Existence of Equilibria in Games with Arbitrary Strategy Spaces and Preferences: A Full Characterization

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Abstract
This paper studies the existence of Nash equilibria when strategy spaces are arbitrary topological spaces and preferences may be nontotal/nontransitive, discontinuous, nonconvex, or nonmonotonic. We define a single condition, recursive diagonal transfer continuity for aggregate payoffs or recursive weak transfer quasi-continuity for individuals’ preferences, which establishes the existence of Nash equilibria in games with arbitrary strategy spaces and preferences without imposing any kind of quasiconcavity-related conditions. Moreover, they are also necessary for the existence of equilibrium in any games. The results obtained can particularly be used to study the existence of equilibria in general economic environments with no linear (convex) structures such as those in market design and matching theories.

Keywords: Nash equilibrium; discontinuous games, non-ordered preferences, arbitrary strategy space, recursive transfer continuity.

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1 Introduction

The notion of Nash equilibrium is probably one of the most important solution concepts in economics in general and game theory in particular, which has wide applications in almost all areas
of economics and in business and other social sciences. The classical existence theorems on Nash equilibrium (e.g. in Nash (1950, 1951), Debreu (1952), Glicksberg (1952), Nikaido and Isoda (1955)) typically assume continuity and quasiconcavity for the payoff functions, in addition to convexity and compactness of strategy spaces. However, in many important economic models, such as those in Bertrand (1883), Hotelling (1929), Milgrom and Weber (1985), Dasgupta and Maskin (1986), and Jackson (2009), payoffs are discontinuous and/or non-quasiconcave, and strategy spaces are nonconvex and/or noncompact.


However, all the existing results impose linear (convex) structures and only provide sufficient conditions for the existence of equilibrium. In order to apply a fixed-point theorem (say, Brouwer, Kakutani, Tarski’s fixed point theorem, or KKM lemma, etc.), they all need to assume some forms of quasiconcavity (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy space. While it may be the convex structure which easily connects economics to mathematics, but in many important situations (such as market design problems and matching problems, particularly in key areas including the problems of college admission, housing allocation, kidney exchange, army programs where commodities or alternatives are invisible so that the choice spaces are discrete), there are no convex structures.

Thus, convexity assumption excludes the possibility of considering discrete games, and consequently seriously limits applicabilities of economic theory. As such, the intrinsic nature of equilibrium has not been fully understood yet. Why does or does not a game have an equilibrium? Are continuity and quasiconcavity both essential to the existence of equilibrium? If so, can continuity and quasiconcavity be combined into one single condition? One can easily find simple examples.

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1 McLennan, Monteiro, and Tourky (2011) and Barelli and Meneghel (2014) recently provide necessary and sufficient conditions for the existence of Nash equilibrium. However, they obtain their existence results under the linear structures.

2 For mixed strategy Nash equilibrium, quasiconcavity is automatically satisfied since the mixed extension has linear payoff functions. Thus only some form of continuity matters for the existence of mixed strategy Nash equilibrium.

3 A good reference for matching theory, see the book by Roth and Sotomayor (1990).
of economic games that have or do not have an equilibrium, but none of them can be used to reveal the existence/non-existence of equilibria in these games. This paper sheds some light in answering these questions.

We fully characterize the existence of pure strategy Nash equilibrium in general games with arbitrary topological strategy spaces\(^4\) that may be discrete or non-convex and payoffs (resp. preferences) that may be discontinuous (resp. discontinuous or nontotal/nontransitive) or do not have any form of quasi-concavity (resp. convexity) or monotonicity. We introduce the notions of recursive transfer continuities, specifically recursive diagonal transfer continuity for aggregate payoffs and recursive weak transfer quasi-continuity for individuals’ preferences, respectively.

It is shown that the single condition, recursive diagonal transfer continuity (resp. recursive weak transfer quasi-continuity) is necessary, and further, under compactness of strategy space, sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs (resp. preferences).\(^5\) We also provide an existence theorem for a strategy space that may not be compact. We show that recursive diagonal transfer continuity (resp. recursive weak transfer quasi-continuity) with respect to a compact set is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary (possibly noncompact or open) strategy spaces and general payoffs (resp. preferences). Recursive diagonal transfer continuity (resp. recursive weak transfer quasi-continuity) defined on respective spaces also permits the existence of symmetric pure strategy Nash equilibria in games with general strategy spaces and payoffs (resp. references).

Recursive diagonal transfer continuity (resp. recursive weak transfer quasi-continuity) strengthens diagonal transfer continuity introduced in Baye, Tian, Zhou’s (1993) (resp. recursive weak transfer quasi-continuity introduced in Nessah (2008)) to allow recursive (sequential) transfers in order to get rid of the diagonal transfer quasiconcavity assumption (resp. the strong diagonal transfer quasiconcavity assumption) so that these conditions turn out to be necessary and sufficient conditions for the existence of equilibria in compact games. As such, no quasiconcavity/monotonicity-related conditions are assumed. These results may be used to studies the existence of equilibrium in general games with no linear (convex) structures such as equilibrium issues in market design theory and matching theory. In the paper, we also provide sufficient conditions for the existence of equilibrium without imposing any form of quasiconcavity.

The remainder of the paper is organized as follows. Section 2 provides basic notation and definitions, and analyze the essence of Nash equilibrium. Section 3 investigates the existence of pure strategy Nash equilibrium by using aggregate payoffs and individuals’ preferences, respectively. We also provide sufficient conditions for recursive transfer continuities. Section 4 extends the

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\(^4\)In particular, the strategy spaces may not be metrizable, locally convex, Hausdorff, or even not regular.

\(^5\)As such, one cannot say that recursive diagonal transfer continuity (resp. recursive weak transfer quasi-continuity) is equivalent to Nash equilibrium.
results to symmetric pure strategy Nash equilibrium. Concluding remarks are offered in Section 5.

2 Preliminaries: Nash Equilibrium and Its Intrinsic Nature

2.1 Notion and Definitions

Let \( I \) be the set of players that is either finite or countably infinite. Suppose that each player \( i \)'s strategy set \( X_i \) is a nonempty subset of a general topological space \( E_i \) that may not be metrizable, locally convex, Hausdorff, or even not regular. Denote by \( X = \prod_{i \in I} X_i \) the Cartesian product of the sets of strategy profiles, equipped with the product topology. For each player \( i \in I \), denote by \( -i \) all other players rather than player \( i \). Also denote by \( X_{-i} = \prod_{j \neq i} X_j \) the Cartesian product of the sets of strategies of players \( -i \). Without loss of generality, suppose player \( i \)'s preference relation is given by the weak preference \(<_i \) defined on \( X \), which may be nontotal or nontransitive.\(^6\) Let \( \succ_i \) denote the asymmetric part of \( \succsim_i \), i.e., \( y \succ_i x \) if and only if \( y \succsim_i x \) but not \( x \succsim_i y \).

A game \( G = (X_i, \succsim_i)_{i \in I} \) is simply a family of ordered tuples \( (X_i, \succsim_i) \).

When \( \succsim_i \) can be represented by a payoff function \( u_i : X \to \mathbb{R} \), the game \( G = (X_i, u_i)_{i \in I} \) is a special case of \( G = (X_i, \succsim_i)_{i \in I} \).

A strategy profile \( x^* \in X \) is a pure strategy Nash equilibrium of a game \( G \) if,

\[
x^* \succsim_i (y_i, x^*_{-i}) \quad \forall i \in I, \forall y_i \in X_i.
\]

2.2 The Essence of Equilibrium and Why the Existing Results are Only Sufficient

Before proceeding to the notions of recursive transfer continuities, we first analyze the intrinsic nature of Nash equilibrium, and why the conventional continuity is unnecessarily strong and most the existing results provide only sufficient but not necessary conditions.

In doing so, we define an “upsetting” (irreflexive) binary relation, denoted by \( \succ \) as follows:

\[
y \succ x \quad \text{iff} \quad \exists i \in I \text{ s.t. } (y_i, x_{-i}) \succsim_i x.
\]

In this case, we say strategy profile \( y \) upsets strategy profile \( x \). It is clear that “\( y \succ x \) for \( x, y \in X \)” is equivalent to “\( x \in X \) is not an equilibrium”. We will use these terms interchangeably. Then, one can easily see that a strategy profile \( x^* \in X \) is a pure strategy Nash equilibrium if and only if there does not exist any strategy \( y \in X \) that upsets \( x^* \). As such, the existence of a Nash equilibrium is equivalent to the existence of a maximal element under this upsetting relation.

\(^6\)The results obtained for weak preferences \( \succsim \), can be also used to get the results for strict preferences \( \succ \). Indeed, from \( \succsim \), we can define a weak preferences \( \succsim_i \) defined on \( X \times X \) as follows: \( y \succsim_i x \) if and only if \( \neg x \succsim_i y \). The preference \( \succsim_i \) defined in such a way is called the completion of \( \succsim \). A preference \( \succsim_i \) is said to be complete if, for any \( x, y \in X \), either \( x \succsim_i y \) or \( y \succsim_i x \). A preference \( \succsim_i \) is said to be total if, for any \( x, y \in X, x \neq y \) implies \( x \succsim_i y \), or \( y \succsim_i x \).
When \( x \in X \) is not a pure strategy Nash equilibrium, then there exists a strategy profile \( y \in X \) such that \( y \succ x \). To ensure the existence of an equilibrium, it usually requires all strategies in a neighborhood \( \mathcal{V}_x \) of \( x \) be upset by some strategy profile \( z \in X \), denoted by \( z \succ \mathcal{V}_x \), i.e., \( z \succ x' \) for all \( x' \in \mathcal{V}_x \). The topological structure of the conventional continuity surely secures this upsetting relation locally at \( x \) by \( y \), i.e., there always exists a neighborhood \( \mathcal{V}_x \) of \( x \) such that \( y \succ \mathcal{V}_x \). As such, no transfers (say, from \( y \) to \( z \)) or switchings (from player \( i \) to \( j \)) are needed for securing this upsetting relation locally at \( x \). However, when \( u_i \) is not continuous, such a topological relation between the upsetting point \( y \) and the neighborhood \( \mathcal{V}_x \) may no longer be true, i.e., we may not have \( y \succ \mathcal{V}_x \). But, if \( y \) can be transferred to \( z \) so that \( z \succ \mathcal{V}_x \), then the upsetting relation \( \succ \) can be secured locally at \( x \). This naturally leads to the following notion of transfer continuity, which is a weak notion of continuity that was first introduced in Tian (1992a), Tian and Zhou (1992), Baye, Tian, and Zhou (1993) and Tian and Zhou (1995) to study preference maximization and the existence of equilibrium.

**Definition 2.1** The upsetting relation \( \succ \) is transfer continuous if, whenever \( y \succ x \) for \( x, y \in X \), there exists a deviation strategy profile \( z \in X \) and a neighborhood \( \mathcal{V}_x \subset X \) of \( x \) such that \( z \succ x' \) for all \( x' \in \mathcal{V}_x \), i.e., the upsetting relation \( \succ \) can be secured locally at \( x \).

When individuals' preferences \( \succ_i \) can be represented by numerical payoff functions \( u_i \) and the number of players is finite, define the aggregator function, \( U : X \times X \to \mathbb{R} \) by

\[
U(y, x) = \sum_{i \in I} u_i(y_i, x_{-i}), \quad \forall (x, y) \in X \times X,
\]

which refers to the aggregate payoff across individuals where for every player \( i \) assuming she or he deviates to \( y_i \) given that all other players follow the strategy profile \( x \). The “upsetting” relation \( \succ \) is then defined as:

\[
y \succ x \quad \text{iff} \quad U(y, x) > U(x, x).
\]

The above definition of transfer continuity in turn immediately reduces to the notion of diagonal transfer continuity introduced by Baye, Tian, and Zhou (1993) for aggregator function, which together with diagonal transfer quasiconcavity, guarantees the existence of pure strategy Nash equilibrium.\(^8\)

**Definition 2.2** A game \( G = (X_i, u_i)_{i \in I} \) is diagonally transfer continuous if, whenever \( U(y, x) > U(x, x) \) for \( x, y \in X \) (i.e., whenever \( x \) is not an equilibrium), there exists another

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\(^7\) \( I \) can be a countably infinite set. In this case, one may define \( U \) according to \( U(y, x) = \sum_{i \in I} \frac{1}{n_i} u_i(y_i, x_{-i}) \).

\(^8\) A game \( G = (X_i, u_i)_{i \in I} \) is diagonally transfer quasiconcave if, for any finite subset \( Y^m = \{y^1, \ldots, y^m\} \subset X \), there exists a corresponding finite subset \( X^m = \{x^1, \ldots, x^m\} \subset X \) such that for any subset \( \{x^{k_1}, x^{k_2}, \ldots, x^{k_s}\} \subset X^m \), \( 1 \leq s \leq m \), and any \( x \in \text{co}\{x^{k_1}, x^{k_2}, \ldots, x^{k_s}\} \) we have \( \min_{1 \leq i \leq s} U(x, y^{k_i}) \leq U(x, x) \).
deviation strategy profile \( z \in X \) and a neighborhood \( \mathcal{V}_x \) of \( x \) such that \( U(z, x') > U(x', x') \) for all \( x' \in \mathcal{V}_x \).

Note that, to secure “upsetting” relation locally at \( x \) by \( z \), it is unnecessary for all players \( i \) to have \( u_i(z_i, \mathcal{V}_{x_i}) > u_i(\mathcal{V}_x) \) (i.e., \( u_i(z_i, x'_{-i}) > u_i(x') \) \( \forall x' \in \mathcal{V}_x \)), but it is enough for just one player. Diagonal transfer continuity in Baye, Tian, and Zhou (1993), better-reply security in Reny (1999), weak transfer continuity in Nessah and Tian (2008), for instance, weaken the conventional continuity along this line.

It may be remarked that, in contrast to better-reply security introduced in Reny (1999) and its various extensions, an advantage of diagonal transfer continuity is that it can be used to prove the existence of mixed strategy equilibrium. Indeed, Prokopovych and Yannelis (2014) complements Baye, Tian, and Zhou (1993) in this respect. Without using diagonal transfer continuity (BTZ, 1993) and transfer lower semicontinuity introduced in (Tian 1992), it is impossible to apply the KKM lemma properly to the mixed strategy equilibrium existence problem.

Moreover, to secure “upsetting” relation locally at \( x \) by \( z \), it is unnecessary to just fix one player so that \( (z_i, \mathcal{V}_{x_i}) \succ_i \mathcal{V}_x \), but players can be switched among them to secure this upsetting relation locally at \( x \). If for every \( x' \in \mathcal{V}_x \), there exists a player \( i \) such that \( (z_i, x'_{-i}) \succ_i x' \), all is done here. In other words, we can secure this “upsetting” relation locally by possibly switching players for every strategy in a neighborhood. This exactly comes up with the notion of weak transfer quasi-continuity introduced by Nessah and Tian (2008), which together with strong diagonal transfer quasiconcavity,\(^9\) guarantees the existence of pure strategy Nash equilibrium.

**Definition 2.3** A game \( G = (X_i, u_i)_{i \in I} \) is said to be weakly transfer quasi-continuous if, whenever \( x \in X \) is not an equilibrium, there exists a strategy profile \( y \in X \) and a neighborhood \( \mathcal{V}_x \) of \( x \) so that for every \( x' \in \mathcal{V}_x \), there exists a player \( i \) such that \( u_i(y_i, x'_{-i}) > u_i(x') \).

Weak transfer quasi-continuity is also called single-deviation property in Reny (2009), which is weaker than the better-reply security. It may be remarked that strong diagonal transfer quasiconcavity cannot be replaced by conventional quasiconcavity for weak transfer quasi-continuity to guarantee the existence of equilibrium. Indeed, Reny (2009) shows with the aid of a three-person game that a game that is compact, quasiconcave, and weakly transfer quasi-continuous need not have a pure strategy Nash equilibrium.\(^10\)

Note that, while the weak transfer quasi-continuity (or lower single-deviation property) explicitly exhibits switchings among agents, the diagonal transfer continuity for aggregator function \( U \)

\(^9\)A game \( G = (X_i, u_i)_{i \in I} \) is said to be strongly diagonal transfer quasiconcave if for any finite subset \( \{y^1, ..., y^m\} \subseteq X \), there exists a corresponding finite subset \( \{x^1, ..., x^m\} \subseteq X \) such that for any subset \( \{x^1, x^2, ..., x^s\} \subseteq \{x^1, x^2, ..., x^m\}, 1 \leq s \leq m \), and any \( x \in co\{x^1, x^2, ..., x^k\} \), there exists \( y \in \{y^1, ..., y^k\} \) so that \( u_i(x, x_{-i}) \leq u_i(x) \) for all \( i \in I \).

\(^10\)However, for two-person games where the strategy sets are subsets of the real line, Prokopovych (2013) shows that the conclusion holds.
internalizes (implicitly allows) the switchings. Such implicit switchings have an advantage that it may become easy to check “upsetting” relations, especially for “complementary discontinuities”, by which a downward jump in one player’s payoff can always be accompanied by an upward jump in another player’s payoff (cf. Maskin and Dasgupta (1986) and Simon (1987)), in addition to having the advantage that it can be used to show the existence of mixed strategy equilibrium as shown in Prokopovych and Yannelis (2014).

Although this “upsetting” relation is ensured, it may not still be sufficient for the existence of equilibrium unless imposing some forms of quasiconcavity, transitivity, or monotonicity. This is why, to make this upsetting relation sufficient for the existence of equilibrium, most the existing results impose additional assumptions such as convexity of strategy spaces and quasiconcavity/transitivity/monotonicity-related conditions of payoffs in order to use a fixed point theorem. But, such kind of combined conditions can only apply to economic models with linear (convex) structures and most of them are only sufficient but may not be necessary for the existence of equilibrium. As mentioned in the introduction, to apply a fixed-point theorem to prove the existence of equilibrium, they all need to assume some forms of quasiconcavity (or transitivity/monotonicity) and continuity of payoffs, in addition to compactness and convexity of strategy spaces.

As such, a single upsetting transfer along may not be enough to guarantee the existence of an equilibrium, some recursive (sequential multiple) upsetting transfers starting from a strategy profile \( y^0 \) may be needed for the existence of equilibrium without imposing any quasiconcavity/monotonicity-related conditions. This is just similar to extending the weak axiom of revealed preference (WARP) to strong axiom of revealed preference (SARP) in order to fully reveal individuals’ preferences. With such recursively upsetting relations, we are able to allow not only sequential transfers, but also the switchings of players in any stage of upsetting transfers and for different strategy profiles in a neighborhood. As a result, such a weak notion of transfer continuity may become both necessary and sufficient for the existence of Nash equilibria for any games. Indeed, we will show that the above intuition and insights turn to be correct. It is the two requirements—securing upsetting relation and recursive transfers that characterize the existence or nonexistence of an equilibrium.

3 Existence of Pure Strategy Nash Equilibria

In this section we investigate the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs/preferences. We will provide two existence theorems. One is based on aggregate payoffs and the other is based on individuals’ payoffs/preferences.
3.1 Existence of Equilibrium in Games with the Aggregate Payoffs

In this subsection we assume that individuals’ preferences can be represented by payoff functions. We consider a mapping of individual payoffs into the aggregator function $U : X \times X \rightarrow \mathbb{R}$. The aggregator function approach is pioneered by Nikaido and Isoda (1955), and is also used by Baye, Tian, and Zhou (1993). Dasgupta and Maskin (1986) use a similar approach to prove the existence of mixed strategy Nash equilibrium in games with discontinuous payoff functions. An advantage of this approach is that it internalizes the switchings among players so that checking “upsetting” relations are relatively easier due to the complementarity that a game generally has.

**Definition 3.1** (Recursive Upsetting) A strategy profile $y^0 \in X$ is said to be **recursively upset by** $z \in X$ if there exists a finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ such that $U(y^1, y^0) > U(y^0, y^0), U(y^2, y^1) > U(y^1, y^1), \ldots, U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$.

For convenience, we say that $y^0$ is **directly upset by** $z$ if $z \succ y^0$ and **indirectly upset by** $z$ when $m > 1$. Recursive upsets says that a strategy profile $y^0$ can be directly or indirectly upset by a strategy profile $z$ through sequential deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}\}$ in a recursive way that $y^0$ is upset by $y^1, y^1$ is upset by $y^2, \ldots$, and $y^{m-1}$ is upset by $z$.

**Definition 3.2** (Recursive Diagonal Transfer Continuity) A game $G = (X_i, u_i)_{i \in I}$ is said to be **recursively diagonal transfer continuous** if, whenever $x$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of $x$ such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for every $z$ that recursively upsets $y^0$.

In words, recursive diagonal transfer continuity implies that, whenever $x$ is not an equilibrium, there exists a starting point $y^0$ such that every recursive upsetting chain $\{y^0, y^1, y^2, \ldots, y^m\}$ disproves the possibility of an equilibrium in a sufficiently small neighborhood of $x$, i.e., all points in the neighborhood are upset by all securing strategy profiles that directly or indirectly upset $y^0$. This implies that, if the game is not recursively diagonal transfer continuous, then there is a nonequilibrium strategy profile $x$ such that for every $y^0 \in X$ and every neighborhood $\mathcal{V}_x$ of $x$, some deviation strategy profiles in the neighborhood cannot be upset by a securing strategy profile $z$ that directly or indirectly upsets $y^0$.

In the definition of recursive diagonal transfer continuity, $x$ is transferred to $y^0$ that could be any point in $X$. We can similarly define $m$-recursive diagonal transfer continuity. A game $G = (X_i, u_i)_{i \in I}$ is **$m$-recursively diagonal transfer continuous** if the phrase “for every $z$ that recursively upsets $y^0$” in the above definition is replaced by “for every $z$ that $m$-recursively upsets $y^0$”. Thus, a game $G = (X_i, u_i)_{i \in I}$ is recursively diagonal transfer continuous if it is $m$-recursively diagonal transfer continuous on $X$ for all $m = 1, 2, \ldots$. Then diagonal transfer continuity, introduced by Baye, Tian, and Zhou (1993) implies 1-recursively diagonal transfer continuity,
and weak transfer quasi-continuity introduced by Nessah and Tian (2008) implies 1-recursive weak transfer quasi-continuity (by letting \( y^0 = x \)), respectively. Since they are in form of single transfer, diagonal transfer continuity or weak transfer quasi-continuity is neither necessary nor sufficient, and thus some form of quasiconcavity such as (strong) diagonal transfer quasiconcavity is needed for the existence of equilibrium as studied in Baye, Tian, and Zhou (1993) and Nessah and Tian (2008).

**Remark 3.1** Under recursive diagonal transfer continuity, when \( U(z, y^{m-1}) > U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}) \), ..., \( U(y^1, y^0) > U(y^0, y^0) \), we have not only \( U(z, V_x) > U(V_x, V_x) \), but also \( U(y^{m-1}, V_x) > U(V_x, V_x) \), ..., \( U(y^1, V_x) > U(V_x, V_x) \). That is, any chain of securing strategy profiles \( \{y^1, y^2, ..., y^{m-1}\} \) obtained by truncating a recursive upsetting chain \( \{y^1, y^2, ..., y^{m-1}, z\} \) is also a recursive upsetting chain, including \( y^1 \).

**Remark 3.2** Recursive diagonal transfer continuity neither implies nor is implied by continuity for games with two or more players.\(^\text{11}\) This becomes clear when one sees recursive diagonal transfer continuity is a necessary and sufficient condition for the existence of pure strategy Nash equilibrium while continuity of the aggregate payoff function is neither a necessary nor sufficient condition for the existence of pure strategy Nash equilibrium.

Before we formally state our main result, we give a brief description and explain why recursive diagonal transfer continuity of an upsetting relation \( \succ \) ensures the existence of pure strategy Nash equilibrium in a compact game. When a game fails to have a pure strategy Nash equilibrium on a compact strategy space \( X \), by recursive diagonal transfer continuity, for every \( x \), there is a starting transfer strategy profile \( y^0 \) such that all points in a neighborhood of \( x \) will be upset by any \( z \) that recursively upsets \( y^0 \). Then there are finite strategy profiles \( \{x^1, x^2, ..., x^n\} \) whose neighborhoods cover \( X \). Thus, all of the points in a neighborhood, say \( V_{x1} \), will be upset by a corresponding deviation profile \( z^1 \), which means \( z^1 \) cannot be in \( V_{x1} \). If it is in some other neighborhood, say, \( V_{x2} \), then it can be shown that \( z^2 \) will upset all strategy profiles in the union of \( V_{x1} \) and \( V_{x2} \) so that \( z^2 \) is not in the union of \( V_{x1} \) and \( V_{x2} \). Suppose \( z^2 \in V_{x3} \). Then we can similarly show that \( z^3 \) is not in the union of \( V_{x1} \), \( V_{x2} \) and \( V_{x3} \). With this recursive process going on, we can finally show that \( z^n \notin V_{x1} \cup V_{x2} \cup ..., \cup V_{xn} \), which means \( z^n \) will not be in the strategy space \( X \), resulting in a contradiction.

Now we are ready to state our main result on the existence of pure strategy Nash equilibrium in games.

\(^{11}\)In one-player games recursive diagonal transfer continuity is equivalent to the player’s utility function possessing a maximum on a compact set, and consequently it implies transfer weak upper continuity introduced in Tian and Zhou (1995), which is weaker than continuity.
Suppose the strategy space $X$ of a game $G = (X_i, u_i)_{i \in I}$ is compact. Then, if the game is recursively diagonal transfer continuous on $X$, it possesses a pure strategy Nash equilibrium.

**Proof.** Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium. Then, for each $x \in X$, there exists $y \in X$ such that $U(y, x) > U(x, x)$, and thus, by recursive diagonal transfer continuity, there exists $y^0$ and a neighborhood $\mathcal{V}_x$ such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ whenever $y^0 \in X$ is recursively upset by $z$, i.e., for every sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ with $U(z, y^{m-1}) > U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$, we have $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. Since there is no equilibrium by the contrapositive hypothesis, $y^0$ is not an equilibrium and thus, by recursive diagonal transfer continuity, such a sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ exists for some $m \geq 1$.

Since $X$ is compact and $X \subseteq \bigcup_{x \in X} \mathcal{V}_x$, there is a finite set $\{x^1, \ldots, x^L\}$ such that $X \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$. For each of such $x^i$, the corresponding initial deviation profile is denoted by $y^{0i}$ so that $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ whenever $y^{0i}$ is recursively upset by $z^i$.

Since there is no equilibrium, for each of such $y^{0i}$, there exists $z^i$ such that $U(z^i, y^{0i}) > U(y^{0i}, y^{0i})$, and then, by 1-recursive diagonal transfer continuity, we have $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$. Now consider the set of securing strategy profiles $\{z^1, \ldots, z^L\}$. Then, $z^i \notin \mathcal{V}_{x^i}$, otherwise, by $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$, we will have $U(z^i, z^i) > U(z^i, z^i)$, a contradiction. So we must have $z^i \notin \mathcal{V}_{x^i}$.

Without loss of generality, suppose $z^1 \in \mathcal{V}_{x^2}$. Since $U(z^2, z^1) > U(z^1, z^1)$ by noting that $z^1 \in \mathcal{V}_{x^2}$ and $U(z^1, y^{01}) > U(y^{01}, y^{01})$, then, by 2-recursive diagonal transfer continuity, we have $U(z^2, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$. Also, $U(z^2, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$. Thus $U(z^2, \mathcal{V}_{x^2}) \cup U(z^2, \mathcal{V}_{x^2}) = U(\mathcal{V}_{x^2} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^2})$, and consequently $z^2 \notin \mathcal{V}_{x^2} \cup \mathcal{V}_{x^2}$.

Again, without loss of generality, suppose $z^2 \in \mathcal{V}_{x^3}$. Since $U(z^3, z^2) > U(z^2, z^2)$ by noting that $z^2 \in \mathcal{V}_{x^3}$, $U(z^2, z^1) > U(z^1, z^1)$, and $U(z^2, y^{01}) > U(y^{01}, y^{01})$, by 3-recursive diagonal transfer continuity, we have $U(z^3, \mathcal{V}_{x^3}) > U(\mathcal{V}_{x^3}, \mathcal{V}_{x^3})$. Also, since $U(z^3, z^2) > U(z^2, z^2)$ and $U(z^2, y^{02}) > U(y^{02}, y^{02})$, by 2-recursive diagonal transfer continuity, we have $U(z^3, \mathcal{V}_{x^3}) > U(\mathcal{V}_{x^3}, \mathcal{V}_{x^3})$. Thus, we have $U(z^3, \mathcal{V}_{x^3}) \cup U(z^2, \mathcal{V}_{x^3}) > U(z^3, \mathcal{V}_{x^3}) \cup U(z^2, \mathcal{V}_{x^3}) \cup \mathcal{V}_{x^3}$, and consequently $z^3 \notin \mathcal{V}_{x^3} \cup \mathcal{V}_{x^3} \cup \mathcal{V}_{x^3}$.

With this recursive process going on, for $k = 3, \ldots, L$, we can show that $z^k \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots \cup \mathcal{V}_{x^k}$, i.e., $z^k$ is not in the union of $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \ldots, \mathcal{V}_{x^k}$. In particular, for $k = L$, we have $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots \cup \mathcal{V}_{x^L}$ and so $z^L \notin X \subseteq \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots \cup \mathcal{V}_{x^L}$, a contradiction. $\blacksquare$

**Remark 3.3** Recursive diagonal transfer continuity is in fact also a necessary condition for any game to possess a pure strategy Nash equilibrium. To see this, first note that, if $x^* \in X$ is a pure strategy Nash equilibrium of a game $G$, we must have $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$,
which is obtained by summing up $u_i(y_i, x^*_i) \leq u_i(x^*) \quad \forall \ y_i \in X_i$ for all players. Let $x^*$ be a pure strategy Nash equilibrium and $U(y, x) > U(x, x)$ for $x, y \in X$. Let $y^0 = x^*$ and $\mathcal{V}_x$ be a neighborhood of $x$. Then, it is impossible to find any securing strategy profile $y^1$ such that $U(y^1, y^0) > U(y^0, y^0)$, and thus the recursive diagonal transfer continuity holds trivially.

Although recursive diagonal transfer continuity is necessary for the existence of a pure strategy Nash equilibrium, but it may not be sufficient for the existence of a pure strategy Nash equilibrium when a strategy space is noncompact. As such, unlike some people thought, recursive diagonal transfer continuity cannot be regarded as being equivalent to the definition of Nash equilibrium without any other restrictions such as the compactness of strategy space. To see this, consider the following counterexample.

**Example 3.1** Consider the following two-person game with $X_1 = X_2 = (0, 1)$ and the payoff functions given by $u_i(x_1, x_2) = x_i \ i = 1, 2$.

The game clearly does not possess a pure strategy Nash equilibrium. However, it is recursively diagonal transfer continuous on $X$.

Indeed, for any two strategy profiles $x, y \in X$ with $U(y, x) > U(x, x)$, choose $\epsilon > 0$ such that $(x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \subset X$. Let $y^0 = (x_1 + \epsilon, 1) \times (x_2 + \epsilon, 1) \in X$ and $\mathcal{V}_x \subseteq (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon)$. Note that the $U(y, x) = y_1 + y_2$. Then, for any finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ with $U(y^1, y^0) > U(y^0, y^0), U(y^2, y^1) > U(y^1, y^0), \ldots, U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, i.e., $z_1 + z_2 > y_1^{m-1} + y_2^{m-1} > \ldots > y_1^1 + y_2^1$, we have $U(z, x') = z_1 + z_2 > y_1^0 + y_2^0 > x'_1 + x'_2$ for all $x' \in \mathcal{V}_x$. Thus, $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$, which means the game is recursively diagonal transfer continuous on $X$.

The above theorem assumes that the strategy space of a game is compact. This may still be a restrictive assumption since strategy space of a game may not be closed or bounded when it is finite dimensional. For instance, it is well known that Walrasian mechanism can be regarded as a generalized game. However, when preferences are strictly monotone, excess demand functions are not well defined for zero prices, and then we cannot use Theorem 3.1 to show the existence of competitive equilibrium.

In the following we show that the compactness of strategy space in Theorem 3.1 can also be relaxed.\(^{12}\) To do so, we first introduce the following stronger version of recursive diagonal transfer continuity.

\(^{12}\)I thank David Rahman for raising this issue to me. Thanks also to Adam Wong for pointing out a misstatement in an earlier version of Theorem 3.2.
**Definition 3.3** Let $B$ be a subset of $X$. A game $G = (X_i, u_i)_{i \in I}$ is said to be recursively diagonal transfer continuous on $X$ with respect to $B$ if, whenever $x$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of $x$ such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in $B$ and (2) $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for every finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$.

Condition (1) in the above definition ensures that if a strategy profile $x$ is not an equilibrium for the game $G = (X_i, u_i)_{i \in I}$, it must not be an equilibrium when the strategy space is constrained to be $B$. Note that, while $\{y^1, \ldots, y^m\}$ are required to be in $B$, $y^0$ is not necessarily in $B$ but can be any point in $X$. Also, when $B = X$, recursive diagonal transfer continuity on $X$ with respect to $B$ reduces to recursive diagonal transfer continuity on $X$. We then have the following theorem that generalizes Theorem 3.1 by relaxing compactness of games.

The following theorem shows the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and an arbitrary payoff function that may be discontinuous or nonquasiconcave.

**Theorem 3.2** If there exists a compact set $B \subseteq X$ such that the game $G = (X_i, u_i)_{i \in I}$ is recursively diagonal transfer continuous on $X$ with respect to $B$, then it possesses a pure strategy Nash equilibrium.

**Proof.** The proof is essentially the same as that of Theorem 3.1 and we just outline the proof here. To show the existence of a pure strategy Nash equilibrium on $X$, it suffices to show that the game possesses a pure strategy Nash equilibrium $x^*$ in $B$ if it is recursively diagonal transfer continuous on $X$ with respect to $B$. Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium in $B$. Then, since the game $G$ is recursively diagonal transfer continuous on $X$ with respect to $B$, for each $x \in B$, there exists $y^0$ and a neighborhood $\mathcal{V}_x$ such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in $B$ and (2) $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for any finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$. Since there is no equilibrium by the contrapositive hypothesis, $y^0$ is not an equilibrium and thus, by recursive diagonal transfer continuity on $X$ with respect to $B$, such a sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ exists for some $m \geq 1$.

Since $B$ is compact and $B \subseteq \bigcup_{x \in X} \mathcal{V}_x$, there is a finite set $\{x^1, \ldots, x^L\} \subseteq B$ such that $B \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$. For each of such $x^i$, the corresponding initial deviation profile is denoted by $y^{0i}$ so that $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ whenever $y^{0i}$ is recursively upset by $z^i$ through any finite subset of securing strategy profiles $\{y^{1i}, \ldots, y^{mi}\} \subset B$ with $y^{mi} = z^i$. Then, by the same argument as
Similarly, recursive diagonal transfer continuity on $X$ with respect to $B$ is also a necessary condition for any game to possess a pure strategy Nash equilibrium. Indeed, suppose $x^*$ is a pure strategy Nash equilibrium. Let $B = \{x^*\}$. Then, the set $B$ is clearly compact. Now, for any $x, y \in X$ such that $U(y, x) > U(x, x)$ for $x, y \in X$, let $y^0 = x^*$ and $V_x$ be a neighborhood of $x$. Since $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$ and $y^0 = x^*$ is a unique element in $B$, it is no other securing strategy profiles $y^1$ such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the game is recursively diagonal transfer continuous on $X$ with respect to $B$.

**Remark 3.4** Similarly, recursive diagonal transfer continuity on $X$ with respect to $B$ is also a necessary condition for any game to possess a pure strategy Nash equilibrium. Indeed, suppose $x^*$ is a pure strategy Nash equilibrium. Let $B = \{x^*\}$. Then, the set $B$ is clearly compact. Now, for any $x, y \in X$ such that $U(y, x) > U(x, x)$ for $x, y \in X$, let $y^0 = x^*$ and $V_x$ be a neighborhood of $x$. Since $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$ and $y^0 = x^*$ is a unique element in $B$, it is no other securing strategy profiles $y^1$ such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the game is recursively diagonal transfer continuous on $X$ with respect to $B$.

**Example 3.2** Consider again Example 3.1. We know the game is recursively diagonal transfer continuous on $X$. However, there does not exist any compact set $B \subset X$ such that the game is recursively diagonal transfer continuous on $X$ with respect to $B$. To see this, let $\bar{y}$ be a maximal element of $U(y, y) = y_1 + y_2$ on $B$. Choosing $x \in X \setminus B$ with $x_1 + x_2 > \bar{y}_1 + \bar{y}_2$, we then have $U(x, x) > U(y', x)$ for all $y' \in B$. Also, for any $y$ with $y_1 + y_2 > x_1 + x_2$, we have $U(y, x) > U(x, x)$. We then cannot find any strategy profile $y^0 \in X$ and a neighborhood $V_x$ of $x$ such that such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in $B$ and (2) $U(z, V_x) > U(V_x, V_x)$ for every deviation profile $z \in B$ that upsets directly or indirectly $y^0$. We show this by considering two cases.

Case 1. $y^0_1 + y^0_2 \geq \bar{y}_1 + \bar{y}_2$. $y^0$ can be upset by a strategy profile $y' \in X \setminus B$ with $y'_1 + y'_2 > y^0_1 + y^0_2$, but it cannot be upset by any strategy profile in $B$.

Case 2. $y^0_1 + y^0_2 < \bar{y}_1 + \bar{y}_2$. For strategy profile $z \in D$ that upsets directly or indirectly $y^0$, we have $U(z, V_x) < U(V_x, V_x)$, but not $U(z, V_x) > U(V_x, V_x)$.

Thus, we cannot find any strategy profile $y^0 \in X$ and a neighborhood $V_x$ of $x$ such that such that (1) whenever $y^0$ is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in $B$ and (2) $U(z, V_x) > U(V_x, V_x)$ for every deviation profile $z$ that recursively upsets $y^0$. Hence, the game is not recursively diagonal transfer continuous on $X$ with respect to $B$. It of course, by Theorem 3.2, there is no pure strategy Nash equilibrium on $X$.

On the other hand, the following example shows that, although the strategy space is an open unit interval, highly discontinuous and nonquasiconcave, we can use Theorem 3.2 to argue the existence of equilibrium.

**Example 3.3** Consider a game with $n = 2$, $X_1 = X_2 = (0, 1)$ that is an open unit interval set, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 
1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\
0 & \text{otherwise}
\end{cases} \quad i = 1, 2.$$
where $Q = \{x \in (0, 1) : x$ is a rational number$\}$. Then the game is not compact nor quasiconcave either. It is not weakly transfer quasi-continuous either (so, as shown in Nessah and Tian (2008), it is not diagonally transfer continuous, better-reply secure, or weakly transfer continuous either). To see this, consider any nonequilibrium $x$ that consists of irrational numbers. Then, for any neighborhood $V_x$ of $x$, choosing $x' \in V_x$ with $x'_1 \in Q$ and $x'_2 \in Q$, we have $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$ and $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$ for any $y \in X$. So the game is not weakly transfer quasi-continuous. Thus, there is no existing theorem that can be applied.

However, it is recursively diagonal transfer continuous on $X$. Indeed, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let $y^0$ be any vector with rational numbers, $B = \{y^0\}$, and $V_x$ be a neighborhood of $x$. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile $y^1$ such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the game is recursively diagonal transfer continuous on $X$ with respect to $B$. Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium. In fact, the set of pure strategy Nash equilibria consists of all rational numbers on $(0, 1)$.

In general, the weaker the conditions in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game. For this reason it is useful to provide some sufficient conditions for recursive diagonal transfer continuity. It is well known that the convexity of preferences can be substituted for transitivity of preferences in maximizing preferences of individuals. The following results show that this is true also for economic games.

**Definition 3.4** (Deviation Transitivity) $G = (X_i, u_i)_{i \in I}$ is said to be deviational transitive if $U(y^2, y^1) > U(y^1, y^1)$ and $U(y^1, y^0) > U(y^0, y^0)$ imply that $U(y^2, y^0) > U(y^0, y^0)$. That is, the upsetting dominance relation is transitive.

We then have the following result without assuming the convexity of strategy space and imposing any form of quasiconcavity.

**Proposition 3.1** Suppose $G = (X_i, u_i)_{i \in I}$ is compact and deviational transitive. If $G$ is 1-recursively diagonal transfer continuous, then there exists a pure strategy Nash equilibrium.

**Proof.** We only need to show that, when $G$ is deviational transitive, 1-recursively diagonal transfer continuity implies $m$-recursively diagonal transfer continuity for $m \geq 1$. Suppose $x$ is not an equilibrium. Then, by 1-recursively diagonal transfer continuity, there exists a strategy profile $y^0 \in X$ and a neighborhood $V_x$ of $x$ such that $U(z, V_x) > U(V_x, V_x)$ whenever $U(z, y^0) > U(y^0, y^0)$ for any $z \in X$.

Now, for any sequence of deviation profiles $\{y^1, \ldots, y^{m-1}, y^m\}$, if $U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$, we then have
by deviation transitivity of \( U \), and thus by 1-recursive diagonal transfer continuity, \( U (y^m, V_x) > U (V_x, V_x) \). Since \( m \) is arbitrary, \( G \) is recursively diagonal transfer continuous.

### 3.2 Existence of Equilibrium in Games with Individuals’ Preferences

The aggregator function approach adopted in the previous subsection captures the idea of using multiple (finite) securing strategies in a way of transferring upsetting relation from one strategy or agent to another strategy or agent so that the upsetting relations can be preserved locally. This approach has a number of advantages such as, it implicitly allows for, or internalizes, the switchings among agents in an upsetting relation so that the proof is relatively simpler, and it is relatively easy to check the upsetting relations due to the complementarity that secures payoffs. The aggregator function approach, however, also has a number of disadvantages. First, we need to assume that the preferences of each player can be represented by a payoff function. Secondly, we need to assume that the number of players is either finite or countable. Thirdly, it is a cardinal approach, but not an ordinal approach. While monotonic transformations preserve individuals’ upsetting relations unchanged, it may not be true after aggregation, i.e., with a mapping by the aggregator function, a deviation strategy profile \( y \) may no longer upset a strategy profile \( x \) after some monotonic transformation although \( y \) upsets \( x \) before the monotonic transformation. Fourthly, the aggregator function approach only reveals the total upsetting relations, it is less clear about individuals’ strategic interactions, and thus it lacks a more natural game theoretical analysis.

Nevertheless, the method developed in this paper does not necessarily need to define the aggregator function \( U \). What matters is the concept of upsetting and multiple (finite) securing strategies. We can also show the existence of equilibrium in terms of individuals’ payoffs or preferences. The individual preference approach overcomes all the shortcomings above and has the following advantages: (1) Preferences may not be represented by a payoff function; (2) the set of players can be arbitrary, (3) monotonic transformations preserve individuals’ upsetting relations, and (4) the analysis reveals more clear individuals’ strategic interactions.

**Definition 3.5** A game \( G = (X_i, \succ_i)_{i \in I} \) is said to be **recursively weakly transfer quasi-continuous** if, whenever \( x \in X \) is not an equilibrium, there exists a strategy profile \( y^0 \in X \) (possibly \( y^0 = x \)) and a neighborhood \( \mathcal{V}_x \) of \( x \) such that for every \( x' \in \mathcal{V}_x \) and every finite set of deviation strategy profiles \( \{y^1, y^2, \ldots, y^{m-1}, z\} \) with \((y^1_{i_1}, y^0_{-i_1}) \succ_{i_1} y^0 \) for some \( i_1 \in I \), \((y^2_{i_2}, y^1_{-i_2}) \succ_{i_2} y^1 \) for some \( i_2 \in I \), \ldots, \((z_{i_m}, y^{m-1}_{-i_m}) \succ_{i_m} y^{m-1} \) for some \( i_m \in I \), there exists player \( i \in I \) such that \((z_i, x'_{-i}) \succ_i x' \).

Note that, the notion of recursive weak transfer quasi-continuity allows the switchings (transfers) among players in the process of recursive upsetting transfers and at every point in the neighborhood \( \mathcal{V}_x \). As such, weak transfer quasi-continuity is a weak form of continuity where a prof-
itable deviation exists not for every point of the neighborhood of $x$ but only for each of the nonequilibrium points belonging to the neighborhood. Similarly, we can define the notions of $m$-recursive weak transfer quasi-continuity and recursive weak transfer quasi-continuity on $X$ with respect to $B$ for $B \subset X$. Note that, if a game is weakly transfer quasi-continuous, it is 1-recursively weakly transfer continuous by letting $y^0 = x$. But, the converse may not be true. Thus, weak transfer quasi-continuity is stronger than 1-recursive weak transfer quasi-continuity.

Now it is worth noticing what is the relationship between recursive weak transfer quasi-continuity defined above and the recursive diagonal transfer continuity of a game defined in the previous subsection. This relationship may help us to simplify the proof, as it will be seen below. By using the upsetting binary relation $\succ$ defined in the previous section, we can define recursive transfer continuity accordingly.

**Definition 3.6** The “upsetting” relation $\succ$ is said to be **recursively transfer continuous** if, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $V_x$ of $x$ such that $z \succ V_x$ for any $z$ that recursively upsets $y^0$.

**Lemma 3.1** Let $\succ$ be the upsetting relation defined by (1). We then have

1. A game $G = (X_i, \succ_i)_{i \in I}$ is recursively weakly transfer quasi-continuous on $X$ if and only if the “upsetting” relation $\succ$ is recursively transfer continuous on $X$.
2. A game $G = (X_i, \succ_i)_{i \in I}$ is weakly transfer quasi-continuous on $X$ if and only if the upsetting relation $\succ$ is transfer continuous on $X$.

The proof is straightforward, and thus it is omitted here.

By Lemma 3.1, we then have the following result that guarantees the existence of pure strategy Nash equilibrium in qualitative games with compact strategy spaces and general preferences.

**Theorem 3.3** Suppose $G = (X_i, \succ_i)_{i \in I}$ is compact. If it is recursively weakly transfer quasi-continuous, then the game $G$ possesses a pure strategy Nash equilibrium on $X$.

Similarly, we can show recursive transfer continuity is also necessary for the existence of Nash equilibrium. Recursive transfer continuity is needed for the existence of equilibrium.

The compactness of strategy space in Theorem 3.3 can also be removed. The following theorem shows the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and preferences that may not be represented by a payoff function, nontotal/nontransitive, non-convex or discontinuous.

**Definition 3.7** Let $B$ be a subset of $X$. A game $G = (X_i, u_i)_{i \in I}$ is said to be **recursively weakly transfer quasi-continuous on $X$ with respect to $B$** if, whenever $x$ is not an equilibrium,
there exists a strategy profile \( y^0 \in X \) (possibly \( y^0 = x \)) and a neighborhood \( V_x \) of \( x \) such that

1. whenever \( y^0 \) is upset by a strategy profile in \( X \setminus B \), it is upset by a strategy profile in \( B \) and
2. for every \( x' \in V_x \) and every finite set of deviation strategy profiles \( \{ y^1, y^2, \ldots, y^{m-1}, z \} \) with \((y^1_{i_1}, y^0_{-i_1}) \succ_i y^0 \) for some \( i_1 \in I \), \((y^2_{i_2}, y^1_{-i_2}) \succ_i y^1 \) for some \( i_2 \in I \), \ldots, \((z_{i_m}, y^{m-1}_{-i_m}) \succ_i y^{m-1} \) for some \( i_m \in I \), there exists player \( i \in I \) such that \((z_i, x'_{-i}) \succ_i x' \).

We then have the following theorem.

**Theorem 3.4** A game \( G = (X_i, \succ_i)_{i \in I} \) has a pure strategy Nash equilibrium if and only if there exists a compact set \( B \subseteq X \) such that the game is recursively weakly transfer quasi-continuous on \( X \) with respect to \( B \).

Similarly, we can provide some new sufficient conditions by using deviation transitivity.

**Definition 3.8** (Deviation Transfer Transitivity) \( G = (X_i, \succ_i)_{i \in I} \) is said to be deviational transfer transitive if for \( y^0, y^1, y^2 \in X \), there is some \( i \in I \) such that \((y^2_i, y^1_{-i}) \succ_i y^1 \) and \((y^1_i, y^0_{-i}) \succ_i y^0 \) imply that \((y^2_i, y^0_{-i}) \succ_i y^0 \).

Similar to Proposition 3.1, replacing strong diagonal transfer quasiconcavity by deviation transfer transitivity, we have the following result without assuming the convexity and convexity of strategy space and imposing any form of quasiconcavity.

**Proposition 3.2** Suppose \( G = (X_i, \succ_i)_{i \in I} \) is compact and deviational transfer transitive. If \( G \) is 1-recursively weak transfer quasi-continuous, then there exists a pure strategy Nash equilibrium point.

It is clear that a sufficient conditions for a two-person game \( G = (X_1, X_2; u_1, u_2) \) to be diagonally monotonic if \( u_i(x_i, x_{-i}) \) is either increasing or decreasing in \( x_i \). Indeed, consider the case where \( u_i \) is increasing in \( x_i \). We need to show \( u_i(z_i, y_{-i}) > u_i(y) \) and \( u_i(y_i, x_{-i}) > u_i(x) \) imply \( u_i(z_i, x_{-i}) > u_i(x) \). For \( z, y, x \in X \), by monotonicity of \( u_i \), we have \( z_i > y_i \) when \( u_i(z_i, y_{-i}) > u_i(y) \) and \( y_i > x_i \) when \( u_i(y_i, x_{-i}) > u_i(x) \), and thus we have \( z_i > y_i > x_i \). Then, we have \( u_i(z_i, x_{-i}) \geq u_i(y_i, x_{-i}) > u_i(x_i, x_{-i}) \), which means \( G \) is deviational transitive.

Notice that better-reply security and its strength by many others\(^{13}\) imply weak transfer quasi-continuity which in turn implies 1-recursive weak transfer quasi-continuity.\(^{14}\) Also, payoff security and better-reply are closely related to transfer lower continuity defined in Tian (1992a) and the transfer reciprocal upper semicontinuity. Indeed, as shown in Lemmas 1 and 2 in Prokopovych (2011), a game is payoff secure if and only if it is transfer lower semicontinuous, and better-reply security is equivalent to the transfer reciprocal upper semicontinuity in payoff secure games.

\(^{13}\)Reny (1999) shows that a game \( G = (X_i, u_i)_{i \in I} \) is better-reply secure if it is payoff secure and reciprocally upper semi-continuous. Bagh and Jofre (2006) further show that \( G = (X_i, u_i)_{i \in I} \) is better-reply secure if it is payoff secure and weakly reciprocal upper semi-continuous.

\(^{14}\)The proofs of these implications can be found in Nessah and Tian (2008).
4 Existence of Symmetric Pure Strategy Nash Equilibria

The techniques developed in the previous section can be used to study the existence of symmetric pure strategy Nash equilibrium. Throughout this section, we assume that the strategy spaces for all players are the same. As such, let \( X_0 = X_1 = \ldots = X_n \). If in addition, \( u_1(y, x, \ldots, x) = u_2(x, y, x, \ldots, x) = \ldots, u_n(x, \ldots, x, y) \) for all \( x, y \in X \), we say that \( G = (X_i, u_i)_{i \in I} \) is a quasi-symmetric game.

**Definition 4.1** A Nash equilibrium \((x_1^*, \ldots, x_n^*)\) of a game \( G \) is said to be symmetric if \( x_1^* = \ldots = x_n^* \).

For convenience, we denote, for each player \( i \), and for all \( x, y \in X_0 \), \( u_i(x, \ldots, y, \ldots, x) \) the function \( u_i \) evaluated at the strategy in which player \( i \) chooses \( y \) and all others choose \( x \).

Define a quasi-symmetric function \( \psi : X_0 \times X_0 \to \mathbb{R} \) by

\[
\psi(y, x) = u_i(x, \ldots, y, \ldots, x). \tag{4}
\]

Since \( G \) is quasi-symmetric, \( x^* \) is a symmetric pure strategy Nash equilibrium if and only if \( \psi(y, x^*) \leq \psi(x^*, x^*) \) for all \( y \in X_i \).

**Definition 4.2** \( \psi : X_0 \times X_0 \to \mathbb{R} \) is said to be recursively diagonal transfer continuous if, whenever \( x \in X \) is not equilibrium, there exists a strategy profile \( y^0 \in X \) (possibly \( y^0 = x \)) and a neighborhood \( V_x \) of \( x \) such that \( \psi(z, V_x) > \psi(V_x, V_x) \) for any \( z \) that recursively upsets \( y^0 \).

We then have the following theorem.

**Theorem 4.1** Suppose a game \( G = (X_i, u_i)_{i \in I} \) is quasi-symmetric and compact. Then it possesses a symmetric pure strategy Nash equilibrium if and only if \( \psi : X_0 \times X_0 \to \mathbb{R} \) is recursively diagonal transfer continuous on \( X \).

**Proof.** The proof is the same as that of Theorem 3.1 provided \( U \) is replaced by \( \psi \), thus it is omitted here. ■

Similar to Proposition 3.1, we have the following proposition.

**Proposition 4.1** Suppose a game \( G = (X_i, u_i)_{i \in I} \) is quasi-symmetric, compact, and devotional transitive. If \( \psi(x, y) \) defined by (4) is 1-recursively diagonal transfer continuous on \( X \), then it possesses a pure strategy symmetric Nash equilibrium.
5 Conclusion

This paper fully characterizes the existence of pure strategy Nash equilibrium in games with arbitrary strategy space and payoffs/preferences. We establish a condition, called recursive diagonal transfer continuity for aggregate payoffs or recursive weak transfer quasi-continuity for individuals’ preferences, which is not only sufficient but also necessary for the existence of pure strategy Nash equilibrium.

The relation of recursive transfer continuities and direct transfer continuities is similar to that of the weak axiom of revealed preference (WARP) and strong axiom of revealed preference (SARP). Directly revealing a preference by WARP is not enough to fully reveal individuals’ preferences, and then one may resort to indirectly revealing a preference by SARP to fully reveal an individual rational behavior. Similarly, diagonal transfer continuity or better-reply security alone is not enough to guarantee the existence of Nash equilibrium, one then may need to use a notion of recursive transfer continuity to fully characterize the existence of equilibrium.

The basic transfer method we adopted here has been already systematically developed in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Baye, Tian, and Zhou (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. These papers, especially Zhou and Tian (1992), develop three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to study the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian (1993) and Tian and Zhou (1995)). Since an “upsetting” binary relation is defined so that the existence of a Nash equilibrium is equivalent to the existence of a maximal element under this relation, the characterization results on the existence of maximal elements under ordered or nontotal/nontransitive preferences, such as in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Rodríguez-Palmero and García-Lapresta (2002) can be used to prove or characterize the existence of Nash equilibria. In particular, incorporating recursive transfers into various transfer continuities can be used to obtain full characterization results for many other solution problems. Indeed, transfer irreflexive lower continuity (TILC) that shares the same feature as recursive transfer continuities has been introduced in Rodríguez-Palmero and García-Lapresta (2002) for studying the existence of maximal elements for irreflexive binary relations.

It is worth remarking that extensions in weakening the conventional continuity that use “securing a payoff” have the basic nature of transfer continuity. In fact, payoff security and better-reply security introduced by Reny (1999) and their extensions by many others actually fall in the forms
of transfer continuity. Indeed, as Prokopovych (2011) shows in Lemma 1 and 2, payoff security is equivalent to the transfer lower semicontinuity that was introduced in Tian (1992a), better-reply security is equivalent to the transfer reciprocal upper semicontinuity in payoff secure games, respectively.

The approach developed in the paper can similarly used to fully characterize the existence of mixed strategy Nash and Bayesian Nash equilibria in games with general strategy spaces and payoffs. It can also allow us to ascertain the existence of equilibria in important classes of economic games. Tian (2012a) show how they can be employed to characterize the existence of competitive equilibrium for economies with excess demand functions. More importantly, he results obtained can be used to study the existence of equilibrium in various general games with no linear (convex) structures such as in market design theory and matching theory. Indeed, Tian (2012b) provides a full characterize on the existence of stable matchings.

Finally, it may be remarked that the main purpose of this paper is not intent to provide conditions that is easy to check, but to characterize the essence of equilibrium in general games. Recursive transfer continuity provides a way of understanding the essence of equilibrium, more than necessarily providing a way to check its existence. It helps us to understand what kind of games can have or cannot have equilibria. In general, the weaker the conditions in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game. In the paper, we also provide some sufficient conditions for the existence of equilibrium in discontinuous and nonconvex games. A potential future work may be attempted to find more sufficient conditions for recursive diagonal transfer continuity.
References


