax-rate-setting problems.

Definition 2.1. A governance arrangement is called **authoritarian governance** if the government moves first to unilaterally determine a tax rate and the capitalist has Δ-delayed information when choosing the investment horizon.

Definition 2.2. A governance arrangement is called **inclusive governance**[11] if the government bargains with the capitalist to cooperatively determine a tax rate and the capitalist has overall information when choosing the investment horizon.

The difference on information structure stems from the observation that strong top-down authority generally leads towards a very low degree of transparency, such as the Soviet Union under Stalin's regime and the episode of China before the implementation of Reform and Opening policy, while inclusive governance induces rational cooperation which calls for a relatively high degree of transparency. That is, information sharing is the prerequisite condition for building up a sustainable relationship of incentive-compatible cooperation. In particular, it is not necessary to let the capitalist have overall information under inclusive governance, we normalize his information delay to zero because only the difference (of information structures) matters when comparing the two governance arrangements. In other words, we can still obtain our major results after relaxing this normalization imposed on inclusive governance.

To demonstrate the validity of inclusive governance, we shall take China as an example. In fact, political centralization, fiscal decentralization, inter-jurisdictional competition and the mechanism governing political turnover in China motivated local governments to form a cooperative relationship with capitalists (or investors), hence attracting sufficient capital investment and guaranteeing fast economic growth during the past several decades (e.g., Montinola et al., 1995; Li and Zhou, 2005; and Xu, 2011). Alternatively, a type of inclusive governance adopted by local governments laid out the self-enforced institutional foundation of China's growth miracle.

[11] In the language of Olson (2000), inclusive governance may be interpreted as the maximization of encompassing interests between the power and citizens. We may also interpret it as a realization of open access orders respecting economically incentive-compatible requirements (see North et al., 2006).

3 Equilibrium Derivation

3.1 Equilibrium under Authoritarian Governance

Under authoritarian governance, events proceed as follows:

Stage 1. The capitalist chooses an exit time $\tau_\Delta \in \mathcal{T}_\Delta$ by solving problem (2).

Stage 2. The government determines a capital-income tax rate by solving

$$\max_{0\le\tau_k\le 1} \mathbb{E}_{t_0}\left(\int_0^{\tau_\Delta} e^{-\rho(t_0+t)}\left\{\underbrace{\varepsilon \ln c(t)}_{\text{utility for the benevolent}} + \underbrace{(1-\varepsilon)\ln[\tau_k(r-\delta)k(t)]}_{\text{utility for the selfish}}\right\} dt\right) \tag{4}$$

subject to (1).

Stage 3. The capitalist chooses a consumption plan by solving problem (3).

In contrast to the political polarization adopted by Alesina and Tabellini (1989) and the normative assumption of a benevolent government, we assume as shown in (4) that the government maximizes a weighted average of utilities of both types of politicians. Indeed, one can interpret it as a kind of political-power balance between the two conflicting groups of politicians, which is commonly seen in both democracies and non-democracies. For instance, it represents a two-party bargaining equilibrium or a realization of political compromise (e.g., Dixit et al., 2000) within the government. In addition, we focus on the taxation policy lack of commitment in the sense that it is determined after the capitalist has chosen an exit time.

Using backward induction, equilibrium is derived and stated in the following lemma.

Lemma 3.1. *Suppose the economy is under authoritarian governance. Then, we have:*

(i) The subgame perfect equilibrium is $\{c^*(t), \tau_k^*\} = \left\{\rho k(t), \frac{\rho(1-\varepsilon)}{r-\delta}\right\}$ *with*

$$k(t) = k_0 \exp\left\{\left[r-\delta-\rho(2-\varepsilon)-\frac{1}{2}\sigma^2\right]t+\sigma B(t)\right\}. \tag{5}$$

(ii) If the capital return rate is bounded as shown in Appendix, then the optimal exit time is $\tau_\Delta^* = \inf\left\{t>0; k(t)=\tilde{k}^*\right\}$ *with* $\tilde{k}^* = \frac{\lambda_1\tilde{\varpi}}{\lambda_1-1}$, *in which*

$$\lambda_1 = \frac{\sigma^2-2\mu+\sqrt{(2\mu-\sigma^2)^2+8\rho\sigma^2}}{2\sigma^2} \tag{6}$$

and

$$\tilde{\varpi} = \varpi e^{-\mu\Delta} \tag{7}$$

where $\mu \equiv r-\delta-\rho(2-\varepsilon) > 0$.

Proof. See Appendix. □

From (5), it is easy to see that capital-income tax rate discourages the capitalist's consumption via negatively distorting his capital accumulation along the entire path. Also, the larger the fraction of selfish politicians or equivalently the smaller the fraction of benevolent politicians, the higher the equilibrium tax rate. Part (ii) confirms the existence and uniqueness of an optimal exit time under mild assumptions. As an optimal stopping rule, the capitalist shall withdraw his capital investment via selling the asset or stock when his capital stock reaches a constant level denoted $\tilde{k}^*$, during which the associated transaction cost has already been taken into account.

If the capitalist has overall information rather than Δ-delayed information in choosing an optimal exit time, then part (ii) of Lemma 3.1 needs to be revised as follows.

Lemma 3.2. *Suppose the economy is under authoritarian governance with the information delay satisfying $\Delta \downarrow 0$. Then, the optimal exit time is $\tau_0^* = \inf\{t > 0; k(t) = k^*\}$ with $k^* = \frac{\lambda_1 \varpi}{\lambda_1 - 1}$, where $k(t)$ and λ_1 are respectively given by (5) and (6).*

Proof. Since the proof is similar to that of Lemma 3.1, we omit it to economize on the space. □

As is obvious, the optimal stopping rule is different from that in Lemma 3.1. The direct implication is that the factor of information availability does matter in determining the optimal exit time. We shall discuss another implication of this factor in Section 5. Furthermore, this lemma offers a useful intermediate case in the sense that we can compare it with Lemma 3.1 to identify the equilibrium effect resulted from different information structures within the same game form, and compare it with the following Lemma 3.3 to identify the equilibrium effect resulted from different game forms under the same information structure.

3.2 Equilibrium under Inclusive Governance

Under inclusive governance, events proceed as follows:

Stage 1. The capitalist chooses an exit time $\tau_0 \in \mathcal{T}_0$ by solving problem (2).

Stage 2. Under rational cooperation, the maximization problem is

$$\max_{c(t)>0, 0\leq\tau_k\leq 1} \mathbb{E}_{t_0}\left(\int_0^{\tau_0} e^{-\rho(t_0+t)}\left\{\underbrace{\ln c(t)}_{\text{utility for the capitalist}} + \overbrace{\varepsilon \ln c(t) + (1-\varepsilon)\ln[\tau_k(r-\delta)k(t)]}^{\text{utility for the government}}\right\} dt\right) \tag{8}$$

subject to (1). That is, (8) defines the collective objective.

Using backward induction, equilibrium is derived and stated in the following lemma.

Lemma 3.3. *Suppose the economy is under inclusive governance. Then, we have:*

(i) The cooperative equilibrium is $\{c^{**}(t), \tau_k^{**}\} = \left\{\frac{\rho(1+\varepsilon)}{2}k(t), \frac{\rho(1-\varepsilon)}{2(r-\delta)}\right\}$ *with*

$$k(t) = k_0 \exp\left[\left(r - \delta - \rho - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right]. \tag{9}$$

(ii) Under both Nash bargaining solution/Shapley value and proportional-distribution allocation principles, the cooperative equilibrium satisfies group rationality, individual rationality, Pareto efficiency and subgame consistency, and neither the capitalist nor the government unilaterally deviates from cooperation.[12]

(iii) If the capital return rate is bounded as shown in Appendix, then the optimal exit time is $\tau_0^{**} = \inf\{t > 0; k(t) = k^{**}\}$ *with* $k^{**} = \frac{h_1 \varpi}{h_1 - 1}$*, in which*

$$h_1 = \frac{\sigma^2 - 2(r-\delta-\rho) + \sqrt{[2(r-\delta-\rho) - \sigma^2]^2 + 8\rho\sigma^2}}{2\sigma^2}. \tag{10}$$

Proof. See Appendix. □

When compared to Lemma 3.1, here the equilibrium tax rate is smaller, the equilibrium consumption rate is smaller, the equilibrium savings rate is higher, and hence the equilibrium expected growth rate of capital accumulation is higher. Nonetheless, note that the relationship between the optimal exit time and the equilibrium speed of capital accumulation is in general not monotone, we cannot predict that one definitely induces a longer investment horizon than does the other one. Another interesting observation arises from comparing (5) with (9), i.e., the composition of politicians matters for the equilibrium capital accumulation under authoritarian governance while it is irrelevant for that under inclusive governance. The reason for this is that the composition of politicians just affects the equilibrium tax rate under authority, whereas it affects both equilibrium consumption and equilibrium tax rate under cooperation and also the two effects offset along the equilibrium path of capital accumulation.

[12] We will define these terms when we prove the lemma in Appendix.

4 The Choice between Authority and Inclusive Governance

4.1 Theoretical Prediction

Let $\mathbb{E}(\tau_\Delta^*)$[13] and $\mathbb{E}(\tau_0^*)$ denote the expected exit times under authoritarian governance for $\Delta > 0$ and $\Delta \downarrow 0$, respectively. Let $\mathbb{E}(\tau_0^{**})$ denote the expected exit time under inclusive governance. The following theorem analyzes the equilibrium choice between authoritarian governance and inclusive governance by using the standard of inducing a later exit time and hence a longer expected investment horizon, providing the same entry time $t_0 = 0$. This theorem carries the central message of our paper.

Theorem 4.1. *For the economy under consideration, we have the following conclusions.*

(i) If $\frac{\varpi}{k_0} > \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta}$*, then* $\mathbb{E}(\tau_\Delta^*) > 0$*; if* $\frac{\varpi}{k_0} > \frac{\lambda_1 - 1}{\lambda_1}$*, then* $\mathbb{E}(\tau_0^*) > 0$*; and if* $\frac{\varpi}{k_0} > \frac{h_1 - 1}{h_1}$*, then* $\mathbb{E}(\tau_0^{**}) > 0$*.*

(ii) $\mathbb{E}(\tau_0^*) > \mathbb{E}(\tau_\Delta^*)$ *for* $\forall \Delta > 0$ *and* $\mathbb{E}(\tau_\Delta^*)$ *is strictly decreasing in* Δ*.*

(iii) If $\Delta < \Delta_1^*$*, in which* $\Delta_1^* > 0$ *is shown in Appendix, then there exists a finite upper bound, denoted by* $\Xi^* > 0$ *and shown in Appendix, of* $\frac{\varpi}{k_0}$ *such that* $\mathbb{E}(\tau_0^{**}) > \mathbb{E}(\tau_\Delta^*)$ *for any* $\frac{\varpi}{k_0} \le \Xi^*$*.*

(iv) If $\frac{\varpi}{k_0} > \Xi^*$*, then there exists a threshold, denoted by* $\Delta_2^* > 0$ *and shown in Appendix, of* Δ *such that*

$$\mathbb{E}(\tau_\Delta^*) \begin{cases} > \mathbb{E}(\tau_0^{**}) & \text{if } \Delta < \Delta_2^*, \\ = \mathbb{E}(\tau_0^{**}) & \text{if } \Delta = \Delta_2^*, \\ < \mathbb{E}(\tau_0^{**}) & \text{if } \Delta > \Delta_2^*. \end{cases}$$

(v) For the same threshold $\Xi^* > 0$*,*

$$\mathbb{E}(\tau_0^*) \begin{cases} > \mathbb{E}(\tau_0^{**}) & \text{if } \frac{\varpi}{k_0} > \Xi^*, \\ = \mathbb{E}(\tau_0^{**}) & \text{if } \frac{\varpi}{k_0} = \Xi^*, \\ < \mathbb{E}(\tau_0^{**}) & \text{if } \frac{\varpi}{k_0} < \Xi^*. \end{cases}$$

Proof. See Appendix. □

Part (i) provides heterogenous conditions that guarantee positive exit times under alternative governance arrangements. We have identified the conditions under which a governance arrangement

[13] For notational simplicity, here we assume the initial time to be $t_0 = 0$, and hence $\mathbb{E}_{t_0}$ is simply written as $\mathbb{E}$.

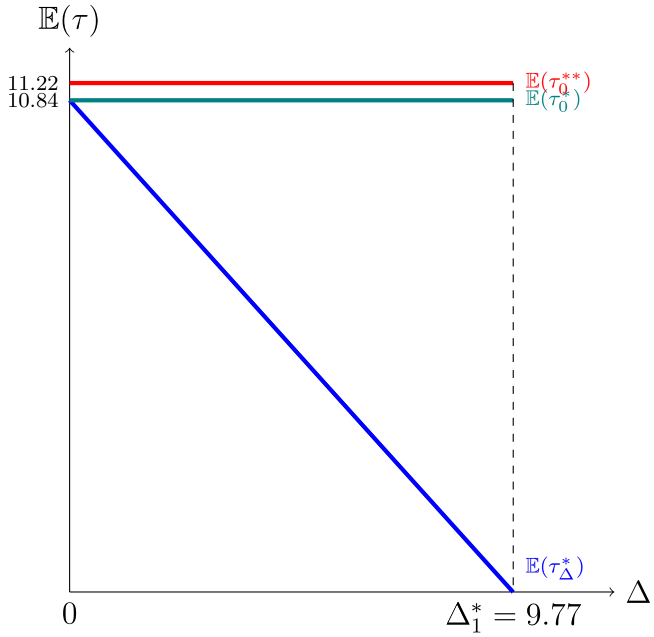

Figure 1: Results (ii) and (iii) of Theorem 4.1: $\varpi/k_0 = 0.695 = \Xi^*$.

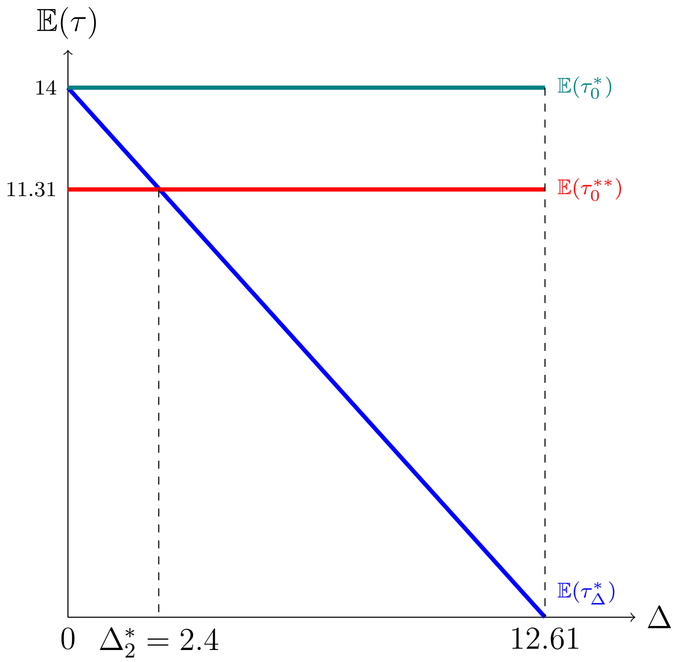

Figure 2: Results (ii) and (iv) of Theorem 4.1: $\varpi/k_0 = 0.71 > 0.695 = \Xi^*$.

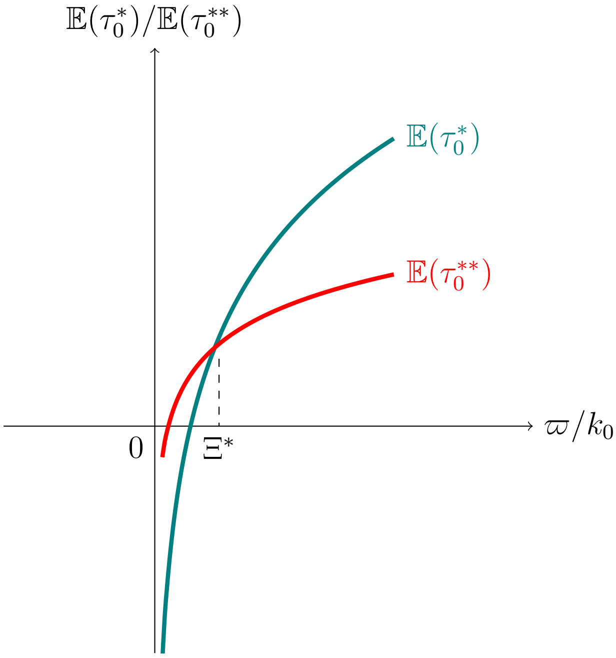

Figure 3: Result (v) of Theorem 4.1.

incentivizes the capitalist to sustain investment for a longer time than does the other one. In what follows, authority is called to dominate inclusive governance if it induces a strictly later exit time than does inclusive governance, and vice versa; authority and inclusive governance are called indifferent if they induce the same expected exit time. Also, one can refer to Figures 1-3 for intuitive observations.

Firstly, if information delay is smaller than a critical value, then there is an upper bound of exit cost such that inclusive governance dominates authority within the bound. Secondly, if exit cost is beyond the upper bound, then we can find another threshold of information delay such that authority dominates inclusive governance below the threshold, authority and inclusive governance are equivalent upon the threshold, while inclusive governance dominates authority above the threshold. Thirdly, for the special case where the information delay under authoritarian governance approaches zero, we find a threshold that is exactly the above established upper bound of exit cost such that authority dominates inclusive governance above the threshold, authority and inclusive governance are equivalent upon the threshold, while inclusive governance dominates authority below the threshold. Particularly, the greater the size of delayed information under authoritarian governance, the earlier the expected exit time, a discouragement effect originated from the delayed information availability. Loosely speaking, our result implies that, ceteris paribus, narrowing information delay can increase the relative advantage of authority while lowering exit cost can increase the relative advantage of inclusive governance. Theorem 4.1, accordingly, provides novel predictions on the connection between governance arrangement and investment endurance.

4.2 Numerical Illustration

Here we carry out numerical analysis of the model, which can help us understand Theorem 4.1 more intuitively. In particular, we use parameter values representative of the U.S. economy. Following the estimation of Poterba (1998), we set $r = 0.086$ which is the average pretax rate of return on capital for the 1990-1996 period. As usually used in real-business-cycle models, we set $\delta = 0.025$, which corresponds to about 10% depreciation per annum, and also the time-discount rate $\rho = 0.03$. By following Poterba (1998) to set the target $\tau_k = 0.42$, namely the average tax rate during 1990-1996, we use the capital income tax equilibrium under authoritarian governance to get that $\varepsilon = 0.146$. We summarize all parameter values in Table 1.

Table 1: Parameter Values

Parameter	Value	Description
r	0.086	Capital return rate
δ	0.025	Depreciation rate
ρ	0.03	Subjective discount factor
σ	0.025	Percentage volatility
ε	0.146	Fraction of benevolent politicians

We next calculate the expected investment horizons for different values of information delay and the ratio of exit cost to initial capital, which are reported in Tables 2-7.

Table 2: Expected Investment Horizons for $t_0 = 0$ and $\Delta = 0.5$

ϖ/k_0	0.68	0.69	0.70	0.705	0.706	0.707	0.708	0.709	0.71	0.72
$\mathbb{E}(\tau_\Delta^*)$	3.43	6.35	9.22	10.65	10.93	11.21	11.50	**11.78**	12.06	14.86
$\mathbb{E}(\tau_0^*)$	3.96	6.88	9.76	11.18	11.47	**11.75**	12.03	12.31	12.59	15.39
$\mathbb{E}(\tau_0^{**})$	10.25	10.74	11.22	11.45	11.50	**11.55**	11.59	**11.64**	11.69	12.15

To guarantee $\mathbb{E}(\tau_\Delta^*) > 0$, it follows from part (i) of Theorem 4.1 that $\frac{\varpi}{k_0} > \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta}$ must be satisfied. In fact, if this condition is satisfied, then we also get that $\mathbb{E}(\tau_0^*) > 0$ and $\mathbb{E}(\tau_0^{**}) > 0$. By using the parameter values in Table 1 and equation (6), we get that $\left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta} = e^{0.005\Delta}/1.5$. Also,

Table 3: Expected Investment Horizons for $t_0 = 0$ and $\Delta = 1$

ϖ/k_0	0.68	0.69	0.70	0.706	0.707	0.708	0.709	0.71	0.711	0.72
$\mathbb{E}(\tau_\Delta^*)$	3.03	5.94	8.82	10.53	10.81	11.10	11.38	11.66	**11.94**	14.46
$\mathbb{E}(\tau_0^*)$	3.96	6.88	9.76	11.47	**11.75**	12.03	12.31	12.59	12.88	15.39
$\mathbb{E}(\tau_0^{**})$	10.25	10.74	11.22	11.50	**11.55**	11.59	11.64	11.69	**11.74**	12.15

Table 4: Expected Investment Horizons for $t_0 = 0$ and $\Delta = 3$

ϖ/k_0	0.69	0.70	0.707	0.71	0.711	0.712	0.713	0.717	0.718	0.72
$\mathbb{E}(\tau_\Delta^*)$	4.20	7.07	9.06	9.91	10.19	10.47	10.75	11.87	**12.15**	12.71
$\mathbb{E}(\tau_0^*)$	6.88	9.76	**11.75**	12.59	12.88	13.16	13.44	14.56	14.84	15.39
$\mathbb{E}(\tau_0^{**})$	10.74	11.22	**11.55**	11.69	11.74	11.78	11.83	12.02	**12.06**	12.15

Table 5: Expected Investment Horizons for $t_0 = 0$ and $\Delta = 5$

ϖ/k_0	0.69	0.70	0.707	0.71	0.72	0.722	0.728	0.729	0.73	0.74
$\mathbb{E}(\tau_\Delta^*)$	1.47	4.35	6.34	7.19	9.99	10.54	12.20	12.47	**12.75**	15.47
$\mathbb{E}(\tau_0^*)$	6.88	9.76	**11.75**	12.59	15.39	15.95	17.60	17.88	18.15	20.87
$\mathbb{E}(\tau_0^{**})$	10.74	11.22	**11.55**	11.69	12.15	12.25	12.52	12.57	**12.61**	13.07

Table 6: Expected Investment Horizons for $t_0 = 0$ and $\Delta = 7$

ϖ/k_0	0.70	0.707	0.71	0.72	0.73	0.735	0.736	0.739	0.74	0.75
$\mathbb{E}(\tau_\Delta^*)$	2.98	4.97	5.81	8.61	11.37	12.74	**13.01**	13.82	14.09	16.78
$\mathbb{E}(\tau_0^*)$	9.76	**11.75**	12.59	15.39	18.15	19.52	19.79	20.60	20.87	23.56
$\mathbb{E}(\tau_0^{**})$	11.22	**11.55**	11.69	12.15	12.61	12.84	**12.89**	13.02	13.07	13.52

Table 7: Expected Investment Horizons for $t_0 = 0$ and $\Delta = 10$

ϖ/k_0	0.702	0.707	0.71	0.72	0.73	0.74	0.745	0.747	0.748	0.75
$\mathbb{E}(\tau_\Delta^*)$	0.77	2.19	3.04	5.83	8.59	11.31	12.66	13.20	**13.46**	14.00
$\mathbb{E}(\tau_0^*)$	10.33	**11.75**	12.59	15.39	18.15	20.87	22.22	22.76	23.02	23.56
$\mathbb{E}(\tau_0^{**})$	11.31	**11.55**	11.69	12.15	12.61	13.07	13.29	13.38	**13.43**	13.52

we can have $e^{0.005\Delta}/1.5 \cong 0.67$ for $\forall\Delta \in [0,2]$, $e^{0.005\Delta}/1.5 \cong 0.68$ for $\Delta = 3$ or 5, $e^{0.005\Delta}/1.5 \cong 0.69$ for $\Delta = 7$, and $e^{0.005\Delta}/1.5 \cong 0.701$ for $\Delta = 10$. This is why we begin with $\varpi/k_0 = 0.68$, 0.69, 0.70 and 0.702 in Tables 2-7. These cases with different possible values of Δ are informative enough for quantitatively illustrating our theoretical prediction.

We obtain the following findings which are consistent with Theorem 4.1. First, we always have $\mathbb{E}(\tau_\Delta^*) < \mathbb{E}(\tau_0^*)$, verifying the discouragement effect of delayed information availability imposed on investment horizon. Second, for any given Δ, expected investment horizon increases as ϖ/k_0 increases, regardless of governance arrangement. Third, the expected investment horizon under authoritarian governance increases much faster with respect to the ratio ϖ/k_0 than that under inclusive governance. Fourth, there is a unique threshold of ϖ/k_0 such that $\mathbb{E}(\tau_0^{**}) > \mathbb{E}(\tau_0^*)$ below the threshold while $\mathbb{E}(\tau_0^{**}) < \mathbb{E}(\tau_0^*)$ above the threshold. Fifth, there is another greater threshold of ϖ/k_0 such that $\mathbb{E}(\tau_0^{**}) > \mathbb{E}(\tau_\Delta^*)$ below the threshold while $\mathbb{E}(\tau_0^{**}) < \mathbb{E}(\tau_\Delta^*)$ above the threshold. And sixth, in general, the larger the value of Δ, the higher the threshold of ϖ/k_0 is required to get $\mathbb{E}(\tau_0^{**}) < \mathbb{E}(\tau_\Delta^*)$.

We hence have two implications: (1) as the difference between the two thresholds (of ϖ/k_0) measures the equilibrium effect originated from the informational difference between the two governance arrangements, the observation that the difference is non-decreasing in Δ means that a bigger informational difference generally creates a bigger equilibrium effect; (2) to make authoritarian governance dominate inclusive governance in inducing a longer expected investment horizon, a bigger Δ must be accompanied by a bigger ϖ/k_0, namely a lower degree of government transparency under authority must be accompanied by a lower degree of capital mobility.

5 Further Discussion

Here we discuss the choice of information delay and exit cost under authoritarian governance. As an authoritarian government, it is expected to have the power and motive to (partially) determine both information delay and exit cost. As we can use Δ to measure the government's hidden information, one natural question is: what's the optimal choice of Δ for the government? To answer this question, we analyze how it affects the government's welfare during $(0, \mathbb{E}(\tau_\Delta^*))$:

$$V_\Delta^G \equiv \mathbb{E}_{t_0}\left[\int_0^{\mathbb{E}(\tau_\Delta^*)} V^G(t,k(t))dt\right],$$

where $V^G(t,k(t)) = e^{-\rho(t_0+t)}\left[\frac{1}{\rho}\ln k(t) + C_4\right]$ with C_4 and $k(t)$ given by (18) and (24) in Appendix.

We then obtain the following proposition.

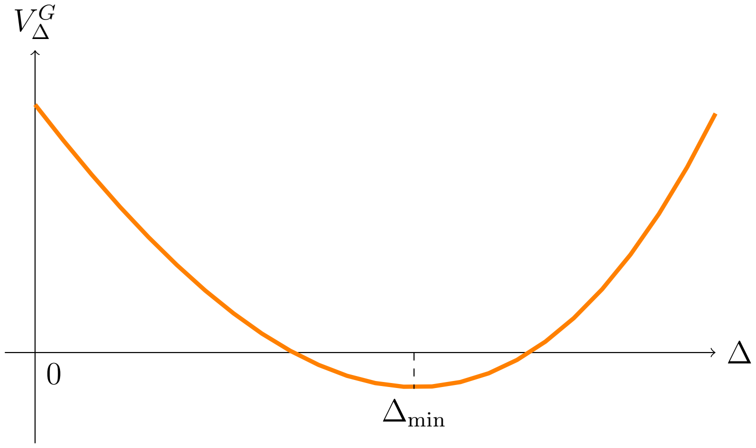

Figure 4: U-shaped Relationship between V_Δ^G and Δ

Proposition 5.1. *For the economy under consideration, we have the following conclusions.*

(i) (U-Shaped Relationship between V_Δ^G and Δ) Suppose $\varpi > \left(\frac{\lambda_1-1}{\lambda_1}\right) e^{-\rho C_4}$. Then there exists a unique critical value, denoted by $\Delta_{\min} = \frac{1}{\mu}\ln\left(\frac{\lambda_1\varpi}{\lambda_1-1}\right) + \frac{\rho}{\mu}C_4 > 0$, of Δ such that

$$\frac{\partial V_\Delta^G}{\partial \Delta} \begin{cases} > 0 & \text{if } \Delta > \Delta_{\min}, \\ = 0 & \text{if } \Delta = \Delta_{\min}, \\ < 0 & \text{if } \Delta < \Delta_{\min}. \end{cases}$$

(ii) For $\forall \Delta > 0$, suppose $k_0 < e^{-\rho C_4}$. Then

$$\frac{\partial V_\Delta^G}{\partial \Delta} \begin{cases} > 0 & \text{if } \varpi \in \left(k_0\left(\frac{\lambda_1-1}{\lambda_1}\right)e^{\mu\Delta}, \left(\frac{\lambda_1-1}{\lambda_1}\right)\exp\left(\mu\Delta - \rho C_4\right)\right), \\ = 0 & \text{if } \varpi = \left(\frac{\lambda_1-1}{\lambda_1}\right)\exp\left(\mu\Delta - \rho C_4\right), \\ < 0 & \text{if } \varpi > \left(\frac{\lambda_1-1}{\lambda_1}\right)\exp\left(\mu\Delta - \rho C_4\right). \end{cases}$$

Proof. See Appendix. □

The U-shaped relationship implies that only when the information advantage is already larger than a threshold can the government benefit from further increasing its hidden information, otherwise doing this hurts the government (see also Figure 4). This argument also relies on a lower bound imposed on the exit cost. In fact, we can find another threshold of exit cost such that the government benefits from further increasing its hidden information only when the exit cost is lower than the threshold, otherwise doing this hurts the government.

If we interpret the exit cost as a kind of transaction cost determined by the government, then the immediate question is: what's the optimal choice of exit cost for the government? Answering this question produces the following proposition.

Proposition 5.2 (U-Shaped Relationship between V_{Δ}^{G} and ϖ)**.** *Suppose* $k_0 < e^{-\rho C_4}$. *Then there exists a unique critical value, denoted as* $\varpi_{\min} = \left(\frac{\lambda_1 - 1}{\lambda_1}\right) \exp\left(\mu\Delta - \rho C_4\right)$, *of* ϖ *such that*

$$\frac{\partial V_{\Delta}^{G}}{\partial \varpi} \begin{cases} > 0 & \text{for } \varpi > \varpi_{\min}, \\ = 0 & \text{for } \varpi = \varpi_{\min}, \\ < 0 & \text{for } \varpi \in \left(k_0 \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta}, \varpi_{\min}\right). \end{cases}$$

Proof. See Appendix. □

The U-shaped relationship implies that only when the exit cost is already larger than a threshold can the government benefit from further increasing it, otherwise doing this hurts the government. Even so, we argue that both Δ and ϖ cannot be set arbitrarily large. There generally exists an upper bound for each parameter, and also these bounds can be endogenously determined. A possible approach is to introduce a shadow (or underground) economy into the current framework and allow the capitalist to freely choose between investing in the current formal economy and investing in the shadow economy. As documented by Schneider and Enste (2000), the shadow economy is not subject to government regulation and hence taxes are avoided. In addition, the associated returns and risks may be different. In consequence, these bounds can be set so that the capitalist is indifferent between the two options, namely they generate the same expected utility and the same terminal capital. We omit detailed calculations to economize on the space.

6 Conclusion

We develop an analytical framework to comparatively study authoritarian governance and inclusive governance in providing incentives for longer investments. In terms of determining a capital income tax rate, authority and inclusive governance represent two types of governance relationships connecting government and investors. We identify explicit conditions enabling us to predict when authority dominates inclusive governance, when inclusive governance dominates authority, and when they are indifferent. Controlling for capital return rate, our results imply that the relative advantage of inclusive governance can be strengthened by lowering the exit cost (or allowing a higher degree of capital mobility) while the relative advantage of authority can be strengthened by increasing the degree of government transparency.

Another important finding is that simply cutting the tax rate is not necessarily sufficient to provide incentives for a capitalist to invest in the current economy for a longer time. For instance, inclusive governance is shown to induce a smaller equilibrium tax rate than does authority while authority may still dominate inclusive governance in sustaining capital investment. The implication is thus that, in addition to capital taxation, institutional factors such as the degree of capital mobility and the degree of government transparency are also relevant in determining the equilibrium expected investment horizon. As a final remark, our results suggest the following order of institutional change for open economies: before liberalizing capital account, government should first establish an inclusive (or business-friendly) governance arrangement with strengthened government transparency.

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Appendix: Proofs

Proof of Lemma 3.1: We will complete it in 4 steps.

Step 1. Solving the problem in stage 3 gives:

Claim 6.1. *The capitalist sets the consumption at time t to be $c^*(t) = \rho k(t)$.*

Proof. We prove that there is a continuously differentiable function $V^C(t, k(t))$ satisfying the Bellman-Isaacs-Fleming partial differential equation:

$$-V_t^C(t,k(t)) - \frac{1}{2}\sigma^2 k^2(t) V_{kk}^C(t,k(t)) = \max_{c(t)>0} \left\{ e^{-\rho(t_0+t)} \ln c(t) + V_k^C(t,k(t))[(1-\tau_k)(r-\delta)k(t) - c(t)] \right\}. \tag{11}$$

Performing the maximization operator gives

$$\frac{1}{c(t)} = e^{\rho(t_0+t)} V_k^C(t,k(t)). \tag{12}$$

We guess that

$$V^C(t,k(t)) = e^{-\rho(t_0+t)}[C_1 \ln k(t) + C_2], \tag{13}$$

in which constants C_1 and C_2 are to be determined. Applying (12) and (13) to (11) and rearranging the algebra result in $C_1 = \frac{1}{\rho}$ and

$$C_2 = -\frac{\sigma^2}{2\rho^2} + \frac{1}{\rho}\ln\rho + \frac{1}{\rho^2}(1-\tau_k)(r-\delta) - \frac{1}{\rho}. \tag{14}$$

□

Step 2. Solving the problem in stage 2 gives:

Claim 6.2. *The government sets the tax rate to be $\tau_k^* = \frac{\rho(1-\varepsilon)}{r-\delta}$.*

Proof. The Bellman equation reads as:

$$\begin{aligned} -V_t^G(t,k(t)) - \frac{1}{2}\sigma^2 k^2(t) V_{kk}^G(t,k(t)) &= \max_{0\le\tau_k\le1} \left\{ e^{-\rho(t_0+t)}\varepsilon \ln[\rho k(t)] \right. \\ &\left. + e^{-\rho(t_0+t)}(1-\varepsilon)\ln[\tau_k(r-\delta)k(t)] + V_k^G(t,k(t))k(t)[(1-\tau_k)(r-\delta)-\rho] \right\}. \end{aligned} \tag{15}$$

Performing the maximization operator gives rise to

$$e^{-\rho(t_0+t)}(1-\varepsilon) = V_k^C(t,k(t))k(t)(r-\delta)\tau_k. \tag{16}$$

If we try

$$V^G(t,k(t)) = e^{-\rho(t_0+t)}[C_3 \ln k(t) + C_4] \tag{17}$$

for constants C_3 and C_4 which are to be determined, then applying (16) and (17) to (15) produces $C_3 = \frac{1}{\rho}$ and

$$C_4 = -\frac{\sigma^2}{2\rho^2} + \frac{1}{\rho}\ln\rho + \frac{1}{\rho^2}(r-\delta-\rho) + \frac{1-\varepsilon}{\rho}\ln\left(\frac{1-\varepsilon}{e}\right). \tag{18}$$

Then, by making use of $C_3 = \frac{1}{\rho}$, (16) and (17) we obtain the desired τ_k^*. □

Step 3. To solve the problem in stage 1, we first put $Z(t) \equiv (t_0 + t, k(t))'$ for $t \geq 0$. Then, it follows from (1) and Claims 6.1 and 6.2 that

$$dZ(t) = \begin{bmatrix} 1 \\ \left(\underbrace{r-\delta-\rho(2-\varepsilon)}_{\equiv\mu>0}\right) k(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma k(t) \end{bmatrix} dB(t), \quad Z(0) = \begin{bmatrix} t_0 \\ k_0 \end{bmatrix}, \tag{19}$$

and the corresponding differential generator is

$$\mathcal{A}\phi(t_0,k_0) = \frac{\partial\phi}{\partial t_0} + \mu k_0\frac{\partial\phi}{\partial k_0} + \frac{1}{2}\sigma^2 k_0^2\frac{\partial^2\phi}{\partial k_0^2}, \quad \forall\phi\in C^2(\mathbb{R}^2). \tag{20}$$

If we try a function ϕ of the form $\phi(t_0,k_0) = e^{-\rho t_0}k_0^\lambda$ for some constant $\lambda\in\mathbb{R}$, then we can get $\mathcal{A}\phi(t_0,k_0) = e^{-\rho t_0}k_0^\lambda\left[-\rho+\mu\lambda+\frac{1}{2}\sigma^2\lambda(\lambda-1)\right]$. By solving equation $\sigma^2\lambda^2 + (2\mu-\sigma^2)\lambda - 2\rho = 0$ we get the unique positive root:

$$\lambda_1 = \frac{\sigma^2 - 2\mu + \sqrt{(2\mu-\sigma^2)^2 + 8\rho\sigma^2}}{2\sigma^2}. \tag{21}$$

If we let $\lambda_1 > 1$, then we should rely on an additional assumption that

$$\rho > \mu. \tag{22}$$

In what follows, we will suppose that condition (22) always holds true. With this value of λ_1 we put

$$\phi(t_0,k_0) = \begin{cases} e^{-\rho t_0}\tilde{C}k_0^{\lambda_1} & \text{if } (t_0,k_0)\in D \\ \psi(t_0,k_0) & \text{if } (t_0,k_0)\notin D \end{cases} \tag{23}$$

for some constant $\tilde{C}$, function $\psi(t_0,k_0)$ and continuation region D, remaining to be determined.

To find a reasonable guess for the continuation region D, we first note that by using Itô formula:

$$k(t) = k_0\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right], \tag{24}$$

which implies that

$$\mathbb{E}^{k_0}[k(\Delta)] = k_0\exp(\mu\Delta) \tag{25}$$

for $\Delta > 0$. Hence, we can rewrite the objective function as

$$\begin{aligned}
\psi(t_0, k_0) &\equiv \mathbb{E}^{t_0,k_0}\left[e^{-\rho(t_0+\Delta)}(k(\Delta) - \varpi)\right] \\
&= e^{-\rho(t_0+\Delta)}\left\{\mathbb{E}^{k_0}[k(\Delta)] - \varpi\right\} \\
&= e^{-\rho(t_0+\Delta)}\left(k_0 e^{\mu\Delta} - \varpi\right) \\
&= e^{-\rho t_0}\exp((\mu-\rho)\Delta)\left(k_0 - \varpi e^{-\mu\Delta}\right) \\
&\equiv e^{-\rho t_0}\Sigma\left(k_0 - \tilde{\varpi}\right),
\end{aligned} \tag{26}$$

where we have used (25) and defined

$$\Sigma \equiv \exp((\mu-\rho)\Delta), \quad \tilde{\varpi} \equiv \varpi e^{-\mu\Delta}. \tag{27}$$

Then, applying (20) to (26) results in $\mathcal{A}\psi(t_0, k_0) = e^{-\rho t_0}\Sigma[(\mu-\rho)k_0 + \rho\tilde{\varpi}]$. Therefore, we have

$$\begin{aligned}
U &\equiv \{(t_0, k_0); \mathcal{A}\psi(t_0, k_0) > 0\} \\
&= \{(t_0, k_0); (\mu-\rho)k_0 + \rho\tilde{\varpi} > 0\} \\
&= \left\{(t_0, k_0); k_0 < \frac{\rho\tilde{\varpi}}{\rho-\mu}\right\},
\end{aligned}$$

where we have used assumption (22).

We now determine the associated continuation region denoted by D. First, note that for $\forall t'$,

$$\begin{aligned}
\psi^*\left(t_0 - t', k_0\right) &= \sup_{\tau}\mathbb{E}_{(t_0-t')}\left[e^{-\rho\tau}\Sigma\left(k(\tau) - \tilde{\varpi}\right)\right] \\
&= \sup_{\tau}\mathbb{E}\left[e^{-\rho(\tau+(t_0-t'))}\Sigma\left(k(\tau) - \tilde{\varpi}\right)\right] \\
&= e^{\rho t'}\sup_{\tau}\mathbb{E}\left[e^{-\rho(\tau+t_0)}\Sigma\left(k(\tau) - \tilde{\varpi}\right)\right] \\
&= e^{\rho t'}\sup_{\tau}\mathbb{E}_{t_0}\left[e^{-\rho\tau}\Sigma\left(k(\tau) - \tilde{\varpi}\right)\right] = e^{\rho t'}\psi^*\left(t_0, k_0\right).
\end{aligned}$$

Then, we can get

$$\begin{aligned}
D + (t', 0) &= \left\{(t+t', k_0)\,; (t, k_0) \in D\right\} \\
&= \left\{(t_0, k_0)\,; (t_0 - t', k_0) \in D\right\} \\
&= \left\{(t_0, k_0)\,; \psi\left(t_0 - t', k_0\right) < \psi^*\left(t_0 - t', k_0\right)\right\} \\
&= \left\{(t_0, k_0)\,; e^{\rho t'}\psi\left(t_0, k_0\right) < e^{\rho t'}\psi^*\left(t_0, k_0\right)\right\} \\
&= \left\{(t_0, k_0)\,; \psi\left(t_0, k_0\right) < \psi^*\left(t_0, k_0\right)\right\} = D,
\end{aligned}$$

which yields that the continuation region D is invariant w.r.t. t in the sense that $D + (t', 0) = D$ for $\forall t'$. In consequence, the connected component of D that contains U must have the form

$$D = \left\{(t_0, k_0); 0 < k_0 < \tilde{k}^*\right\} \tag{28}$$

for some $\tilde{k}^*$ such that $U \subseteq D$, i.e.,

$$\tilde{k}^* \geq \frac{\rho\tilde{\varpi}}{\rho - \mu}. \tag{29}$$

Indeed, we can even argue that D cannot have any other components, and we prove this claim by means of contradiction. Suppose that U' is another component of D and it is disjoint from U, then we should have $\mathcal{A}\psi < 0$ in U' and so, if $Z(0) \in U'$, it follows from the Dynkin's Formula that

$$\mathbb{E}_{Z(0)}\left[\psi\left(Z(\tau)\right)\right] = \psi\left(Z(0)\right) + \mathbb{E}_{Z(0)}\left[\int_0^\tau \mathcal{A}\psi\left(Z(t)\right)dt\right] < \psi\left(Z(0)\right)$$

for all exit times τ bounded by the exit time from a k-bounded strip in U'. By this we can apply the Existence Theorem for Optimal Stopping (see, Øksendal, 2003) to conclude that $\psi^*\left(Z(0)\right) = \psi\left(Z(0)\right)$, which hence leads to $U' = \emptyset$, an empty set.

Hence, by (23), (26) and (28) we now put

$$\phi(t_0, k_0) = \begin{cases} e^{-\rho t_0}\tilde{C}k_0^{\lambda_1} & \text{if } 0 < k_0 < \tilde{k}^* \\ e^{-\rho t_0}\Sigma\left(k_0 - \tilde{\varpi}\right) & \text{if } \tilde{k}^* \leq k_0 \end{cases} \tag{30}$$

for some constant $\tilde{C} > 0$, to be determined. We, w.o.l.g, guess that the value function is C^1 at $k_0 = \tilde{k}^*$, which gives the following "high-contact" (or smooth-fit) conditions: $\tilde{C}(\tilde{k}^*)^{\lambda_1} = \Sigma\left(\tilde{k}^* - \tilde{\varpi}\right)$ (continuity at $k_0 = \tilde{k}^*$) and $\tilde{C}\lambda_1(\tilde{k}^*)^{\lambda_1 - 1} = \Sigma$ (differentiability at $k_0 = \tilde{k}^*$). It is easy to obtain the solutions:

$$\tilde{k}^* = \frac{\lambda_1\tilde{\varpi}}{\lambda_1 - 1}, \quad \tilde{C} = \frac{\Sigma}{\lambda_1}(\tilde{k}^*)^{1-\lambda_1}. \tag{31}$$

It remains to verify that with these values of $\tilde{k}^*$ and $\tilde{C}$ the function ϕ given by (30) satisfies all the conditions (i)-(xi) of Theorem 3.2 (Integro-variational inequalities for optimal stopping, pp.53-54) of Øksendal and Sulem (2009). To this end, first note that (i) and (ix) hold by construction of ϕ. Moreover, $\phi = \psi$ outside D. Accordingly, to verify (ii) we only need to prove that $\phi \geq \psi$ on D, i.e., that

$$\tilde{C}k_0^{\lambda_1} \geq \Sigma\left(k_0 - \tilde{\varpi}\right) \quad \text{for } 0 < k_0 < \tilde{k}^*. \tag{32}$$

Define the difference by $\zeta(k_0) \equiv \tilde{C}k_0^{\lambda_1} - \Sigma\left(k_0 - \tilde{\varpi}\right)$. By our chosen values of $\tilde{C}$ and $\tilde{k}^*$ in (31) we have $\zeta(\tilde{k}^*) = \zeta'(\tilde{k}^*) = 0$. Additionally, due to $\lambda_1 > 1$ by (21)-(22), $\zeta''(k_0) = \tilde{C}\lambda_1(\lambda_1 - 1)k_0^{\lambda_1 - 2} > 0$ for $0 < k_0 < \tilde{k}^*$. Consequently, $\zeta(k_0) > 0$ for $0 < k_0 < \tilde{k}^*$ and (32) holds true, and hence (ii) is verified.

For (iii), note that the boundary of set D is given by $\partial D = \left\{(t_0, k_0); k_0 = \tilde{k}^*\right\}$, we hence have

$$\mathbb{E}^{Z(0)}\left[\int_0^\infty \mathbb{I}_{\partial D}(Z(t))dt\right] = \int_0^\infty P^{k_0}\left[k(t) = \tilde{k}^*\right]dt = 0,$$

where $\mathbb{I}_{\partial D}(\cdot)$ denotes an indicator function. Also, by our construction of D and ϕ, it is trivial to see that ∂D is a Lipschitz surface and $\phi \in C^2(\mathbb{R} \times (0,\infty) \setminus \partial D)$ has locally bounded derivatives near ∂D, namely (iv) and (v) always hold true. In addition, it is straightforward to verify that (vii) holds based on our construction of ϕ.

For (vi), namely $\mathcal{A}\phi \leq 0$ on $\mathbb{R} \times (0,\infty) \setminus \partial D$, we know that outside D we have $\phi(t_0, k_0) = e^{-\rho t_0}\Sigma\left(k_0 - \tilde{\varpi}\right)$ and therefore $\mathcal{A}\phi = e^{-\rho t_0}\Sigma[(\mu-\rho)k_0 + \rho\tilde{\varpi}]$, which combines with (22) reveals that $(\mu-\rho)k_0 + \rho\tilde{\varpi} \leq 0$ for all $k_0 \geq \tilde{k}^*$ is equivalent to $\tilde{k}^* \geq \frac{\rho\tilde{\varpi}}{\rho-\mu}$. This is completely consistent with requirement (29). Hence, combining it with (31) leads us to

$$\frac{\lambda_1\tilde{\varpi}}{\lambda_1 - 1} \geq \frac{\rho\tilde{\varpi}}{\rho-\mu} \;\Leftrightarrow\; \lambda_1 \leq \frac{\rho}{\mu}. \tag{33}$$

To check if (x) holds true, i.e., $\tau_\Delta^* = \tau_D \equiv \inf\{t > 0; k(t) \notin D\} < \infty$ a.s., we consider the solution of $k(t)$ given by (24). By applying the law of iterated logarithm for Brownian motion we conclude that if $\mu > \frac{1}{2}\sigma^2$, then $\lim_{t\to\infty} k(t) = \infty$ a.s., and in particular $\tau_\Delta^* = \tau_D < \infty$ almost surely. Here, for (viii) to hold it suffices that (xi) holds true. In what follows, we provide conditions under which (xi) holds. Since by applying Heine-Borel theorem and Weierstrass theorem we know that ϕ is bounded on compact set $[0, \tilde{k}^*]$, it suffices to verify that $\{e^{-\rho\tau_\Delta}k(\tau_\Delta)\}_{\tau_\Delta\in\mathcal{T}_\Delta}$ is uniformly integrable. For this to be true it suffices that there exists a constant $W > 0$ such that

$$\mathbb{E}\left[e^{-2\rho\tau_\Delta}k^2(\tau_\Delta)\right] \leq W \quad \text{for } \forall\tau_\Delta \in \mathcal{T}_\Delta. \tag{34}$$

Since we have from (24) that $\mathbb{E}\left[e^{-2\rho\tau_\Delta}k^2(\tau_\Delta)\right] = k_0^2\mathbb{E}\left[\exp\left\{\left[2(\mu-\rho)+\sigma^2\right]\tau_\Delta\right\}\right]$, we can conclude that if

$$2(\mu-\rho) + \sigma^2 \leq 0, \tag{35}$$

then (34) holds, and hence (xi) holds as well.

Now, we summarize what we have proved:

Claim 6.3. *Suppose (22), (33), (35) and $\mu > \frac{1}{2}\sigma^2$ hold true for $\mu \equiv r - \delta - \rho(2-\varepsilon) > 0$. Then, with λ_1, $\tilde{C}$ and $\tilde{k}^*$ given by (21) and (31) the function ϕ given by (30) coincides with the value function Φ_Δ of our problem, and $\tau_\Delta^* = \tau_D \equiv \inf\left\{t > 0; k(t) = \tilde{k}^*\right\}$ is an optimal exit time, where D is the continuation region given by (28).*

Step 4. To complete the proof, we need the following result.

Claim 6.4. *Suppose the capital return rate is restricted as in the following proof, then the conditions used in Claim 6.3 hold true.*

Proof. First, we have $\mu > \frac{1}{2}\sigma^2 \Leftrightarrow r > \delta + \rho(2-\varepsilon) + \frac{1}{2}\sigma^2 \equiv r_{\min}$. Since it is easy to show that (22) implies (33), we just need to show that $\mu < \rho \Leftrightarrow r < \delta + \rho(2-\varepsilon) + \rho \equiv r_{\max}$. Also, note that (35) yields $\frac{1}{2}\sigma^2 \leq \rho - \mu$, we hence have $r_{\min} < r_{\max}$. As a consequence, the required conditions hold true as long as $r \in (r_{\min}, r_{\max})$. $\square$

Therefore, we obtain the subgame perfect equilibrium outcome by combining these results. Q.E.D.

Proof of Lemma 3.3: We shall complete it in 4 steps.

Step 1. Solving the problem in stage 2 gives:

Claim 6.5. *The cooperative equilibrium is* $\{c^{**}(t), \tau_k^{**}\} = \left\{\frac{\rho(1+\varepsilon)}{2}k(t), \frac{\rho(1-\varepsilon)}{2(r-\delta)}\right\}$ *for the value function* $J(t,k(t)) = e^{-\rho(t_0+t)}[C_5 \ln k(t) + C_6]$, *in which* $C_5 = \frac{2}{\rho}$ *and* $C_6 = \frac{2(r-\delta-\rho)-\sigma^2}{\rho^2} + \frac{1+\varepsilon}{\rho}\ln\left[\frac{\rho(1+\varepsilon)}{2}\right] + \frac{1-\varepsilon}{\rho}\ln\left[\frac{\rho(1-\varepsilon)}{2}\right]$.

Proof. Omitted. $\square$

Step 2. To justify cooperation, we will show that group rationality, individual rationality and subgame consistency are satisfied. In addition, no one will unilaterally deviate from cooperation under some given Pareto optimal payoff allocation principles.

Step 2a. Consider first the non-cooperative case, we obtain the following claim:

Claim 6.6. *The Markovian-feedback Nash equilibrium is* $\{\hat{c}(t), \hat{\tau}_k\} = \left\{\rho k(t), \frac{\rho(1-\varepsilon)}{r-\delta}\right\}$ *with value functions* $J^C(t,k(t)) = e^{-\rho(t_0+t)}[C_7 \ln k(t) + C_8]$ *and* $J^G(t,k(t)) = e^{-\rho(t_0+t)}[C_9 \ln k(t) + C_{10}]$, *in which* $C_7 = C_9 = \frac{1}{\rho}$, $C_8 = -\frac{\sigma^2}{2\rho^2} + \frac{1}{\rho}\ln\rho + \frac{r-\delta}{\rho^2} - \frac{2-\varepsilon}{\rho}$ *and* $C_{10} = -\frac{\sigma^2}{2\rho^2} + \frac{\varepsilon}{\rho}\ln\rho + \frac{r-\delta-\rho}{\rho^2} + \frac{1-\varepsilon}{\rho}\ln\left[\frac{\rho(1-\varepsilon)}{e}\right]$.

Proof. Omitted. $\square$

Step 2b. Applying Claim 6.5 to (1), the trajectory of capital accumulation along the cooperative equilibrium is thus expressed as

$$k^{**}(t) = k_0 \exp\left[\left(r - \delta - \rho - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right]. \tag{36}$$

Definition 6.1. Group rationality is satisfied if $J(t,k^{**}(t)) > J^C(t,k^{**}(t)) + J^G(t,k^{**}(t))$ along the cooperative trajectory $\{k^{**}(t)\}_{t=0}^{\tau_0}$.

Claim 6.7. *Group rationality is satisfied for the cooperative equilibrium.*

Proof. It follows from Claims 6.5 and 6.6 that we only need to confirm that $C_6 > C_8 + C_{10}$. In fact, we have

$$C_6 > C_8 + C_{10} \ \Leftrightarrow \ [2-(1-\varepsilon)]^{2-(1-\varepsilon)}e^{2(1-\varepsilon)} > 4. \tag{37}$$

Define $p \equiv 1-\varepsilon$, then $0 < p < 1$ based on our specification. Consider function $f(p) \equiv (2-p)^{2-p}e^{2p}$, it is easy to obtain $\frac{\partial \ln f(p)}{\partial p} = \ln\left(\frac{e}{2-p}\right) > 0$, which implies that $\inf_{0<p<1} \ln f(p) = \lim_{p\downarrow 0} \ln f(p) = \ln 4$. Thus, $f(p) > 4$ always holds true for $0 < p < 1$, which means that (37) holds and hence group rationality is satisfied. ☐

Step 2c. Here, we shall get a subgame consistent payoff distribution procedure (PDP).

Let Γ_t^{**} denote the set of reliable values of $k^{**}(t)$ at time t generated by (36). For notational consistency, we use k_t^{**} to represent a generic element of set Γ_t^{**}. Also, let vector $\xi(t') \equiv \left[\xi^C(t'), \xi^G(t')\right]$ denote the instantaneous payoff at time $t' \in (0, \tau_0)$. In particular, along cooperative trajectory $\{k_t^{**}\}_{t=0}^{\tau_0}$ we put the following value functions:

$$\nu^{(t_0)i}\left(t', k_{t'}^{**}\right) \equiv \mathbb{E}_{t'}\left[\int_{t'}^{\tau_0} e^{-\rho(z-t')}\xi^i(z)dz \,\middle|\, k(t') = k_{t'}^{**}\right]$$

and

$$\nu^{(t_0)i}\left(t, k_t^{**}\right) \equiv \mathbb{E}_{t}\left[\int_{t}^{\tau_0} e^{-\rho(z-t)}\xi^i(z)dz \,\middle|\, k(t) = k_{t}^{**}\right]$$

for $i \in \{C, G\}$, $k_{t'}^{**} \in \Gamma_{t'}^{**}$, $k_t^{**} \in \Gamma_t^{**}$ and $t \geq t' \geq t_0 \geq 0$.

Definition 6.2. The vector $\nu^{(t_0)}\left(t', k_{t'}^{**}\right) \equiv \left[\nu^{(t_0)C}\left(t', k_{t'}^{**}\right), \nu^{(t_0)G}\left(t', k_{t'}^{**}\right)\right]$ is a **valid imputation** for $t' \in (0, \tau_0)$ and $k_{t'}^{**} \in \Gamma_{t'}^{**}$ if it satisfies requirements:

(1) It is a **Pareto optimal** imputation vector;

(2) **Individual rationality**, i.e., $\nu^{(t_0)i}\left(t', k_{t'}^{**}\right) \geq J^i\left(t', k_{t'}^{**}\right)$ for $i \in \{C, G\}$.

In particular, Pareto optimality is straightforwardly satisfied by the cooperative maximization problem given by equation (8).

Additionally, we need more notations:

$$\gamma^{(t_0)i}\left(t'; t', k_{t'}^{**}\right) \equiv \mathbb{E}_{t'}\left[\int_{t'}^{\tau_0} e^{-\rho(z-t')}\xi^i(z)dz \,\middle|\, k(t') = k_{t'}^{**}\right] = \nu^{(t_0)i}(t', k_{t'}^{**})$$

and

$$\gamma^{(t_0)i}\left(t'; t, k_t^{**}\right) \equiv \mathbb{E}_{t}\left[\int_{t}^{\tau_0} e^{-\rho(z-t')}\xi^i(z)dz \,\middle|\, k(t) = k_{t}^{**}\right]$$

for $i \in \{C, G\}$ and $t \geq t' \geq t_0 \geq 0$. Noting the following property:

$$\gamma^{(t_0)i}\left(t'; t, k_t^{**}\right) \equiv e^{-\rho(t-t')}\mathbb{E}_t\left[\int_t^{\tau_0} e^{-\rho(z-t)}\xi^i(z)\,dz \,\middle|\, k(t) = k_t^{**}\right] = e^{-\rho(t-t')}\gamma^{(t_0)i}(t; t, k_t^{**}) \tag{38}$$

for $i \in \{C, G\}$ and $k_t^{**} \in \Gamma_t^{**}$, we hence have the following definition.

Definition 6.3. A solution imputation is said to satisfy **subgame consistency** if it satisfies condition (38).

That is, subgame consistency requires that the extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal.

Claim 6.8. *An instantaneous payment at time $t' \in (0, \tau_0)$ equaling*

$$\xi^i(t') = -\nu_t^{(t_0)i}\left(t', k_{t'}^{**}\right) - \frac{1}{2}\sigma^2\left(k_{t'}^{**}\right)^2\nu_{kk}^{(t_0)i}\left(t', k_{t'}^{**}\right) - \nu_k^{(t_0)i}\left(t', k_{t'}^{**}\right)k_{t'}^{**}(r - \delta - \rho)$$

for $i \in \{C, G\}$ and $k_{t'}^{**} \in \Gamma_{t'}^{**}$ yields a subgame consistent solution imputation.

Proof. We omit it as it is similar to the proof of Theorem 5.8.3 in Yeung and Petrosyan (2006). □

Step 2d. We first define two commonly used PDP, then we prove some desired properties.

Definition 6.4. An allocation principle is called **Nash bargaining solution/Shapley value** if at time t_0 the imputation assigned to player i is

$$\nu^{(t_0)i}(t_0, k_0) = J^i(t_0, k_0) + \frac{1}{2}\left[J(t_0, k_0) - \sum_{j\in\{C,G\}} J^j(t_0, k_0)\right]$$

for $i \in \{C, G\}$; and at time $t' \in (0, \tau_0)$, the imputation assigned to player i is

$$\nu^{(t_0)i}\left(t', k_{t'}^{**}\right) = J^i\left(t', k_{t'}^{**}\right) + \frac{1}{2}\left[J\left(t', k_{t'}^{**}\right) - \sum_{j\in\{C,G\}} J^j\left(t', k_{t'}^{**}\right)\right]$$

for $i \in \{C, G\}$ and $k_{t'}^{**} \in \Gamma_{t'}^{**}$.

It is easy to verify that Nash bargaining solution and Shapley value coincide with each other when there are just two players in the game.

Definition 6.5. An allocation principle is called **proportional distribution** if at time t_0 the imputation assigned to player i is

$$\nu^{(t_0)i}(t_0, k_0) = \frac{J^i(t_0, k_0)}{\sum_{j\in\{C,G\}} J^j(t_0, k_0)} J(t_0, k_0)$$

for $i \in \{C, G\}$; and at time $t' \in (0, \tau_0)$, the imputation assigned to player i is

$$\nu^{(t_0)i}\left(t', k_{t'}^{**}\right) = \frac{J^i\left(t', k_{t'}^{**}\right)}{\sum_{j \in \{C,G\}} J^j\left(t', k_{t'}^{**}\right)} J\left(t', k_{t'}^{**}\right)$$

for $i \in \{C, G\}$ and $k_{t'}^{**} \in \Gamma_{t'}^{**}$.

Claim 6.9. *Both Nash bargaining solution/Shapley value and proportional distribution can provide us with a valid imputation.*

Proof. A trivial application of Claim 6.7. □

Claim 6.10. *Both Nash bargaining solution/Shapley value and proportional distribution principle meet subgame consistency.*

Proof. Since the equilibrium feedback strategies are Markovian in the sense that they just depend on current state and current time, one can readily observe that

$$\begin{pmatrix} c^{**(t_0)}\left(t, k_t^{**}\right) \\ \tau_k^{**(t_0)}\left(t, k_t^{**}\right) \end{pmatrix} = \begin{pmatrix} c^{**(t')}\left(t, k_t^{**}\right) \\ \tau_k^{**(t')}\left(t, k_t^{**}\right) \end{pmatrix}$$

for $t_0 \leq t' \leq t < \tau_0$ and $k^{**}(t) \equiv k_t^{**} \in \Gamma_t^{**}$. In addition, by using this property we can get that $J^{(t_0)C}\left(t', k_{t'}^{**}\right) = e^{-\rho(t'-t_0)} J^{(t')C}\left(t', k_{t'}^{**}\right)$, $J^{(t_0)G}\left(t', k_{t'}^{**}\right) = e^{-\rho(t'-t_0)} J^{(t')G}\left(t', k_{t'}^{**}\right)$ and $J^{(t_0)}\left(t', k_{t'}^{**}\right) = e^{-\rho(t'-t_0)} J^{(t')}\left(t', k_{t'}^{**}\right)$, in which the LHS measure the expected present values of non-cooperative and cooperative payoffs in time interval $[t', \tau_0)$ when $k^{**}(t') = k_{t'}^{**}$ and the game starts from time $t_0 \leq t'$.

For Nash bargaining solution/Shapley value, we then have

$$\begin{aligned} \nu^{(t_0)i}\left(t', k_{t'}^{**}\right) &= J^{(t_0)i}\left(t', k_{t'}^{**}\right) + \frac{1}{2}\left[J^{(t_0)}\left(t', k_{t'}^{**}\right) - \sum_{j \in \{C,G\}} J^{(t_0)j}\left(t', k_{t'}^{**}\right)\right] \\ &= e^{-\rho(t'-t_0)}\left\{J^{(t')i}\left(t', k_{t'}^{**}\right) + \frac{1}{2}\left[J^{(t')}\left(t', k_{t'}^{**}\right) - \sum_{j \in \{C,G\}} J^{(t')j}\left(t', k_{t'}^{**}\right)\right]\right\} \\ &= e^{-\rho(t'-t_0)} \nu^{(t')i}\left(t', k_{t'}^{**}\right) \end{aligned}$$

for $i \in \{C, G\}$, $t_0 \leq t' < \tau_0$ and $k_{t'}^{**} \in \Gamma_{t'}^{**}$. Similarly, for the proportional distribution principle,

$$\begin{aligned} \nu^{(t_0)i}\left(t', k_{t'}^{**}\right) &= \frac{J^{(t_0)i}\left(t', k_{t'}^{**}\right)}{\sum_{j \in \{C,G\}} J^{(t_0)j}\left(t', k_{t'}^{**}\right)} J^{(t_0)}\left(t', k_{t'}^{**}\right) \\ &= \frac{e^{-\rho(t'-t_0)} J^{(t')i}\left(t', k_{t'}^{**}\right)}{\sum_{j \in \{C,G\}} e^{-\rho(t'-t_0)} J^{(t')j}\left(t', k_{t'}^{**}\right)} e^{-\rho(t'-t_0)} J^{(t')}\left(t', k_{t'}^{**}\right) \\ &= e^{-\rho(t'-t_0)}\left[\frac{J^{(t')i}\left(t', k_{t'}^{**}\right)}{\sum_{j \in \{C,G\}} J^{(t')j}\left(t', k_{t'}^{**}\right)} J^{(t')}\left(t', k_{t'}^{**}\right)\right] \\ &= e^{-\rho(t'-t_0)} \nu^{(t')i}\left(t', k_{t'}^{**}\right) \end{aligned}$$

for $i \in \{C, G\}$, $t_0 \le t' < \tau_0$ and $k_{t'}^{**} \in \Gamma_{t'}^{**}$. Therefore, the required assertion follows. $\square$

Claim 6.11. *Under Nash bargaining solution/Shapley value and proportional distribution principle, neither the capitalist nor the government will unilaterally deviate from cooperation.*

Proof. We first consider the case under Nash bargaining solution/Shapley value. At date $t \ge t_0$, if no one deviates from cooperation, the payoff allocation is

$$\nu^i\left(t, k^{**}(t)\right) = J^i\left(t, k^{**}(t)\right) + \frac{1}{2}\left[J\left(t, k^{**}(t)\right) - \sum_{j \in \{C,G\}} J^j\left(t, k^{**}(t)\right)\right]$$

for $i \in \{C, G\}$. It follows from Claim 6.7 that $\nu^i\left(t, k^{**}(t)\right) > J^i\left(t, k^{**}(t)\right)$ for all $i \in \{C, G\}$. If the capitalist unilaterally deviates from cooperation, he gets payoff $J^C\left(t, \hat{k}(t)\right) = e^{-\rho(t_0+t)}\left[C_7 \ln \hat{k}(t) + C_8\right]$ with C_7 and C_8 given in Claim 6.6, and $\hat{k}(t)$ a solution of

$$d\hat{k}(t) = \left\{r - \delta - \frac{\rho}{2}\left[(1-\varepsilon) + 2\right]\right\}\hat{k}(t)dt + \sigma\hat{k}(t)dB(t),$$

which compares with (36) shows $k^{**}(t) > \hat{k}(t)$ for $\forall t$. As we have $J^C\left(t, k^{**}(t)\right) = e^{-\rho(t_0+t)}[C_7 \ln k^{**}(t) + C_8]$ with the same C_7 and C_8, it's immediate that $J^C\left(t, \hat{k}(t)\right) < J^C\left(t, k^{**}(t)\right) < \nu^C\left(t, k^{**}(t)\right)$. On the other hand, if the government unilaterally deviates from cooperation, it gets payoff $J^G\left(t, \tilde{k}(t)\right) = e^{-\rho(t_0+t)}\left[C_9 \ln \tilde{k}(t) + C_{10}\right]$ with C_9 and C_{10} given by Claim 6.6, and $\tilde{k}(t)$ a solution of

$$d\tilde{k}(t) = \left[r - \delta - \rho\left(1 + \frac{1-\varepsilon}{2}\right)\right]\tilde{k}(t)dt + \sigma\tilde{k}(t)dB(t),$$

which shows $k^{**}(t) > \tilde{k}(t) = \hat{k}(t)$ for $\forall t$. Then, it is easy to obtain $J^G\left(t, \tilde{k}(t)\right) < J^G\left(t, k^{**}(t)\right) < \nu^G\left(t, k^{**}(t)\right)$. To sum up, unilateral deviation always results in less payoff, hence neither the capitalist nor the government will unilaterally deviate from cooperation.

For the case under proportional distribution principle, since by Claim 6.7 the payoff allocation under cooperation satisfies

$$\nu^i\left(t, k^{**}(t)\right) = \frac{J^i\left(t, k^{**}(t)\right)}{\sum_{j \in \{C,G\}} J^j\left(t, k^{**}(t)\right)} J\left(t, k^{**}(t)\right) > J^i(t, k^{**}(t))$$

for $i \in \{C, G\}$, no one will unilaterally deviate from cooperation following the same reason shown above, which is hence omitted to economize on the space. $\square$

Step 3. Solving the problem in stage 1 gives rise to:

Claim 6.12. *Suppose* $r-\delta<2\rho$, $h_1 \le \frac{\rho}{r-\delta-\rho}$, $r-\delta-\rho-\frac{1}{2}\sigma^2>0$ *and* $2(r-\delta)-4\rho+\sigma^2\le 0$ *hold true for* $h_1=\frac{\sigma^2-2(r-\delta-\rho)+\sqrt{[2(r-\delta-\rho)-\sigma^2]^2+8\rho\sigma^2}}{2\sigma^2}$, $k^{**}=\frac{h_1\varpi}{h_1-1}$ *and* $\bar{C}=\frac{1}{h_1}(k^{**})^{1-h_1}$. *Then, we can derive function*

$$\phi(t_0,k_0)=\begin{cases} e^{-\rho t_0}\bar{C}k_0^{h_1} & \text{if } 0<k_0<k^{**} \\ e^{-\rho t_0}\left(k_0-\varpi\right) & \text{if } k^{**}\le k_0 \end{cases}$$

such that it coincides with value function Φ_0 *of our problem, and* $\tau_0^{**}=\tau_D\equiv\inf\{t>0;k(t)=k^{**}\}$ *is an optimal exit time with continuation region* $D=\{(t_0,k_0);0<k_0<k^{**}\}$.

Proof. It is similar to that of Claim 6.2 and hence omitted. □

Step 4. To complete the proof, we need the following result.

Claim 6.13. *Suppose the capital return rate is restricted as shown in the following proof, then the conditions used in Claim 6.12 hold true.*

Proof. First, we have $r-\delta-\rho-\frac{1}{2}\sigma^2>0 \Leftrightarrow r>\delta+\rho+\frac{1}{2}\sigma^2\equiv\widetilde{r}_{\min}$. In addition, since $2(r-\delta)-4\rho+\sigma^2\le 0$ implies $r-\delta<2\rho$, we just show that $2(r-\delta)-4\rho+\sigma^2\le 0\Leftrightarrow r\le\delta+2\rho-\frac{1}{2}\sigma^2\equiv\widetilde{r}_{\max}$. Also, it is easy to show that $h_1\le\frac{\rho}{r-\delta-\rho}$ is implied by $r-\delta<2\rho$. In consequence, we just need $\rho>\sigma^2$ to ensure that $\widetilde{r}_{\min}<\widetilde{r}_{\max}$. To conclude, the required conditions hold true as long as $r\in(\widetilde{r}_{\min},\widetilde{r}_{\max}]$. □

The proof is, therefore, complete. Q.E.D.

Proof of Theorem 4.1: We shall complete it in 4 steps.

Step 1. By using Lemma 3.1, we have

$$\tilde{k}^*=k(\tau_\Delta^*)=k_0\exp\left\{\left[r-\delta-\rho(2-\varepsilon)-\frac{1}{2}\sigma^2\right]\tau_\Delta^*+\sigma B(\tau_\Delta^*)\right\},$$

which yields

$$\ln\left(\frac{\tilde{k}^*}{k_0}\right)=\left[r-\delta-\rho(2-\varepsilon)-\frac{1}{2}\sigma^2\right]\mathbb{E}(\tau_\Delta^*),$$

i.e.,

$$\mathbb{E}(\tau_\Delta^*)=\frac{\ln\left(\frac{\tilde{k}^*}{k_0}\right)}{r-\delta-\rho(2-\varepsilon)-\frac{1}{2}\sigma^2}. \tag{39}$$

Similarly, it follows from Lemma 3.2 that

$$\mathbb{E}(\tau_0^*)=\frac{\ln\left(\frac{k^*}{k_0}\right)}{r-\delta-\rho(2-\varepsilon)-\frac{1}{2}\sigma^2}. \tag{40}$$

Since $\tilde{\varpi} < \varpi$ by (7), it is easy to see that $\tilde{k}^* < k^*$ and hence $\mathbb{E}(\tau_\Delta^*) < \mathbb{E}(\tau_0^*)$. Moreover, to make $\mathbb{E}(\tau_\Delta^*) > 0$ we require that $k_0 < \tilde{k}^*$, which implies that $\frac{\varpi}{k_0} > \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta}$ for $\forall \Delta > 0$.

Step 2. We now proceed to show that $k^* < k^{**}$ for $\forall \varpi > 0$. Define a function $l(x) \equiv -x + \sqrt{x^2 + 8\rho\sigma^2}$ for $x > 0$, by which we obtain $l'(x) = -1 + \frac{x}{\sqrt{x^2+8\rho\sigma^2}} < 0$ for $\forall x$, i.e., $l(x)$ is a strictly decreasing function with respect to x. Then, by comparing (6) with (10) we immediately get $\lambda_1 > h_1$. As a result, $k^* = \frac{\lambda_1 \varpi}{\lambda_1 - 1} < \frac{h_1 \varpi}{h_1 - 1} = k^{**}$ for $\forall \varpi > 0$.

Step 3. It follows from Lemma 3.3 that

$$\mathbb{E}(\tau_0^{**}) = \frac{\ln\left(\frac{k^{**}}{k_0}\right)}{r - \delta - \rho - \frac{1}{2}\sigma^2}. \tag{41}$$

Combining (39) with (41) and using the definition of $\tilde{k}^*$ show that

$$\begin{aligned}
\frac{\mathbb{E}(\tau_\Delta^*)}{\mathbb{E}(\tau_0^{**})} < 1 \;\Leftrightarrow\; & \frac{\ln\left(\frac{k^*}{k_0}\right) - \mu\Delta}{\ln\left(\frac{k^{**}}{k_0}\right)} < \frac{\mu - \frac{1}{2}\sigma^2}{r - \delta - \rho - \frac{1}{2}\sigma^2} \\
\Leftrightarrow\; & \Delta > \frac{\ln\left(\frac{k^*}{k_0}\right) - \ln\left(\frac{k^{**}}{k_0}\right)\left(\frac{\mu - \frac{1}{2}\sigma^2}{r-\delta-\rho-\frac{1}{2}\sigma^2}\right)}{\mu},
\end{aligned}$$

in which we also impose the assumption that $\frac{\varpi}{k_0} > \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta}$. First, note that

$$\ln\left(\frac{k^*}{k_0}\right) \le \ln\left(\frac{k^{**}}{k_0}\right)\left(\frac{\mu - \frac{1}{2}\sigma^2}{r - \delta - \rho - \frac{1}{2}\sigma^2}\right) \Leftrightarrow \frac{\varpi}{k_0} \le \left(\frac{h_1}{h_1 - 1}\right)^{\frac{\mu - \frac{1}{2}\sigma^2}{\rho(1-\varepsilon)}} \left(\frac{\lambda_1 - 1}{\lambda_1}\right)^{\frac{r-\delta-\rho-\frac{1}{2}\sigma^2}{\rho(1-\varepsilon)}} \equiv \Xi^*$$

and

$$\left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta} < \Xi^* \Leftrightarrow \Delta < \underbrace{\left(\frac{1}{\mu}\right)\left[\frac{\mu - \frac{1}{2}\sigma^2}{\rho(1-\varepsilon)}\right] \ln\left(\frac{\frac{h_1}{h_1-1}}{\frac{\lambda_1}{\lambda_1-1}}\right)}_{>0} \equiv \Delta_1^*,$$

then it is immediate that $\mathbb{E}(\tau_\Delta^*) < \mathbb{E}(\tau_0^{**})$ for any $\Delta < \Delta_1^*$ and $\frac{\varpi}{k_0} \le \Xi^*$, as desired. Otherwise, we consider the case with $\frac{\varpi}{k_0} > \Xi^*$. Noting that

$$\Delta > \frac{\ln\left(\frac{k^*}{k_0}\right) - \ln\left(\frac{k^{**}}{k_0}\right)\left(\frac{\mu - \frac{1}{2}\sigma^2}{r-\delta-\rho-\frac{1}{2}\sigma^2}\right)}{\mu} \Leftrightarrow \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta} > \left(\frac{h_1 - 1}{h_1}\right)^{\frac{\mu - \frac{1}{2}\sigma^2}{r-\delta-\rho-\frac{1}{2}\sigma^2}} \left(\frac{\varpi}{k_0}\right)^{\frac{\rho(1-\varepsilon)}{r-\delta-\rho-\frac{1}{2}\sigma^2}},$$

$$\frac{\varpi}{k_0} > \left(\frac{h_1 - 1}{h_1}\right)^{\frac{\mu - \frac{1}{2}\sigma^2}{r-\delta-\rho-\frac{1}{2}\sigma^2}} \left(\frac{\varpi}{k_0}\right)^{\frac{\rho(1-\varepsilon)}{r-\delta-\rho-\frac{1}{2}\sigma^2}} \Leftrightarrow \frac{\varpi}{k_0} > \frac{h_1 - 1}{h_1},$$

$$\Xi^* \Big/ \frac{h_1 - 1}{h_1} = \left(\frac{h_1}{h_1 - 1} \Big/ \frac{\lambda_1}{\lambda_1 - 1}\right)^{\frac{r-\delta-\rho-\frac{1}{2}\sigma^2}{\rho(1-\varepsilon)}} > 1$$

and also

$$\frac{\varpi}{k_0} > \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu\Delta} \Leftrightarrow \Delta < \frac{1}{\mu} \ln \left[\left(\frac{\lambda_1}{\lambda_1 - 1}\right) \left(\frac{\varpi}{k_0}\right) \right],$$

then we claim that there exists a lower bound, written as

$$\Delta_2^* \equiv \frac{\ln\left(\frac{k^*}{k_0}\right) - \ln\left(\frac{k^{**}}{k_0}\right)\left(\frac{\mu - \frac{1}{2}\sigma^2}{r - \delta - \rho - \frac{1}{2}\sigma^2}\right)}{\mu} > 0,$$

of Δ such that $\mathbb{E}(\tau_\Delta^*) < \mathbb{E}(\tau_0^{**})$ for any $\Delta \in \left(\Delta_2^*, \frac{1}{\mu} \ln \left[\left(\frac{\lambda_1}{\lambda_1 - 1}\right)\left(\frac{\varpi}{k_0}\right)\right]\right)$. Using the above calculation, we can also have $\mathbb{E}(\tau_\Delta^*) = \mathbb{E}(\tau_0^{**})$ for $\Delta = \Delta_2^*$ and $\mathbb{E}(\tau_\Delta^*) > \mathbb{E}(\tau_0^{**})$ for any $\Delta < \Delta_2^*$, as required.

Step 4. Making use of (40) and (41) reveals that

$$\frac{\mathbb{E}(\tau_0^*)}{\mathbb{E}(\tau_0^{**})} = \frac{\ln\left(\frac{\lambda_1}{\lambda_1 - 1}\right) + \ln\left(\frac{\varpi}{k_0}\right)}{\ln\left(\frac{h_1}{h_1 - 1}\right) + \ln\left(\frac{\varpi}{k_0}\right)} \left(\frac{r - \delta - \rho - \frac{1}{2}\sigma^2}{\mu - \frac{1}{2}\sigma^2}\right).$$

Thus, by rearranging the terms, we have

$$\frac{\mathbb{E}(\tau_0^*)}{\mathbb{E}(\tau_0^{**})} \leq 1 \;\Leftrightarrow\; \ln\left(\frac{\varpi}{k_0}\right) \leq \frac{\left(\frac{\mu - \frac{1}{2}\sigma^2}{r - \delta - \rho - \frac{1}{2}\sigma^2}\right) \ln\left(\frac{h_1}{h_1 - 1}\right) - \ln\left(\frac{\lambda_1}{\lambda_1 - 1}\right)}{1 - \left(\frac{\mu - \frac{1}{2}\sigma^2}{r - \delta - \rho - \frac{1}{2}\sigma^2}\right)},$$

which gives rise to $\mathbb{E}(\tau_0^*) \leq \mathbb{E}(\tau_0^{**}) \Leftrightarrow \frac{\varpi}{k_0} \leq \Xi^*$. In the meantime, note that we need the constraint $\frac{\varpi}{k_0} > \frac{\lambda_1 - 1}{\lambda_1}$ to make $\mathbb{E}(\tau_0^*) > 0$. Since it is easy to verify that $\Xi^* > \frac{\lambda_1 - 1}{\lambda_1}$, we hence have $\mathbb{E}(\tau_0^*) < \mathbb{E}(\tau_0^{**})$ for any $\frac{\varpi}{k_0} \in \left(\frac{\lambda_1 - 1}{\lambda_1}, \Xi^*\right)$, $\mathbb{E}(\tau_0^*) = \mathbb{E}(\tau_0^{**})$ for $\frac{\varpi}{k_0} = \Xi^*$, and also $\mathbb{E}(\tau_0^*) > \mathbb{E}(\tau_0^{**})$ for any $\frac{\varpi}{k_0} > \Xi^*$, as desired in (v). Q.E.D.

Proof of Proposition 5.1: By using (17), (18) and (24), it is easy to verify that

$$\mathbb{E}_{t_0}\left[\int_0^{\mathbb{E}(\tau_\Delta^*)} |V^G(t, k(t))| dt\right] = \int_{\omega \in \Omega} \int_0^{\mathbb{E}(\tau_\Delta^*)} |V^G(t, k(t))| dt dP(\omega) < \infty$$

for any $\mathbb{E}(\tau_\Delta^*) < \infty$, thus applying Fubini's Theorem implies that

$$V_\Delta^G \equiv \mathbb{E}_{t_0}\left[\int_0^{\mathbb{E}(\tau_\Delta^*)} V^G(t, k(t)) dt\right] = \int_0^{\mathbb{E}(\tau_\Delta^*)} \mathbb{E}_{t_0}[V^G(t, k(t))] dt.$$

By applying the formula of integration by parts, we thus have

$$V_\Delta^G = -\left[\left(\frac{\mu - \frac{1}{2}\sigma^2}{\rho^2}\right)\mathbb{E}(\tau_\Delta^*) + \frac{\mu - \frac{1}{2}\sigma^2 + \rho \ln k_0 + \rho^2 C_4}{\rho^3}\right] e^{-\rho\mathbb{E}(\tau_\Delta^*)} + \frac{\mu - \frac{1}{2}\sigma^2 + \rho \ln k_0 + \rho^2 C_4}{\rho^3},$$

where we, without any loss of generality, normalize t_0 to zero for notational simplicity. Applying the chain rule gives rise to

$$\frac{\partial V_\Delta^G}{\partial \Delta} = \frac{\partial V_\Delta^G}{\partial \mathbb{E}(\tau_\Delta^*)} \frac{\partial \mathbb{E}(\tau_\Delta^*)}{\partial \Delta} = -\left(\frac{\mu}{\mu - \frac{1}{2}\sigma^2}\right)\left[\left(\frac{\mu - \frac{1}{2}\sigma^2}{\rho}\right)\mathbb{E}(\tau_\Delta^*) + \frac{\ln k_0}{\rho} + C_4\right] e^{-\rho\mathbb{E}(\tau_\Delta^*)},$$

which combines with (39) and $\Delta_{\min} = \frac{1}{\mu} \ln\left(\frac{\lambda_1 \varpi}{\lambda_1 - 1}\right) + \frac{\rho}{\mu} C_4 > 0 \Leftrightarrow \varpi > \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{-\rho C_4}$ give rise to the required assertion in part (i). Making use of the chain rule and (39) again, we obtain

$$\left.\frac{\partial^2 V_\Delta^G}{\partial \Delta^2}\right|_{\Delta=\Delta_{\min}} = \left.\frac{\partial\left(\partial V_\Delta^G / \partial \Delta\right)}{\partial \mathbb{E}(\tau_\Delta^*)} \frac{\partial \mathbb{E}(\tau_\Delta^*)}{\partial \Delta}\right|_{\Delta=\Delta_{\min}} > 0,$$

which hence confirms the uniqueness of such a critical value of Δ. So, we complete the proof of part (i). For part (ii), we just need to note that

$$\frac{\partial V_\Delta^G}{\partial \Delta} > 0 \Leftrightarrow \varpi < \left(\frac{\lambda_1 - 1}{\lambda_1}\right) \exp(\mu \Delta - \rho C_4),$$

$$\mathbb{E}(\tau_{\Delta_{\min}}^*) = \left(\frac{1}{\mu - \frac{1}{2}\sigma^2}\right) \left[\ln\left(\frac{1}{k_0}\right) - \rho C_4\right] > 0 \Leftrightarrow k_0 < e^{-\rho C_4}$$

and also for all $\Delta > 0$, $\mathbb{E}(\tau_\Delta^*) > 0 \Leftrightarrow \varpi > k_0 \left(\frac{\lambda_1 - 1}{\lambda_1}\right) e^{\mu \Delta}$. Q.E.D.

Proof of Proposition 5.2: Applying Fubini's Theorem, the formula of integration by parts, the chain rule and (39), we obtain

$$\frac{\partial V_\Delta^G}{\partial \varpi} = \frac{\partial V_\Delta^G}{\partial \mathbb{E}(\tau_\Delta^*)} \frac{\partial \mathbb{E}(\tau_\Delta^*)}{\partial \varpi} = \frac{1}{\left(\mu - \frac{1}{2}\sigma^2\right) \varpi} \left[\left(\frac{\mu - \frac{1}{2}\sigma^2}{\rho}\right) \mathbb{E}(\tau_\Delta^*) + \frac{\ln k_0}{\rho} + C_4\right] e^{-\rho \mathbb{E}(\tau_\Delta^*)},$$

where we have normalized t_0 to zero for notational simplicity. Letting $\partial V_\Delta^G / \partial \varpi = 0$, we thus get the critical value as $\varpi_{\min} = \left(\frac{\lambda_1 - 1}{\lambda_1}\right) \exp\left(\mu \Delta - \rho C_4\right) > 0$. Note that

$$\frac{\partial\left(\partial V_\Delta^G / \partial \varpi\right)}{\partial \mathbb{E}(\tau_\Delta^*)} = \frac{1}{\varpi} \left[\frac{1}{\rho} - \mathbb{E}(\tau_\Delta^*) - \frac{\ln k_0 + \rho C_4}{\mu - \frac{1}{2}\sigma^2}\right] e^{-\rho \mathbb{E}(\tau_\Delta^*)}$$

and

$$\mathbb{E}(\tau_\Delta^*)\big|_{\varpi=\varpi_{\min}} = -\frac{\ln k_0 + \rho C_4}{\mu - \frac{1}{2}\sigma^2} > 0 \Leftrightarrow k_0 < e^{-\rho C_4},$$

we hence have

$$\left.\frac{\partial^2 V_\Delta^G}{\partial \varpi^2}\right|_{\varpi=\varpi_{\min}} = \left.\frac{\partial\left(\partial V_\Delta^G / \partial \varpi\right)}{\partial \mathbb{E}(\tau_\Delta^*)} \frac{\partial \mathbb{E}(\tau_\Delta^*)}{\partial \varpi}\right|_{\varpi=\varpi_{\min}} = \frac{1}{\rho\left(\mu - \frac{1}{2}\sigma^2\right) {\varpi_{\min}}^2} \exp\left[\frac{\rho(\ln k_0 + \rho C_4)}{\mu - \frac{1}{2}\sigma^2}\right] > 0,$$

which therefore confirms the required assertion. Q.E.D.