Full Characterizations of Minimax Inequality, Fixed-Point Theorem, Saddle-Point Theorem, and KKM Principle in Arbitrary Topological Spaces

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Abstract

This paper provides necessary and sufficient conditions for the existence of solutions of some important problems from optimization and nonlinear analysis by replacing two typical conditions — continuity and quasiconcavity with a unique condition, weakening topological vector spaces to arbitrary topological spaces that may be discrete, continuum, non-compact or non-convex. We establish a single condition, γ-*recursive transfer lower semicontinuity*, which fully characterizes the existence of γ-equilibrium of minimax inequality without imposing any restrictions on topological space. The result is then used to provide full characterizations on fixed-point theorem, saddle-point theorem, and KKM principle.

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1 Introduction

Ky Fan minimax inequality [1, 2] is probably one of the most important results in mathematical sciences in general and nonlinear analysis in particular, which is mutually equivalent to many important basic mathematical theorems such as the classical Knaster Kuratowski Mazurkiewicz (KKM) lemma, Sperner's lemma, Brouwer's fixed point theorem, and Kakutani fixed point theorem [3-5]. It has also become a crucial tool in proving many existence problems in various fields, especially in variational inequality problems, mathematical programming, partial differential equations, impulsive control, equilibrium problems in economics, various optimization problems, saddle points, fixed points, coincidence points, intersection points, and complementarity problems, etc.

The classical Ky Fan minimax inequality typically assumes *lower semicontinuity* and *quasiconcavity* for the functions, in addition to convexity and compactness in Hausdorff topological vector spaces. However, in many situations, these assumptions may not be satisfied [3]. The function under consideration may not be lower semicontinuous and/or quasiconcave, and choice spaces may be topological vector spaces. Similar situations are true for fixed point theorems, saddle point theorems, coincidence theorems, and intersection theorems (including various forms of FKKM theorems). As such, much work has been dedicated to weakening these conditions as in [6-35] and the references therein, among which some seek to weaken the quasiconcavity/semicontinuity of function, or drop convexity/compactness of choice sets, while others seek to weaken Hausdorff topological vector spaces to topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces, generalized convex spaces, and other types of spaces.

However, almost all the existing results only provide sufficient conditions for the existence of equilibrium. They are also based on strong topological structures, especially topological vector spaces. Besides, in order to have these existence results, some forms of convexity/lattice and continuity of functions are assumed. While it may be the convex/lattice structures that easily connect optimization or existence problems to mathematics, in many important situations such as those with discrete choice sets, there are no convex or lattice structures. As such, the intrinsic nature of the existence of solution in general case has not been fully understood yet. Why does or does not a problem have a solution? Are both lower semicontinuity and quasiconcavity (or their weaker forms) essential to the existence of equilibrium?

This paper provides complete solutions to the problem of minimax inequality and other related problems by replacing the assumptions concerning continuity and quasi-concavity with a unique condition, passing from topological vector spaces to arbitrary topological spaces that may be discrete, continuum, non-compact or non-convex, and the function that may not be lower semicontinuous or does not impose any form of convexity-related condition. We define a single condition, γ-*recursive transfer lower semicontinuity*, which fully characterizes the existence of equilibrium of minimax inequality without imposing any kind of convexity or any restrictions on topological space.

It is shown that γ-recursive transfer lower semicontinuity is necessary, and further, under compactness, sufficient for the existence of equilibrium of minimax inequalities for general topological strategy spaces and functions. We also provide a complete solution for the case of any arbitrary choice space that may not be compact. We show that γ-recursive transfer lower semicontinuity with respect to a compact set D is necessary and sufficient for the existence of equilibrium of minimax inequalities for arbitrary (possibly noncompact or open) topological spaces and general functions.

Since minimax inequality provides the foundation for many of the modern essential results in diverse areas of mathematical sciences, the results not only fully characterize the existence of solution to minimax inequality, but also introduce new techniques and methods for studying other optimization problems and generalize/characterize some basic mathematics results such as the FKKM theorem, fixed point theorem, saddle point theorem, variational inequalities, and coincidence theorem, etc. As illustrations, we show how they can be employed to provide full characterizations on the existence of fixed points, saddle points, and intersection points. The method of proof adopted to obtain our main results is also new and elementary — neither fixed-point nor KKM-principle approach.

The remainder of this paper is organized as follows. Section 2 states the basic notation and definitions. Section 3 provides necessary and sufficient conditions for the existence of solution to the Ky Fan minimax inequality for an arbitrary topological space. Then, we develop necessary and sufficient conditions for the existence of fixed points, saddle points as well as intersection points in Sections 4-6.

2 Notation and Definitions

Let X be a topological space and $D \subseteq X$. Denote the collections of all subsets, convex hull, closure, and interior of the set D by 2^D, $co\, D$, $cl\, D$, and $int\, D$, respectively. A function $f : X \to R \cup \{\pm\infty\}$ is *lower semicontinuous* on X if for each point x', $\liminf_{x\to x'} f(x) \geq f(x')$, or equivalently, if its epigraph epi$f \equiv \{(x, a) \in X \times R : f(x) \leq a\}$ is a closed subset of $X \times R$. A function $f : X \to R \cup \{\pm\infty\}$ is *upper semicontinuous* on X if $-f$ is lower semicontinuous on X. f is continuous on X if f is both upper and lower semicontinuous on X.

Let X be a convex subset of a topological vector space. A function $f : X \to R$ is quasiconcave on X if for any y^1, y^2 in X and any $\theta \in [0, 1]$, $\min\{f(y^1), f(y^2)\} \leq f(\theta y^1 + (1 - \theta)y^2)$, and f is quasiconvex on X if $-f$ is quasiconcave on X. A function $f : X \times X \to R$ is diagonally quasiconcave in y if for any finite points $y^1, \ldots, y^m \in X$ and any $y \in co\{y^1, \ldots, y^m\}$, $\min_{1\leq k\leq m} f(y, y^k) \leq f(y, y)$ (cf. [35]). A function $f : X \times X \to R$ is γ-diagonally quasiconcave in y if for any $y^1, \ldots, y^m \in X$ and $y \in co\{y^1, \ldots, y^m\}$, $\min_{1\leq k\leq m} f(y, y^k) \leq \gamma$. A correspondence $F : X \rightrightarrows X$ is said to be *FS convex on X if for every finite subset $\{x^1, x^2, \ldots, x^m\}$ of X, we have co* $\{x^1, x^2, \ldots, x^m\} \subseteq \bigcup_{j=1}^m F(x^j)$.[1] A correspondence $F : X \rightrightarrows X$ is said to be *SS convex* if $x \notin co\, F(x)$ for all $x \in X$.[2] It is easily shown that a function $\phi : X \times X \to R \cup \{\pm\infty\}$ is γ-diagonally quasiconcave in x if and only if the correspondence $F : X \rightrightarrows X$, defined by $F(x) = \{y \in X : \phi(x, y) \leq \gamma\}$ for all $x \in X$, is FS convex on X.

3 Full Characterization of Ky Fan Minimax Inequality

We begin by stating the notion of γ-equilibrium for minimax inequality problem.

[1] The FS is for Fan [2] and Sonnenschein [36].

[2] The SS is for Shafer and Sonnenschein [37].

DEFINITION 3.1 Let X be a topological space and $\gamma \in R$. A function ϕ: $X \times X \to R \cup \{\pm\infty\}$ is said to have a γ-*equilibrium* on X if there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$.

Ky Fan in [1, 2] provides the following classical result on minimax inequality problem.

THEOREM 3.1 (KY FAN MINIMAX INEQUALITY) *Let X be a compact convex set in a Hausdorff topological vector space. Let ϕ: $X \times X \to R$ be a function such that*

(a) $\phi(x, x) \leq 0$ for all $x \in X$;

(b) ϕ is lower semicontinuous in y;

(c) ϕ is quasiconcave in x.

Then ϕ possesses a 0-equilibrium $y^* \in X$.

Ky Fan minimax inequality has then been extended in various ways: Quasi-concavity is weakened to diagonal quasiconcavity, γ-diagonal quasiconcavity, or transfer (γ-diagonal) quasiconcavity; lower semicontinuity is weakened to transfer lower semicontinuity or γ-transfer lower semicontinuity; compactness is weakened to noncompactness; and Hausdorff topological vector space is weakened to topological vector space, Lassonde type convex space, Horvath type H-space, generalized convex space, etc. (cf. Fan [4], Allen [6], Ansari et al. [7], Cain and González [8], Chebbi [9], Ding [14], Georgiev and Tanaka [15], Iusem and Soca [17], Karamardian [18], Lignola [21], Lin and Chang [22], Lin and Tian [25], Nessah and Tian [26], Tian [28, 29], Yuan [33], and Zhou and Chen [35] among others). Yet there are no full characterization results available and the topological spaces are assumed to be topological vector spaces.

In the following, we establish a single condition that is necessary and sufficient for the existence of solution to a minimax inequality defined on an arbitrary topological space. We begin with the notion of γ-transfer lower semicontinuity introduced by [28].

DEFINITION 3.2 Let X be a topological space. A function ϕ: $X \times X \to R \cup \{\pm\infty\}$ is said to be γ-*transfer lower semicontinuous* in y if for all $x \in X$ and $y \in X$, $\phi(x, y) > \gamma$ implies that there exists some point $z \in X$ and some neighborhood $\mathcal{N}(y)$ of y such that $\phi(z, y') > \gamma$ for all $y' \in \mathcal{N}(y)$.

We now introduce the notion of recursive diagonal transfer continuity.

DEFINITION 3.3 Let X be a topological space. A function ϕ: $X \times X \to R \cup \{\pm\infty\}$ is said to be γ-*recursively transfer lower semicontinuous* in y if, whenever $\phi(x, y) > \gamma$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ of y such that $\phi(z, \mathcal{V}_y) > \gamma$ for any sequence of points $\{z^1, \dots, z^{m-1}, z\}$ with $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, …, $\phi(z^1, z^0) > \gamma$, $m = 1, 2, \ldots$. Here $\phi(z, \mathcal{V}_y) > \gamma$ means that $\phi(z, y') > \gamma$ for all $y' \in \mathcal{V}_y$.

REMARK 3.1 In the definition of γ-recursive transfer lower semicontinuity, y is transferred to z^0 that could be any point in X. Under γ-recursive transfer lower semicontinuity, when $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, …, $\phi(z^1, z^0) > \gamma$, we have not only $\phi(z, \mathcal{V}_y) > \gamma$, but also $\phi(z^{m-1}, \mathcal{V}_y) > \gamma$, …, $\phi(z^1, \mathcal{V}_y) > \gamma$.

Similarly, we can define m-γ-recursive transfer lower semicontinuity in y. A function ϕ is m-γ-*recursively transfer lower semicontinuous in* y when the number of points in the sequence is m. Thus, a function ϕ is γ-recursively transfer lower semicontinuous in y if it is m-γ-recursively transfer lower semicontinuous in y for all $m = 1, 2 \ldots$.

Now we are ready to state our main result on the existence of γ-equilibrium of minimax inequality defined on a general topological space.

THEOREM 3.2 *Let X be a compact topological space, $\gamma \in R$, and $\phi : X \times X \to R \cup \{\pm\infty\}$ be a function with $\phi(x, x) \leq \gamma$ for all $x \in X$. Then ϕ possesses a γ-equilibrium if and only if it is γ-recursively transfer lower semicontinuous in y.*

PROOF. *Sufficiency* ($\Leftarrow$). Suppose y is not a γ-equilibrium point. Then there is an $x \in X$ such that $\phi(x, y) > \gamma$. Then, by γ-recursive transfer lower semicontinuity of ϕ in y, for each $y \in X$, there exists a point z^0 and a neighborhood $\mathcal{V}_y$ such that $\phi(z, \mathcal{V}_y) > \gamma$ for any sequence of recursive points $\{z^1, \ldots, z^{m-1}, z\}$ with $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, ..., $\phi(z^1, z^0) > \gamma$. Since there is no γ-equilibrium by the contrapositive hypothesis, z^0 is not a γ-equilibrium, and thus, by γ-recursive transfer lower semicontinuity in y, such a sequence of recursive points $\{z^1, \ldots, z^{m-1}, z\}$ exists for some $m \geq 1$.

Since X is compact and $X \subseteq \bigcup_{y \in X} \mathcal{V}_y$, there is a finite set $\{y^1, \ldots, y^T\}$ such that $X \subseteq \bigcup_{i=1}^{T} \mathcal{V}_{y^i}$. For each of such y^i, the corresponding initial point is denoted by z^{0i} so that $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$ whenever z^{0i} is γ-recursively upset by z^i. Since there is no γ-equilibrium, for each of such z^{0i}, there exists z^i such that $\phi(z^i, z^{0i}) > \gamma$, and then, by 1-$\gamma$-recursive transfer lower semicontinuity, we have $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$. Now consider the set of points $\{z^1, \ldots, z^T\}$. Then, $z^i \notin \mathcal{V}_{y^i}$; otherwise, by $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$, we will have $\phi(z^i, z^i) > \gamma$, a contradiction to the fact that $\phi(x, x) \leq \gamma$ for all $x \in X$. Thus we must have $z^1 \notin \mathcal{V}_{y^1}$.

Without loss of generality, suppose $z^1 \in \mathcal{V}_{y^2}$. Since $z^1 \in \mathcal{V}_{y^2}$ and $\phi(z^1, z^{01}) > \gamma$, $\phi(z^2, z^1) > \gamma$. Then, by 2-γ-recursive transfer lower semicontinuity, we have $\phi(z^2, \mathcal{V}_{y^1}) > \gamma$. Also, $\phi(z^2, \mathcal{V}_{y^2}) > \gamma$. Thus $\phi(z^2, \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2}) > \gamma$, and consequently $z^2 \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2}$. Again, without loss of generality, we suppose $z^2 \in \mathcal{V}_{y^3}$. Since $\phi(z^3, z^2) > \gamma$ by noting that $z^2 \in \mathcal{V}_{y^3}$, $\phi(z^2, z^1) > \gamma$, and $\phi(z^1, z^{01}) > \gamma$, then, by 3-$\gamma$-recursive transfer lower semicontinuity, we have $\phi(z^3, \mathcal{V}_{y^1}) > \gamma$. Also, since $\phi(z^3, z^2) > \gamma$ and $\phi(z^2, z^{02}) > \gamma$, by 2-$\gamma$-recursive transfer lower semicontinuity, we have $\phi(z^3, \mathcal{V}_{y^2}) > \gamma$. Thus, $\phi(z^3, \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \cup \mathcal{V}_{y^3}) > \gamma$, and consequently $z^3 \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \cup \mathcal{V}_{y^3}$.

With this recursive process going on, we can show that $z^k \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \cup \ldots, \cup \mathcal{V}_{y^k}$, i.e., z^k is not in the union of $\mathcal{V}_{y^1}, \mathcal{V}_{y^2}, \ldots, \mathcal{V}_{y^k}$ for $k = 1, 2, \ldots, T$. In particular, for $k = T$, we have $z^T \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \ldots \cup \mathcal{V}_{y^T}$ and thus $z^T \notin X \subseteq \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \ldots \cup \mathcal{V}_{y^T}$, a contradiction. Then there exists $y^* \in X$ such that $(x, y^*) \leq \gamma$ for all $x \in X$, and thus y^* is a γ-equilibrium point of the minimax inequality.

Necessity ($\Rightarrow$). Suppose y^* is a γ-equilibrium and $\phi(x, y) > \gamma$ for $x, y \in X$. Let $z^0 = y^*$ and $\mathcal{V}_y$ be a neighborhood of y. Since $\phi(x, y^*) \leq \gamma$ for all $x \in X$, it is impossible to find any finite sequence $\{z^1, \ldots, z^m\}$ such that $\phi(z^1, z^0) > \gamma, \ldots, \phi(z^m, z^{m-1}) > \gamma$. Thus, the γ-recursive transfer lower semicontinuity holds trivially. ■

Although γ-recursive transfer lower semicontinuity is necessary for the existence of solution to the problem, it may not be sufficient for the existence of γ-equilibrium when a choice space X is noncompact. To see this, consider the following counterexample.

EXAMPLE 3.1 Let $X =]0, 1[$ and ϕ: $X \times X \to R \cup \{\pm\infty\}$ be defined by $\phi(x, y) = x - y$.

The minimax inequality clearly does not possess a 0-equilibrium. However, it is 0-recursively transfer lower semicontinuous in y. Indeed, for any two points $x, y \in X$ with $\phi(x, y) = x - y > 0$, choose $\epsilon > 0$ such that $[y - \epsilon, y + \epsilon] \subset X$. Let $z^0 = y + \epsilon \in X$ and $\mathcal{V}_y \subseteq [y - \epsilon, y + \epsilon]$. Then, for any finite set $\{z^1, \ldots, z^{m-1}, z\}$ with $\phi(z^1, z^0) = z^1 - z^0 > 0$, $\phi(z^2, z^1) = z^2 - z^1 > 0$, ..., $\phi(z, z^{m-1}) = z - z^{m-1} > 0$, i.e., $z > z^{m-1} > \ldots > z^0$, we have $\phi(z, y') = z - y' > z^0 - y' > 0$ for all $y' \in \mathcal{V}_y$. Thus, $\phi(z, \mathcal{V}_y) > 0$, which means ϕ is 0-recursively transfer lower semicontinuous in y.

Nevertheless, Theorem 3.2 can be extended to any topological choice space. To do so, we introduce the following version of γ-recursive transfer lower semicontinuity.

DEFINITION 3.4 Let X be a topological space and $D \subseteq X$. A function ϕ: $X \times X \to R \cup \{\pm\infty\}$ is said to be γ-recursively transfer lower semicontinuous in y with respect to D if, whenever $\phi(x, y) > \gamma$ for $x \in X$ and $y \in D$, there is a point $z^0 \in D$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ of y such that (1) $\phi(x', y) > \gamma$ for some $x' \in D$, and (2) $\phi(z^m, \mathcal{V}_y) > \gamma$ for any finite subset $\{z^1, \ldots, z^m\} \subseteq D$ with $\phi(z^m, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, ..., $\phi(z^1, z^0) > \gamma$.

The following theorem fully characterizes the existence of solution to minimax inequalities for arbitrary topological spaces.

THEOREM 3.3 *Let X be a topological space, $\gamma \in R$, and $\phi : X \times X \to R \cup \{\pm\infty\}$ be a function with $\phi(x, x) \leq \gamma$ for all $x \in X$. Then there is a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$ if and only if there exists a compact subset $D \subseteq X$ such that ϕ is γ-recursively transfer lower semicontinuous in y with respect to D.*

PROOF. *Sufficiency* ($\Leftarrow$). The proof is essentially the same as that of sufficiency in Theorem 3.2 and we just outline it here. We first show that there exists a γ-equilibrium y^* in D. Suppose, by way of contradiction, that there is no γ-equilibrium in D. Then, by γ-recursive transfer lower semicontinuity in y with respect to D, for each $y \in D$, there exists $z^0 \in D$ and a neighborhood $\mathcal{V}_y$ such that $\phi(z^m, \mathcal{V}_y) > \gamma$ for any finite subset of points $\{z^1, \ldots, z^m\} \subseteq D$ with $\phi(z^m, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, ..., $\phi(z^1, z^0) > \gamma$. Since there is no γ-equilibrium in D by the contrapositive hypothesis, z^0 is not a γ-equilibrium point in D and thus, by γ-recursive transfer lower semicontinuity in y with respect to D, such a sequence of recursive points $\{z^1, \ldots, z^{m-1}, z^m\}$ exists for some $m \geq 1$.

Since D is compact and $D \subseteq \bigcup_{y \in X} \mathcal{V}_y$, there is a finite set $\{y^1, \ldots, y^T\} \subseteq D$ such that $D \subseteq \bigcup_{i=1}^T \mathcal{V}_{y^i}$. For each of such y^i, the corresponding initial point is denoted by z^{0i} so that $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$ whenever $\phi(z^i, z^{0i}) > \gamma$ for any finite subset $\{z^{1i}, \ldots, z^{mi}\} \subseteq D$ with $z^{mi} = z^i$. Then, by the same argument as in the proof of Theorem 3.2, we will obtain that z^k is not in the union of $\mathcal{V}_{y^1}, \mathcal{V}_{y^2}, \ldots, \mathcal{V}_{y^k}$ for $k = 1, 2, \ldots, T$. For $k = T$, we have $z^T \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \ldots \cup \mathcal{V}_{y^T}$ and thus $z^T \notin D \subseteq \bigcup_{i=1}^T \mathcal{V}_{y^i}$, a contradiction. Thus, there exists a point $y^* \in D$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in D$.

We now show that y^* must be a γ-equilibrium in X. Suppose not. Then there is $x \in X$ with $\phi(x, y^*) > \gamma$, and thus, by γ-recursive transfer lower semicontinuity in y with respect to D, there exists $x' \in D$ with $\phi(x', y^*) > \gamma$, which means y^* is not a γ-equilibrium in D, a contradiction.

Necessity ($\Rightarrow$). Suppose y^* is a γ-equilibrium. Let $D = \{y^*\}$. Then, the set D is clearly compact. Now, let $z^0 = y^*$ and $\mathcal{V}_{y^*}$ be a neighborhood of y^*. Since $\phi(x, y^*) \leq \gamma$ for all $x \in X$ and $z^0 = y^*$ is a unique element in D, there is no point $z^1 \in D$ such that $\phi(z^1, z^0) > \gamma$ or $\phi(x', y^*) > \gamma$ for some $x' \in D$. Hence, ϕ is γ-recursively transfer continuous in y with respect to D. ■

COROLLARY 3.1 (GENERALIZED KY FAN'S MINIMAX INEQUALITY) *Let X be a topological space, ϕ: $X \times X \to R \cup \{\pm\infty\}$ be a function, and $\gamma := \sup_{y \in X} \phi(y, y)$.*[3] *Then there is a point $y^* \in X$ such that $\phi(x, y^*) \leq \sup_{y \in X} \phi(y, y), \forall x \in X$ if and only if there exists a compact subset $D \subseteq X$ such that ϕ is γ-recursively transfer lower semicontinuous in y with respect to D.*

[3]When $\gamma = \sup_{y \in X} \phi(y, y) = +\infty$, any point in X is clearly a γ-equilibrium with $\gamma = +\infty$.

Theorem 3.3 and Corollary 3.1 thus strictly generalize all the existing results on the minimax inequality such as those in Fan [1, 2, 4], Allen [6], Ansari et al. [7], Chebbi [9], Ding [14], Lignola [21], Lin and Chang [22], Nessah and Tian [26], Tian [28], Yuan [33], Zhou and Chen [35].

The following example about game theory shows that, although the strategy space is an open set and the payoff function is highly discontinuous and nonquasiconcave, we can use Theorem 3.3 to assert the existence of Nash equilibrium.

EXAMPLE 3.2 Consider a game with $n = 2$, $X = X_1 \times X_2 = (0,1) \times (0,1)$ that is an open unit interval set, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, 2,$$

where $\mathbb{Q} = \{x \in (0,1) : x \text{ is a rational number}\}$.

Let $U(x, y) = u_1(x_1, y_2) + u_2(y_1, x_2)$. Define a function $\phi : X \times X \to R$ by

$$\phi(x, y) = U(x, y) - U(y, y).$$

It is clear $\phi(x, x) \leq 0$ for all $x \in X$. Then ϕ is neither (0-transfer) lower semicontinuous in y nor (diagonally) quasiconcave in x. However, it is 0-recursively transfer lower semicontinuous in y. Indeed, suppose $\phi(x, y) > 0$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let z^0 be a vector with rational coordinates, $D = \{z^0\}$, and $\mathcal{V}_y$ be a neighborhood of y. Since $\phi(x, z^0) \leq 0$ for all $x \in X$, it is impossible to find any strategy profile z^1 such that $\phi(z^1, z^0) > 0$. Hence, ϕ is 0-recursively transfer lower semicontinuous in y with respect to D. Therefore, by Theorem 3.3, there exists $\bar{y} \in X$ such that $\phi(x, \bar{y}) \leq 0$ for all $x \in X$. In particular, letting $x_1 = \bar{y}_1$ and keeping x_2 varying, we have $u_2(\bar{y}_1, x_2) \leq u_2(\bar{y}_1, \bar{y}_2), \forall x_2 \in X_2$, and letting $x_2 = \bar{y}_2$ and keeping x_1 varying, we have $u_1(x_1, \bar{y}_2) \leq u_1(\bar{y}_1, \bar{y}_2), \forall x_1 \in X_1$. Hence, this game possesses a Nash equilibrium. In fact, the set of Nash equilibria consists of all rational coordinates on $(0, 1)$.

4 Full Characterization of a Fixed Point

This section provides necessary and sufficient conditions for the existence of a fixed point of a function defined on a set that may be finite, continuum, nonconvex, or noncompact.

Let X be a topological space. A correspondence $F : X \rightrightarrows X$ has a fixed point $x \in X$ if $x \in F(x)$. If F is a single-valued function f, then a fixed point x of f is characterized by $x = f(x)$.

We first recall the notion of diagonal transfer continuity introduced by [38].

DEFINITION 4.1 A function $\varphi : X \times X \to R \cup \{\pm\infty\}$ is diagonally transfer continuous in y if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z \in X$ and a neighborhood $\mathcal{V}_y \subset X$ of y such that $\varphi(z, y') > \varphi(y', y')$ for all $y' \in \mathcal{V}_y$.

Similarly, we can define the notion of recursive diagonal transfer continuity.

DEFINITION 4.2 (**Recursive Diagonal Transfer Continuity**) A function φ: $X \times X \to R \cup \{\pm\infty\}$ is said to be *recursively diagonally transfer continuous* in y if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ of y such that $\varphi(z, y') > \varphi(y', y')$ for all $y' \in \mathcal{V}_y$ and for any finite subset $\{z^1, \ldots, z^m\} \subseteq X$ with $z^m = z$ and $\varphi(z, z^{m-1}) > \varphi(z^{m-1}, z^{m-1})$, $\varphi(z^{m-1}, z^{m-2}) > \varphi(z^{m-2}, z^{m-2})$, …, $\varphi(z^1, z^0) > \varphi(z^0, z^0)$ for $m \geq 1$.

THEOREM 4.1 (FIXED POINT THEOREM) *Let X be a nonempty and compact subset of a metric space (E, d) and $f : X \to X$ be a function. Then, f has a fixed point if and only if the function φ: $X \times X \to R \cup \{\pm\infty\}$, defined by $\varphi(x, y) = -d(x, f(y))$, is recursively diagonally transfer continuous in y.*

PROOF. Define $\phi : X \times X \to R$ by $\phi(x, y) = d(y, f(y)) - d(x, f(y))$. Then $\phi(x, x) \leq 0$ for all $x \in X$. We can easily see ϕ is 0-recursively transfer lower semicontinuous in y if and only if φ is recursively diagonally transfer continuous in y. Then, by Theorem 3.2, there exists $\bar{y}$ such that $\phi(x, \bar{y}) \leq 0, \forall x \in X$, or equivalently $d(\bar{y}, f(\bar{y})) \leq d(x, f(\bar{y})), \forall x \in X$ if and only if φ is recursively diagonally transfer continuous in y. In particular, letting $\bar{x} = f(\bar{y})$, we have $d(\bar{y}, f(\bar{y})) \leq$

$d(f(\bar{y}), f(\bar{y})) = 0$ and thus $\bar{y} = f(\bar{y})$. Therefore, f has a fixed point if and only if the function $-d(x, f(y))$ is recursively diagonally transfer continuous in y. ■

Theorem 4.1 can also be generalized by relaxing the compactness of X.

DEFINITION 4.3 A function φ: $X \times X \to R \cup \{\pm\infty\}$ is said to be *recursively diagonally transfer continuous* in y with respect to D if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x \in X$ and $y \in D$, there exists a point $z^0 \in D$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ of y such that (1) $\varphi(x', y) > \varphi(y, y)$ for some $x' \in D$, and (2) $\varphi(z, y') > \varphi(y', y')$ for all $y' \in \mathcal{V}_y$ and for any finite subset $\{z^1, \dots, z^m\} \subseteq X$ with $z^m = z$ and $\varphi(z, z^{m-1}) > \varphi(z^{m-1}, z^{m-1})$, $\varphi(z^{m-1}, z^{m-2}) > \varphi(z^{m-2}, z^{m-2})$, …, $\varphi(z^1, z^0) > \varphi(z^0, z^0)$ for $m \geq 1$.

THEOREM 4.2 *Let X be a nonempty subset of a metric space (E, d) and $f : X \to X$ be a function. Then, f has a fixed point if and only if there exists a compact set $D \subseteq X$ such that $-d(x, f(y))$ is recursively diagonally transfer continuous in y with respect to D.*

PROOF. The proof is the same as in Theorem 3.3, and it is omitted here. ■

Theorem 3.3 and Corollary 3.1 strictly generalize many existing fixed point theorems in the literature, such as the well-known Browder fixed point theorem and Tarski fixed point theorem in [39], as well as those in Fan [2, 4], Halpern [40, 41], Halpern and Bergman [42], Reich [43], Istrăţescu [44], Tian [45] and the references therein.

5 Full Characterization of a Saddle Point

The saddle point theorem is an important tool in variational problems and game theory. Much work has been dedicated to weakening its existence conditions. However, almost all these results assume that functions under consideration are defined on topological vector spaces. In this section, we present some results on saddle points without imposing any form of convexity conditions.

DEFINITION 5.1 Let X be a topological space and ϕ: $X \times X \to R \cup \{\pm\infty\}$ a function. A pair $(\overline{x}, \overline{y})$ in $X \times X$ is called a saddle point of ϕ in $X \times X$ if $\phi(\overline{x}, y) \leq \phi(\overline{x}, \overline{y}) \leq \phi(x, \overline{y})$ for all $x \in$

X and $y \in X$.

The following is the classical result on saddle points.

THEOREM 5.1 (VON NEUMANN THEOREM) *Let X be nonempty, compact and convex subset in a Hausdorff locally convex topological vector space E, and $\phi : X \times X \to R \cup \{\pm\infty\}$ be a function. Suppose that*

(a) ϕ is lower semicontinuous and quasiconvex in y;

(b) ϕ is upper semicontinuous and quasiconcave in x.

Then, ϕ has a saddle point.

Also, a lot of work has been done by weakening the conditions of semicontinuity and/or quasiconcavity/quasiconvexity of von Neumann Theorem. Our results characterize the existence of a saddle point for a general topological space without assuming any kind of quasiconvexity or quasiconcavity.

DEFINITION 5.2 Let X be a topological space. A function ϕ: $X \times X \to R \cup \{\pm\infty\}$ is said to be γ-recursively transfer upper semicontinuous in x if $-\phi$ is $-\gamma$-recursively transfer lower semicontinuous in x. We can similarly define γ-recursive transfer upper semicontinuity in x with respect to $D \subseteq X$.

THEOREM 5.2 *Let X be a compact topological space, $\gamma \in R$, and ϕ: $X \times X \to R \cup \{\pm\infty\}$ be a function with $\phi(x, x) = \gamma$ for all $x \in X$. Then there exists a saddle point $(\bar{x}, \bar{y}) \in X \times X$ if and only if ϕ is γ-recursively transfer upper semicontinuous in x and $-\gamma$-recursively transfer lower semicontinuous in y.*

PROOF. Applying Theorem 3.2 to $\phi(x, y)$, we have the existence of $\bar{y} \in X$ such that $\phi(x, \bar{y}) \leq \gamma, \forall x \in X$. Let $\psi(x, y) = -\phi(y, x)$. Then $\psi(x, x) = -\gamma$ for all $x \in X$. Also, since ϕ is γ-recursively transfer upper semicontinuous in x, ψ is $-\gamma$-recursively transfer lower semicontinuous in x. Applying Theorem 3.2 again to $\psi(x, y)$, we have the existence of $\bar{x} \in X$ such that $\phi(\bar{x}, y) \geq \gamma, \forall y \in X$. Combining these inequalities, we have $\phi(\bar{x}, \bar{y}) \leq \gamma$ and $\phi(\bar{x}, \bar{y}) \geq \gamma$, and therefore

$\phi(\bar{x}, \bar{y}) = \gamma$. Thus, $(\bar{x}, \bar{y})$ is a saddle point satisfying $\phi(\overline{x}, y) \leq \phi(\overline{x}, \overline{y}) \leq \phi(x, \overline{y})$ for all $x \in X$ and $y \in X$. ■

Theorem 5.2 can also be generalized by relaxing the compactness of X.

THEOREM 5.3 *Let X be a topological space, $\gamma \in R$, and ϕ: $X \times X \to R \cup \{\pm\infty\}$ be a function with $\phi(x, x) = \gamma$ for all $x \in X$. Then there exists a saddle point $(\bar{x}, \bar{y}) \in X \times X$ if and only if there exist two compact sets D_1 and D_2 in X such that ϕ is γ-recursively transfer upper semicontinuous in x with respect to D_1 and γ-recursively transfer lower semicontinuous in y with respect to D_2.*

PROOF. The proof is the same as in Theorem 3.3, and it is omitted here. ■

6 Characterization on KKM Principle

Now we use Theorems 3.2 and 3.3 to generalize the FKKM theorem that is a generalization of KKM lemma [46]. We begin by stating the following KKM principle due to [4].

THEOREM 6.1 (FKKM THEOREM) *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. Let $F : X \rightrightarrows Y$ be a correspondence such that*

(a) for each $x \in X$, $F(x)$ is a relatively closed subset of Y;

(b) F is FS-convex on X;

(c) there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Also, this theorem has been generalized in various forms in the literature ([34]). In the following, we provide a full characterization on nonemptiness of intersection points of sets in FKKM theorem where F is a correspondence mapping from X to X.

THEOREM 6.2 *Let X be a compact topological space and $F : X \rightrightarrows X$ be a correspondence with $x \in F(x)$ for all $x \in X$. Then $\bigcap_{x \in X} F(x) \neq \emptyset$ if and only if the function ϕ: $X \times X \to R \cup \{\pm\infty\}$ defined by*

$$\phi(x,y) = \begin{cases} \gamma & \text{if } (x,y) \in G \\ +\infty & \text{otherwise} \end{cases}$$

is γ-recursively transfer lower semicontinuous in y. Here $\gamma \in R$ and $G = \{(x,y) \in X \times X : y \in F(x)\}$.

PROOF. By Theorem 3.2, there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$ if and only if ϕ is γ-recursively transfer lower semicontinuous in y. However, by construction, $\bigcap_{x \in X} F(x) \neq \emptyset$ if and only if there exists a point $x^* \in X$ such that $\phi(x, x^*) \leq \gamma$ for all $x \in X$. ■

Similarly, we can drop the compactness of X, and have the following theorem.

THEOREM 6.3 *Let X be a topological space and $F : X \rightrightarrows X$ be a correspondence with $x \in F(x)$ for all $x \in X$. Then $\bigcap_{x \in X} F(x) \neq \emptyset$ if and only if there exists a compact subset $D \subseteq X$ such that ϕ: $X \times X \to R \cup \{\pm\infty\}$ defined by*

$$\phi(x,y) = \begin{cases} \gamma & \text{if } (x,y) \in G \\ +\infty & \text{otherwise} \end{cases}$$

is γ-recursively transfer lower semicontinuous in y with respect to D. Here $\gamma \in R$ and $G = \{(x,y) \in X \times X : y \in F(x)\}$.

7 Conclusions

The existing results on the existence of solutions for some important problems from optimization and nonlinear analysis are based on continuity-related and convexity-related conditions, in addition to the assumption of topological vector spaces. Besides, these results only provide sufficient conditions for the existence problems and no full characterization results have been given in the literature. This paper fills this gap by replacing the assumptions concerning continuity and convexity with a single condition that is necessary and sufficient for the existence of solution in minimax inequalities, fixed points, saddle points, or intersection points defined on arbitrary topological spaces that may be discrete, continuum, non-compact or non-convex.

The basic transfer method is systematically developed in [28, 38, 47-49] for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in discontinuous games. These papers, especially Zhou and Tian [49], have developed three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to study the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian [47] and Tian and Zhou [48]).

The notion of recursive transfer continuity extends usual transfer continuity from single transfer to allowing recursive (sequential) transfers. Incorporating recursive transfers into various transfer continuities can also be used to obtain full characterization results for many other solution problems. Indeed, transfer irreflexive lower continuity (TILC) that shares the same feature as recursive transfer continuities has been introduced in [50] for studying the existence of maximal elements for irreflexive binary relations. [51, 52] provides full characterizations on the existence of Nash equilibrium in general games with arbitrary (topological) strategy spaces and competitive equilibrium, respectively.

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