The Blocking Lemma and Group Strategy-Proofness in Many-to-Many Matchings

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Abstract

This paper considers group strategy-proofness in many-to-many two-sided matching problems. We first show that the Blocking Lemma holds for a many-to-many matching model under quota-saturability condition and max-min preference criterion that is stranger than substitutability of preferences. This result extends the Blocking Lemma for one-to-one matching and for many-to-one matching to many-to-many matching problem. It is then shown that the deferred acceptance mechanism is group strategy-proof for agents on the proposing side under max-min preference criterion and quota-saturability condition. Neither the Blocking Lemma nor the group incentive compatibility can be guaranteed if the preference condition is weaker than max-min criterion.

Keywords: Many-to-many matching; Blocking Lemma; Max-min preferences; Deferred acceptance algorithm; Group strategy-proofness

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1 Introduction

Many-to-many matching models study assignment problems where agents can be divided into two disjoint sets: the set of firms and the set of workers. Each firm wishes to hire a set

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of workers, and each worker wishes to work for a set of firms. Firms have preferences over
the possible sets of workers, and workers have preferences over the possible sets of firms. The
assumption that workers may work in more than one firm is not unusual. A physician may have
a medical position at a hospital and a teaching position at some university. A faculty member in
college may have a part-time position in different places. A many-to-many assignment problem
is to match each agent (a firm or worker) with a subset of agents from the other side of the
market. If a firm hires a worker, we say that the two agents form a partnership. A set of
partnerships is called a matching.

The many-to-many matching problem is a natural extension of the college admissions prob-
lem of Gale and Shapley (1962) to the general quota case. The notions with respect to the
college admissions problem are commonly generalized to the many-to-many matching model.
For a matching problem, the concept of stability is of primary importance. A matching is stable
(pairwise-stable) if all partnerships occur between acceptable partners (individual rationality)
and there is no unmatched worker-firm pair mutually prefer each other to their assigned partners
(pairwise blocking). For the marriage problem and the college admissions problem, Gale and
Shapley’s deferred acceptance algorithms yield stable matchings. Moreover, the stable matching
produced by the deferred acceptance procedure is also optimal for applicants. That is, every
applicant is at least as well off under the assignment given by the deferred acceptance procedure
as he would be under any other stable assignment. Roth (1984) adapts the deferred acceptance
algorithm to the many-to-many matching market and obtains the corresponding optimal stable
assignment. Hatfield and Kominers (2014) consider contract design and stability in many-to-
many matching and they show that substitutable preferences are sufficient to guarantee the
existence of stable outcomes. In contrast to results for the setting of many-to-one matching
with contracts, they also show that if any agent’s preferences are not substitutable, then the
existence of a stable outcome cannot be guaranteed.

The incentive compatibility of matching has interested researchers extensively. For one-to-
one matching problem, Roth (1982) investigates the marriage problem and obtain that the men
(resp. women)-optimal matching is strategy-proof for men (resp. women).\footnote{Dubins and Freedman (1981) show that, under the men (resp. women)-proposing deferred acceptance algorithm, there exists no coalition of men (resp. women) that can simultaneously improve the assignment of all its members if those outside the coalition state their true preferences. This result implies the property of strategy-proofness.} For a unified model, Hatfield and Milgrom (2005) study the incentive property for matching with contracts. They
obtain that the doctor-optimal matching is strategy-proof for doctors under very weak preference
assumption (hospitals’ preferences satisfy substitutability and the law of aggregate demand). Under the same framework, Hatfield and Kojima (2009) show that the doctor-optimal matching is group strategy-proof for doctors. For many-to-one matching problem, Roth (1985) studies the college admissions problem and shows that, when colleges have responsive preferences, the colleges-optimal matching may not be strategy-proof for colleges, while the students-optimal matching is strategy-proof for students. It seems that we need a stronger condition to guarantee the incentive compatibility for agents with multi-unit demands.

On the incentives issue of many-to-many matchings, Sotomayor (2004) considers implementation of pairwise-stable matchings in subgame perfect Nash equilibrium in the sense that the set of subgame perfect Nash equilibrium outcomes of the proposed mechanism coincides with the set of pairwise-stable matchings. It is further shown that the core always contains the equilibrium outcomes under the maximin preference restriction. However, the solution concept of subgame perfect Nash equilibrium imposes a strong informational requirement that agents know each other about their preferences.

For truthful implementation, Bao and Balinski (2000) propose the max-min criterion and assert that their reduction algorithm is stable and strategy-proof for agents on one side of the matching market if all agents’ preferences satisfy max-min criterion. However, Hatfield et al. (2014) show that the max-min preference criterion is not sufficient for the existence of a stable and strategy-proof matching mechanism. Thus, Bao and Balinski’s (2000) result is incomplete. As such, it is still an important open question on the strategy-proofness for agents with multi-unit demands.

The present paper considers group strategy-proofness in many-to-many two-sided matching problems. In doing so, we first extends the Blocking Lemma to the case of many-to-many matchings. For one-to-one and many-to-one matching problems, the Blocking Lemma is an important instrumental result, which identifies a particular blocking pair for any non-stable and individually rational matching that is preferred by some agents of one side of the market to their optimal stable matching. Its interest lies in the fact that it has been used to derive some key conclusions on matching. Using the Blocking Lemma for one-to-one matching, Gale and Sotomayor (1985) give a short proof for the group strategy-proofness of the deferred acceptance algorithm. For many-to-one matching, the Blocking Lemma holds under responsive preference profile. The responsiveness seems too restrictive to fulfill. For a weak preference restriction,

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2There may be no SPNE outcome in the core when the preferences of the agents are not maximin as shown in Sotomayor (2004).

3Gale and Sotomayor attribute the formulation of the lemma to J.S. Huang.

4Roth (1985) introduces responsiveness of preference relations for college admission problems. Specifically,
Martínez et al. (2010) show that the corresponding Blocking Lemma for workers (who have unit demand) holds under substitutable and quota-separable preference. They also note that the Blocking Lemma for firms (which have multi-unit demand) does not hold even under responsive preference.

A natural question is then what restriction on preferences can enable the Blocking Lemma for agents with multi-unit demand to meet. We show that max-min preference restriction, together with an additional condition—quota-saturability—which says that in the market there are enough acceptable workers (or workers’ quotas) such that each firm can hire as many as their quota acceptable workers if they want, establishes the Blocking Lemma for many-to-many matchings.

This result is then used to investigate group strategy-proofness of the deferred acceptance mechanism for many-to-many matchings. We show that, if agents have max-min preferences and there are enough acceptable workers, then the deferred acceptance algorithm is group strategy-proof for agents on the proposing side. While all of the existing incentive compatibility results are for agents with unit demand, our incentive compatibility result is a try for strategy-proofness of agents with multi-unit demands. We show by example (Example 3) that both the Blocking Lemma and the group incentive compatibility for many-to-many matching may not be true when the preference condition is weaker than the max-min criterion.

As we will show, while both max-min criterion and responsiveness condition imply substitutability introduced by Kelso and Crawford (1982) and commonly assumed in matching literature, max-min criterion is not implied by nor implies responsiveness. It is the max-min preferences that have both properties of substitutability and complementarity that enable us to obtain the desired results.

The remainder of the present paper is organized as follows. We present some preliminaries responsiveness means that, for any two subsets of workers that differ in only one worker, a firm prefers the subset containing the most-preferred worker. Formally, we say a firm $f$’s preference relation is responsive if for any $w_1, w_2$ and any $S$ such that $w_1, w_2 \notin S$ and $|S| < q_f$, we have $S \cup \{w_1\} P(f) S \cup \{w_2\}$ if and only if $\{w_1\} P(f) \emptyset$, where $w_1, w_2$ are the partners of $f$ and $S$ is a set of partners of $f$. It is easy to obtain that the responsiveness is stronger than the substitutability.

Barberà et al. (1991) propose another concept of separable preference (being different from that used by Sotomayor, 1999), which has been extensively used in matching models. See, for instance, Alkan (2001), Dutta and Massó (1997), Ehlers and Klaus (2003), Martínez et al. (2000), Martínez et al. (2001), Martínez et al. (2004b), Papai (2000), and Sönmez (1996). Based on this condition, Martínez et al. (2010) propose a new concept called quota-separability. Formally, A firm $f$’s preference relation $P(f)$ over sets of workers is quota $q_f$-separable if: (i) for all $S \subseteq W$ such that $|S| < q_f$ and $w \notin S$, it implies that $(S \cup \{w\}) P(f) S$ if and only if $\{w\} P(f) \emptyset$; (ii) $\emptyset P(f) S$ for all $S$ such that $|S| > q_f$.  

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on the formal model in the next section. In Section 3 we study the Blocking Lemma for a many-to-many matching model. In Section 4 we study the incentive compatibility of the firms-proposing deferred acceptance algorithm. We conclude in Section 5. All proofs are provided in the Appendix.

2 Preliminaries

2.1. The model

For concreteness, we use the language of firms-workers matching model. The agents in our model consist of two disjoint sets, the set of firms $F = \{f_1, \cdots, f_n\}$ and the set of workers $W = \{w_1, \cdots, w_m\}$. Generic agents are denoted by $v \in V \equiv F \cup W$ while generic firms and workers are denoted by $f$ and $w$, respectively. Each firm wishes to hire a set of workers, and each worker wishes to work for a set of firms. Each $f \in F$ has a strict, complete and transitive preference relation $\succ_f$ over $W \cup \{\emptyset\}$, and each $w \in W$ has a strict, complete and transitive preference relation $\succ_w$ over $F \cup \{\emptyset\}$. The notation $\emptyset$ denotes the prospect of a position remaining unfilled.

We call a firm $f$ is acceptable to $w$ if $f \succ_w \emptyset$, and a worker $w$ is acceptable to $f$ if $w \succ_f \emptyset$. The preference relations of firms and workers can be represented by order lists. For example, $\succ_f: w_2, w_1, \emptyset, w_3, \cdots$ denotes that firm $f$ prefers to hire $w_2$ rather than $w_1$, that it prefers to hire either one of them rather than leave a position unfilled, and that all other workers are unacceptable, in the sense that it would be preferable to leave a position unfilled rather than to fill it with, say, worker $w_3$. Similarly, for the preference relation of a worker, $\succ_w: f_1, f_3, f_2, \emptyset, \cdots$ indicates that the only positions the worker would accept are those offered by $f_1, f_3$ and $f_2$, in that order. We will write $f_i \succ_w f_j$ to indicate that worker $w$ prefers $f_i$ to $f_j$, and $f_i \succeq_w f_j$ to indicate that either $f_i \succ_w f_j$ or $f_i = f_j$. Similarly, we can give corresponding notations on preference relations of firms.

We are allowing for the possibility that worker $w$ may prefer to remain unemployed rather than work for an unacceptable firm and that firm $f$ may prefer not to hire any worker rather than hire any unacceptable worker. Each agent $v$ has a quota $q_v$ which is the maximum number of partnerships she or it may enter into. Let $q \equiv (q_v)_{v \in V}$ denote the vector of quotas.

**Definition 1** A matching is a mapping $\mu : F \cup W \to 2^{F \cup W}$ such that

1. $\mu(f) \in 2^W$ and $|\mu(f)| \leq q_f$ for all $f \in F$,
2. $\mu(w) \in 2^F$ and $|\mu(w)| \leq q_w$ for all $w \in W$, 

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(3) \( w \in \mu(f) \) if and only if \( f \in \mu(w) \) for all \( f \in F \) and \( w \in W \).

We denote by \( \mu(f) \equiv \{ w \in W | f \in \mu(w) \} \) the set of workers who are assigned to \( f \) and \( \mu(w) \equiv \{ f \in F | w \in \mu(f) \} \) the set of firms which are assigned to \( w \). We will denote by \( \min(\mu(v)) \) the least preferred agent of \( v \) in \( \mu(v) \).

Note that, for \( v \in F \cup W \), we stipulate \( |\mu(v)| = 0 \) if \( \mu(v) = \emptyset \). For this condition we say agent \( v \) is single.

For any two matchings \( \mu \) and \( \mu' \), the notation \( \mu P(\mu) \mu' \) means \( \mu(f)P(\mu(f))\mu'(f) \), \( \mu R(\mu) \mu' \) means \( \mu(f)\mu'(f) \), \( \mu R(F) \mu' \) means \( \mu R(f)\mu' \) for all \( f \in F \), and, \( \mu P(F) \mu' \) means \( \mu R(F)\mu' \) and \( \mu \neq \mu' \).

A matching \( \mu \) is blocked by an individual \( i \in F \cup W \) if there exists some player \( j \in \mu(i) \) such that \( \emptyset \succ i j \). A matching is individually rational if it is not blocked by any individual. A matching \( \mu \) is blocked by a pair \( (f, w) \in F \times W \) if they are not matched together under \( \mu \), and further

1. \( f \) is acceptable to \( w \) and \( w \) is acceptable to \( f \),
2. \( |\mu(f)| < q_f \) or \( w \succ_f w' \) for some \( w' \in \mu(f) \),
3. \( |\mu(w)| < q_w \) or \( f \succ_w f' \) for some \( f' \in \mu(w) \).

**Definition 2** A matching \( \mu \) is stable if it is not blocked by any individual nor any firm-worker pair.

This definition is known as “pairwise-stability”. Throughout this paper, we adopt the concept of stability of matching as pairwise-stability.

For many-to-many matching problems, since every agent (a firm or worker) wishes to match with a subset of agents from the other side of the market, we also need to specify each firm \( f \)’s (resp. worker \( w \)’s) preference relation, denoted by \( P(f) \) (resp. \( P(w) \)), over the set of potential partner groups \( 2^W \) (resp. \( 2^F \)). For each \( v \in V \), we assume the preference relation \( P(v) \) is strict and transitive, but the completeness is not required. Notice that, over the set of singleton subsets, the preference relation \( P(v) \) coincides with \( \succ_v \). That is, \( \{w_1\}P(f)\{w_2\} \) if and only if \( w_1 \succ_f w_2 \) and \( \{f_1\}P(w)\{f_2\} \) if and only if \( f_1 \succ_w f_2 \) for any \( f, f_1, f_2 \in F \) and \( w, w_1, w_2 \in W \). We can also present \( P(f) \) and \( P(w) \) by order lists. Since only acceptable sets of partners will matter, we only write the lists involving acceptable partners. For instance,

\[
P(f) : w_1 w_2, w_1, w_2, w_3, \text{ and}
P(w) : f_1 f_3, f_1, f_2
\]
indicate that \( \{w_1, w_2\} P(f) \{w_1\} P(f) \{w_2\} P(f) \{w_3\} P(f) \emptyset \) and \( \{f_1, f_3\} P(w) \{f_1\} P(w) \{f_2\} P(w) \emptyset \), respectively.

Some restrictions on firm \( f \)'s preference relation \( P(f) \) are usually imposed, among which substitutability is often adopted.\(^6\) In the language of firms-workers matching model, substitutability of firm \( f \)'s preferences requires: “if hiring \( w \) is optimal when certain workers are available, hiring \( w \) must still be optimal when a subset of workers are available.” Formally,

**Definition 3** An agent \( f \)'s preference relation \( P(f) \) satisfies *substitutability* if, for any sets \( S \) and \( S' \) with \( S \subseteq S' \) and \( w \in S, w \in Ch(S', P(f)) \) implies \( w \in Ch(S, P(f)) \), where \( Ch(S, P(f)) \) denotes agent \( f \)'s most-preferred subset of \( S \) according to \( f \)'s preference relation \( P(f) \).

In literature, there is a different way to define substitutability. For instance, Hatfield and Milgrom (2005) and Hatfield and Kominers (2014) define the substitutability by rejection function. Specifically, the definition of substitutability by rejection function can be expressed as: \( f \)'s preference relation \( P(f) \) satisfies substitutability if for any sets \( S \) and \( S' \) with \( S \subseteq S' \subseteq W \), we have \( R(S, P(f)) \subseteq R(S', P(f)) \), where \( R(S, P(f)) \equiv S \setminus Ch(S, P(f)) \) denotes agent \( f \)'s rejection function. It is easy to check that these two ways are in fact equivalent.\(^7\)

The substitutable preference condition, (together with other condition, such as law of aggregate demand), ensures the stability of matching problem and incentive compatibility of agents with unit demand. However, for incentive compatibility of agents with multi-unit demand, we need a slightly stronger preference condition. Throughout this paper, we assume that each agent \( v \)'s preference relation \( P(v) \) satisfies the max-min criterion.\(^8\) This condition was first proposed by Baïou and Balinski (2000). For completeness, we state it as follows.


\(^7\)To see this, suppose that, for any sets \( S \) and \( S' \) with \( S \subseteq S' \) and \( w \in S, w \in Ch(S', P(f)) \) implies \( w \in Ch(S, P(f)) \). For any \( w' \in R(S, P(f)) \), we then have \( w' \notin S \) and \( w' \notin Ch(S, P(f)) \). By assumption we have \( w' \notin Ch(S', P(f)) \). Since \( w' \in S \subseteq S' \), we have \( w' \in R(S', P(f)) \). Thus, \( R(S, P(f)) \subseteq R(S', P(f)) \). On the other hand, suppose that \( R(S, P(f)) \subseteq R(S', P(f)) \) for any sets \( S \) and \( S' \) with \( S \subseteq S' \subseteq W \). For \( w \in S \), if \( w \in Ch(S', P(f)) \), then \( w \notin R(S', P(f)) \). By assumption we have \( w \notin R(S, P(f)) \). Since \( w \in S \), we have \( w \in Ch(S, P(f)) \). Thus, \( w \in Ch(S', P(f)) \) implies \( w \in Ch(S, P(f)) \).

\(^8\)Similar conditions were studied by Echenique and Oviedo (2006), Kojima (2007) and Sotomayor (2004).
**Definition 4 (Max-Min Criterion)** The firm $f$’s preference relation $P(f)$ is said to satisfy the max-min criterion iff for any two sets of acceptable workers $S_1, S_2 \subseteq W$ with $|S_1| \leq q_f$ and $|S_2| \leq q_f$,  

(i) The strict preference relation $P(f)$ over $W$ is defined as: $S_1P(f)S_2$ if and only if $S_2$ is a proper subset of $S_1$, or, $|S_1| \geq |S_2|$ and $\min(S_1) \succ_f \min(S_2)$ (i.e., $f$ strictly prefers the least preferred worker in $S_1$ to the least preferred worker in $S_2$), where $\min(S_1)$ denotes the least preferred worker of $f$ in $S_1$;  

(ii) The weak preference relation, denoted by $R(f)$, is defined as: $S_1R(f)S_2$ if and only if $S_1P(f)S_2$ or $S_1 = S_2$.  

The preference relation $P(w)$ of worker $w \in W$ satisfies the max-min criterion if the corresponding condition is met.  

It is interesting to point out that max-min preferences over groups of workers can display either substitutability or complementarity effect, depending on whether $S_2$ is a proper subset of $S_1$. While responsiveness clearly implies substitutability, the max-min criterion also implies substitutability. To see that max-min criterion is stronger than substitutability, we assume firm $f$’s preference relation $P(f)$ satisfies max-min criterion. Let $S$ and $S'$ be sets of workers, with $S \subseteq S'$. Suppose $w \in Ch(S' \cup \{w\}, P(f))$. We shall prove $w \in Ch(S \cup \{w\}, P(f))$. Consider the following two cases:  

Case (I). $|Ch(S' \cup \{w\}, P(f))| < q_f$. The condition $w \in Ch(S' \cup \{w\}, P(f))$ implies that $w$ is acceptable to $f$. Then, by the definition of max-min criterion, we have $w \in Ch(S \cup \{w\}, P(f))$.  

Case (II). $|Ch(S' \cup \{w\}, P(f))| = q_f$. We have $|Ch(S' \cup \{w\}, P(f))| = q_f$ according to max-min criterion. Since $S \cup \{w\} \subseteq S' \cup \{w\}$, we have $\min(Ch(S' \cup \{w\}, P(f))) \succ_f \min(Ch(S \cup \{w\}, P(f)))$. The condition $w \in Ch(S' \cup \{w\}, P(f))$ implies $w \succ_f \min(Ch(S' \cup \{w\}, P(f)))$. Thus we infer $w \succ_f \min(Ch(S \cup \{w\}, P(f)))$.  

Thus, when $S_2$ is a proper subset of $S_1$, the max-min criterion displays substitutability effect. When $S_2$ and $S_1$ have no inclusion relationship, the max-min criterion then will display complementarity effect. Indeed, when $|S_2| \leq |S_1| \leq q_f$, the value $u_f(S_1) = \min_{w \in S_1} \{u_f(w)\}$ and $u_f(S_2) = \min_{w \in S_2} \{u_f(w)\}$. Then $S_1P(f)S_2$ if and only if $\min(S_1) \succ_f \min(S_2)$, i.e., a firm’s welfare over a group of workers is increased only if the least preferred worker’s overall quality in the group becomes higher, which means firms’ preferences over a group of workers are complementary.  

However, while max-min criterion and responsiveness both imply substitutability, there is no implication relationship between max-min preferences and responsive preferences. Indeed, firstly,  

\*Sotomayor (2004) gives a similar illustration.
max-min criterion is not stronger than responsiveness. For example, suppose \( w_1 \succ_f w_2 \succ_f w_3 \) and \( q_f = 2 \). Then \( \{w_1, w_3\} P(f)\{w_2, w_3\} \) if \( P(f) \) is responsive. However, \( \{w_1, w_3\} P(f)\{w_2, w_3\} \) does not hold if \( P(f) \) satisfies max-min criterion. Secondly, responsiveness is not stronger than max-min criterion. For example, we assume \( w_1 \succ_f w_2 \succ_f w_3 \succ_f w_4 \) and \( q_f = 2 \). Then \( \{w_2, w_3\} P(f)\{w_1, w_4\} \) if \( P(f) \) satisfies max-min criterion. However, \( \{w_2, w_3\} P(f)\{w_1, w_4\} \) does not necessarily hold if \( P(f) \) is responsive.

Denote the preferences profile of all agents by \( P \equiv (P(v))_{v \in V} \), and a firms-worker matching problem by a four-tuple \((F, W; q; P)\). Given a matching market \((F, W; q; P)\), it is assumed that a firm can provide at most one position to any given worker.\(^{10}\)

2.2. Deferred acceptance algorithm

The deferred acceptance algorithm was first proposed by Gale and Shapley (1962) to find a stable assignment for the marriage problem (one-to-one matching) and college admissions problem (many-to-one matching). If every player has max-min preference, then the deferred acceptance algorithm can be used to find a stable assignment for a many-to-many matching.

Specifically, the Firms-Proposing Deferred Acceptance Algorithm proceeds as follows:

**Step 1.** *(a)*. Each firm \( f \) proposes to its most-preferred \( q_f \) acceptable workers (and if it has fewer acceptable choices than \( q_f \), then it proposes to all its acceptable choices).

*(b)*. Each worker \( w \) then places on her waiting list the \( q_w \) firms which rank highest, or all firms if there are fewer than \( q_w \) firms, and rejects the rest.

In general, at

**Step k.** *(c)*. Any firm \( f \) that was rejected at step \((k - 1)\) by any worker proposes to its most-preferred \( q_f \) acceptable workers who have not yet rejected it (and if there are fewer than \( q_f \) remaining acceptable workers, then it proposes to all).

*(d)*. Each worker \( w \) selects the top \( q_w \) — or all firms if there are fewer than \( q_w \) firms—from among the new firms and those on her waiting list, puts them on her new waiting list, and rejects the rest.

Since no firm proposes twice to the same worker, this algorithm always terminates in a finite number of steps. The algorithm terminates when there are no more rejections. Each worker is matched with firms on her waiting list in the last step.

It can be shown that, under the max-min preference restriction, the matching produced by the firms-proposing deferred acceptance algorithm, denoted by \( \mu_F \), is stable (see, for instance, Jiao and Tian, 2015). Symmetrically, the matching produced by the workers-proposing de-\(^{10}\)For matching with contracts, Hatfield and Milgrom (2005) make a similar assumption.
ferred acceptance algorithm, denoted by $\mu_W$, is also stable. Furthermore, we have the following proposition that shows $\mu_F$ is the (unique) optimal stable matching for firms.

**Proposition 1** For the firms-workers matching market $(F,W;q;P)$, if every agent has max-min preference, then $\mu_F$ is the optimal stable assignment for the firms; that is, for any other stable matching $\mu$, $\mu_F(f) R(f) \mu(f)$ for every $f \in F$.

The proof of Proposition 1 is given in the Appendix. Symmetrically, $\mu_W$ is the unique optimal stable assignment for the workers.

### 3 The Blocking Lemma

For the firms-workers matching model with max-min preferences, the Blocking Lemma for firms can be expressed as: if the set of firms that strictly prefer an individually rational matching $\mu$ to $\mu_F$ is nonempty, then there must exist a blocking pair $(f, w)$ of $\mu$ with the property that, under the matching $\mu$, $w$ works for a firm strictly preferring $\mu$ to $\mu_F$, and $f$ considers $\mu_F$ being at least as good as $\mu$. Formally,

**Definition 5** (The Blocking Lemma) Let $\mu$ be any individually rational matching and $F'$ be the nonempty set of all firms that prefer $\mu$ to $\mu_F$. We say that the Blocking Lemma (for firms) holds if there exists $f \in F \setminus F'$ and $w \in \mu(F')$ such that the pair $(f, w)$ blocks $\mu$.

Gale and Sotomayor (1985) prove the Blocking Lemma for one-to-one matching. For many-to-one matching problem, the Blocking Lemma (for agents with unit demand) can be easily obtained by the decomposition lemma (see Roth and Sotomayor, 1990) if the responsive preference condition is satisfied. Recently, Martínez et al. (2010) prove the same result under a weak preference condition (i.e., substitutable and quota-separable preference profiles). They also note that the Blocking Lemma for agents with multi-unit demand does not hold even under responsive preference profiles. In fact, the following example shows that, under any reasonable preference assumption (including responsiveness, separability and even max-min criterion), we cannot expect to obtain the Blocking Lemma for agents with multi-unit demand.

**Example 1** There are two firms $f_1, f_2$ with $q_{f_1} = 1, q_{f_2} = 2$, and two workers $w_1, w_2$ with $q_{w_1} = 1 = q_{w_2}$. The preferences are as follows:

- $P(w_1) : f_1, f_2$
- $P(f_1) : w_2, w_1$
- $P(w_2) : f_2, f_1$
- $P(f_2) : w_1, w_2, w_1, w_2$. 


The matching produced by the firms-proposing deferred acceptance algorithm is:

$$
\mu_F = \begin{pmatrix}
  f_1 & f_2 \\
  w_1 & w_2
\end{pmatrix}.
$$

An individually rational matching $\mu$ is:

$$
\mu = \begin{pmatrix}
  f_1 & f_2 \\
  w_2 & w_1
\end{pmatrix}.
$$

One can see that $P(f_2)$ satisfies responsiveness, separability and max-min criterion. It is easy to check that both $f_1$ and $f_2$ prefer $\mu$ to $\mu_F$. Thus $F' = \{f_1, f_2\}$ is nonempty and $F' = F$. The conclusion of the Blocking Lemma does not hold.

In Example 1, we can see that firm $f_2$’s preference relation is commonly reasonable, but the number of workers is less than the aggregate quota of firms. This may potentially cause the failure of the Blocking Lemma. To avoid this possible environment, we propose the following condition which says that in the market there are enough acceptable workers (or workers’ quotas) such that each firm can hire as many as their quota acceptable workers if they want. Formally,

**Definition 6** *(Quota-Saturability Condition)* For the firms-workers matching problem, we say that the firms-quota-saturability condition holds if there are enough acceptable workers such that each firm $f$ can be assigned $q_f$ acceptable workers under the firm-optimal stable matching.

Similarly, we can define the workers-quota-saturability condition. If we assume that each firm and each worker are acceptable to each other, then the firms-quota-saturability condition can be represented as $\sum_{w \in W} q_w \geq \sum_{f \in F} q_f$ and the workers-quota-saturability condition can be represented as $\sum_{f \in F} q_f \geq \sum_{w \in W} q_w$.

With the above preparation, we now state our main result in the following theorem.

**Theorem 1** For the firms-workers matching model, if every agent has max-min preference and the firms-quota-saturability condition is satisfied, then the Blocking Lemma for firms holds.

The proof of Theorem 1 is given in the Appendix. Symmetrically, the Blocking Lemma for workers holds if every agent has max-min preference and the workers-quota-saturability condition is satisfied.

Note that both the quota-saturability condition and the max-min criterion are crucial for the Blocking Lemma. In Example 1, the firms-quota-saturability condition is not satisfied and the Blocking Lemma fails. The following example shows the importance of max-min criterion even if the firms-quota-saturability condition is satisfied.
Example 2 There are three firms $f_1, f_2, f_3$ with $q_{f_1} = 2, q_{f_2} = 1, q_{f_3} = 1$, and four workers $w_1, w_2, w_3, w_4$ with $q_{w_1} = q_{w_2} = q_{w_3} = q_{w_4} = 1$. The preferences are as follows:

$P(f_1) : w_1w_2, w_1w_3, w_1w_4, w_2w_3, w_2w_4, w_3w_4, w_1, w_2, w_3, w_4$

$P(f_2) : w_3, w_1, w_2, w_4$

$P(f_3) : w_1, w_2, w_3, w_4$

$P(w_1) : f_2, f_1, f_3$

$P(w_2) : f_3, f_1, f_2$

$P(w_3) : f_1, f_2, f_3$

$P(w_4) : f_1, f_3, f_2$.

The matching produced by the firms-proposing deferred acceptance algorithm is:

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4 & w_1 & w_2 \end{pmatrix}.$$

An individually rational matching $\mu$ is as follows:

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2w_4 & w_2 & w_1 \end{pmatrix}.$$

One can see that the firms-quota-saturability condition holds, but firm $f_1$’s preference does not satisfy the max-min criterion. It is easy to check that all of the three firms prefer $\mu$ to $\mu_F$, so $F' = \{f_1, f_2, f_3\}$ is nonempty and $F' = F$. Therefore, the Blocking Lemma does not hold. This example shows that, if the max-min criterion is not satisfied, the Blocking Lemma does not hold even if the firms-quota-saturability condition holds and each agent has responsive preference.

This example also shows that the max-min criterion cannot be weakened to substitutability criterion for the Blocking Lemma, as we can check that $P(f_1)$ is responsive and responsiveness implies substitutability. Furthermore, we note that the Blocking Lemma fails to hold even if we slightly weaken the requirement of max-min preference. To see this, we consider the following example.

Example 3 There are three firms $f_1, f_2, f_3$ with $q_{f_1} = 3, q_{f_2} = 1, q_{f_3} = 1$, and five workers $w_1, w_2, w_3, w_4, w_5$. The preferences are as follows:

$\succ_{w_1} : f_2, f_1, f_3, \emptyset$ \quad $\succ_{w_5} : f_1, f_3, f_2, \emptyset$

$\succ_{w_2} : f_3, f_1, f_2, \emptyset$ \quad $\succ_{f_1} : w_1, w_2, w_3, w_4, w_5, \emptyset$

$\succ_{w_3} : f_1, f_2, f_3, \emptyset$ \quad $\succ_{f_2} : w_3, w_1, \cdots$

$\succ_{w_4} : f_1, f_3, f_2, \emptyset$ \quad $\succ_{f_3} : w_4, w_2, \cdots$

Then the firms-optimal matching is

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4w_5 & w_1 & w_2 \end{pmatrix}.$$
For firm $f_1$’s preference over groups of workers, we slightly weaken the requirement of max-min criterion.

Case I: We assume that $f_1$ prefers $\{w_1, w_2\}$ to $\{w_3, w_4, w_5\}$. Sometimes, such assumption seems to be reasonable. One can easily check that this assumption violates the max-min criterion, as $|\{w_1, w_2\}| \geq |\{w_3, w_4, w_5\}|$ does not hold although $\min(\{w_1, w_2\}) \succ_{f_1} \min(\{w_3, w_4, w_5\})$. Then, for the following individually rational matching

$$\mu_1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2 & w_3 & w_4 \end{pmatrix},$$

we can see that all of the three firms prefer $\mu$ to $\mu_F$, so $F' = \{f_1, f_2, f_3\}$ is nonempty and $F' = F$. Therefore, the Blocking Lemma does not hold.

Case II: We assume that $f_1$ prefers $\{w_1, w_2, w_5\}$ to $\{w_3, w_4, w_5\}$. Generally, such assumption is natural and reasonable. One can easily check that this assumption violates the max-min criterion, as $\min(\{w_1, w_2, w_5\}) \succ_{f_1} \min(\{w_3, w_4, w_5\})$ does not hold although $|\{w_1, w_2, w_5\}| \geq |\{w_3, w_4, w_5\}|$. Then, for the following individually rational matching

$$\mu_2 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2w_5 & w_3 & w_4 \end{pmatrix},$$

we can see that all of the three firms prefer $\mu$ to $\mu_F$, so $F' = \{f_1, f_2, f_3\}$ is nonempty and $F' = F$. Therefore, the Blocking Lemma does not hold.

This example indicates that one cannot expect to obtain the Blocking Lemma for agents with multi-unit demand under a preference restriction which is weaker than max-min criterion.

4 Strategy-Proofness of the Deferred Acceptance Algorithm

We know that the Blocking Lemma is an important instrumental result for one-to-one and many-to-one matching problem. For one-to-one matching, Gale and Sotomayor (1985) give a short proof for the group strategy-proofness of the deferred acceptance algorithm with the help of the Blocking Lemma. Under responsive preference, we can extend the corresponding result to the many-to-one matching (strategy-proofness for agents with unit demand). In this section, we show the strategy-proofness of the firms-proposing deferred acceptance algorithm, which heavily relies on the Blocking Lemma obtained in the previous section.

Given the firms $F$ and workers $W$, a mechanism $\varphi$ is a function from any stated preferences profile and quota-vector $q$ to a matching. A mechanism is stable iff the outcome of that mech-
anism, denoted by $\varphi(F, W; q; P)$, is a stable matching for any reported preferences profile and $q$.

**Definition 7** A mechanism $\varphi$ is *group strategy-proof for firms* iff, at every preference profile $P$, there is no group of firms $F' \subset F$ and, a preference profile $(P'(f))_{f \in F'}$ and quota-vector $(q'_{f})_{f \in F'}$ such that every $f \in F'$ strictly prefers $\varphi(F, W; (q'_{f})_{f \in F'}, (P_{f})_{f \in F}, (P'(f))_{f \in F'}; (P'(f))_{f \in F\setminus F'})$ to $\varphi(F, W; q; P)$.

Now we state the group incentive compatibility for many-to-many matching as follows.

**Theorem 2** For the firms-workers matching model, if every agent has max-min preference and the firms-quota-saturability condition is satisfied, then the firms-proposing deferred acceptance algorithm is group strategy-proof for firms.

We note that this group incentive compatibility heavily depends on the Blocking Lemma. Since the conditions of max-min preference and firms-quota-saturability are crucial for the Blocking Lemma, they play an important role for the group strategy-proofness of firms-proposing deferred acceptance algorithm. As in the case for the Blocking Lemma, one cannot expect to obtain the group incentive compatibility for firms under a preference restriction which is weaker than max-min criterion. Specifically, we continue to see Example 3.

**Example 3 (Continued)** Consider the sets of firms and workers, and their preferences and quotas as given in Example 3. The firms-optimal matching is

$$
\mu_{F} = \begin{pmatrix}
    f_1 & f_2 & f_3 \\
    w_3 & w_4 & w_5 & w_1 & w_2 \\
\end{pmatrix}.
$$

For firm $f_1$’s preference over groups of workers, we slightly weaken the requirement of max-min criterion.

Case I: We assume that $f_1$ prefers $\{w_1, w_2\}$ to $\{w_3, w_4, w_5\}$, which violates the max-min criterion. Then firm $f_1$ can receive a strictly better assignment by misreporting its preferences and quota. That is, if it reports $q_{f_1} = 2$ and $w_1, w_2$ are its only acceptable workers, then the matching produced by the firms-proposing deferred acceptance algorithm is:

$$
\mu_{1} = \begin{pmatrix}
    f_1 & f_2 & f_3 \\
    w_1 & w_2 & w_3 & w_4 \\
\end{pmatrix},
$$

and $f_1$ will become strictly better off than to report truthfully. Consequently, $\mu_{F}$ is not strategy-proof.
Case II: We assume that $f_1$ prefers $\{w_1, w_2, w_5\}$ to $\{w_3, w_4, w_5\}$, which violates the max-min criterion. Then firm $f_1$ can receive a strictly better assignment by misreporting its preferences and quota. That is, if it reports $q_{f_1} = 3$ and $w_1, w_2, w_5$ are its only acceptable workers, then the matching produced by the firms-proposing deferred acceptance algorithm is:

$$
\mu_2 = \begin{pmatrix}
  f_1 & f_2 & f_3 \\
  w_1 w_2 w_5 & w_3 & w_4
\end{pmatrix},
$$

and $f_1$ will become strictly better off than to report truthfully. Consequently, $\mu_F$ is not strategy-proof.

5 Conclusion

We obtain the Blocking Lemma and the group incentive compatibility for agents with multi-unit demand under max-min criterion and quota-saturability condition. The corresponding results fail to hold even if we just slightly weaken the requirement of max-min criterion. The max-min criterion is stronger than substitutability criterion. Even if someone might consider max-min criterion as a restricted condition, our results may still be regarded as an impossibility result. That is, the Blocking Lemma and the group incentive compatibility for many-to-many matching fail to hold under any preference condition weaker than max-min criterion. As such, in either way of explanations, these results are interesting.
Appendix

Proof of Proposition 1. Define a firm \( f \) and a worker \( w \) to be achievable for each other in a matching problem \( (F,W;q;P) \) if \( f \) and \( w \) are matched at some stable assignment. We will show that in the deferred acceptance algorithm with firms proposing, no firm is ever rejected by an achievable worker. Consequently the stable matching \( \mu_F \) is produced by assigning each firm to its most-preferred achievable workers. Then \( \mu_F \) is the optimal stable matching for every firm.

The proof will be done by induction. We first note that at step 1 no firm is rejected by its achievable worker. Suppose not. There exists some firm \( f \) which is rejected by one of its achievable workers \( w \). \( w \) rejecting \( f \) implies there are \( q_w \) firms that proposed to \( w \) at step 1 (we denote the set of these \( q_w \) firms by \( S(q_w) \)), which are better than \( f \) to \( w \). This means that each firm \( \tilde{f} \in S(q_w) \) treats \( w \) as one of its \( q_{\tilde{f}} \)-first-best choices, that is, on \( \tilde{f} \)'s preference list there are fewer than \( q_{\tilde{f}} \) workers, who are better than \( w \) to \( \tilde{f} \). Then \( \tilde{f} \) always wants to be matched with \( w \). Since \( w \) and \( \tilde{f} \) are achievable for each other, there exists some stable matching \( \mu \) such that \( \tilde{f} \in \mu(w) \). Considering \( |\mu(w)| \leq q_w = |S(q_w)| \), \( f \in \mu(w) \) and \( f \notin S(q_w) \), we know that there must exist some firm, say, \( f' \), satisfying \( f' \in S(q_w) \setminus \mu(w) \). Since \( f' \succ_w f \), \( f \in \mu(w) \), \( f' \notin \mu(w) \) and \( f' \) treats \( w \) as one of its \( q_{f'} \)-first-best choices, we know that \( \mu \) is blocked by \( (f',w) \), which contradicts the stability of \( \mu \).

Suppose that up to a given step, say, step \( k \), in the procedure of the deferred acceptance algorithm, no firm has yet been rejected by any worker who is achievable for it. We want to show that, at step \((k + 1)\), no firm is rejected by any of its achievable workers. Suppose a worker \( w \) rejects some firm \( f \) at step \((k + 1)\). If she rejects \( f \) as unacceptable, then \( w \) is not achievable for \( f \), and we are done. If \( f \) is acceptable to \( w \), then \( w \) rejects \( f \) because she has better (than \( f \)) choices in hand. By max-min criterion, there must be \( q_w \) firms which are better than \( f \) to \( w \) on her waiting list. We denote the set of these \( q_w \) firms by \( \tilde{S}(q_w) \). One can show that \( w \) is not achievable for \( f \).

Specifically, we know that any firm \( \tilde{f} \in \tilde{S}(q_w) \) proposes to \( w \) at step \((k + 1)\) or at some previous step. The inductive assumption implies that \( \tilde{f} \) has not been rejected by any worker who is achievable to it before step \((k + 1)\). Then on \( \tilde{f} \)'s preference list there are fewer than \( q_{\tilde{f}} \) workers who are achievable to \( \tilde{f} \) and are better than \( w \) for \( \tilde{f} \). We suppose \( w \) and \( f \) are achievable for each other. By definition there exists some stable matching \( \tilde{\mu} \) such that \( f \in \tilde{\mu}(w) \). Since \( |\tilde{\mu}(w)| \leq q_w, |\tilde{S}(q_w)| = q_w, f \in \tilde{\mu}(w) \) and \( f \notin \tilde{S}(q_w) \), we can infer that \( \tilde{S}(q_w) \setminus \tilde{\mu}(w) \) is nonempty. There exists some firm, say, \( \tilde{f}_0 \in \tilde{S}(q_w) \setminus \tilde{\mu}(w) \). It is easy to see that \( \tilde{\mu} \) is blocked by \( (\tilde{f}_0, w) \),
which contradicts the stability of \( \tilde{\mu} \). The proof is completed. \( \square \)

**Proof of Theorem 1.** We prove the theorem by considering two cases.

**Case I.** \( \mu(F') \neq \mu_F(F') \).

For any firm \( \tilde{f} \in F' \), by the quota-saturability condition, we know \( |\mu_F(\tilde{f})| = q_{\tilde{f}} \). The fact that \( \tilde{f} \) prefers \( \mu \) to \( \mu_F \) implies \( |\mu(\tilde{f})| \geq |\mu_F(\tilde{f})| = q_{\tilde{f}} \) according to the max-min criterion. Together with \( |\mu(\tilde{f})| \leq q_{\tilde{f}} \), we obtain \( |\mu(\tilde{f})| = q_{\tilde{f}} \). Then it holds \( |\mu(F')| = \sum_{f \in F'} |\mu(f)| = \sum_{f \in F'} |\mu_F(f)| = |\mu_F(F')| \).

(i) \( \mu(F') \) is a proper subset of \( \mu_F(F') \).

We construct two sets as follows. Let \( M_1 = \{(f, w) \in \mu | f \in F', w \in \mu(F')\} \) and \( M_2 = \{(f, w) \in \mu_F | f \in F', w \in \mu(F')\} \). It holds \( M_1 = (M_1 \cap M_2) \cup (M_1 \setminus M_2) \). One can easily check that \( |M_1| = |\mu(F')| \) and \( |M_2| < |\mu_F(F')| \) by the constructions of \( M_1 \) and \( M_2 \), and the assumption of \( \mu(F') \) being a proper subset of \( \mu_F(F') \). Together with \( |\mu(F')| = |\mu(F')| \), we have \( |M_2| < |M_1| \). Then \( M_1 \setminus M_2 \neq \emptyset \). For each pair \( (\tilde{f}, \tilde{w}) \in M_1 \setminus M_2 \), one can know that \( \tilde{w} \succ_{\tilde{f}} \min(\mu_F(\tilde{f})) \) by the quota-saturability condition, max-min criterion, and \( \mu(\tilde{f}) \succ_{\tilde{f}} \mu_F(\tilde{f}) \).

Then \( \tilde{f} \) must propose to \( \tilde{w} \) in the procedure of the deferred acceptance algorithm of \( \mu_F \). By the assumption of individual rationality of \( \mu \), \( (\tilde{f}, \tilde{w}) \in \mu \) indicates that \( \tilde{f} \) is acceptable to \( \tilde{w} \). Then \( (\tilde{f}, \tilde{w}) \notin \mu_F \) implies that \( \tilde{w} \) has \( q_{\tilde{w}} \) choices which are better than \( \tilde{f} \) and then rejects \( \tilde{f} \).

We have \( |\mu_F(\tilde{w})| = q_{\tilde{w}} \). Thus, \( \tilde{f} \in \mu(\tilde{w}) \) means that there must exist some \( f' \in F \) such that \( (f', \tilde{w}) \in \mu_F \setminus \mu \). Now we claim the following statement.

**Claim:** There is at least one pair \( (f_1, w_1) \in M_1 \setminus M_2 \) and one firm \( f_2 \in F \setminus F' \) such that \( f_2 \in \mu_F(w_1) \setminus \mu(w_1) \).

We prove the claim by contradiction. Suppose not. Then for any \( (\tilde{f}, \tilde{w}) \in M_1 \setminus M_2 \), there always exists some firm \( f \in F' \) such that \( f \in \mu_F(\tilde{w}) \setminus \mu(\tilde{w}) \). That is, every pair \( (\tilde{f}, \tilde{w}) \in M_1 \setminus M_2 \) corresponds to one pair \( (f, \tilde{w}) \in M_2 \setminus M_1 \). This implies \( |M_2 \setminus M_1| \geq |M_1 \setminus M_2| \), and hence \( |M_2| = |(M_1 \setminus M_2) \cup (M_1 \cap M_2)| \geq |(M_2 \setminus M_1) \cup (M_1 \cap M_2)| = |M_1| \), which contradicts \( |M_2| < |M_1| \).

The proof of the claim is completed.

For \( w_1 \) and \( f_2 \) as given in the claim, we show that \( (f_2, w_1) \) is a blocking pair of \( \mu \) satisfying the Blocking Lemma. Indeed, one can know that \( f_2 \) and \( w_1 \) are unmatched under \( \mu \) by \( f_2 \notin \mu(w_1) \). We have shown that the condition \( (f_1, w_1) \in M_1 \setminus M_2 \) implies that, in the procedure of the deferred acceptance algorithm of \( \mu_F \), \( f_1 \) proposes to \( w_1 \) but \( w_1 \) has \( q_{w_1} \) choices which are better than \( f_1 \) and then rejects \( f_1 \). Combining \( f_2 \in \mu_F(w_1) \), one has \( f_2 \succ_{w_1} f_1 \). However, \( (f_1, w_1) \in M_1 \) indicates that \( f_1 \) and \( w_1 \) are matched together under \( \mu \). On the other hand,
considering \( f_2 \in F \setminus F' \), we know that \( \mu_F(f_2)R(f_2)\mu(f_2) \). If \( |\mu(f_2)| < q_{f_2} \), then \((f_2, w_1)\) blocks \( \mu \). If \( |\mu(f_2)| = q_{f_2} \), the max-min criterion implies \( \min(\mu_F(f_2)) \geq_{f_2} \min(\mu(f_2)) \). Together with \( w_1 \in \mu_F(f_2) \) and \( w_1 \notin \mu(f_2) \), we can obtain \( w_1 \succ_{f_2} \min(\mu(f_2)) \). Thus, \((f_2, w_1)\) blocks \( \mu \).

\((ii)\) \( \mu(F') \) is not a proper subset of \( \mu_F(F') \).

In view of \( \mu(F') \neq \mu_F(F') \), one can obtain \( \mu(F') \setminus \mu_F(F') \neq \emptyset \). Choose \( w \) in \( \mu(F') \setminus \mu_F(F') \) such that \( w \in \mu(f') \) for some \( f' \in F' \). The individual rationality of \( \mu \) implies that \( f' \) is acceptable to \( w \). Since \( f' \) prefers \( \mu(f') \) to \( \mu_F(f') \), \( w \in \mu(f') \) implies \( w \succ_{f'} \min(\mu(f')) \succ_{f'} \min(\mu_F(f')) \) by the max-min criterion. This indicates that, in the procedure of the deferred acceptance algorithm, \( f' \) had proposed to \( w \) before it proposed to \( \min(\mu_F(f')) \). Since \( f' \) is acceptable to \( w \), \( w \notin \mu_F(f') \) implies that \( w \) has \( q_w \) choices which are better than \( f' \) and then rejects \( f' \). Then we have \( |\mu_F(w)| = q_w \). Combining conditions \( f' \in \mu(w) \), \( f' \notin \mu_F(w) \), \( |\mu_F(w)| = q_w \) and \( |\mu(w)| \leq q_w \), one can infer that there must exist some firm, say, \( f \in \mu_F(w) \), being unmatched with \( w \) under \( \mu \).

It is easy to see that \((f, w)\) blocks \( \mu \). Firstly, we know that \( f' \in \mu(w) \) and \( f \notin \mu(w) \). \( f \in \mu_F(w) \) and \( w \) rejecting \( f' \) in the procedure of the deferred acceptance algorithm indicate that \( f \succ_w f' \). Secondly, by \( w \notin \mu_F(F') \) and \( w \in \mu_F(f) \) we know \( f \in F \setminus F' \), and consequently \( \mu FR(f)\mu \). If \( |\mu(f)| < q_f \), we are done. If \( |\mu(f)| = q_f \), the max-min criterion implies \( \min(\mu_F(f)) \succ_f \min(\mu(f)) \). Together with \( w \in \mu_F(f) \) and \( w \notin \mu(f) \), we can obtain \( w \succ_f \min(\mu(f)) \). So \((f, w)\) blocks \( \mu \) with \( f \in F \setminus F' \) and \( w \in \mu(F') \).

Case II. \( \mu(F') = \mu_F(F') \equiv W' \).

For any \( w' \in W' \), the notation \( \mu(w') \cap F' \) will denote the set of firms which are matched to \( w' \) and belong to \( F' \).

\((iii)\) There exists some worker \( w \in W' \) such that \( |\mu(w) \cap F'| > |\mu_F(w) \cap F'| \).

We can find some firm, say, \( f' \in F' \), such that \( f' \in [\mu(w) \cap F'] \setminus [\mu_F(w) \cap F'] \). Thus \( w \in \mu(f') \setminus \mu_F(f') \). \( f' \) preferring \( \mu \) to \( \mu_F \) implies \( \min(\mu(f')) \succ_{f'} \min(\mu_F(f')) \). Thus we have \( w \succ_{f'} \min(\mu_F(f')) \). This indicates that \( f' \) must propose to \( w \) in the procedure of the deferred acceptance algorithm of \( \mu_F \). \( f' \notin \mu_F(w) \) implies that \( |\mu_F(w)| = q_w \) and \( w \) rejects \( f' \). By \( |\mu(w) \cap F'| > |\mu_F(w) \cap F'| \), we know \( |\mu_F(w) \cap F'| < q_w \). Hence, there is at least one firm, say, \( f \in F \setminus F' \), satisfying \( f \in \mu_F(w) \setminus \mu(w) \). Clearly, \( f \succ_w f' \). Also by \( f \in F \setminus F' \), we have \( \mu FR(f)\mu \).

Taking a similar argument as given in \((i)\) or \((ii)\), it is easy to show \((f, w)\) blocks \( \mu \).

\((iv)\) There exists some worker \( w' \in W' \) such that \( |\mu(w') \cap F'| < |\mu_F(w') \cap F'| \).

By conditions \( \mu(F') = \mu_F(F') \) and \( \sum_{f \in F'} |\mu(f)| = \sum_{f \in F'} |\mu_F(f)| \), we infer that there must be some worker \( w \in W' \) such that \( |\mu(w) \cap F'| > |\mu_F(w) \cap F'| \). The proof is done by \((iii)\).
(v) \(|\mu(w') \cap F'| = |\mu_F(w') \cap F'|\) for all \(w' \in W'\).

We assume that, in the deferred acceptance algorithm of \(\mu_F\), \(w\) is the last worker in \(W'\) to receive a proposal from a firm in \(F'\) (if there are more than one such workers, we choose anyone among them). For the sake of simplicity, we assume that this proposal was offered by firm \(f' \in F'\) at step \(k\). Then by \(\mu_F(F') = W'\) we know that, in the procedure of the deferred acceptance algorithm of \(\mu_F\), each firm \(\tilde{f} \in F'\) is always matched with \(q_{\tilde{f}}\) workers and there is no worker who will reject any firm of \(F'\) from step \(k\) to the termination of the algorithm. By the assumption, we also know that \((w, f') \in \mu_F\) and \(w = \min(\mu_F(f'))\).

Since \(f'\) prefers \(\mu\) to \(\mu_F\), it implies \(\min(\mu(f')) \succ_{\mu} \min(\mu_F(f'))\) by the max-min criterion. Then we know that \(w = \min(\mu_F(f')) \notin \mu(f')\). Also, since \(w \in \mu_F(F')\) and \(\mu(F') = \mu_F(F')\), we have \(w \in \mu(F')\). Together with \(|\mu(w) \cap F'| = |\mu_F(w) \cap F'|\), we obtain that there is some firm \(f'' \in F'\) such that \(w \in \mu(f'') \setminus \mu_F(f'')\). \(f''\) preferring \(\mu\) to \(\mu_F\) implies \(\min(\mu(f'')) \succ_{f''} \min(\mu_F(f''))\).

Thus we have \(w \succ_{f''} \min(\mu_F(f''))\). This indicates that \(f''\) must propose to \(w\) in the procedure of the deferred acceptance algorithm of \(\mu_F\). \(f'' \notin \mu_F(w)\) implies that \(|\mu_F(w)| = q_w\) and \(w\) rejects \(f''\). The supposition that \(w\) rejects \(f''\) at some step before step \(k\) indicates that \(w\) already has \(q_w\) (better than \(f''\)) choices from step \((k - 1)\). Since \(f'\) proposes to \(w\) and is accepted by \(w\) at step \(k\) and \(w\) does not reject any worker belonging to \(F'\) at this step, \(w\) must reject some firm \(f \in F \setminus F'\) and accept \(f'\) at step \(k\).

Then, (a) If \((f, w) \notin \mu\), we can prove \((f, w)\) is a blocking pair of \(\mu\). Indeed, on the one hand, since \(w\) rejects \(f''\) before she rejects \(f\) in the deferred acceptance algorithm of \(\mu_F\), we have \(f \succ_w f''\). However, \((f, w) \notin \mu\) and \((f'', w) \in \mu\). On the other hand, \(f \in F \setminus F'\) implies \(\mu_F R(f) \mu\). If \(|\mu(f)| < q_f\), then \((f, w)\) blocks \(\mu\). If \(|\mu(f)| = q_f\), \(\mu_F R(f) \mu\) implies \(\min(\mu_F(f)) \succeq_f \min(\mu(f))\). Since \(w\) rejects \(f\) in the procedure of the deferred acceptance algorithm of \(\mu_F\), we have \(w \succ_f \min(\mu_F(f))\) and in turn \(w \succ_f \min(\mu(f))\). So \((f, w)\) blocks \(\mu\) with \(f \in F \setminus F'\) and \(w \in \mu(F')\).

(b) If \((f, w) \in \mu\), there must exist at least one firm, say, \(f_0 \in F \setminus F'\) satisfying \(f_0 \in \mu_F(w) \setminus \mu(w)\), as \(|\mu_F(w)| = q_w\), \(|\mu(w) \cap F'| = |\mu_F(w) \cap F'|\), \(f \in F \setminus F', f \notin \mu_F(w)\) and \(f \in \mu(w)\). Then we can show \((f_0, w)\) is a blocking pair of \(\mu\). To see this, on the one hand, since \(f_0 \in \mu_F(w)\) and \(w\) rejects \(f''\) in the deferred acceptance algorithm of \(\mu_F\), we have \(f_0 \succ_w f''\). However, \((f_0, w) \notin \mu\) and \((f'', w) \in \mu\). On the other hand, \(f_0 \in F \setminus F'\) implies \(\mu_F R(f_0) \mu\). If \(|\mu(f_0)| < q_{f_0}\), then \((f_0, w)\) blocks \(\mu\). If \(|\mu(f_0)| = q_{f_0}\), \(\mu_F R(f_0) \mu\) implies \(\min(\mu_F(f_0)) \succeq_{f_0} \min(\mu(f_0))\). Together with \(w \in \mu_F(f_0)\) and \(w \notin \mu(f_0)\), we can obtain \(w \succ_{f_0} \min(\mu(f_0))\). Thus, \((f_0, w)\) blocks \(\mu\). The proof is completed. □

Proof of Theorem 2. Let \(\mu_F\) be the assignment produced by the firms-proposing deferred
acceptance algorithm under the true preferences and true quotas \((F, W; q; P)\). We suppose some nonempty subset \(F'\) of \(F\) misstates their preferences and quotas and is strictly better off under matching \(\mu\), produced by the firms-proposing deferred acceptance algorithm according to the reported preferences and quotas \((F, W; (q'_f)_{f \in F'}, (q_f)_{f \in F \setminus F'}; (P'(f))_{f \in F'}, (P(f))_{f \in F \setminus F'})\), than under \(\mu_F\) with respect to \((F, W; q; P)\). We note that \(\mu\) is stable with respect to the misreporting environment. If \(\mu\) is not individually rational, it must be the case that for some \(f' \in F'\), \(f'\) is matched to an unacceptable worker or matched with more mates than its quota. This contradicts \(\mu P(f') \mu_F\). Assume now \(\mu\) is individually rational. By the Blocking Lemma, there exists \((f, w)\) that blocks \(\mu\), with \(f \in F \setminus F'\) and \(w \in W\) under the true preferences and quotas \((F, W; q; P)\). Since \(f\) and \(w\) do not misstate their preferences and quotas, they are exactly the same in \((F, W; q; P)\) and \((F, W; (q'_f)_{f \in F'}, (q_f)_{f \in F \setminus F'}; (P'(f))_{f \in F'}, (P(f))_{f \in F \setminus F'})\). So \((f, w)\) blocks \(\mu\) under \((F, W; (q'_f)_{f \in F'}, (q_f)_{f \in F \setminus F'}; (P'(f))_{f \in F'}, (P(f))_{f \in F \setminus F'})\). However, we know \(\mu\) is stable with respect to the misreporting environment. This contradiction completes the proof. \(\Box\)
References


