The College Admissions Problem Reconsidered

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Abstract

This paper revisits the college admissions problem and studies the efficiency, incentive, and monotonicity for colleges. We show that max-min criterion that is stronger than substitutability, together with the quota-saturability that requires having enough acceptable applicants, guarantees weak Pareto efficiency and strategy-proofness for colleges under the colleges-proposing deferred acceptance algorithm. Moreover, we introduce a new notion of max-min criterion, called W-max-min criterion, which together with the quota-saturability condition, ensures that the colleges-proposing deferred acceptance algorithm is not only weakly Pareto efficient and strategy-proof, but also monotone for colleges.

Keywords: College admissions; Pareto efficiency; Strategy-proofness; Max-min preferences; Deferred acceptance algorithm; Monotonicity
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1 Introduction

Gale and Shapley (1962, henceforth GS) originally studied the college admissions problem. This problem generalizes the marriage matching model in such a way that colleges have preferences over students and students have preferences over colleges; each college c can accept at most a certain number q_c of students and each student can enroll in at most one college. For

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the college admissions problem, two assignment criteria are concerned primarily: stability and optimality. The stability requires that there exist no coalition of students and colleges to block the assignment. The optimality means that every agent (college or student) is at least as well off under the given stable assignment as he would be under any other stable assignment. GS proposed the deferred acceptance algorithm that always yields an optimal stable matching for agents on the proposing side.

For the efficiency and incentives, the stability implies Pareto efficiency even for one-to-one matching (if the preference relation is strict). Roth (1982) showed that there exists no mechanism that is both stable and strategy-proof. Indeed, strategy-proofness is not only incompatible with stability but also Pareto efficiency and individual rationality. Alcalde and Barberá (1994) obtained that there exists no mechanism that is Pareto efficient, individually rational, and strategy-proof. For relatively weak expectation, there are some positive results. Roth (1982) investigated the marriage problem and obtained that the men (resp. women)-optimal matching is weakly Pareto efficient and strategy-proof for men (resp. women).[1]

For a unified model, Hatfield and Milgrom (2005) studied the incentive property for matching with contracts. They obtained that the doctor-optimal matching is strategy-proof for doctors under very weak preference assumption (hospitals' preferences satisfy substitutes and the law of aggregate demand). Under the same framework as above, Hatfield and Kojima (2009) showed that the doctor-optimal matching is group strategy-proof and weakly Pareto efficient for doctors. For the college admissions problem, Roth (1985) showed that, when colleges have responsive preferences, the colleges-optimal matching may be neither weakly Pareto efficient nor strategy-proof for colleges, while the students-optimal matching is weakly Pareto efficient and strategy-proof for students.

Baïou and Balinski (2000) introduced the notion of max-min preferences and presented the reduction algorithm by graphic approach. They studied the many-to-many matching problem and asserted that, if every agent has max-min preference, the reduction algorithm produces a stable assignment that satisfies weak Pareto efficiency, strategy-proofness and monotonicity for agents on one side of the market. However, even for many-to-one matchings, Hatfield et al. (2014) constructed an example to show that the condition of max-min preferences is not sufficient for the existence of a stable mechanism that satisfies weak Pareto efficiency, strategy-

[1] Dubins and Freedman (1981) showed that, under the men (resp. women)-proposing deferred acceptance algorithm, there exists no coalition of men (resp. women) that can simultaneously improve the assignment of all its members if those outside the coalition state their true preferences. This result implies the property of strategy-proofness.

proofness or monotonicity. Thus, the above-mentioned result of Baïou and Balinski (2000) is incomplete. Therefore, it is still an important open question on the efficiency, incentive, and monotonicity for agents with multi unit demand.

In this paper we consider efficiency, incentive, and monotonicity of many-to-one matching problems. We propose an additional condition which was originally introduced by GS. In fact, Gale and Shapley (1962) made the following assumption for the college admissions model: *"There are enough applicants to assign each college precisely as many as its quota students."* Although this assumption did not play a crucial role in the analysis of GS, it seems to be a natural and reasonable condition. We call it the *quota-saturability* condition. When each college is assigned to as many as its quota acceptable applicants, the colleges side is saturated.

We will show that, if colleges have max-min preferences and there are enough acceptable students, then the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient and strategy-proof for colleges. We also deal with the issue of monotonicity for the college admissions problem and obtain that the colleges-proposing deferred acceptance algorithm is not only weakly Pareto efficient and strategy-proof, but also monotone for colleges if the quota-saturability condition and the W-max-min preference criterion are satisfied.

Specifically, the contribution of this paper is threefold: Firstly, as mentioned above, we propose the quota-saturability condition and show that the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient and strategy-proof for colleges if the max-min preference criterion and the quota-saturability condition are satisfied. If the quota-saturability condition is not satisfied, Hatfield et al. (2014) showed that there exists no stable mechanism that satisfies weak Pareto efficiency or strategy-proofness for colleges under the max-min preference. On the other hand, if the max-min preference criterion does not hold, Roth (1985) showed that the colleges-optimal matching may be neither weakly Pareto efficient nor strategy-proof for colleges even under the assumptions of quota-saturability condition and responsive preferences.

It may be remarked that while max-min criterion and responsiveness condition both implies substitutability introduced by Kelso and Crawford (1982) and commonly assumed in matching literature, it can be easily checked that max-min criterion is not implied by nor implies responsiveness. It is the max-min preferences that have both properties of substitutability and complementary enable us to obtain the desired results possible. We also note that one likely cannot expect to obtain the desired efficiency and incentive properties for colleges under a preference restriction which is weaker than max-min criterion.

Secondly, we extend the "blocking lemma" for marriage problem (one-to-one matching) obtained by Gale and Sotomayor (1985) to the college admissions problem (many-to-one matching). For many-to-one matching, the blocking lemma for agents with unit demand holds under responsive preference profile. For a weak preference restriction, Martínez et al. (2010) show that the corresponding result holds under substitutable and quota-separable preference. They also note that the blocking lemma for agents who have multi-unit demand does not hold even under responsive preference. We will show that the blocking lemma for colleges (which with multi-unit demand) holds under max-min preference restriction.

Thirdly, we introduce the notion of W-max-min preference and prove that the colleges-proposing deferred acceptance algorithm is not only weakly Pareto efficient and strategy-proof, but also monotone for colleges if the W-max-min criterion and the quota-saturability condition are satisfied. For the college admissions problem, Balinski and Sönmez (1999) proved that the students-proposing deferred acceptance algorithm is monotone for students. Essentially, the result obtained by Balinski and Sönmez is about agents who have unit demand. We extend their result to the case of agents with multiple demand.

The remainder of this paper is organized as follows. The next section presents some preliminaries on the formal model. In Section 3 we study the efficiency and incentive properties for colleges. We deal with the issue of monotonicity for colleges in Section 4. In Section 5 we note by an example that the efficiency and incentive properties fail even if we slightly relax preference requirement imposed by max-min criterion. We conclude in Section 6. All proofs are provided in the Appendix.

2 The Model

Our model follows the framework of Roth and Sotomayor (1989, 1990). There are two finite and disjoint sets, $C = \{c_1, \cdots, c_{|C|}\}$ and $S = \{s_1, \cdots, s_{|S|}\}$, of colleges and students, respectively, where the notation $|A|$ denotes the number of elements of the set A. Each student has preferences $\succ_s$ over the set of colleges C and the outside option — the null college $\varnothing$, and each college has preferences $\succ_c$ over the set of students S and the prospect of having its seat unfilled, also denoted by $\varnothing$. Student s is *acceptable* to college c iff $s \succ_c \varnothing$, and college c is acceptable to student s iff $c \succ_s \varnothing$. We assume these preferences are complete, transitive and strict, so they may be represented by order lists. For example, $\succ_c$: $s_2, s_1, \varnothing, s_3, \cdots$ denotes that college c prefers to enroll s_2 rather than s_1, that it prefers to enroll either one of them rather than leave a position unfilled, and that all other students are unacceptable, in the sense that it would be preferable to leave a position unfilled rather than fill it with, say, student s_3. Similarly, for the preferences of a student, $\succ_s$: $c_1, c_3, c_2, \varnothing, \cdots$ indicates that the only positions the student would accept are those offered by c_1, c_3 and c_2, in that order. We will write $c_i \succ_s c_j$ to indicate that student s prefers c_i to c_j, and $c_i \succeq_s c_j$ to indicate that either $c_i \succ_s c_j$ or $c_i = c_j$. Similarly, we can give corresponding notions on preferences of colleges. Each college c has a *quota* q_c which is the maximum number of students for which it has places. Let $q \equiv (q_c)_{c \in C}$ denote the vector of college quotas.

We denote the preferences profile of all colleges by $\succ_C \equiv (\succ_c)_{c \in C}$, and the preferences profile of all students by $\succ_S \equiv (\succ_s)_{s \in S}$. The preferences profile of all players is denoted by $\succ \equiv (\succ_C, \succ_S)$. We denote a college admissions problem by a four-tuple $(C, S; q; \succ)$. We also write it as $(C, S; q; \succ_C, \succ_S)$.

For colleges, since they may be matched with different sets of students, we also need to consider their preferences over groups of students.[2] We assume these preferences are transitive, but the completeness is not required. Particularly, throughout this paper we assume the following properties hold (see, e.g., Konishi and Ünver, 2006a):

(i) *Weak monotonicity in population:* For every college c, any subset of acceptable students $G \in 2^S$ with $|G| < q_c$ and any student $s \notin G$, c prefers $G \cup \{s\}$ to G if and only if s is acceptable to c.[3]

(ii) For any $G \in 2^S$, if $|G| > q_c$ or G contains any unacceptable students, then c prefers having all its positions unfilled to admitting G.

We will specify the colleges-proposing deferred acceptance algorithm below. These two assumptions provide rationality for the procedure of that algorithm: colleges always want to match with as many as possible (within their quotas) acceptable students and never propose to any unacceptable student.

2.1 Matching between Colleges and Students

Definition 2.1 A *matching* is a correspondence $\mu : C \cup S \rightrightarrows 2^{C \cup S \cup \{\varnothing\}}$ such that

(1) $\mu(c) \subseteq S \cup \{\varnothing\}$ and $|\mu(c)| \leq q_c$ for all $c \in C$,

(2) $\mu(s) \subseteq C \cup \{\varnothing\}$ and $|\mu(s)| \leq 1$ for all $s \in S$,

(3) $s \in \mu(c)$ if and only if $\mu(s) = \{c\}$ for all $c \in C$ and $s \in S$.[4]

[2] Without confusion, we abuse notations: For any $i \in S$ (resp. $i \in C$), $j, k \in C \cup \{\varnothing\}$ (resp. $j, k \in S \cup \{\varnothing\}$), the preference relation $\{j\} \succ_i \{k\}$ is also denoted as $j \succ_i k$.

[3] College's preferences satisfy strong monotonicity in population iff $\forall c \in C, \forall G, G' \in 2^S$, $|G'| < |G| \leq q_c$ implies $G \succ_c G'$ (see, e.g., Konishi and Ünver, 2006b). Obviously, strong monotonicity implies weak monotonicity.

[4] For $i \in S$, $j \in C \cup \{\varnothing\}$, we also write the notation $\mu(i) = \{j\}$ as $\mu(i) = j$ if it is not confused.

Note that, for all $i \in C \cup S$, we stipulate $|\mu(i)| = 0$ if $\mu(i) = \{\varnothing\}$. We will use the notation $\min(\mu(c))$ to denote the least preferred student of c in the set $\mu(c)$.

A matching μ is *blocked by an individual* $i \in C \cup S$ iff there exists some player $j \in \mu(i)$ such that $\varnothing \succ_i j$. A matching is *individually rational* iff it is not blocked by any individual. A matching μ is *blocked by a pair* $(c, s) \in C \times S$ iff

(1) c is acceptable to s and s is acceptable to c,

(2) $|\mu(c)| < q_c$ or $s \succ_c s'$ for some $s' \in \mu(c)$, and

(3) $c \succ_s \mu(s)$.

Definition 2.2 A matching μ is *stable* iff it is not blocked by any individual or any college-student pair.

2.2 Deferred Acceptance Algorithm

The *deferred acceptance algorithm* was first proposed by GS to find a stable assignment for the marriage problem (one-to-one matching) and college admissions problem (many-to-one matching). Specifically, the *Colleges-Proposing Deferred Acceptance Algorithm* proceeds as follows:

Step 1. (a). Each college c proposes to its top q_c acceptable students (if c has fewer acceptable choices than q_c, then it proposes to all its acceptable choices).

(b). Each student s then places the best college among those proposed to her on her waiting list, and rejects the rest.

In general, at

Step k. (c). Any college c who was rejected at step $(k-1)$ by any student proposes to its most-preferred q_c acceptable students who have not yet rejected it (if there are fewer than q_c remaining acceptable students, then it proposes to all).

(d). Each student s selects the best one from among the new colleges and that on her waiting list, puts it on her new waiting list, and rejects the rest.

Since no college proposes twice to the same student, this algorithm always terminates in a finite number of steps. The algorithm terminates when there are no more rejections. Each student is matched with the college on his waiting list in the last step.

The colleges-proposing deferred acceptance algorithm yields an assignment denoted by μ_C. GS showed that μ_C is a stable matching and it is optimal for every college. That is, for any $c \in C$, there exists no other stable matching μ such that $\mu \succ_c \mu_C$.

3 Pareto Efficiency and Strategy-proofness for Colleges

In this section we study the Pareto efficiency and strategy-proofness for colleges. Roth (1985) constructed two examples to show that, if the colleges have responsive preferences over groups of students,[5] then the colleges-proposing deferred acceptance algorithm may be neither weakly Pareto efficient nor strategy-proof for colleges. Baïou and Balinski (2000) introduced the concept of max-min preference[6] and investigated efficiency and incentives properties for many-to-many matching. They provided valuable ideas, although some of their conclusions are not completely correct. More recently, Hatfield et al. (2014) obtained some negative results. Specifically, in the context of college admissions problem, results of Hatfield et al. (2014) show that there exists no stable mechanism that satisfies weak Pareto efficiency and strategy-proofness for colleges even though colleges have max-min preferences over groups of students. We first state the definition of max-min preference as follows.

Definition 3.1 (Max-Min Criterion) The preference relation of $c \in C$ is said to satisfy the *max-min criterion* iff for any two sets of acceptable students $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$,

(i) The strict preference relation $\succ_c$ over groups of students is defined as: $G_1 \succ_c G_2$ if and only if G_2 is a proper subset of G_1, or, $|G_1| \geq |G_2|$ and $\min(G_1) \succ_c \min(G_2)$ (i.e., c strictly prefers the least preferred student in G_1 to the least preferred student in G_2), where $\min(G_i)$ denotes the least preferred student of c in G_i;

(ii) The weak preference relation $\succeq_c$ over groups of students is defined as: $G_1 \succeq_c G_2$ if and only if $G_1 \succ_c G_2$ or $G_1 = G_2$.[7]

It is interesting to point out that max-min preferences over groups of students can display either substitutability or complementarity effect, depending on whether G_2 is a proper subset of G_1. As we know, for many-to-one or many-to-many matching problem, substitutability is often adopted.[8] While responsibility clearly implies substitutability, the max-min criterion also

[5]The preferences of college c are responsive iff whenever $G \in 2^S$ and $i, j \in S$ such that $|G| < q_c$ and $i, j \notin G$, then $i \succ_c j$ implies $G \cup \{i\} \succ_c G \cup \{j\}$.

[6]Similar conditions were studied by Echenique and Oviedo (2006), Kojima (2007) and Sotomayor (2004).

[7]Without confusion, we abuse notations: For college c, we denote its preferences over groups of students and over individual students by the same notations $\succ_c$ and $\succeq_c$.

[8]In the language of the college admissions model, substitutability of college c's preferences requires: "if admitting s is optimal when certain students are available, admitting s must still be optimal when a subset of students are available." Formally, an agent c's preference relation $\succ_c$ satisfies substitutability if, for any sets S and S' with $S \subseteq S'$, $s \in Ch(S' \cup \{s\}, \succ_c)$ implies $s \in Ch(S \cup \{s\}, \succ_c)$, where $Ch(S \cup \{s\}, \succ_c)$ denotes agent c's most-preferred

implies substitutability. To see that max-min criterion is stronger than substitutability, we assume that college c's preference relation $\succ_c$ satisfies max-min criterion. Let S and S' be sets of students, with $S \subseteq S'$. Suppose $s \in Ch(S' \cup \{s\}, \succ_c)$. We shall prove $s \in Ch(S \cup \{s\}, \succ_c)$. Consider the following two cases:

Case (I): $|Ch(S \cup \{s\}, \succ_c)| < q_c$. The condition $s \in Ch(S' \cup \{s\}, \succ_c)$ implies that s is acceptable to c. Then, by the definition of max-min criterion, we have $s \in Ch(S \cup \{s\}, \succ_c)$.

Case (II): $|Ch(S \cup \{s\}, \succ_c)| = q_c$. We have $|Ch(S' \cup \{s\}, \succ_c)| = q_c$ according to max-min criterion. Since $S \cup \{s\} \subseteq S' \cup \{s\}$, we have $\min(Ch(S' \cup \{s\}, \succ_c)) \succeq_c \min(Ch(S \cup \{s\}, \succ_c))$. The condition $s \in Ch(S' \cup \{s\}, \succ_c)$ implies $s \succeq_c \min(Ch(S' \cup \{s\}, \succ_c))$. Thus we infer $s \succeq_c \min(Ch(S \cup \{s\}, \succ_c))$. By max-min criterion, $s \in Ch(S \cup \{s\}, \succ_c)$.

Thus, when G_2 is a proper subset of G_1, the max-min criterion displays substitutability effect. When G_2 and G_1 have no belonging relationship, the max-min criterion then will display complementarity effect. Indeed, when $|G_2| \leq |G_1| \leq q_c$, the value $u_c(G_1) = \min_{s \in G_1}\{u_c(s)\}$ and $u_c(G_2) = \min_{s \in G_2}\{u_c(s)\}$.[9] Then $G_1 \succ_c G_2$ if and only if $\min(G_1) \succ_c \min(G_2)$, i.e., a college's welfare over a group of students is increased only if the least preferred student's overall quality in the group becomes higher, which means colleges' preferences over a group of students are complementary.

However, while max-min criterion and responsiveness both imply substitutability, there is no implication relationship between max-min preferences and responsive preferences. Indeed, the responsiveness does not imply the max-min criterion. For instance, suppose that college c's preferences over individual students are given by $s_1 \succ_c s_2 \succ_c s_3$, $q_c = 2$ and that the preference list over groups of two candidates is given by $\{s_1, s_2\} \succ_c \{s_1, s_3\} \succ_c \{s_2, s_3\}$. Then it is clear that the responsiveness is satisfied, but $\{s_1, s_3\} \succ_c \{s_2, s_3\}$ does not imply $\min\{s_1, s_3\} \succ_c \min\{s_2, s_3\}$, which violates the max-min criterion.

The max-min criterion does not imply the responsiveness either. For instance, now suppose $s_1 \succ_c s_2 \succ_c s_3$, $q_c = 2$, and that the preferences over groups of two candidates are given by $\{s_1, s_2\} \succ_c \{s_1, s_3\}$, $\{s_1, s_2\} \succ_c \{s_2, s_3\}$ and, college c cannot compare $\{s_1, s_3\}$ and $\{s_2, s_3\}$.

subset of $S \cup \{c\}$ according to c's preference relation $\succ_c$. Kelso and Crawford (1982) introduce the substitutability condition on hospital preferences in a matching model with wages. Roth (1984) adapts the deferred acceptance algorithm to the many-to-many matching model with substitutable preference and obtains the corresponding optimal stable matching. Martínez et al. (2004), Echenique and Oviedo (2006) and Klaus and Walzl (2009) study the stability problem of many-to-many matching under substitutable preference. Hatfield and Kojima (2008), Hatfield and Kojima (2009), Hatfield and Kojima (2010) and Hatfield and Milgrom (2005) study the matching with contracts under substitutable hospitals' preferences.

[9]Sotomayor (2004) gave a similar illustration.

Then it is clear that the max-min criterion is satisfied, but it violates the responsiveness (otherwise one would have $\{s_1, s_3\} \succ_c \{s_2, s_3\}$, contradicting the hypothesis that college c cannot compare $\{s_1, s_3\}$ and $\{s_2, s_3\}$).

A matching μ is said to be *weakly colleges-efficient* iff there exists no individually rational matching μ' such that $\mu'(c) \succ_c \mu(c)$ for all $c \in C$. A weakly students-efficient matching is defined analogously.

For any two matchings μ and μ', the notions $\mu \succ_c \mu'$ means $\mu(c) \succ_c \mu'(c)$, $\mu \succeq_c \mu'$ means $\mu(c) \succeq_c \mu'(c)$, $\mu \succeq_C \mu'$ means $\mu \succeq_c \mu'$ for every $c \in C$, and $\mu \succ_C \mu'$ means $\mu \succeq_C \mu'$ and $\mu \succ_c \mu'$ for some college c.

Given the colleges C and students S, a *mechanism* φ is a function from any stated preferences profile $\succ$ and quota-vector q to a matching. A mechanism is *stable* iff the outcome of that mechanism, denoted by $\varphi(C, S; q; \succ)$, is a stable matching for any reported $\succ$ and q. A mechanism is *weakly colleges (resp. students)-efficient* iff it always selects a weakly colleges (resp. students)-efficient matching for every preference profile. A mechanism is *colleges (resp. students)-strategy-proof* iff at every preference profile, no college (resp. student) can receive a strictly better assignment by misrepresenting its preferences.

Now we borrow the example of Abdulkadiroğlu and Sönmez (2010) to show that, under any reasonable preference assumption, there exists no stable matching mechanism satisfying weak colleges-efficiency or colleges-strategy-proofness.

Example 1 This example is essentially the same as that used by Hatfield et al. (2014): there are two colleges c_1, c_2 with $q_{c_1} = 2, q_{c_2} = 1$, and two students s_1, s_2. The preferences are as follows:

$\succ_{s_1}$: c_1, c_2, $\varnothing$ $\quad$ $\succ_{c_1}$: $\{s_1, s_2\}$, $\{s_2\}$, $\{s_1\}$, $\{\varnothing\}$

$\succ_{s_2}$: c_2, c_1, $\varnothing$ $\quad$ $\succ_{c_2}$: $\{s_1\}$, $\{s_2\}$, $\{\varnothing\}$

The only stable matching for this problem is:

$$\mu = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \{s_2\} \end{pmatrix}.$$

For another matching

$$\mu' = \begin{pmatrix} c_1 & c_2 \\ \{s_2\} & \{s_1\} \end{pmatrix},$$

we can see that both c_1 and c_2 prefer μ' to μ. This means that there exists no stable mechanism being of weakly colleges-efficient. In addition, if college c_1 reports that s_2 is its only acceptable student, other agents's preference relations keep unchanged, then the only stable matching is

μ'. Thus under any stable matching mechanism college c_1 can become strictly better off by misreporting its preference. There exists no stable mechanism being of colleges-strategy-proof.

In Example 1, we can see that college c_1's preference relation is commonly reasonable, but the number of students is less than the aggregate quota of colleges. This may potentially cause the failure of the existence of stable mechanism satisfying colleges-efficiency or colleges-strategy-proofness. To avoid this possible environment, we propose the following condition which says that in the market there are enough acceptable students such that each college can admit as many as its quota acceptable students if they want.

Definition 3.2 (Quota-Saturability) For the college admissions problem, we say *the quota-saturability condition* holds iff there are enough acceptable students such that each college c can be assigned q_c acceptable students under the college-optimal stable matching.

We will show that the max-min criterion, together with quota-saturability condition, ensures that the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient and strategy-proof for colleges. We first present the following lemma.

Lemma 1 *Suppose that the quota-saturability condition is satisfied. Let μ be a stable assignment. If for the least preferred student $\min(\mu(c))$ of every $c \in C$, there exists another college c' which prefers $\min(\mu(c))$ to one of its mates $\mu(c')$, then there exists a stable assignment μ^* with $\mu^* \succ_C \mu$.*

This lemma actually says that, for a stable assignment such that each college is matched with as many as its quota students, if each least preferred student has another college which want to be matched with her, then there exists another stable assignment such that no college becomes worse off and at least one college is strictly better off. Intuitively, Since each least preferred student has another college which want to be matched with her, it is expected to find a cycle consisting of the same number of least preferred students and colleges such that each least preferred student is followed by a college which wishes to be matched with her. Then let every least preferred student enroll into the college which followed her. By this way, one can get the desirable matching.

We note that Lemma 1 was first claimed by Baïou and Balinski (2000, henceforth BB) without assuming the quota-saturability condition. However, the conclusion of Lemma 1 fails to fulfil if the quota-saturability condition is not satisfied. Specifically, we consider the setting as given in Example 1.

Example 1 (Continued) Consider the sets of colleges and students, and their preferences and quotas are as given in Example 1. The only stable matching for this problem is:

$$\mu = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \{s_2\} \end{pmatrix}.$$

For c_1, $\min(\mu(c_1)) = s_1$ is not matched to c_2 and c_2 prefers $\min(\mu(c_1)) = s_1$ to $s_2 \in \mu(c_2)$. For c_2, $\min(\mu(c_2)) = s_2$ is not matched to c_1 and c_1 prefers $\min(\mu(c_2)) = s_2$ to $s_1 \in \mu(c_1)$. There exists no other stable assignment. This indicates that we cannot obtain the conclusion of Lemma 1 if the quota-saturability condition is not satisfied.

With Lemma 1, we have the following theorem on weak Pareto efficiency.

Theorem 1 *Suppose the quota-saturability condition holds and the preference of every college satisfies the max-min criterion. Then, the colleges-proposing deferred acceptance algorithm is weakly colleges-efficient.*

Theorem 1 shows that the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient for colleges under the max-min preference and quota-saturability condition. If the colleges have responsive preferences over groups of students, the corresponding result does not hold even if the quota-saturability condition is satisfied (see Proposition 1 of Roth, 1985). We know that the max-min criterion is unrelated to the responsiveness. Thus, Theorem 1 lies in obtaining the desired result under a preference condition that is not stronger or weaker than responsive preference.

To obtain the strategy-proofness property, we give the following “blocking lemma”, which extends the “blocking lemma” for marriage problem obtained by Gale and Sotomayor (1985).

Lemma 2 *Suppose the quota-saturability condition holds and the preference of every college satisfies the max-min criterion. Let μ be any individually rational matching and C' be all colleges who prefer μ to μ_C. If C' is nonempty, then there is some $c' \in C'$ and a pair $(c, s) \in C \backslash C' \times \mu(c')$ that blocks μ.*

Note that both quota-saturability condition and max-min preference are crucial for the conclusion of Lemma 2. We illustrate this by the following two examples.

Example 1 (Continued) Consider the sets of colleges and students, and their preferences and quotas are as given in Example 1. The matching produced by the colleges-proposing deferred acceptance algorithm is:

$$\mu_C = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \{s_2\} \end{pmatrix}.$$

An individually rational matching μ is:

$$\mu = \left(\begin{array}{cc} c_1 & c_2 \\ \{s_2\} & \{s_1\} \end{array} \right).$$

One can see that the max-min criterion is satisfied, but the quota-saturability condition does not hold. It is easy to check that both c_1 and c_2 prefer μ to μ_C. Thus $C' = \{c_1, c_2\}$ is nonempty and $C' = C$. The conclusion of Lemma 2 does not hold. Hence, this example shows that, if the quota-saturability condition does not hold, the max-min criterion alone is not sufficient for the conclusion in Lemma 2. Thus, the quota-saturability condition cannot be dispensed with.

Example 2 There are three colleges c_1, c_2, c_3 with $q_{c_1} = 2, q_{c_2} = 1, q_{c_3} = 1$, and four students s_1, s_2, s_3, s_4. The preferences are as follows:

$\succ_{c_1}$: $\{s_1, s_2\}$, $\{s_1, s_3\}$, $\{s_1, s_4\}$, $\{s_2, s_3\}$, $\{s_2, s_4\}$, $\{s_3, s_4\}$, $\{s_1\}$, $\{s_2\}$, $\{s_3\}$, $\{s_4\}$, $\{\varnothing\}$

$\succ_{c_2}$: $\{s_3\}$, $\{s_1\}$, $\{s_2\}$, $\{s_4\}$, $\{\varnothing\}$ $\quad$ $\succ_{c_3}$: $\{s_1\}$, $\{s_2\}$, $\{s_3\}$, $\{s_4\}$, $\{\varnothing\}$

$\succ_{s_1}$: c_2, c_1, c_3, $\varnothing$ $\quad$ $\succ_{s_2}$: c_3, c_1, c_2, $\varnothing$

$\succ_{s_3}$: c_1, c_2, c_3, $\varnothing$ $\quad$ $\succ_{s_4}$: c_1, c_3, c_2, $\varnothing$

The matching produced by the colleges-proposing deferred acceptance algorithm is:

$$\mu_C = \left(\begin{array}{ccc} c_1 & c_2 & c_3 \\ \{s_3, s_4\} & \{s_1\} & \{s_2\} \end{array} \right).$$

An individually rational matching μ is as follows:

$$\mu = \left(\begin{array}{ccc} c_1 & c_2 & c_3 \\ \{s_2, s_4\} & \{s_3\} & \{s_1\} \end{array} \right).$$

One can see that the quota-saturability condition holds, but college c_1's preference does not satisfy the max-min criterion. It is easy to check that all of the three colleges prefer μ to μ_C, so $C' = \{c_1, c_2, c_3\}$ is nonempty and $C' = C$. Therefore, the conclusion of Lemma 2 does not hold. This example shows that, if the max-min criterion is not satisfied, we cannot achieve the conclusion of Lemma 2 even if the quota-saturability condition holds.

With the help of the above "blocking lemma", we have the following theorem on strategy-proofness.

Theorem 2 *Suppose that the quota-saturability condition holds and the preference of every college satisfies the max-min criterion. Then the colleges-proposing acceptance algorithm is colleges-strategy-proof.*

Combining Theorems 1 and 2 together, we know that the algorithm is weakly Pareto efficient and strategy-proof for colleges under the max-min preference and quota-saturability condition. If colleges have responsive preferences over groups of students, neither efficiency nor strategy-proofness holds even if the quota-saturability condition is satisfied (see Propositions 1 and 2 of Roth, 1985).

4 Monotonicity for Colleges

In this section we consider the issue of monotonicity for the college admissions problem. We want to find the conditions under which the colleges-proposing deferred acceptance algorithm is not only weakly Pareto efficient and strategy-proof, but also monotone for colleges. The monotonicity means that an agent is weakly better off if she becomes more preferred by players on the opposite side. For completeness, we first introduce the following concept originally proposed by Balinski and Sönmez (1999):

Definition 4.1 A preference relation $\succ'_s$ is *an improvement for a college* c *over* $\succ_s$ iff

(1) $c \succ_s c'$ implies $c \succ'_s c'$ for all $c' \in C \cup \{\varnothing\}$ and

(2) $c_1 \succ'_s c_2$ if and only if $c_1 \succ_s c_2$ for all $c_1, c_2 \in (C \cup \{\varnothing\}) \backslash \{c\}$.

Loosely speaking, an improved preference relation means c becomes more preferred by s. A student preference profile $\succ'_S$ is an improvement for c over $\succ_S$ iff for every s, $\succ'_s$ is an improvement for c over $\succ_s$. That is, c becomes more preferred by all students.

Definition 4.2 A mechanism φ is *colleges-monotone* iff, whenever for any preference profile $\succ$, any $c \in C$, and any students preference profiles $\succ_S$ and $\succ'_S$, $\succ'_S$ is an improvement for c over $\succ_S$, then c weakly prefers $\varphi(\succ'_S, \succ_C)$ to $\varphi(\succ_S, \succ_C)$.

That is, the outcome of a mechanism is weakly better off for a college if that college becomes more preferred by all students. A students-monotone mechanism is also defined in the same way.

Balinski and Sönmez (1999) proposed the concept of "respecting improvement". Baïou and Balinski (2000) used the notion of monotonicity to express the same meaning. For the college admissions problem, Balinski and Sönmez (1999) proved that the students-proposing deferred acceptance algorithm is monotone for students. That is, under the students-proposing deferred acceptance algorithm, a student will enroll in a weakly better college if she becomes more preferred by all colleges. Baïou and Balinski (2000) studied the many-to-many matching under the assumption of max-min preference. They asserted that the reduction algorithm is also monotone for agents on one of the two matching sides. However, Hatfield et al. (2014) showed that the condition of max-min preference is not sufficient for the existence of a stable mechanism that satisfies the monotonicity property. Hence the above result of Baïou and Balinski (2000) is also incomplete. In fact, even though the preferences of colleges are max-min and the quota-saturability condition is satisfied, there may not exist a stable colleges-monotone mechanism. To see this, consider the following example:

Example 3 There are three colleges c_1, c_2, c_3 with $q_{c_1} = 2, q_{c_2} = 1, q_{c_3} = 1$, and four students s_1, s_2, s_3, s_4. The preferences are as follows:

$\succ_{s_1}$: $c_2,\ c_1,\ c_3,\ \varnothing$ $\qquad$ $\succ_{c_1}$: $\{s_1, s_2\}, \cdots,\ \{s_3\},\ \{s_4\},\ \{\varnothing\}$

$\succ_{s_2}$: $c_3,\ c_1,\ c_2,\ \varnothing$ $\qquad$ $\succ_{c_2}$: $\{s_4\},\ \{s_1\}, \cdots$

$\succ_{s_3}$: $c_1,\ c_2,\ c_3,\ \varnothing$ $\qquad$ $\succ_{c_3}$: $\{s_2\},\ \{s_4\}, \cdots$

$\succ_{s_4}$: $c_3,\ c_2,\ c_1,\ \varnothing$

Under $(C, S; q; \succ_C, \succ_S)$, the only stable matching for this problem is:[10]

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_3\} & \{s_4\} & \{s_2\} \end{pmatrix}.$$

Now suppose $\succ'_{s_2}$: $c_1,\ c_3,\ c_2,\ \varnothing$, and $\succ'_{s_i} = \succ_{s_i}$ for $i = 1, 3, 4$. Then $\succ'_S$ is an improvement for c_1 over $\succ_S$. Under $(C, S; q; \succ_C, \succ'_S)$, the only stable matching for this problem is:[11]

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_2, s_3\} & \{s_1\} & \{s_4\} \end{pmatrix}.$$

In this example, the quota-saturability condition is satisfied, but under max-min preference (resp. responsive preference), we cannot obtain $\mu'(c_1) \succeq_{c_1} \mu(c_1)$,[12] although c_1 becomes (weakly) more preferred by all students. Thus, there does not exist any stable college-monotone mechanism.

A natural question is under what conditions the monotonicity property can also be guaranteed. To this end, we propose the following preference criterion, which, together with the quota-saturability condition, ensures that the colleges-proposing deferred acceptance algorithm is not only weakly Pareto efficient and strategy-proof, but also monotone for colleges.

[10]The uniqueness of the stable matching under $(C, S; q; \succ_C, \succ_S)$ is shown in the Appendix.

[11]The uniqueness of the stable matching under $(C, S; q; \succ_C, \succ'_S)$ is shown in the Appendix.

[12]Under responsive preference, we have $\mu(c_1) = \{s_1, s_3\} \succ_c \{s_2, s_3\} = \mu'(c_1)$. Under max-min preference, colleges c_1 cannot compare $\mu(c_1) = \{s_1, s_3\}$ and $\mu'(c_1) = \{s_2, s_3\}$.

Definition 4.3 (W-Max-Min Criterion) The preference relation of $c \in C$ is said to satisfy the *W-max-min criterion* iff the following conditions are met: for any two sets of acceptable students $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$,

(i) The weak preference relation $\succeq_c$ over groups of students is defined as: If G_2 is a subset of G_1 or, $|G_1| \geq |G_2|$ and $\min(G_1) \succeq_c \min(G_2)$ (i.e., c weakly prefers[13] the least preferred student in G_1 to the least preferred student in G_2), then $G_1 \succeq_c G_2$;

(ii) If $G_1 \succeq_c G_2$ and $G_2 \succeq_c G_1$, then G_1 and G_2 are indifferent for c, denoted by $G_1 \sim_c G_2$. If $G_1 \succeq_c G_2$ and $G_2 \not\succeq_c G_1$, then c strictly prefers G_1 to G_2, denoted by $G_1 \succ_c G_2$.

Here the notion of "W-max-min criterion" refers to the max-min criterion that is defined in terms of weak preferences over group of students to distinguish from the conventional notion of max-min criterion that is defined in terms of strict preferences over group of students. Note that, for any $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$, $G_1 \succ_c G_2$ under the W-max-min criterion is equivalent to $G_1 \succ_c G_2$ under the max-min criterion. The difference between the max-min criterion and the W-max-min criterion lies in: (1) Under the max-min criterion, $G_1 \succeq_c G_2$ and $G_2 \succeq_c G_1$ imply $G_1 = G_2$, while under W-max-min criterion $G_1 \succeq_c G_2$ and $G_2 \succeq_c G_1$ imply $G_1 = G_2$ or, $|G_1| = |G_2|$ and $\min(G_1) = \min(G_2)$. (2) For any $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$, under the max-min criterion, c may not compare G_1 and G_2. It may be remarked that, together with the transitivity of preferences, the preference relation between G_1 and G_2 is complete under the W-max-min criterion, that is, either $G_1 \succeq_c G_2$ or $G_2 \succeq_c G_1$.

We note that there is cohesive coincidence between the W-max-min criterion and the deferred acceptance algorithm. We illustrate this point by a simple example. Assume there five students $s_1, \cdots, s_5$, college c's preference relation is given by $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_4 \succ_c s_5$ and the quota $q_c = 2$. Then the assignment outcomes produced by the deferred acceptance algorithm and the W-max-min preference relations can be compared as in Table 1.

We then have the following result.

Theorem 3 *Suppose that the W-max-min criterion and the quota-saturability condition are satisfied. Then the colleges-proposing deferred acceptance algorithm is not only weakly colleges-efficient and colleges-strategy-proof, but also monotone for colleges.*

Theorem 3 indicates that, if the condition of max-min preference in Theorems 1-2 is replaced by W-max-min criterion condition, then the colleges-proposing deferred acceptance algorithm

[13] As c's preference over individual students is strict, "weakly prefers" means that $\min(G_1) \succ_c \min(G_2)$ or $\min(G_1) = \min(G_2)$.

Table 1: Comparison between DA-algorithm and W-max-min preference relation

The satisfactory grade of c	The last student c proposes to in the DA algorithm	c's partners	W-max-min preference relation
1	s_2	$\{s_1, s_2\}$	$\{s_1, s_2\}$ is the first-best choice of c
2	s_3	$\{s_1, s_3\}$ or $\{s_2, s_3\}$	$\{s_1, s_3\} \sim_c \{s_2, s_3\}$
3	s_4	$\{s_1, s_4\}$ or $\{s_2, s_4\}$ or $\{s_3, s_4\}$	$\{s_1, s_4\} \sim_c \{s_2, s_4\} \sim_c \{s_3, s_4\}$
4	s_5	$\{s_1, s_5\}$ or $\{s_2, s_5\}$ or $\{s_3, s_5\}$ or $\{s_4, s_5\}$	$\{s_1, s_5\} \sim_c \{s_2, s_5\} \sim_c \{s_3, s_5\} \sim_c \{s_4, s_5\}$

guarantees not only weak Pareto efficiency and strategy-proofness, but also monotonicity for colleges.

5 Discussion

As we discussed in Section 3, max-min preferences contain both substitutability and complementarity effects, which enable us to get the desirable properties such as weak Pareto efficiency, strategy-proofness, and monotonicity. An question is then whether the max-min criterion can be weakened. The answer is unlikely. Indeed, we note that both the weak Pareto efficiency and strategy-proofness properties fail even if we slightly weaken the requirement of max-min preference. Specifically, we consider the following example.

Example 4 There are three colleges c_1, c_2, c_3 with $q_{c_1} = 2, q_{c_2} = 1, q_{c_3} = 1$, and five students s_1, s_2, s_3, s_4, s_5. The preferences are as follows:

$\succ_{s_1}$: $c_2,\ c_1,\ c_3,\ \varnothing$ $\qquad \succ_{s_5}$: $c_1,\ c_3,\ c_2,\ \varnothing$

$\succ_{s_2}$: $c_3,\ c_1,\ c_2,\ \varnothing$ $\qquad \succ_{c_1}$: $s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ \varnothing$

$\succ_{s_3}$: $c_1,\ c_2,\ c_3,\ \varnothing$ $\qquad \succ_{c_2}$: $s_3,\ s_1,\ \cdots$

$\succ_{s_4}$: $c_1,\ c_3,\ c_2,\ \varnothing$ $\qquad \succ_{c_3}$: $s_4,\ s_2,\ \cdots$

Under $(C, S; q; \succ_C, \succ_S)$, the college-optimal matching is

$$\mu_C = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_3, s_4, s_5\} & \{s_1\} & \{s_2\} \end{pmatrix}.$$

For college c_1's preference over groups of students, we slightly weaken the requirement of max-min criterion.

Case I: We assume that c_1 prefers $\{s_1, s_2\}$ to $\{s_3, s_4, s_5\}$. Sometimes, such assumption seems to be reasonable. One can easily check that this assumption violates the max-min criterion, as $|\{s_1, s_2\}| \geq |\{s_3, s_4, s_5\}|$ does not hold although $\min(\{s_1, s_2\}) \succ_{c_1} \min(\{s_3, s_4, s_5\})$. Then, under

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \{s_3\} & \{s_4\} \end{pmatrix},$$

every college obtains improvement relative to μ_C. Thus μ_C is not weakly Pareto efficient. In addition, for this case, college c_1 can receive a strictly better assignment by misreporting its preferences and quota. That is, if it reports $q_{c_1} = 2$ and s_1, s_2 are its only acceptable students, then it will become strictly better off than to report truthfully. Consequently, μ_C is not strategy-proof.

Case II: We assume that c_1 prefers $\{s_1, s_2, s_5\}$ to $\{s_3, s_4, s_5\}$. Generally, such assumption is natural and reasonable. One can easily check that this assumption violates the max-min criterion, as $\min(\{s_1, s_2, s_5\}) \succ_{c_1} \min(\{s_3, s_4, s_5\})$ does not hold although $|\{s_1, s_2, s_5\}| \geq |\{s_3, s_4, s_5\}|$. Then, under

$$\mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2, s_5\} & \{s_3\} & \{s_4\} \end{pmatrix},$$

every college obtains improvement relative to μ_C. Thus μ_C is not weakly Pareto efficient. For this case, college c_1 can also receive a strictly better assignment by misreporting its preferences. That is, if it reports s_1, s_2 and s_5 are its only acceptable students, then it will become strictly better off than to report truthfully. Thus μ_C is not strategy-proof.

This example indicates that one unlikely expect to obtain the desired efficiency and incentive properties for colleges under a preference restriction which is weaker than max-min criterion.

6 Conclusion

We investigate the efficiency, incentive, and monotonicity properties for colleges. We first show that the colleges-proposing deferred acceptance algorithm is weakly Pareto-efficient for colleges if the quota-saturability condition holds and every college has max-min preference. We then extend the "blocking lemma" for marriage problem (one-to-one matching) obtained by Gale and Sotomayor to the college admissions problem (many-to-one matching), and consequently obtain the strategy-proofness property of the colleges-proposing deferred acceptance algorithm. We also define the notion of W-max-min criterion and show that the W-max-min criterion, together with quota-saturability condition, ensures that the colleges-proposing deferred acceptance algorithm is weakly Pareto-efficient, strategy-proof and monotone for colleges.

For marriage problem (one-to-one matching), Roth (1982) showed that the men-proposing deferred acceptance algorithm is weakly Pareto efficient and strategy-proof for men. The strategy-proofness property is also obtained by Dubins and Freedman (1981). However, there exists no straightforward approach to extend the corresponding results to the case of many-to-one matching. Roth (1985) showed that the college admissions problem is not equivalent to the marriage problem: the colleges-proposing deferred acceptance algorithm may be neither weakly Pareto efficient nor strategy-proof for colleges if colleges have responsive preferences. In some sense, the results of this paper provide complements for efficiency and incentives properties of many-to-one matching. Whether these efficiency and incentives properties can be extended to the case of many-to-many matching is an interesting problem for future investigations.

Appendix

Proof of Lemma 1. We first note that the quota-saturability condition and the stability of μ imply $|\mu(c)| = q_c$ for every $c \in C$. For simplicity, let $s_c \equiv \min(\mu(c))$, $\alpha \equiv \{s_c|c \in C\}$ and $\beta \equiv \{(c, s_c)|c \in C\}$. For each $s_c \in \alpha$, by assumption, there is at least one $c' \in C$ not matched with s_c who prefers s_c to one of its mates $\mu(c')$. Let c^* be the college in this set that s_c prefers most, and denote by $\alpha^* \equiv \{c^*|c \in C\}$ and $\beta^* \equiv \{(c^*, s_c)|c \in C\}$.

By the stability of μ, we have $c \succ_{s_c} c^*$ for every $c \in C$, as $s_c \notin \mu(c^*)$ and c^* prefers s_c to one of its mates $\mu(c^*)$. By the construction of α^*, we know that $s_c \succ_{c^*} s_{c^*}$ for every $c^* \in \alpha^*$. Since $|\alpha| = |C|$ is finite, there must exist a directed cycle among a subset, γ, of $\alpha \cup \alpha^*$. Particularly, γ can be expressed as $(c, s_c, c^*, s_{c^*}, (c^*)^*, s_{(c^*)^*}, \cdots, c, s_c)$ such that in the sequence every member prefers her predecessor to her successor. We denote by $\gamma^* \equiv \{(c, s_c)|c \in \gamma\} \cup \{(c^*, s_c)|c \in \gamma\}$.

Define μ^* to be μ except for the pairs belonging to γ^*, where those of β^* are taken instead of those of β:

$$(c, s) \in \mu^* \qquad \text{if} \;\; (c, s) \in \mu \backslash \gamma^* \;\; \text{or} \;\; (c, s) \in \beta^* \cap \gamma^*.$$

By construction, we know that, under μ^* and μ, every player is matched with the same number of opposite side players. Thus μ^* is an assignment. It is also stable. Specifically, we suppose $(c, s) \notin \mu^*$, then either $(c, s) \notin \mu \cup \mu^*$ or $(c, s) \in \mu \backslash \mu^*$. For the first case, $(c, s) \notin \mu$ implies that, either there exists some $c' \in C$ being matched to s under μ such that $c' \succ_s c$, or there are q_c students, whom c prefers to s, being matched to c under μ. By the construction of β^* and μ^*, together with the condition $(c, s) \notin \beta^* \cap \gamma^*$, we obtain that, either there exists some $\widetilde{c} \in C$ being matched to s under μ^* such that $\widetilde{c} \succ_s c$, or there are q_c students who are matched to c under μ^* and are preferred to s by c. For the second case, $(c, s) = (c, s_c) \in \beta \cap \gamma^*$, and thus by the construction of μ^*, there are q_c students who are matched to c under μ^* and are preferred to s_c by c. Finally, the construction of μ^* implies that $\mu^* \succ_C \mu$. The proof is completed. □

Proof of Theorem 1. We argue by contradiction. Suppose there exists some individually rational matching μ such that $\mu \succ_c \mu_C$ for every $c \in C$. Since there are enough acceptable students in the market, $|\mu_C(c)| = q_c$ for any $c \in C$ according to the procedure of the deferred acceptance algorithm. By the max-min criterion, $\mu \succ_c \mu_C$ implies $|\mu(c)| = q_c$ and $\min(\mu(c)) \succ_c \min(\mu_C(c))$ for every $c \in C$.

We assert that, if $(c, s) \in \mu_C \backslash \mu$, then there exists some $c' \in C$ such that $(c', s) \in \mu \backslash \mu_C$. Suppose not, then there exists some pair $(c, s) \in \mu_C \backslash \mu$ and $\mu(s) \backslash \mu_C(s) = \{\varnothing\}$. This implies $0 = |\mu(s)| < |\mu_C(s)| = 1$. There must exist $\widetilde{s} \in S$ such that $1 = |\mu(\widetilde{s})| > |\mu_C(\widetilde{s})| = 0$, as

$|\mu_C| = \sum_{c=1}^{c=|C|} q_c = |\mu|$ and $\sum_{s=1}^{s=|S|} |\mu(s)| = |\mu| = |\mu_C| = \sum_{s=1}^{s=|S|} |\mu_C(s)|$. Then we can find some college, say, $\widetilde{c} \in C$, such that $(\widetilde{c}, \widetilde{s}) \in \mu \backslash \mu_C$. Since $\widetilde{s} \succ_{\widetilde{c}} \min(\mu_C(\widetilde{c}))$ and $|\mu_C(\widetilde{s})| = 0$, μ_C is blocked by $(\widetilde{c}, \widetilde{s})$, which contradicts the stability of μ_C.

Particularly, for every $c \in C$, $\mu \succ_c \mu_C$ implies $(c, \min(\mu_C(c))) \in \mu_C \backslash \mu$. Then, there exists some $c' \in C$ such that $(c', \min(\mu_C(c))) \in \mu \backslash \mu_C$. $(c', \min(\mu_C(c))) \in \mu$ and $\mu \succ_{c'} \mu_C$ imply that $\min(\mu_C(c)) \succ_{c'} \min(\mu_C(c'))$. Thus we obtain that, for the stable matching μ_C and every $c \in C$, $\min(\mu_C(c))$ is not matched to some college c' who prefers $\min(\mu_C(c))$ to $\min(\mu_C(c'))$. By Lemma 1, there exists a stable assignment μ' such that $\mu' \succ_C \mu_C$, which contradicts the optimality of μ_C. The proof is completed. $\square$

Proof of Lemma 2. Case I: $\mu(C') \neq \mu_C(C')$. For any college $\widetilde{c} \in C'$, the assumption that $\widetilde{c}$ prefers μ to μ_C implies $|\mu(\widetilde{c})| = q_{\widetilde{c}}$, as the quota-saturability condition implies $|\mu_C(\widetilde{c})| = q_{\widetilde{c}}$ and the max-min criterion holds. Then we have $|\mu(C')| \equiv \sum_{c \in C'} |\mu(c)| = \sum_{c \in C'} |\mu_C(c)| \equiv |\mu_C(C')|$, and thus, by $\mu(C') \neq \mu_C(C')$, we have $\mu(C') \backslash \mu_C(C') \neq \emptyset$. Choose s in $\mu(C') \backslash \mu_C(C')$ such that $s \in \mu(c')$ for some $c' \in C'$. Since c' prefers $\mu(c')$ to $\mu_C(c')$, we obtain $|\mu(c')| = |\mu_C(c')| = q_{c'}$ and $\min(\mu(c')) \succ_{c'} \min(\mu_C(c'))$. Then $s \in \mu(c')$ implies $s \succ_{c'} \min(\mu_C(c'))$. This indicates that, in the procedure of the deferred acceptance algorithm, c' must have proposed to s before it proposes to $\min(\mu_C(c'))$. $s \notin \mu_C(c')$ implies that s has another choice which is better than c' and s rejects c', so $\mu_C(s) \succ_s c'$. Since $s \in \mu(c')$, the college $c = \mu_C(s)$ is unmatched with s under μ. It is easy to see that (c, s) blocks μ. Firstly, $c \succ_s c'$ and $c' = \mu(s)$. Secondly, if $|\mu(c)| < q_c$, the fact that c and s are acceptable to each other implies (c, s) blocks μ; if $|\mu(c)| = q_c$, $s \notin \mu_C(C')$ and $c = \mu_C(s)$ imply $c \notin C'$, then the preference relation $\mu \succ_c \mu_C$ does not hold. Thus $\min(\mu_C(c)) \succeq_c \min(\mu(c))$. Then, by the condition $s \in \mu_C(c) \backslash \mu(c)$, we have $s \succ_c \min(\mu(c))$. Hence (c, s) blocks μ.

Case II: $\mu(C') = \mu_C(C') = S'$. Let s be the last student in S' to receive a proposal from an acceptable college, say, c', of C' in the deferred acceptance algorithm such that $s = \min(\mu_C(c'))$ (if there is more than one such student, we choose anyone among them). Suppose this happens at step k. Since c' prefers μ to μ_C, $\min(\mu(c')) \succ_{c'} \min(\mu_C(c'))$ by the max-min criterion. Then we have $s \notin \mu(c')$. Thus $s \in \mu(C')$ implies that there is some other college $c'' \in C'$ such that $s \in \mu(c'') \backslash \mu_C(c'')$. The assumption that c'' prefers μ to μ_C implies $\min(\mu(c'')) \succ_{c''} \min(\mu_C(c''))$. Hence we have $s \succ_{c''} \min(\mu_C(c''))$. This indicates that c'' must propose to s in the procedure of the deferred acceptance algorithm. $s \notin \mu_C(c'')$ implies that s accepts a better college and rejects c''. According to the deferred acceptance algorithm, we know that student s has at least one better choice on her waiting list when she rejects c'' at some step sooner than step k (as no students reject any college in C' after step $(k-1)$). Since c' proposes to s and is accepted by s at step k, we know it must be the case that s rejects some college $c \in C \backslash C'$ and accepts c' at this step. It is easy to see that c and s are acceptable to each other and $c \succ_s c''$. Taking a similar argument as given in Case I, we obtain (c, s) blocks μ. The proof is completed. $\square$

Proof of Theorem 2. Let μ_C be the assignment produced by the colleges-proposing deferred acceptance algorithm under the true preferences and true quotas $(C, S; q; \succ_C, \succ_S)$. Suppose some nonempty subset C' of C misstates its preferences and quotas and is strictly better off under the matching μ, produced by the colleges-proposing deferred acceptance algorithm according to the reported preferences and quotas $(C, S; q'; \succ'_C, \succ_S)$, than under μ_C with respect to the true environment $(C, S; q; \succ_C, \succ_S)$. By Theorem 1, we know $C' \neq C$. Note that μ is stable under the reported environment $(C, S; q'; \succ'_C, \succ_S)$. If μ is not individually rational, it must be the case that for some $c' \in C'$, c' is matched to an unacceptable student or matched with more students than its quota. This contradicts $\mu \succ_{c'} \mu_C$. Thus μ must be individually rational. Also, by Lemma 2, there exists $(c, s) \in [C \backslash C'] \times S$ that blocks μ with respect to the true preferences and quotas $(C, S; q; \succ_C, \succ_S)$. Since c and s do not misstate their preferences and quotas, they are exactly the same in both $(C, S; q; \succ_C, \succ_S)$ and $(C, S; q'; \succ'_C, \succ_S)$. Thus (c, s) blocks μ with respect to $(C, S; q'; \succ'_C, \succ_S)$. However, we know μ is stable under $(C, S; q'; \succ'_C, \succ_S)$, as μ is produced by the colleges-proposing deferred acceptance algorithm according to $(C, S; q'; \succ'_C, \succ_S$). This contradiction completes the proof. $\square$

Uniqueness of Stable Matching under $(C, S; q; \succ_C, \succ_S)$. If for some matching μ, $|\mu(c_1)| = 1$, it must be unstable. We consider the matching μ such that $|\mu(c_1)| = 2$. If $s_4 \in \mu(c_1)$, then (c_2, s_4) blocks μ, it is unstable. If $s_2 \in \mu(c_1)$, then (c_3, s_2) blocks μ, it is unstable. If $\mu(c_2) = \{s_2\}$, then (c_3, s_2) blocks μ, it is unstable. Hence the only stable matching is: $\mu(c_1) = \{s_1, s_3\}, \mu(c_2) = \{s_4\}$ and $\mu(c_3) = \{s_2\}$. $\square$

Uniqueness of Stable Matching under $(C, S; q; \succ_C, \succ'_S)$. If for some matching μ, $|\mu(c_1)| = 1$, it must be unstable. Consider the matching μ such that $|\mu(c_1)| = 2$. If $s_4 \in \mu(c_1)$, then (c_2, s_4) blocks μ, it is unstable. If $s_2 \notin \mu(c_1)$, then (c_1, s_2) blocks μ, it is unstable. Now consider the case $s_2 \in \mu(c_1)$ but $s_4 \notin \mu(c_1)$. If $\mu(c_2) = \{s_4\}$, then (c_3, s_4) blocks μ, it is unstable. Thus the stable assignment satisfies: $s_2 \in \mu(c_1)$ and $\mu(c_3) = \{s_4\}$. If $s_1 \in \mu(c_1)$, then (c_2, s_1) blocks μ. The only stable matching is: $\mu(c_1) = \{s_2, s_3\}, \mu(c_2) = \{s_1\}$ and $\mu(c_3) = \{s_4\}$. $\square$

Proof of Theorem 3. We can prove the weak Pareto efficiency and strategy-proofness for colleges by repeating the proofs of Lemmas 1-2 and Theorems 1-2 step by step. Thus, here we only need to prove the monotonicity for colleges.

Suppose $\succ'_S$ be an improvement for college c over $\succ_S$. Let μ_C and μ'_C be the two assignments produced by the colleges-proposing algorithm under $(C, S; q; \succ_C, \succ_S)$ and $(C, S; q; \succ_C, \succ'_S)$, respectively. We want to show that $\mu'_C(c) \succeq_c \mu_C(c)$. By the quota-saturability condition, we know $|\mu'_C(c)| = |\mu_C(c)| = q_c$. According to the W-max-min criterion, it is sufficient to prove $\min(\mu'_C(c)) \succeq_c \min(\mu_C(c))$.

We assert that, under $(C, S; q; \succ_C, \succ'_S)$, the college c only needs to propose to students s with $s \succeq_c \min(\mu_C(c))$ on c's preference list, and then the college c will be assigned with q_c students.

Indeed, consider the step, say, step k, in the deferred acceptance algorithm, at which c proposes to at least one student and is not rejected by any student (including students who are proposed to in this step and who place c on their waiting lists at some previous step), and also there are no students who reject c from step k to the termination of the algorithm. That is, after step k, the college c always has q_c matched students and never needs to propose to other students. Let S_k denote the set of students to whom the college c newly proposes at step k, and $\min(S_k)$ denote the least preferred student of c in S_k. For step k, we check the following two cases.

Case I: $\min(S_k) \succeq_c \min(\mu_C(c))$. Obviously, the college c must be matched with q_c students who are weakly preferred to $\min(\mu_C(c))$. The proof is done.

Case II: $\min(\mu_C(c)) \succ_c \min(S_k)$.

(i) $\min(\mu_C(c)) \notin S_k$. We want to prove that c has been accepted by q_c students belonging to $\mu'_C(c)$ before step k, which contradicts the assumption on step k. Thus this case is impossible.

To see this, for any $s \in \mu_C(c)$, if $s \in \mu'_C(c)$, then student $s \in \mu_C(c)$ corresponds to herself $s \in \mu'_C(c)$. If $s \notin \mu'_C(c)$, then it must be the case that college c proposes to s, but s rejects c. The reason is the following:

(1): s rejects c because s enrolls in another college $c^{(1)}$ such that $c^{(1)} \succ'_s c$. Since $\succ'_s$ is an improvement for c over $\succ_s$, it implies $c^{(1)} \succ_s c$. The stability of μ_C and $s \in \mu_C(c)$ imply that $\mu_C(c^{(1)})$ contains $q_{c^{(1)}}$ students who are better than s according to the preferences of $c^{(1)}$. $s \in \mu'_C(c^{(1)})$ indicates that $c^{(1)}$ must propose to s under μ'_C. The underlying reason must be that there exists some student $s^{(1)} \in \mu_C(c^{(1)})$ who rejects $c^{(1)}$ under μ'_C.

(2): $s^{(1)}$ rejects $c^{(1)}$ because she enrolls in another college $c^{(2)}$ such that $c^{(2)} \succ'_{s^{(1)}} c^{(1)}$. Since $\succ'_{s^{(1)}}$ is an improvement for c over $\succ_{s^{(1)}}$, it implies $c^{(2)} \succ_{s^{(1)}} c^{(1)}$. The stability of μ_C and $s^{(1)} \in \mu_C(c^{(1)})$ imply that $\mu_C(c^{(2)})$ contains $q_{c^{(2)}}$ students who are better than $s^{(1)}$ according to the preferences of $c^{(2)}$. $s^{(1)} \in \mu'_C(c^{(2)})$ indicates that $c^{(2)}$ must propose to $s^{(1)}$ under μ'_C. The

underlying reason must be that there exists some student $s^{(2)} \in \mu_C(c^{(2)})$ who rejects $c^{(2)}$ under μ'_C.

(n): With this process going on, we must find at some step a college $c^{(n)}$ and a student $s^{(n)}$ such that $s^{(n)} \in \mu_C(c^{(n)}) \cap \mu'_C(c)$ and $c \succ'_{s^{(n)}} c^{(n)}$, as every college has no empty seat under μ_C and $s \in \mu_C(c)\backslash\mu'_C(c)$.

Thus, for any student $s \in \mu_C(c)\backslash\mu'_C(c)$, we can always find a corresponding student $s^{(n)} \in \mu'_C(c)\backslash\mu_C(c)$. The above analysis indicates that $s^{(n)}$ accepts c previous to that s rejects c, so $s^{(n)}$ has accepted c before step k. Therefore, we obtain that c has been accepted by q_c students belonging to $\mu'_C(c)$ before step k, which violates our assumption on step k.

(ii) $\min(\mu_C(c)) \in S_k$. Then every student in $\mu_C(c)\backslash S_k$ corresponds to a student belonging to $\mu'_C(c)\backslash S_k$ (the proof as given in (i)), and, $|\mu_C(c) \cap S_k| < |S_k|$ and $S_k \subseteq \mu'_C(c)$ imply $|\mu_C(c)| < |\mu'_C(c)|$, which contradicts $|\mu_C(c)| = q_c = |\mu'_C(c)|$. Thus, this case is also impossible.

The combination of (i) and (ii) implies Case II is impossible. We complete the proof. $\square$

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