Efficiency and Respecting Improvements in College Admissions

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Abstract

This paper studies the efficiency and respecting improvements in college admissions. We show that, when colleges have the extended max-min preference over groups of students and the students' side of the market is "thick" enough (as indicated by quota-saturability condition), the college-proposing deferred acceptance algorithm is weakly Pareto efficient for colleges. We then propose a new preference condition called the W-max-min criterion and show that the college-proposing deferred acceptance algorithm respects improvements for colleges if the college-quota-saturability condition is satisfied and each college has the W-max-min preference over groups of students.

Keywords: College admissions; Pareto efficiency; max-min preferences; deferred acceptance algorithm; respecting improvements.
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1 Introduction

Gale and Shapley (1962, henceforth GS) originally studied the many-to-one two-sided matching college admissions problem that extends the one-to-one two-sided matching model in such a way that colleges have preferences over students and students have preferences over colleges;

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each college c can accept a certain number of students with a quota q_c and each student can enroll in at most one college. For the college admissions problem, two assignment criteria are concerned primarily in GS: stability and optimality. The stability requires that there exists no coalition of students and colleges to block the assignment. The optimality (for agents on one side of the market) means that every agent (college or student) is at least as well off under the given stable assignment as he would be under any other stable assignment. GS proposed the deferred acceptance algorithm that always yields an optimal stable matching for agents on the proposing side.

Efficiency is a leading desiderata in two-sided matching market design for well-known reasons. Roth (1982) investigated the marriage problem and obtained that the men (resp. women)-optimal matching is weakly Pareto efficient for men (resp. women). For a unified model, Hatfield and Milgrom (2005) studied the matching with contracts. Matching problems with transfers and without transfers can be embedded this unified model. When hospitals' preferences are substitutes and satisfy the law of aggregate demand, Hatfield and Kojima (2009) showed that the doctor-optimal matching is weakly Pareto efficient for doctors. However, the corresponding weak Pareto efficiency for hospitals do not hold. Even for a special case that the college admissions problem, Roth (1985) showed that, when colleges have responsive preferences, the college-proposing deferred acceptance algorithm may be not weakly Pareto efficient for colleges (colleges have multi-unit demand), while the student-proposing deferred acceptance algorithm is weakly Pareto efficient for students (students have unit demand).

An important open question is: under what restricted domain on preferences is there a stable mechanism such that the weak efficiency for agents with multi-unit demand are guaranteed? Baïou and Balinski (2000) attempted to answer this question by introducing the notion of max-min preferences and presenting the reduction algorithm by graphic approach. The max-min criterion indicates that colleges always want to admit as many and preferable acceptable students as possible within their quotas.

Baïou and Balinski (2000) claimed that, for many-to-many matching, the reduction algorithm results in a stable assignment that satisfies weak Pareto efficiency and respecting improvements for agents on one side of the market if every agent has max-min preference.[1] Unfortunately, even

[1]Balinski and Sönmez (1999) first proposed the concept of "respecting improvement". Baïou and Balinski (2000) used the notion of monotonicity to express the same meaning. For the college admissions problem, Balinski and Sönmez (1999) proved that the student-proposing deferred acceptance algorithm respects improvements for students. Loosely speaking, respecting improvements means that an agent is weakly better off if she becomes more preferred by players on the opposite side. Then respecting improvements for students means that a student

for many-to-one matching, Hatfield et al. (2014) constructed counter-examples to show that the condition of max-min preferences is not sufficient for the existence of a stable mechanism that satisfies weak Pareto efficiency or respecting improvements for agents with multi-unit demand. Thus, the above-mentioned result of Baïou and Balinski (2000) is incorrect. As such, it is still an unanswered question on the weak efficiency and respecting improvements for agents with multi-unit demand.

In this paper we intend to investigate the efficiency and respecting improvements for agents with multi-unit demand in many-to-one matchings. We consider the college admissions model and obtain that the college-proposing deferred acceptance algorithm is weakly Pareto efficient (for colleges) if colleges have the extended max-min preference over groups of students and the college-quota-saturability condition is satisfied. We propose a new preference condition called the W-max-min criterion. Then it is shown that the college-proposing deferred acceptance algorithm respects improvements for colleges if colleges have the W-max-min preference over groups of students and the college-quota-saturability condition is satisfied. For the college admissions problem, Balinski and Sönmez (1999) obtained that the students-proposing deferred acceptance algorithm respects improvements for students. Essentially, the result obtained by Balinski and Sönmez is about agents who have unit demand. We extend their result to the case for agents with multiple demand. Strategy-proofness of the deferred acceptance mechanism under the extended max-min preference criterion and quota-saturability condition is also shown in Jiao and Tian (2017).

The extended max-min criterion is a slight variation on the max-min criterion in such a way that admitting more acceptable students is always more preferable to colleges, which means that colleges should use their capacity of resources as they can. The basic idea on quota-saturability, which means that the students' side of the market is sufficiently "thick", is not new. Indeed, Gale and Shapley (1962) made the assumption for the college admissions problem: "There are enough applicants to assign each college precisely as many students as its quota." Although this assumption on the size of the student market side did not play a crucial role in the analysis of GS, it is a reasonable and critical condition for Pareto efficiency and respecting improvements of the colleges-proposing deferred acceptance algorithm. When each college is assigned to as many acceptable applicants as its quota, the colleges' side is saturated. It is easy to show that, if the quota-saturability condition fails, there may not exist any stable mechanism that is weakly Pareto efficient for colleges (see Example 1 below). Therefore, the quota-saturability condition

will enroll in a (weakly) better college if she becomes more preferred by all colleges.

cannot be dispensed with in order to establish the college-efficiency for the college admissions model.

The main contribution of this paper is twofold: Firstly, we obtain the weak Pareto efficiency for agents with multi-unit demand in many-to-one matching markets, which extends the corresponding result for agents with unit demand (see Roth, 1982 and 1985). We show that the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient for colleges if the extended max-min preference criterion and the quota-saturability condition are satisfied. If the quota-saturability condition is not satisfied, Hatfield et al. (2014) showed that there exists no stable mechanism that satisfies weak Pareto efficiency for colleges under the max-min preference. On the other hand, Roth (1985) showed that the colleges-optimal matching may not be weakly Pareto efficient for colleges even under the assumptions of quota-saturability condition and responsive preferences.

Secondly, we obtain the property of respecting improvements for agents with multi-unit demand in many-to-one matching markets, which extends the corresponding result for agents with unit demand (see Balinski and Sönmez, 1999). We introduce the notion of W-max-min preference and show that the colleges-proposing deferred acceptance algorithm not only is weakly Pareto efficient, but also respects improvements for colleges if the W-max-min criterion and the quota-saturability condition are satisfied. We extend the result of respecting improvements for agents with unit demand (obtained by Balinski and Sönmez, 1999) to the case for agents with multiple demand.

The remainder of this paper is organized as follows. Section 2 presents preliminaries on the formal model. In Section 3 we study the efficiency for agents with multi-unit demand. We deal with the issue of respecting improvements for agents with multi-unit demand in Section 4. In Section 5 we note by an example that the efficiency fails even if we slightly relax the preference requirement imposed by the extended max-min criterion. We conclude in Section 6. Several technical proofs are provided in the Appendix.

2 The Model

For concreteness, we use the language of college admissions problem, though our results will apply to general many-to-one matching markets.

Our model follows the framework of Roth and Sotomayor (1989, 1990). There are two finite and disjoint sets, $C = \{c_1, \cdots, c_{|C|}\}$ and $S = \{s_1, \cdots, s_{|S|}\}$, of colleges and students respectively, where the notation $|A|$ denotes the number of elements of the set A. Each student has preferences

$\succ_s$ over the set of colleges C and the outside option — the null college $\varnothing$, and each college has preferences $\succ_c$ over the set of students S and the prospect of having its seat unfilled, also denoted by $\varnothing$. Student s is *acceptable* to college c iff $s \succ_c \varnothing$, and college c is acceptable to student s iff $c \succ_s \varnothing$. We assume these preferences are complete, transitive and strict, so they may be represented by order lists. For example, $\succ_c$: $s_2, s_1, \varnothing, s_3, \cdots$ denotes that college c prefers to enroll s_2 rather than s_1, that it prefers to enroll either one of them rather than leave a position unfilled, and that all other students are unacceptable, in the sense that it would be preferable to leave a position unfilled rather than fill it with, say, student s_3. Similarly, for the preferences of a student, $\succ_s$: $c_1, c_3, c_2, \varnothing, \cdots$ indicates that the only positions the student would accept are those offered by c_1, c_3 and c_2, in that order. We will write $c_i \succ_s c_j$ to indicate that student s prefers c_i to c_j, and $c_i \succeq_s c_j$ to indicate that either $c_i \succ_s c_j$ or $c_i = c_j$. Similarly, we can give corresponding notions on preferences of colleges. Each college c has a *quota* q_c which is the maximum number of students for which it has places. Let $q \equiv (q_c)_{c \in C}$ denote the vector of college quotas.

We denote the preferences profile of all colleges by $\succ_C \equiv (\succ_c)_{c \in C}$, and the preferences profile of all students by $\succ_S \equiv (\succ_s)_{s \in S}$. The preferences profile of all players is denoted by $\succ \equiv (\succ_C, \succ_S)$. We denote a college admissions problem by a four-tuple $(C, S; q; \succ)$. We also write it as $(C, S; q; \succ_C, \succ_S)$.

For colleges, since they may be matched with different sets of students, we also need to consider their preferences over groups of students.[2] We assume these preferences are transitive, but the completeness is not required. Particularly, throughout this paper we assume the following properties hold (see, e.g., Konishi and Ünver, 2006a):

(i) *Weak monotonicity in population:* For every college c, any subset of acceptable students $G \in 2^S$ with $|G| < q_c$ and any student $s \notin G$, c prefers $G \cup \{s\}$ to G if and only if s is acceptable to c.[3]

(ii) For any $G \in 2^S$, if $|G| > q_c$ or G contains any unacceptable students, then c prefers having all its positions unfilled to admitting G, i.e., $\varnothing \succ_c G$.

We will specify the colleges-proposing deferred acceptance algorithm below. These two assumptions provide rationality for the procedure of that algorithm: colleges always want to match with as many (within their quotas) acceptable students as possible and never propose to any

[2]Without confusion, we abuse notations: For any $i \in S$ (resp. $i \in C$), $j, k \in C \cup \{\varnothing\}$ (resp. $j, k \in S \cup \{\varnothing\}$), the preference relation $\{j\} \succ_i \{k\}$ is also denoted as $j \succ_i k$.

[3]College's preferences satisfy strong monotonicity in population if $\forall c \in C, \forall G, G' \in 2^S$, $|G'| < |G| \leq q_c$ implies $G \succ_c G'$ (see, e.g., Konishi and Ünver, 2006b). Obviously, strong monotonicity implies weak monotonicity.

unacceptable student.

Baïou and Balinski (2000) introduce the following preference condition of max-min criterion.

Definition 2.1 (Max-Min Criterion) The preference relation of $c \in C$ is said to satisfy the *max-min criterion* if for any two sets of acceptable students $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$,

(i) The strict preference relation $\succ_c$ over groups of students is defined as: $G_1 \succ_c G_2$ if and only if $|G_1| \geq |G_2|$ and $\min(G_1) \succ_c \min(G_2)$ (i.e., c strictly prefers the least preferred student in G_1 to the least preferred student in G_2), where $\min(G_i)$ denotes the least preferred student of c in G_i;

(ii) The weak preference relation $\succeq_c$ over groups of students is defined as: $G_1 \succeq_c G_2$ if $G_1 \succ_c G_2$ or $G_1 = G_2$.[4]

Jiao and Tian (2017) extended the max-min preference relation over groups of students by assuming that any group of students G_1 is always comparable with, and actually strictly preferable to, its proper subgroup G_2. Formally, we present the following notion of extended max-min criterion.

Definition 2.2 (Extended Max-Min Criterion) The preference relation of $c \in C$ is said to satisfy the *extended max-min criterion* if for any two sets of acceptable students $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$,

(i) The strict preference relation $\succ_c$ is defined as: $G_1 \succ_c G_2$ if and only if either G_2 is a proper subset of G_1, or $|G_1| \geq |G_2|$ and $\min(G_1) \succ_c \min(G_2)$;

(ii) The weak preference relation $\succeq_c$ over groups of students is defined as: $G_1 \succeq_c G_2$ if and only if $G_1 \succ_c G_2$ or $G_1 = G_2$.

We say that the *extended max-min criterion* is satisfied if the preference of every college satisfies the extended max-min criterion.

Thus, we modify the conventional max-min criterion by adding that $G_1 \succ G_2$ if G_2 is a proper subset of G_1, i.e., admitting more acceptable students is always more preferable for colleges, which is clearly very reasonable for capacity utilization of resources. As such, for the extended max-min criterion, "max" indicates that colleges always want to match with as many

[4]Without confusion, we abuse notations: For college c, we denote its preferences over groups of students and over individual students by the same notations $\succ_c$ and $\succeq_c$.

students as possible, and "min" indicates that colleges focus on the worst student in ranking different sets of students and colleges would like to match with as preferable students (in the sense of preferences over individual student) as possible. It is easy to check there is no implication relationship between the extended max-min criterion and the conventional one.

For many-to-one or many-to-many matching problem, substitutable preferences and responsive preferences are often adopted.[5] It is well known that responsiveness implies substitutability. Here we should point out that the extended max-min criterion also implies substitutability.[6] In addition, we note that there is no implication relationship between extended max-min preferences and responsive preferences. Indeed, the responsiveness does not imply the extended max-min criterion. For instance, suppose that college c's preferences over individual students are given by $s_1 \succ_c s_2 \succ_c s_3$, $q_c = 2$ and that the preference list over groups of two candidates is given by $\{s_1, s_2\} \succ_c \{s_1, s_3\} \succ_c \{s_2, s_3\}$. Then it is clear that the responsiveness is satisfied, but $\{s_1, s_3\} \succ_c \{s_2, s_3\}$ violates the extended max-min criterion.

The extended max-min criterion does not imply the responsiveness either. For instance, now suppose $s_1 \succ_c s_2 \succ_c s_3$, $q_c = 2$, and that the preferences over groups of two candidates are given by $\{s_1, s_2\} \succ_c \{s_1, s_3\}$, $\{s_1, s_2\} \succ_c \{s_2, s_3\}$ and college c cannot compare $\{s_1, s_3\}$ and $\{s_2, s_3\}$. Then it is clear that the extended max-min criterion is satisfied, but it violates the responsiveness (otherwise one would have $\{s_1, s_3\} \succ_c \{s_2, s_3\}$, contradicting the hypothesis that college c cannot compare $\{s_1, s_3\}$ and $\{s_2, s_3\}$).

2.1 Matching between Colleges and Students

Definition 2.3 A *matching* is a correspondence $\mu : C \cup S \to 2^{C \cup S \cup \{\varnothing\}}$ such that

(1) $\mu(c) \subseteq S \cup \{\varnothing\}$ and $|\mu(c)| \leq q_c$ for all $c \in C$,

[5] In the language of the college admissions model, substitutability of college c's preferences requires: "if admitting s is optimal when certain students are available, admitting s must still be optimal when a subset of students are available." Formally, an agent c's preference relation $\succ_c$ satisfies substitutability if, for any sets S and S' with $S \subseteq S'$, $s \in Ch(S' \cup \{s\}, \succ_c)$ implies $s \in Ch(S \cup \{s\}, \succ_c)$, where $Ch(S \cup \{s\}, \succ_c)$ denotes agent c's most-preferred subset of $S \cup \{c\}$ according to c's preference relation $\succ_c$. Kelso and Crawford (1982) introduced the substitutability condition on hospital preferences in a matching model with wages. Roth (1984) adapted the deferred acceptance algorithm to the many-to-many matching model with substitutable preference and obtained the corresponding optimal stable matching. Martínez et al. (2004), Echenique and Oviedo (2006) and Klaus and Walzl (2009) studied the stability problem of many-to-many matching under substitutable preference. Hatfield and Kojima (2008, 2009, 2010) and Hatfield and Milgrom (2005) studied the matching with contracts under substitutable hospitals' preferences.

[6] See Jiao and Tian (2015) for detailed discussion.

(2) $\mu(s) \subseteq C \cup \{\varnothing\}$ and $|\mu(s)| \leq 1$ for all $s \in S$,

(3) $s \in \mu(c)$ if and only if $\mu(s) = \{c\}$ for all $c \in C$ and $s \in S$.[7]

Note that, for all $i \in C \cup S$, we stipulate $|\mu(i)| = 0$ if $\mu(i) = \{\varnothing\}$. For $\mu(c) \neq \{\varnothing\}$, we will use the notation $\min(\mu(c))$ to denote the least preferred student of c in the set $\mu(c)$. For a matching μ, if $s \in \mu(c)$ or $\mu(s) = \{c\}$, we also write it as $(c, s) \in \mu$.

A matching μ is *blocked by an individual* $i \in C \cup S$ if there exists some player $j \in \mu(i)$ such that $\varnothing \succ_i j$. A matching is *individually rational* if it is not blocked by any individual. A matching μ is *blocked by a pair* $(c, s) \in C \times S$ if

(1) c is acceptable to s and s is acceptable to c,

(2) $|\mu(c)| < q_c$ or $s \succ_c \min(\mu(c))$, and

(3) $c \succ_s \mu(s)$.

Definition 2.4 A matching μ is *stable* if it is not blocked by any individual or any college-student pair.

2.2 Deferred Acceptance Algorithm

The *deferred acceptance algorithm* was first proposed by GS to find a stable assignment for the marriage problem (one-to-one matching) and college admissions problem (many-to-one matching). Specifically, the *Colleges-Proposing Deferred Acceptance Algorithm* proceeds as follows:

Step 1. (a). Each college c proposes to its top q_c acceptable students (if c has fewer acceptable choices than q_c, then it proposes to all its acceptable choices).

(b). Each student s then places the best college among those proposed to her on her waiting list, and rejects the rest.

In general, at

Step k. (c). Any college c that was rejected at step $(k-1)$ by any student proposes to its most-preferred q_c acceptable students who have not yet rejected it (if there are fewer than q_c remaining acceptable students, then it proposes to all).

(d). Each student s selects the best one from among the new colleges and that on her waiting list, puts it on her new waiting list, and rejects the rest.

[7] For $i \in S$, $j \in C \cup \{\varnothing\}$, we also write the notation $\mu(i) = \{j\}$ as $\mu(i) = j$ if it is not confused.

Since no college proposes twice to the same student, this algorithm always terminates in a finite number of steps. The algorithm terminates when there are no more rejections. Each student is matched with the college on his waiting list in the last step.

The colleges-proposing deferred acceptance algorithm yields an assignment denoted by μ_C. GS showed that μ_C is a stable matching and it is optimal for every college if colleges have responsive preferences (or colleges have extended max-min preferences, see Jiao and Tian, 2017). That is, for any $c \in C$, there exists no other stable matching μ such that $\mu \succ_c \mu_C$.

3 Weak Pareto Efficiency for Colleges

In this section we study the weak Pareto efficiency for colleges. Roth (1985) constructed an example to show that, if the colleges have responsive preferences over groups of students,[8] then the colleges-proposing deferred acceptance algorithm may be not weakly Pareto efficient for colleges. Baïou and Balinski (2000) introduced the concept of max-min preference[9] and attempted to investigate efficiency for many-to-many matching. Although they provided valuable ideas, their claims are not completely correct. Indeed, Hatfield et al. (2014) showed that there exists no stable mechanism that satisfies weak Pareto efficiency for colleges even though colleges have max-min preferences over groups of students.

A matching μ is said to be *weakly college-efficient* if there exists no individually rational matching μ' such that $\mu'(c) \succ_c \mu(c)$ for all $c \in C$. A weakly students-efficient matching is defined analogously.

For any two matchings μ and μ', the notions $\mu \succ_c \mu'$ means $\mu(c) \succ_c \mu'(c)$, $\mu \succeq_c \mu'$ means $\mu(c) \succeq_c \mu'(c)$, $\mu \succeq_C \mu'$ means $\mu \succeq_c \mu'$ for every $c \in C$, and $\mu \succ_C \mu'$ means $\mu \succeq_C \mu'$ and $\mu \succ_c \mu'$ for some college c.

Given the colleges C and students S, a *mechanism* φ is a function from any stated preferences profile $\succ$ and quota-vector q to a matching. A mechanism φ is *stable* if its outcome function, denoted by $\varphi(C, S; q; \succ)$, is a stable matching for any reported $\succ$ and q. A mechanism φ is *weakly college (resp. student)-efficient* if it always selects a weakly colleges (resp. students)-efficient matching for every preference profile.

Now we borrow the example of Abdulkadiroğlu and Sönmez (2010) to show that, under any reasonable preference assumption, there exists no stable matching mechanism satisfying weak

[8]The preferences of college c are responsive if whenever $G \in 2^S$ and $i, j \in S$ such that $|G| < q_c$ and $i, j \notin G$, $i \succ_c j$ implies $G \cup \{i\} \succ_c G \cup \{j\}$.

[9]Similar conditions were studied by Echenique and Oviedo (2006), Kojima (2007) and Sotomayor (2004).

college-efficiency.

Example 1 This example is essentially the same as that used by Hatfield et al. (2014): there are two colleges c_1, c_2 with $q_{c_1} = 2, q_{c_2} = 1$, and two students s_1, s_2. The preferences are as follows:

$\succ_{s_1}$: $c_1,\ c_2,\ \varnothing$ $\qquad$ $\succ_{c_1}$: $\{s_1, s_2\},\ \{s_2\},\ \{s_1\},\ \{\varnothing\}$

$\succ_{s_2}$: $c_2,\ c_1,\ \varnothing$ $\qquad$ $\succ_{c_2}$: $\{s_1\},\ \{s_2\},\ \{\varnothing\}$.

The only stable matching for this problem is:

$$\mu = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \{s_2\} \end{pmatrix}.$$

For another matching

$$\mu' = \begin{pmatrix} c_1 & c_2 \\ \{s_2\} & \{s_1\} \end{pmatrix},$$

we can see that both c_1 and c_2 prefer μ' to μ. This means there exists no stable mechanism that is weakly college-efficient.

In Example 1, we can see that college c_1's preference relation is commonly reasonable, but the number of students is less than the aggregate quota of colleges. This may potentially cause the failure of the existence of stable mechanism satisfying college-efficiency. To avoid this possible environment, Jiao and Tian (2017) proposed the quota-saturability condition which says that the students' side of the market is sufficiently "thick" such that each college can admit as many acceptable students as its quota if it wants. That is, in the matching market, from the perspective of colleges the student resource is not scarce. Specifically,

Definition 3.1 (Quota-Saturability) For the college admissions problem, let $\widetilde{S}$ be a subset of S such that every college $c \in C$ and every student $s \in \widetilde{S}$ are acceptable to each other. We say *the quota-saturability condition* holds if there exists some $\widetilde{S} \subseteq S$ such that $|\widetilde{S}|$ is thick enough to satisfy $|\widetilde{S}| \geq \sum_{c \in C} q_c$, i.e., the number of available and acceptable students is not less than the aggregate quota of colleges.

In other words, quota-saturability means that there are enough available and acceptable students such that each college c can be assigned q_c acceptable students if it wants. An intuitive interpretation of quota-saturability would be a certain "excess supply" of students in the market that allows all colleges to fill their quotas if they want.

For the college admissions model, we know that $|\mu(c)| \leq q_c$ for each college $c \in C$. If the quota-saturability condition holds and colleges have extended max-min preferences, then every college wants to admit as many students as its quota and there is a sufficiently thick number of students in the market. We can infer that the matching μ must be unstable if $|\mu(c)| < q_c$ for some $c \in C$ because c has empty position and there are applicants who want to enter college c. Therefore, one can obtain that $|\mu(c)| = q_c$ for any $c \in C$ if μ is stable.

We want to show that the extended max-min criterion, together with quota-saturability condition, ensures that the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient for colleges. To do so, we first present the following lemma.

Lemma 1 *Suppose that the quota-saturability condition is satisfied. Let μ be a stable assignment. If for the least preferred student $\min(\mu(c))$ of every $c \in C$, there exists another college c' that prefers $\min(\mu(c))$ to one of its mates $\mu(c')$, then there exists a stable assignment μ^* with $\mu^* \succ_C \mu$.*

This lemma actually says that, for a stable assignment such that each college is matched with as many students as its quota, if each least preferred student has another college that wants to be matched with her, then there exists another stable assignment such that no college becomes worse off and at least one college is strictly better off. Intuitively, since each least preferred student has another college that wants to be matched with her, it is expected to find a cycle consisting of the same number of least preferred students and colleges such that each least preferred student is followed by a college that wishes to be matched with her. Then let every least preferred student enroll into the college that followed her. In this way, one can get the desirable matching.

Lemma 1 was first claimed by Baïou and Balinski (2000, henceforth BB) without assuming the quota-saturability condition. However, the conclusion of Lemma 1 fails to hold if the quota-saturability condition is not satisfied. Specifically, we consider the setting as given in Example 1.

Example 1 (Continued) Consider the sets of colleges and students, and their preferences and quotas as given in Example 1. The only stable matching for this problem is:

$$\mu = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \{s_2\} \end{pmatrix}.$$

For c_1, $\min(\mu(c_1)) = s_1$ is not matched to c_2 and c_2 prefers $\min(\mu(c_1)) = s_1$ to $s_2 \in \mu(c_2)$. For c_2, $\min(\mu(c_2)) = s_2$ is not matched to c_1 and c_1 prefers $\min(\mu(c_2)) = s_2$ to $s_1 \in \mu(c_1)$. There exists no other stable assignment. This indicates that we cannot obtain the conclusion of Lemma 1 if the quota-saturability condition is not satisfied.

With Lemma 1, we have the following theorem on weak Pareto efficiency.

Theorem 1 *Suppose the quota-saturability condition and the extended max-min criterion are satisfied. Then the colleges-proposing deferred acceptance algorithm is weakly colleges-efficient.*

Theorem 1 shows that the colleges-proposing deferred acceptance algorithm is weakly Pareto efficient for colleges under the extended max-min preference and quota-saturability condition. If the colleges have responsive preferences over groups of students, the corresponding result does not hold even if the quota-saturability condition is satisfied (see Proposition 1 of Roth, 1985). We know that the extended max-min criterion is unrelated to the responsiveness. Thus, Theorem 1 lies in obtaining the desired result under a preference condition that is not stronger than responsive preference.

4 Respecting Improvements for Colleges

In this section we consider the issue of respecting improvements for colleges in the college admissions problem. We want to find the conditions under which the colleges-proposing deferred acceptance algorithm respects improvements for colleges. The respecting improvements means that an agent is weakly better off if she becomes more preferred by players on the opposite side. For completeness, we first introduce the following concept originally proposed by Balinski and Sönmez (1999):

Definition 4.1 A preference relation $\succ'_s$ is *an improvement for a college c over $\succ_s$* if

(1) $c \succ_s c'$ implies $c \succ'_s c'$ for all $c' \in C \cup \{\varnothing\}$ and

(2) $c_1 \succ'_s c_2$ if and only if $c_1 \succ_s c_2$ for all $c_1, c_2 \in (C \cup \{\varnothing\}) \backslash \{c\}$.

Loosely speaking, an improved preference relation means c becomes more preferred by s. A student preference profile $\succ'_S$ is an improvement for c over $\succ_S$ if for every s, $\succ'_s$ is an improvement for c over $\succ_s$. That is, c becomes more preferred by all students.

Definition 4.2 A mechanism φ is said to *respect improvements for colleges* if, for any preference profile $\succ$, any $c \in C$, and any student preference profiles $\succ_S$ and $\succ'_S$, if $\succ'_S$ is an improvement for c over $\succ_S$, then c weakly prefers $\varphi(\succ'_S, \succ_C)$ to $\varphi(\succ_S, \succ_C)$.

That is, the outcome of a mechanism is weakly better off for a college if that college becomes more preferred by all students.

Balinski and Sönmez (1999) proposed the concept of "respecting improvement" and proved that the students-proposing deferred acceptance algorithm respects improvements for students. That is, under the students-proposing deferred acceptance algorithm, a student will enroll in a weakly better college if she becomes more preferred by all colleges. Baïou and Balinski (2000) studied the many-to-many matching under the assumption of max-min preference. They asserted that the reduction algorithm respects improvements for agents on one of the two matching sides. However, Hatfield et al. (2014) showed that the condition of max-min preference is not sufficient for the existence of a stable mechanism that satisfies the property of respecting improvements. Hence the above result of Baïou and Balinski (2000) is incorrect. In fact, even though colleges have max-min preferences and the quota-saturability condition is satisfied, there may not exist a stable mechanism satisfying the property of respecting improvements. To see this, consider the following example:

Example 2 There are three colleges c_1, c_2, c_3 with $q_{c_1} = 2, q_{c_2} = 1, q_{c_3} = 1$, and four students s_1, s_2, s_3, s_4. The preferences are as follows:

$\succ_{s_1}$: $c_2,\ c_1,\ c_3,\ \varnothing$ $\qquad$ $\succ_{c_1}$: $\{s_1, s_2\}, \cdots,\ \{s_3\},\ \{s_4\},\ \{\varnothing\}$

$\succ_{s_2}$: $c_3,\ c_1,\ c_2,\ \varnothing$ $\qquad$ $\succ_{c_2}$: $\{s_4\},\ \{s_1\}, \cdots$

$\succ_{s_3}$: $c_1,\ c_2,\ c_3,\ \varnothing$ $\qquad$ $\succ_{c_3}$: $\{s_2\},\ \{s_4\}, \cdots$

$\succ_{s_4}$: $c_3,\ c_2,\ c_1,\ \varnothing$.

Under $(C, S; q; \succ_C, \succ_S)$, the only stable matching for this problem is:[10]

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_3\} & \{s_4\} & \{s_2\} \end{pmatrix}.$$

Now suppose $\succ'_{s_2}$: $c_1,\ c_3,\ c_2,\ \varnothing$, and $\succ'_{s_i} = \succ_{s_i}$ for $i = 1, 3, 4$. Then $\succ'_S$ is an improvement for c_1 over $\succ_S$. Under $(C, S; q; \succ_C, \succ'_S)$, the only stable matching for this problem is:[11]

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_2, s_3\} & \{s_1\} & \{s_4\} \end{pmatrix}.$$

In this example, the quota-saturability condition is satisfied, but under the extended max-min preference (resp. responsive preference), we cannot obtain $\mu'(c_1) \succeq_{c_1} \mu(c_1)$,[12] although c_1 becomes (weakly) more preferred by all students. Thus, there does not exist any stable mechanism respecting improvements for colleges.

[10] The uniqueness of the stable matching under $(C, S; q; \succ_C, \succ_S)$ is shown in the Appendix.

[11] The uniqueness of the stable matching under $(C, S; q; \succ_C, \succ'_S)$ is shown in the Appendix.

[12] Under responsive preference, we have $\mu(c_1) = \{s_1, s_3\} \succ_c \{s_2, s_3\} = \mu'(c_1)$. Under extended max-min preference, college c_1 cannot compare $\mu(c_1) = \{s_1, s_3\}$ and $\mu'(c_1) = \{s_2, s_3\}$.

A natural question is under what conditions the property of respecting improvements for colleges can be guaranteed. Here we propose the following preference criterion, which, together with the quota-saturability condition, ensures that the colleges-proposing deferred acceptance algorithm not only is weakly Pareto efficient, but also respects improvements for colleges.

Definition 4.3 (W-Max-Min Criterion) The preference relation of $c \in C$ is said to satisfy the *W-max-min criterion* if the following conditions are met: for any two sets of acceptable students $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$,

(i) The weak preference relation $\succeq_c$ over groups of students is defined as: If G_2 is a subset of G_1 or, $|G_1| \geq |G_2|$ and $\min(G_1) \succeq_c \min(G_2)$ (i.e., c weakly prefers[13] the least preferred student in G_1 to the least preferred student in G_2), then $G_1 \succeq_c G_2$;

(ii) If $G_1 \succeq_c G_2$ and $G_2 \succeq_c G_1$, then G_1 and G_2 are indifferent for c, denoted by $G_1 \sim_c G_2$. If $G_1 \succeq_c G_2$ and $G_2 \not\succeq_c G_1$, then c strictly prefers G_1 to G_2, denoted by $G_1 \succ_c G_2$.

We say that the *W-max-min criterion* is satisfied if the preference of every college satisfies the W-max-min criterion.

Here the notion of "W-max-min criterion" refers to the max-min criterion, which is defined in terms of weak preferences over groups of students to distinguish from the notion of (extended) max-min criterion that is defined in terms of strict preferences over groups of students. Note that, for any $G_1, G_2 \in 2^S$ with $|G_1| \leq q_c$ and $|G_2| \leq q_c$, $G_1 \succ_c G_2$ under the W-max-min criterion is equivalent to $G_1 \succ_c G_2$ under the extended max-min criterion. The difference between the extended max-min criterion and the W-max-min criterion lies in: under the extended max-min criterion, $G_1 \succeq_c G_2$ and $G_2 \succeq_c G_1$ imply $G_1 = G_2$, while under W-max-min criterion $G_1 \succeq_c G_2$ and $G_2 \succeq_c G_1$ imply $G_1 = G_2$ or $|G_1| = |G_2|$ and $\min(G_1) = \min(G_2)$.

We note that there is a close consistency relationship between the W-max-min criterion and the deferred acceptance algorithm. We illustrate this point by a simple example. Assume there are five students $s_1, \cdots, s_5$, college c's preference relation is given by $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_4 \succ_c s_5$ and the quota $q_c = 2$. Then the assignment outcomes produced by the deferred acceptance algorithm and the W-max-min preference relations can be compared as in Table 1.

We then have the following result.

[13] As c's preference over individual students is strict, "weakly prefers" means that $\min(G_1) \succ_c \min(G_2)$ or $\min(G_1) = \min(G_2)$.

Table 1: Comparison between DA-algorithm and W-max-min preference relation

The satisfaction grade of c	The last student c proposes to in DA	c's partners	W-max-min preference relation
1	s_2	$\{s_1, s_2\}$	$\{s_1, s_2\}$ is the first-best choice of c
2	s_3	$\{s_1, s_3\}$ or $\{s_2, s_3\}$	$\{s_1, s_3\} \sim_c \{s_2, s_3\}$
3	s_4	$\{s_1, s_4\}$ or $\{s_2, s_4\}$ or $\{s_3, s_4\}$	$\{s_1, s_4\} \sim_c \{s_2, s_4\} \sim_c \{s_3, s_4\}$
4	s_5	$\{s_1, s_5\}$ or $\{s_2, s_5\}$ or $\{s_3, s_5\}$ or $\{s_4, s_5\}$	$\{s_1, s_5\} \sim_c \{s_2, s_5\} \sim_c \{s_3, s_5\} \sim_c \{s_4, s_5\}$

Theorem 2 *Suppose that the W-max-min criterion and the quota-saturability condition are satisfied. Then the college-proposing deferred acceptance algorithm not only is weakly college-efficient, but also respects improvements for colleges.*

Theorem 2 indicates that, if the extended max-min criterion in Theorems 1 is replaced by W-max-min criterion, then the college-proposing deferred acceptance algorithm guarantees not only weak Pareto efficiency, but also respecting improvements for colleges.

5 Discussion

As discussed above, the extended max-min and W-max-min preferences enable us to get the desirable properties such as weak Pareto efficiency and respecting improvements for colleges. A question is then whether the extended max-min criterion can be weakened. The answer is negative. Indeed, we note that the weak Pareto efficiency fails even if we slightly weaken the requirement of extended max-min preference. Specifically, we consider the following example.

Example 3 There are three colleges c_1, c_2, c_3 with $q_{c_1} = 3, q_{c_2} = 1, q_{c_3} = 1$, and five students s_1, s_2, s_3, s_4, s_5. The preferences are as follows:

$\succ_{s_1}$: $c_2,\ c_1,\ c_3,\ \varnothing$ $\quad\quad$ $\succ_{s_5}$: $c_1,\ c_3,\ c_2,\ \varnothing$

$\succ_{s_2}$: $c_3,\ c_1,\ c_2,\ \varnothing$ $\quad\quad$ $\succ_{c_1}$: $s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ \varnothing$

$\succ_{s_3}$: $c_1,\ c_2,\ c_3,\ \varnothing$ $\quad\quad$ $\succ_{c_2}$: $s_3,\ s_1,\ \cdots$

$\succ_{s_4}$: $c_1,\ c_3,\ c_2,\ \varnothing$ $\quad\quad$ $\succ_{c_3}$: $s_4,\ s_2,\ \cdots$

Under $(C, S; q; \succ_C, \succ_S)$, the colleges-optimal matching is

$$\mu_C = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_3, s_4, s_5\} & \{s_1\} & \{s_2\} \end{pmatrix}.$$

For college c_1's preference over groups of students, we slightly weaken the requirement of extended max-min criterion.

Case I: We assume that c_1 prefers $\{s_1, s_2\}$ to $\{s_3, s_4, s_5\}$. Sometimes, such assumption seems to be reasonable. One can easily check that this assumption violates the extended max-min criterion, as $|\{s_1, s_2\}| \geq |\{s_3, s_4, s_5\}|$ does not hold although $\min(\{s_1, s_2\}) \succ_{c_1} \min(\{s_3, s_4, s_5\})$. Then, under

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \{s_3\} & \{s_4\} \end{pmatrix},$$

every college obtains improvement relative to μ_C. Thus μ_C is not weakly Pareto efficient.

Case II: We assume that c_1 prefers $\{s_1, s_2, s_5\}$ to $\{s_3, s_4, s_5\}$. Generally, such assumption is natural and reasonable. One can easily check that this assumption violates the extended max-min criterion, as $\min(\{s_1, s_2, s_5\}) \succ_{c_1} \min(\{s_3, s_4, s_5\})$ does not hold although $|\{s_1, s_2, s_5\}| \geq |\{s_3, s_4, s_5\}|$. Then, under

$$\mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2, s_5\} & \{s_3\} & \{s_4\} \end{pmatrix},$$

every college obtains improvement relative to μ_C. Thus μ_C is not weakly Pareto efficient.

This example indicates that one cannot expect to obtain the desired efficiency for colleges under a preference restriction weaker than the extended max-min criterion.

6 Conclusion

Using the well-known college admissions model, we investigate the efficiency and respecting improvements for agents with multi-unit demand in many-to-one matching markets. As long as colleges' preferences satisfy the extended max-min criterion and the students' side of the market is sufficiently thick, we get the weak Pareto efficiency of the college-proposing deferred acceptance algorithm. We then define the notion of W-max-min criterion and show that the W-max-min criterion, together with quota-saturability condition, ensures that the college-proposing deferred acceptance algorithm is weakly Pareto-efficient and respects improvements for colleges. Jiao and Tian (2017) also show that the deferred acceptance mechanism is strategy-proof for agents on the proposing side under the extended max-min preference criterion and quota-saturability condition

The results of this paper provide a complement for efficiency and respecting improvements for agents with multi-unit demand in many-to-one matchings. Whether these results can be extended to the case of many-to-many matching is an interesting problem for future investigation.

Appendix

Proof of Lemma 1. We first note that the quota-saturability condition and the stability of μ imply $|\mu(c)| = q_c$ for every $c \in C$. For simplicity, let $s_c \equiv \min(\mu(c))$, $\alpha \equiv \{s_c | c \in C\}$ and $\beta \equiv \{(c, s_c) | c \in C\}$. For each $s_c \in \alpha$, by assumption, there is at least one college $c' \in C$ not matched with s_c who prefers s_c to one of its mates $\mu(c')$. We denote the set of all such colleges as $\tilde{C}$. Let c^* be the college in $\tilde{C}$ that s_c prefers most, and denote by $\alpha^* \equiv \{c^* | c \in C\}$ and $\beta^* \equiv \{(c^*, s_c) | c \in C\}$.

By the stability of μ, we have $c \succ_{s_c} c^*$ for every $c \in C$, as $s_c \notin \mu(c^*)$ and c^* prefers s_c to one of its mates $\mu(c^*)$. By the construction of α^*, we know that $s_c \succ_{c^*} s_{c^*}$ for every $c^* \in \alpha^*$. Since $|\alpha| = |C|$ is finite, there must exist a directed cycle among a subset, γ, of $\alpha \cup \alpha^*$ (See Figure 1 for a graphic characterization of the directed cycle). Particularly, γ can be expressed as $(c, s_c, c^*, s_{c^*}, (c^*)^*, s_{(c^*)^*}, \cdots, c, s_c)$ such that in the sequence every member prefers her predecessor to her successor. We denote by $\gamma^* \equiv \{(c, s_c) | c \in \gamma\} \cup \{(c^*, s_c) | c \in \gamma\}$.

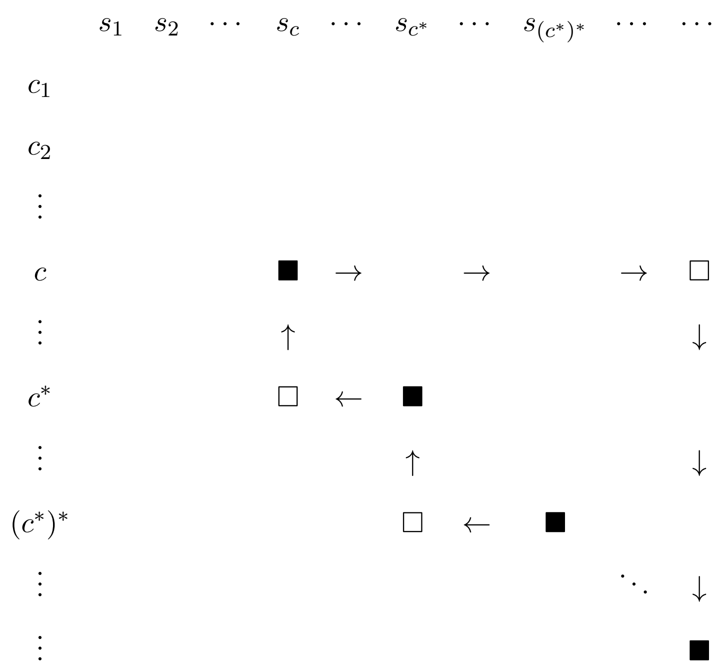

Fig. 1. Graphic characterization of the directed cycle

Note: This is a directed graph of $|C| \times |S|$ grids. ■ denotes that its corresponding column-player (college) and row-player (student) are matched together under μ, for instance, $(c, s_c) \in \mu$. □ denotes that its corresponding column-player (college) and row-player (student) are unmatched under μ, for instance, $(c^*, s_c) \notin \mu$. The arrows characterize the players' preferences. At each row (resp. column), its row-player (resp. column-player) prefers the predecessor to the successor along the arrow's direction, for instance, s_c prefers c to c^* and c^* prefers s_c to s_{c^*}.

Define μ^* to be μ except for the pairs belonging to γ^*, where those of β^* are taken instead of those of β:

$$(c,s) \in \mu^* \qquad \text{if} \;\; (c,s) \in \mu \backslash \gamma^* \;\; \text{or} \;\; (c,s) \in \beta^* \cap \gamma^*.$$

By construction, we know that, under μ^* and μ, every player is matched with the same number of opposite side players. Thus μ^* is an assignment. It is also stable. Specifically, we suppose $(c,s) \notin \mu^*$, then either $(c,s) \notin \mu \cup \mu^*$ or $(c,s) \in \mu \backslash \mu^*$. For the first case, $(c,s) \notin \mu$ implies that, either there exists some $c' \in C$ being matched to s under μ such that $c' \succ_s c$, or there are q_c students, whom c prefers to s, being matched to c under μ. By the construction of β^* and μ^*, together with the condition $(c,s) \notin \beta^* \cap \gamma^*$, we obtain that either there exists some $\widetilde{c} \in C$ being matched to s under μ^* such that $\widetilde{c} \succ_s c$, or there are q_c students who are matched to c under μ^* and are preferred to s by c. For the second case, $(c,s) = (c,s_c) \in \beta \cap \gamma^*$, and thus by the construction of μ^*, there are q_c students who are matched to c under μ^* and are preferred to s_c by c. Finally, the construction of μ^* implies that $\mu^* \succ_C \mu$. The proof is completed. □

Proof of Theorem 1. We argue by contradiction. Suppose there exists some individually rational matching μ such that $\mu \succ_c \mu_C$ for every $c \in C$. Since there are enough acceptable students in the market, $|\mu_C(c)| = q_c$ for any $c \in C$ according to the procedure of the deferred acceptance algorithm. By the extended max-min criterion, $\mu \succ_c \mu_C$ implies $|\mu(c)| = q_c$ and $\min(\mu(c)) \succ_c \min(\mu_C(c))$ for every $c \in C$.

We assert that, if $(c,s) \in \mu_C \backslash \mu$, then there exists some $c' \in C$ such that $(c',s) \in \mu \backslash \mu_C$. Suppose not, then there exists some pair $(c,s) \in \mu_C \backslash \mu$ and $\mu(s) \backslash \mu_C(s) = \{\varnothing\}$. This implies $0 = |\mu(s)| < |\mu_C(s)| = 1$. There must exist $\widetilde{s} \in S$ such that $1 = |\mu(\widetilde{s})| > |\mu_C(\widetilde{s})| = 0$, as $\sum_{c=1}^{c=|C|} |\mu_C(c)| = \sum_{c=1}^{c=|C|} q_c = \sum_{c=1}^{c=|C|} |\mu(c)|$ and $\sum_{s=1}^{s=|S|} |\mu(s)| = \sum_{c=1}^{c=|C|} |\mu(c)| = \sum_{c=1}^{c=|C|} |\mu_C(c)| = \sum_{s=1}^{s=|S|} |\mu_C(s)|$. Then we can find some college, say, $\widetilde{c} \in C$, such that $(\widetilde{c}, \widetilde{s}) \in \mu \backslash \mu_C$. Since $\widetilde{s} \succ_{\widetilde{c}} \min(\mu_C(\widetilde{c}))$ and $|\mu_C(\widetilde{s})| = 0$, μ_C is blocked by $(\widetilde{c}, \widetilde{s})$, which contradicts the stability of μ_C.

Particularly, for every $c \in C$, $\mu \succ_c \mu_C$ implies $(c, \min(\mu_C(c))) \in \mu_C \backslash \mu$. Then there exists some $c' \in C$ such that $(c', \min(\mu_C(c))) \in \mu \backslash \mu_C$. $(c', \min(\mu_C(c))) \in \mu$ and $\mu \succ_{c'} \mu_C$ imply that $\min(\mu_C(c)) \succ_{c'} \min(\mu_C(c'))$. Thus we obtain that, for the stable matching μ_C and every $c \in C$, $\min(\mu_C(c))$ is not matched to some college c' that prefers $\min(\mu_C(c))$ to $\min(\mu_C(c'))$. By Lemma 1, there exists a stable assignment μ' such that $\mu' \succ_C \mu_C$, which contradicts the optimality of μ_C. The proof is completed. □

The following two proofs are to claims made in Example 2.

Uniqueness of Stable Matching under $(C, S; q; \succ_C, \succ_S)$**.** If for some matching μ,

$|\mu(c_1)| = 1$, it must be unstable. We consider the matching μ such that $|\mu(c_1)| = 2$. If $s_4 \in \mu(c_1)$, then (c_2, s_4) blocks μ, it is unstable. If $s_2 \in \mu(c_1)$, then (c_3, s_2) blocks μ, it is unstable. If $\mu(c_2) = \{s_2\}$, then (c_3, s_2) blocks μ, it is unstable. Hence the only stable matching is: $\mu(c_1) = \{s_1, s_3\}, \mu(c_2) = \{s_4\}$ and $\mu(c_3) = \{s_2\}$. $\square$

Uniqueness of Stable Matching under $(C, S; q; \succ_C, \succ'_S)$. If for some matching μ, $|\mu(c_1)| = 1$, it must be unstable. Consider the matching μ such that $|\mu(c_1)| = 2$. If $s_4 \in \mu(c_1)$, then (c_2, s_4) blocks μ, it is unstable. If $s_2 \notin \mu(c_1)$, then (c_1, s_2) blocks μ, it is unstable. Now consider the case $s_2 \in \mu(c_1)$ but $s_4 \notin \mu(c_1)$. If $\mu(c_2) = \{s_4\}$, then (c_3, s_4) blocks μ, it is unstable. Thus the stable assignment satisfies: $s_2 \in \mu(c_1)$ and $\mu(c_3) = \{s_4\}$. If $s_1 \in \mu(c_1)$, then (c_2, s_1) blocks μ. The only stable matching is: $\mu(c_1) = \{s_2, s_3\}, \mu(c_2) = \{s_1\}$ and $\mu(c_3) = \{s_4\}$. $\square$

Proof of Theorem 2. We can prove the weak Pareto efficiency for colleges by repeating the proofs of Lemma 1 and Theorem 1 step by step. Thus, here we only need to show the property of respecting improvements for colleges.

Suppose $\succ'_S$ be an improvement for college c over $\succ_S$. Let μ_C and μ'_C be the two assignments produced by the colleges-proposing algorithm under $(C, S; q; \succ_C, \succ_S)$ and $(C, S; q; \succ_C, \succ'_S)$, respectively. We want to show that $\mu'_C(c) \succeq_c \mu_C(c)$. By the quota-saturability condition, we know $|\mu'_C(c)| = |\mu_C(c)| = q_c$. According to the W-max-min criterion, it is sufficient to prove $\min(\mu'_C(c)) \succeq_c \min(\mu_C(c))$.

We assert that, under $(C, S; q; \succ_C, \succ'_S)$, college c only needs to propose to students s with $s \succeq_c \min(\mu_C(c))$ on c's preference list, and then college c will be assigned with q_c students.

Indeed, consider the step, say, step k, in the deferred acceptance algorithm, at which c proposes to at least one student and is not rejected by any student (including students who are proposed to in this step and who place c on their waiting lists at some previous step), and also there are no students who reject c from step k to the termination of the algorithm. That is, after step k, college c always has q_c matched students and never needs to propose to other students. Let S_k denote the set of students to whom college c newly proposes at step k, and $\min(S_k)$ denote the least preferred student of c in S_k. For step k, we check the following two cases.

Case I: $\min(S_k) \succeq_c \min(\mu_C(c))$. Obviously, college c must be matched with q_c students who are weakly preferred to $\min(\mu_C(c))$. The proof is done.

Case II: $\min(\mu_C(c)) \succ_c \min(S_k)$.

We consider the following two subcases:

Subcase (i): $\min(\mu_C(c)) \notin S_k$.

We want to prove that c has been accepted by q_c students belonging to $\mu'_C(c)$ before step k, which contradicts the assumption on step k. Thus this case is impossible.

To see this, for any $s \in \mu_C(c)$, if $s \in \mu'_C(c)$, then student $s \in \mu_C(c)$ corresponds to herself $s \in \mu'_C(c)$. If $s \notin \mu'_C(c)$, then it must be the case that college c proposes to s, but s rejects c. The reason is the following:

(1): s rejects c because s enrolls in another college $c^{(1)}$ such that $c^{(1)} \succ'_s c$. Since $\succ'_s$ is an improvement for c over $\succ_s$, it implies $c^{(1)} \succ_s c$. The stability of μ_C and $s \in \mu_C(c)$ imply that $\mu_C(c^{(1)})$ contains $q_{c^{(1)}}$ students who are better than s according to the preferences of $c^{(1)}$. $s \in \mu'_C(c^{(1)})$ indicates that $c^{(1)}$ must propose to s under μ'_C. The underlying reason must be that there exists some student $s^{(1)} \in \mu_C(c^{(1)})$ who rejects $c^{(1)}$ under μ'_C (if there are more than one such students, we check each one of them and, there exists at least one such student satisfying the following condition).

(2): $s^{(1)}$ rejects $c^{(1)}$ because she enrolls in another college $c^{(2)}$ such that $c^{(2)} \succ'_{s^{(1)}} c^{(1)}$. Since $\succ'_{s^{(1)}}$ is an improvement for c over $\succ_{s^{(1)}}$, it implies $c^{(2)} \succ_{s^{(1)}} c^{(1)}$. The stability of μ_C and $s^{(1)} \in \mu_C(c^{(1)})$ imply that $\mu_C(c^{(2)})$ contains $q_{c^{(2)}}$ students who are better than $s^{(1)}$ according to the preferences of $c^{(2)}$. $s^{(1)} \in \mu'_C(c^{(2)})$ indicates that $c^{(2)}$ must propose to $s^{(1)}$ under μ'_C. The underlying reason must be that there exists some student $s^{(2)} \in \mu_C(c^{(2)})$ who rejects $c^{(2)}$ under μ'_C.

(n): With this process going on, we must find at some step a college $c^{(n)}$ and a student $s^{(n)}$ such that $s^{(n)} \in \mu_C(c^{(n)}) \cap \mu'_C(c)$ and $c \succ'_{s^{(n)}} c^{(n)}$, as every college has no empty seat under μ_C and $s \in \mu_C(c) \backslash \mu'_C(c)$.

Thus, for any student $s \in \mu_C(c) \backslash \mu'_C(c)$, we can always find a corresponding student $s^{(n)} \in \mu'_C(c) \backslash \mu_C(c)$. The above analysis indicates that $s^{(n)}$ accepts c previous to s rejecting c, so $s^{(n)}$ has accepted c before step k. Therefore, we obtain that c has been accepted by q_c students belonging to $\mu'_C(c)$ before step k, which violates our assumption on step k.

Subcase (ii): $\min(\mu_C(c)) \in S_k$.

For this subcase we can show that every student in $\mu_C(c) \backslash S_k$ corresponds to a student belonging to $\mu'_C(c) \backslash S_k$ (the proof as given in Subcase (i)), and, $|\mu_C(c) \cap S_k| < |S_k|$ and $S_k \subseteq \mu'_C(c)$ imply $|\mu_C(c)| < |\mu'_C(c)|$, which contradicts $|\mu_C(c)| = q_c = |\mu'_C(c)|$. Thus, this case is also impossible.

The combination of Subcases (i) and (ii) implies Case II is impossible. We complete the proof. □

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