General exchange rules with single-peaked preferences

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Abstract

Following Bade (2019), we study the exchange in Shapley-Scarf housing markets when agents have single-peaked preferences. We propose a general class of mechanisms, called the r-neighborhood mechanisms, which are group strategy-proof, individually rational and Pareto efficient. In an r-neighborhood mechanism, every agent progressively points to her most preferred house within distance r of her current house, according to the linear order on houses that underlies agents' single-peaked

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preferences. We show that r-neighborhood mechanisms include Gale's Top Trading Cycles mechanism and Bade's Crawler mechanism as extreme cases, and among r-neighborhood mechanisms, the 1-neighborhood mechanism (i.e., the crawler and its dual) is the only obviously strategy-proof mechanism.

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1 Introduction

We study the Shapley-Scarf housing market, where each agent is endowed with a house and demands exactly one house (Shapley and Scarf, 1974). It is well-known that for these markets, Gale's Top Trading Cycles (TTC) mechanism always selects the unique matching in the core (Roth and Postlewaite, 1977), and it is the only mechanism that is individually rational, Pareto efficient and strategy-proof (Ma, 1994).

Bade (2019) is the first to study housing markets with single-peaked preferences. Consider that there is an objective linear order on the houses: All the houses are located on a street, from the left end to the right end. Agents' preferences are strict and single-peaked: There is an ideal house on the street for each agent. A house h should be worse than $h^{'}$ to an agent, if they locate on the same side of the ideal point, and h is further away from that point than $h^{'}$. Bade (2019) proposes a new mechanism call the crawler. In the crawler, each agent points to her ideal houses. Among all the agents who point to the occupied houses or to the right side of the occupied houses, we find the rightmost agent and assign her ideal houses to her. All agents who currently occupy houses between this agent's choice and the house she vacated "crawl" to the next house on the left. This process is repeated until all agents are matched. Bade (2019) shows that the crawler, like TTC mechanism, is individually rational, efficient and strategy-proof with single-peaked preferences.

Namely, Ma's characterization no longer holds on this restricted domain. Then, how are TTC mechanism and the crawler related to each other? We start with the observation that the process of the crawler is equivalent to letting agents trade progressively with their most preferred direct neighbors. In this framework, we can regard the crawler and the TTC mechanism as extreme cases: agents can only trade with the direct neighbors in the crawler and they can choose the most preferred house among the whole set of houses in the TTC mechanism. The following question arises: Are there other mechanisms also satisfying individual rationality, Pareto efficiency and strategy-proofness in this setting?

We propose a large class of individually rational, Pareto efficient and strategy-proof mechanisms, called r-neighborhood mechanisms, for housing markets with single-peaked preferences. These mechanisms incorporate the TTC mechanism and the crawler as extreme cases. More generally, as houses are lined up on a street, we assume that the distance between two adjacent remaining houses is exactly one. We can easily visualize situations in which each agent is only able to see houses within distance r of her current house, called an r-neighborhood. As a result, she will only wish for her most preferred house in the r-neighborhood. The r-neighborhood mechanisms allow each agent to progressively trade for her most preferred house in her r-neighborhood. Then the TTC mechanisms is a N-neighborhood mechanism, where N denotes the number of agents, and the crawler is a 1-neighborhood mechanism.

In Theorem 1, we show that, like the TTC mechanism and the crawler, r-neighborhood mechanisms are individually rational, Pareto efficient and group strategy-proof. We further generalize the r-neighborhood mechanisms to a larger class of mechanisms called neighborhood mechanisms, in which the sizes of neighborhoods depend on the identity of agents and the occupied houses. In Proposition 2, we show that the neighborhood mechanisms are also individually rational, Pareto efficient and group strategy-proof. These results tell us that there are many individually rational, Pareto efficient and group strategy-proof mechanisms other than the TTC mechanism with single-peaked preferences, and this is in sharp contrast with Ma (1994)'s characterization that TTC is the unique individually rational, Pareto efficient and strategy-proof mechanism in Shapley-Scarf housing markets.

Obvious dominance is a desirable property proposed by Li (2017). Obviously dominant strategies can be seen as optimal by a cognitively limited agent, who does not understand how each strategy's outcome depends on unobserved contingencies. Bade (2019) shows that the crawler can be implemented in obviously dominant strategies. In Theorem 2 we show that among all the r-neighborhood mechanisms, the 1-neighborhood mechanism, which is outcome equivalent to the crawler, is the only mechanism that is implementable in obviously dominant strategies.

In the neighborhood mechanisms, the key assumption that agents are only able to or allowed to point to a restricted range of objects has solid theoretical and empirical foundation. In most theoretical models, there is a trade-off between the benefit and the cost of further objects. As a result, agents would only choose the objects in a limited distance. The cost includes transport cost (see e.g., Hotelling, 1929 or Pal, 1998) or the cost of search(see e.g.,Stigler, 1961 or Ioannides, 1975). Empirical studies also find that distance is one of the most important determinant of the decision of agents and examine the impact of spacial distance on the preferences of consumers(see e.g. Berdegué et al., 2006 or He et al., 2019). Also, the assumption is related to literature about the regulations which restrict the objects available for agents, such as the Nearby Enrollment Policy(see e.g., Black, 1999 and Bayer et al., 2007).

Besides Shapley and Scarf (1974)'s classical work, this paper is most closely related to Ma (1994)'s characterization about the TTC mechanism and the crawler in Bade (2019). This paper is also closely related to Tamura and Hosseini (2022) and Liu (2021). Tamura and Hosseini (2022) show the equivalence between the crawler and the dual crawler, and the equivalence between the crawler from random endowments and the random priority rule. They also find that the crawler is invariant to the order over the object set that preserves single-peakedness. Liu (2021) introduces another class of individually rational, efficient and strategy-proof mechanisms, where only a subset of houses is available for exchange in each step and the next subset of available houses may depend on the exchanges performed previously. The relationship between the two classes of mechanisms will be discussed in detail in Section 4. This paper is related to other works that study mechanisms under single-peaked preferences. Damamme et.al. (2015) provide an algorithm which is Pareto efficient on the single-peaked domain, and Mandal and Roy (2021) present a impossibility result and characterize all strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof assignment rules on a minimally rich single-peaked domain as hierarchical exchange rules. Our paper is also related to the literature which study the properties of Gale's TTC mechanism, such as Pápai (2000) and Sethuraman (2016). Regarding obviously strategy-proofness of trading mechanisms, this paper is related to the works of Li (2017), Troyan (2019) and Pycia and Troyan (2022).

The rest of this paper is organized as follows. In Section 2, we introduce the basic setup, Gale's Top Trading Cycles mechanism and the crawler. We then propose the r-neighborhood mechanisms and present the related results in Section 3. We discuss the further generalization of these mechanisms and the relationship with other related mechanisms in Section 4. Section 5 concludes. All proofs are relegated to Appendix A.

2 Model

2.1 Notations

We study the allocation problem of a Shapley-Scarf housing market (Shapley and Scarf (1974)). In a housing market, each agent is endowed with an indivisible object called a house and agents trade with each other. Let $I = \{1, 2, \cdots, N\}$ denote the set of agents, and let $H = \{h_1, h_2, \cdots, h_N\}$ denote the set of houses. Each agent $i \in I$ is endowed with the house h_i, and has a strict preference P_i over H with symmetric extension R_i. Given a preference P_i, if agent i strictly prefers (weakly prefers, reps.) h_j to h_k, we write $h_j P_i h_k$ ($h_j R_i h_k$, resp.). A preference is a linear order if it is antisymmetric. Given an agent i, let P_{-i} denote the profile of all other agents' preferences. And given a coalition of agents $C \subseteq I$, let P_C and P_{-C} denote the preference profile of the coalition C and the preference profile of all other agents out of C, respectively.

Following Bade (2019), we further assume that agents have single-peaked preferences over houses. Suppose the houses are located on a street that stretches from its left end to its right end. House h_1 is located at the left end and h_N is located at the right end, and if $i < j$ then h_i is located at the left side of h_j. Then a strict preference P_i is **single-peaked** if there exists a house h_{i^*} such that $h_j P_i h_k$ holds if either $k < j \leq i^*$ or $i^* \geq j > k$.

A **matching**(or allocation) μ is a bijection $\mu : N \to H$. If $\mu(i) = h_j$ then agent i is matched with house h_j. A matching μ is **individually rational** if $\mu(i) R_i h_i$ for all i. A matching ν **(Pareto) dominants** a matching μ if $\nu(i) R_i \mu(i)$ for all agent i and $\nu(j) P_j \mu(j)$ for some agent j. A matching μ is **Pareto efficient** if there exists no matching ν that Pareto dominants μ. For notational convenience, we sometimes use $(i_1 h_1, i_2 h_2, \cdots, i_N h_N)$ to present a matching μ where $\mu(i_k) = h_k$ for all $k = 1, 2, \cdots, N$.

A **direct revelation mechanism**, or simply a **mechanism**, f selects a matching $f(P)$ for each preference profile P. Let $f_i(P)$ denote the house that agent i receives in $f(P)$. A mechanism f is **(Pareto) efficient** (**individually rational**, respectively), if for any preference profile P, the matching selected is Pareto efficient (individually rational, respectively). A mechanism f is **strategy-proof** if $f(P_i, P_{-i}) R_i f(P_i^{'}, P_{-i})$ holds for all $P_i, P_i^{'}$ and P_{-i}. The mechanism f is **group strategy-proof** if for any preference profile P, there exist no coalition $C \subseteq I$ and another preference profile $P_C^{'}$ such that for all $i \in C$, $f(P_C^{'}, P_{-C}) R_i f(P_C, P_{-C})$ holds and for some $j \in C$, $f(P_C^{'}, P_{-C}) P_i f(P_C, P_{-C})$ holds. And the mechanism f is **non-bossy** if for all P, i and $P_i^{'}$, if $f_i(P) = f_i(P_i^{'}, P_{-i})$ then $f(P) = f(P_i^{'}, P_{-i})$. Pápai (2000) shows that a mechanism f is group strategy-proof if and only if it is strategy-proof and non-bossy.

2.2 Gale's Top Trading Cycles and the Crawler

Shapley and Scarf (1974) propose the Gale's Top Trading Cycles(TTC) mechanism and show that the TTC mechanism is efficient and individually rational. Roth (1982) shows that the TTC mechanism is strategy-proof. The TTC mechanism operates as follows.

Step 1. Each agent points to the owner of her most preferred house. Due to finiteness, there exists at least one cycle (including self-cycles) and cycles don't intersect. Let agents in cycles trade and remove them.

Step $k, k \geq 2$. Repeat Step 1 with the remaining agents until all are removed.

Ma (1994) shows that the TTC mechanism is the unique IR, efficient and strategy-proof mechanism on the domain of all linear preferences.

Bade (2019) proposes a new mechanism call **the crawler** and shows that the crawler is also individually rational, Pareto efficient and strategy-proof under single-peaked preferences. The crawler operates as follows:

Step 1. Each agent points to her most preferred house in the remaining houses. Among all the agents who point to the current house or point to the right side, the rightmost owner is matched and removed with the house which she points to. Each current occupant of a house between the house vacated and the house removed moves to the next house on the left and regards the new house as new endowment.

Step $k, k \geq 2$**.** Repeat Step 1 with the remaining agents until all are matched.

Since the agents' preferences are single-peaked, there are only three kinds of agents on the street: agents want to move leftwards, want to move rightwards, and already occupy the most preferred house. Among all the agents who want to move rightwards or already occupy the most preferred house, we find the rightmost agent, assign her the most preferred unmatched house and remove them. All agents between the house vacated and the house removed "crawl" to the next house on the left. This process is repeated N times until all agents are matched.

The dual crawler is the dual mechanism, in which we find the leftmost agent preferring to moving leftwards or preferring the current house most. Then we allocate her most preferred house to her. Then agents between the two houses "crawl" to the next house on the right side. Bade (2019) shows that both the crawler and the dual crawler are individually rational, Pareto efficient and strategy-proof.

3 The r-neighborhood mechanism

3.1 Definition

We observe that Bade (2019)'s crawler mechanism essentially only allows each agent to trade for adjacent houses. Inspired by the crawler, we propose general mechanisms in which, progressively, each agent is only allowed to trade for houses within the r-neighborhood of her occupied house.

To begin with, we define the r-neighborhood of an house. An **r-neighborhood** of a houses h is the set of the nearest r houses in each side of h. Formally, giving a house h, a set of houses $\bar{H}$ and an integer r, define a set of houses $Nr(h, \bar{H}; r) = \{h^{'} |$ At most $r - 1$ houses locate between h and $h^{'}\}$ as the r-neighborhood of h on $\bar{H}$. That is, the r-neighborhood $Nr(h, \bar{H}; r)$ contains a house $h^{'}$ if and only if either $|\{\hat{h} \in \bar{H} : h < \hat{h} < h^{'}\}| < r$ or $|\{\hat{h} \in \bar{H} : h^{'} < \hat{h} < h\}| < r$ holds.

For each integer $r \geq 1$, the **r-neighborhood mechanism** is defined as follows.

Step 0. Initialize with $H_1 = H$.

Step 1. Let each agent point to the owner of the most preferred house in the r-neighborhood $Nr(h_i, H_1; r)$. There exists at least one cycle (including self-cycle) and cycles do not intersect. Let agents in cycles trade. Then if an agent occupies her most preferred house in H_1, assign the current house to the agent and remove them. If all agents are removed, terminate. If not, let H_2 be the set of remaining house. Then go to step 2.

Step $k, k \geq 2$. If an agent points to an owner who does not trade in step $k - 1$, let the agent point to the same owner as in step $k - 1$. Otherwise, let this agent point to the owner of her most preferred house in the r-neighborhood $Nr(h, H_k; r)$ with respect to the current house h. There exists at least one cycle (including self-cycle) and cycles do not intersect. Let agents in cycles trade. Then if an agent occupies her most preferred house in H_k, assign the current house to the agent and remove them. If all agents are removed, terminate. If not, let H_{k+1} be the set of remaining houses. Then go to step $k+1$.

That is, in step 1 of the r-neighborhood mechanism, all houses are unmatched, and we can define r-neighborhoods on the whole set H. Each agent points to the most preferred house in the r-neighborhood with respect to the endowment and trades if she is in a cycle. If an agent has already get her most preferred house, she is matched and removed with this house. All the unremoved agents regard the current house as the new endowment, and go to the next step. In a following step k, we define the r-neighborhood on the set of unmatched houses H_k. The target of an agent depends on whether the previous target trades in step $k-1$: If the previous target of an agent is not involved in a trade in step $k-1$, she should point to the same agent as in the last step. This restriction is essential for the strategy-proofness of the mechanisms. If the previous target of an agent is involved in a trade in step $k-1$, the agent can point to the most preferred house in an r-neighborhoods with respect to the remaining houses H_k and the current house. Then agents in cycles trade, and those occupy the most preferred houses are removed. The remaining agents regard the new house as the endowment and then go to the next step. By definition of the single-peaked preference, the preference on a subset $H_k \subseteq H$ is still single-peaked. So an agent has a unique peak in each step. The algorithm terminates when all agents are removed.

As houses are lined up on a street, we can visualize situations in which each agent is only able to see the nearest r houses of her current house, and as a result, she will only be able to choose her most preferred house in the r-neighborhood. The r-neighborhood mechanisms allow each agent to progressively trade for her most preferred house in her r-neighborhood.

The following example illustrates how the r-neighborhood mechanisms unify the TTC mechanism, the crawler and the dual crawler. It also shows that there exist r-neighborhood mechanisms different from these mechanisms.

Example 1. Suppose the set of agents is $I = \{1, 2, 3, 4\}$. Define a preference profile P in which agent $1, 2, 3$ prefer the house on the right side and agent 4 prefers the left side, i.e., we have $h_4 P_i h_3 P_i h_2 P_i h_1$ for $i = 1, 2, 3$ and $h_1 P_4 h_2 P_4 h_3 P_4 h_4$ for agent 4. The following table illustrates the preference profile.

P_1	P_2	P_3	P_4
h_4	h_4	h_4	h_1
h_3	h_3	h_3	h_2
h_2	h_2	h_2	h_3
h_1	h_1	h_1	h_4

In the TTC mechanism and the 4-neighborhood mechanism, all the agents choose their most preferred houses among $\{h_1, h_2, h_3, h_4\}$. So agent 1, 2 and 3 point to h_4, and agent 4 points to h_1. Then agent 1 and 4 trade in a cycle, and receive h_4 and h_1 respectively. Then agent 3 and 2 are in self-cycles and matched with h_3 and h_2 successively in the following two steps. The final matching is $(4h_1, 2h_2, 3h_3, 1h_4)$.

In the crawler, no agent initially occupies the most preferred house. Agent 1, 2 and 3 want to move rightwards, and agent 3 is the rightmost one. So agent 3 is screened out in the first step. She is matched with h_4 and agent 4 "crawls" to h_3. In the following step, agent 2 is screened out and is matched with h_3, and agent 4 crawls to h_2. Then agent 1 is screened out and is matched with h_2, and agent 4 finally receives h_1. So the final matching of the crawler is $(4h_1, 1h_2, 2h_3, 3h_4)$. Note that the process above is outcome equivalent to the process that agent 4 sequentially trade with agent 3, agent 2 and agent 1, and finally receives h_1.

We can reproduce the crawler with the 1-neighborhood mechanism. In the 1-neighborhood mechanism, agent 1 chooses between $\{h_1, h_2\}$, and points to h_2. Agent 2 chooses among $\{h_1, h_2, h_3\}$ and points to h_3. Similarly, agent 3 points to h_4 and agent 4 points to h_3 in the first step. Then agent 3 and 4 trades, and agent 3 is matched and removed with h_4. In the second step, agent 1 should point to h_2 since agent 2 is not in a cycle in step 1. Agent 2 prefers h_3 most and points to it. Agent 4 (with h_3) chooses between $\{h_2, h_3\}$ and points to h_2. So agent 2 and agent 4 trade. Then agent 2 is matched and removed with h_3. In the third step, agent 1 and agent 4 (with h_2) point to each other and trade in a cycle. Then all are removed. So the matching $(4h_1, 1h_2, 2h_3, 3h_4)$ is selected, which is the same as the matching selected by the crawler.

As for the dual crawler, agent 4 is the only agent who wants to move leftwards, and no agent already occupies her most preferred house. So agent 4 receives house h_1 in the first step, and agent 1, 2 and 3 crawls to the next right house. Then agent 3 (with h_4) is the only agent occupying the most preferred unmatched house and no agent wants to move leftwards. So agent 3 receives h_4. Similar steps occurs with agent 2 and 1 successively in the following steps, and the matching selected is $(4h_1, 1h_2, 2h_3, 3h_4)$, which is the same as the matching from the crawler and the 1-neighborhood mechanism.

We can reproduce the TTC mechanism and the crawler with r-neighborhood mechanisms by letting $r \geq 3$ and $r = 1$, respectively. And if $r = 2$, the 2-neighborhood mechanism selects different matching from the TTC mechanism and the crawler. In the first step of 2-neighborhood mechanism, agents 1 can choose among $\{h_1, h_2, h_3\}$ and points to h_3; agent 4 can choose among $\{h_2, h_3, h_4\}$ and points to h_2; the other two agents can choose among $\{h_1, h_2, h_3, h_4\}$ and point to h_4. Therefore agent 2 and agent 4 exchange their houses first, and agent 2 is matched and removed with her most preferred house h_4. In the second step, agent 1 should still point to h_3 since agent 3 is not in a cycle in step 1. Agent 4 (with h_2) and agent 3 chooses among $\{h_1, h_2, h_3\}$ and points to h_1 and h_3 respectively. Then agent 3 is matched and removed with h_3. In the following step, agent 1 (with h_1) and agent 4 (with h_2) point to each other and trade in a cycle. All agents are removed, and the matching $(4h_1, 1h_2, 3h_3, 2h_4)$ is selected, which is different from the matchings from the TTC mechanism and the crawler.

In Example 1, the r-neighborhoods select the same matching with the TTC mechanism and the crawler by letting $r = 4$ and $r = 1$ respectively. Besides, the dual crawler is also reproduced with $r = 1$ in our framework.

3.2 Results

Proposition 1 states the equivalent results between the r-neighborhood mechanisms, the TTC mechanism, the crawler and the dual crawler.

Proposition 1. *If the preferences of agents are single-peaked, the following statements hold:*

- *The TTC mechanism and the N-neighborhood mechanism are outcome equivalent.*
- *The crawler, the dual crawler and the 1-neighborhood mechanism are outcome equivalent.*

The first statement of Proposition 1 is implied by the definition of the N-neighborhood mechanisms and the TTC mechanism. The equivalence between the crawler and the 1-neighborhood mechanism is implied by the observation that each step of the crawler can be replicated by letting agents trade with the direct neighbors sequentially. This observation is also true between the dual crawler and the 1-neighborhood mechanism. So the second statement implies the equivalence between the crawler and the dual crawler, which is a main result in Tamura and Hosseini (2022).

The class of r-neighborhood mechanisms is a large class of mechanisms. The main result is that all the r-neighborhood mechanisms are individually rational, Pareto efficient and group strategy-proof.

Theorem 1. *If a mechanism f is an r-neighborhood mechanism, then it is individually rational, Pareto efficient and group strategy-proof.*

Theorem 1 ensures that the r-neighborhood mechanisms maintain the good properties of the TTC mechanism and the crawler and greatly generalize the class of such mechanisms with single-peaked preferences.

3.3 Obviously strategy-proofness

Some implementations of strategy-proof mechanisms can be recognised more easily. Li (2017) define obvious dominance to verify the strategies that would be chosen even if agents do not understand how each strategy's outcome depends on unobserved contingencies. A strategy is obviously dominant if, for any deviation, at any information set where the two strategies first diverge, the best outcome under the deviation is no better than the worst outcome under the dominant strategy. A mechanism is **obviously strategy-proof (OSP)** if it has an equilibrium in obviously dominant strategies. Bade (2019) shows that the crawler is obviously strategy-proof. Proposition 1 shows that the crawler is outcome equivalent to the 1-neighborhood mechanism, and below we show that 1-neighborhood mechanism is the only OSP-implementable mechanism in the class of r-neighborhood mechanisms.

An **extensive-form game** G is a rooted tree representing a set of **history**. If h is a continuation history of $h^{'}$, we denote $h^{'} \subseteq h$, and $h^{'}$ is a **subhistory** of h. A history with no continuation history is **terminal**. Each terminal history is associated with a matching. At every history h, an agents has a set of **action** $A(h)$. A strategy S_i chooses an action at every information set for agent i. An **extensive-form mechanism** is an extensive-form game G together with a profile of strategies $(S_i)_{i \in I}$. Following Li (2017), a strategy S_i **obviously dominant** another strategy $S_i^{'}$ if at the earliest history at which S_i and $S_i^{'}$ diverge, the worst payoff of agent i from playing S_i is at least as good as the best possible payoff of playing $S_i^{'}$. A profile $(S_i(\cdot))_{i \in I}$ of strategy is obviously dominant if for any agent i and preference P_i, the strategy $S_i(P_i)$ obviously dominants any other strategies with respect to P_i. An extensive-form mechanism is **obviously strategy-proof** if there exists a profile of strategies $(S_i(\cdot))_{i \in I}$ that is obviously dominant. Let $G(S(P))$ denotes the matching selected with the extensive-form game G, the profile of strategies $(S_i(\cdot))_{i \in I}$ and the preference profile P. A (direct revelation) mechanism f is OSP-implementable if there exist an extensive-form mechanism G and a profile of obviously dominant strategies $(S_i(\cdot))_{i \in I}$ such that $G(S(P)) = f(P)$ for all P. We say that G **OSP-implement** f and f is **obviously implemented** in obviously dominant strategies $(S_i(P_i))_{i \in I}$.

Strategy-proof mechanisms may not be OSP-implementable. Li (2017) shows that Gale's Top Trading Cycles mechanism with at least three agents cannot be implemented in obviously strategies with the domain of all linear preferences. With single-peaked preferences, Bade (2019) shows that the TTC mechanism with at least 4 agents is not OSP-implementable, and the crawler can be implemented in obviously dominant strategies.

Although r-neighborhood mechanisms has already greatly expanded the set of individually rational, Pareto efficient and group strategy-proof mechanisms, most r-neighborhood mechanisms are not OSP-implementable. The following theorem shows that the 1-neighborhood mechanism is the only OSP r-neighborhood mechanism.

Theorem 2. *If $|I| \geq 4$ and f is an r-neighborhood mechanism, then f can be implemented in obviously dominate strategies if and only if it is a 1-neighborhood mechanism.*

Theorem 2 implies that the 1-neighborhood mechanism, which is outcome equivalent to the crawler, is the only OSP-implementable mechanism in the class of r-neighborhood mechanisms when there are at least 4 agents. When $|I| \leq 3$, an r-neighborhood mechanism is either equivalent to the crawler or equivalent to the TTC mechanism, and in each case the r-neighborhood mechanism can be implemented in obviously dominant strategies(Bade, 2019).

4 Extension

4.1 Generalization

In an r-neighborhood mechanism, the size of r-neighborhoods is constant for all agents. We can further generalize the r-neighborhood mechanisms if the sizes of neighborhoods depend on the agents, the houses and the directions.

For an agent i and a house h_j, let positive integers l_i^j, r_i^j denote the left and right size for a neighborhood respectively. Given a set of houses $\bar{H}$, let $Nr(h_j, \bar{H}; l_i^j, r_i^j)$ denote the **neighborhood** with size l_i^j, r_i^j for agent i with h_j. The neighborhood $Nr(h_j, \bar{H}; l_i^j, r_i^j)$ contains a house $h^{'}$ if and only if either $|\{\hat{h} \in \bar{H} : h^{'} < \hat{h} < h_j\}| < l_i^j$ or $|\{\hat{h} \in \bar{H} : h_j < \hat{h} < h^{'}\}| < r_i^j$ holds. The generalized r-neighborhood mechanisms, which is simply called the **neighborhood mechanisms**, operates as follows.

Step 0. Initialize with $H_1 = H$.

Step 1. Let each agent i point to the owner of the most preferred house in the neighborhood $Nr(h_i, H_1; l_i^i, r_i^i)$. There exists at least one cycle (including self-cycle) and cycles do not intersect. Let agents in cycles trade. Then if an agent occupies her most preferred house in H_1, assign the current houses to the agent and remove them. If all agents are removed, terminate. If not, let H_2 be the set of remaining houses. Then go to step 2.

Step $k, k \geq 2$. If an agent points to an owner who does not trade in step $k-1$, let the agent point to the same owner as in step $k-1$. Otherwise, let this agent point to the owner of her most preferred house in the r-neighborhood $Nr(h_i, H_k; l_i^i, r_i^i)$ with respect to the current house h. There exists at least one cycle (including self-cycle) and cycles do not intersect. Let agents in cycles trade. Then if an agent occupies her most preferred house in H_k, assign the current house to the agent and remove them. If all agents are removed, terminate. If not, let H_{k+1} be the set of remaining houses. Then go to step $k+1$.

In the neighborhood mechanisms, agents may point into different neighborhoods at different houses, and the neighborhood mechanisms allow different agents to have different views in the same house. In each step k, we first define the neighborhoods with respect to the unmatched houses H_k. The target of an agent still depends on the previous target in the step $k-1$: If the previous target trades in step $k-1$, the agent points to the most preferred house in the neighborhood with respect to the current house; Otherwise, this agent points the same target as in the step $k-1$. This restriction is essential for the strategy-proofness of the neighborhood mechanisms.

We can also visualize situations in which an agent is only able to see houses in the neighborhood. Compared with the r-neighborhood mechanisms, the range of visible houses is determined by the identity of the agent and the occupied houses. An agent will only wish for her most preferred house in the neighborhood, and each agent progressively trades for the most preferred houses in the neighborhoods.

So r-neighborhood mechanisms are special cases of neighborhood mechanisms. Proposition 2 shows that neighborhood mechanisms preserve the good properties of the r-neighborhood mechanisms.

Proposition 2. *If a mechanism f is a neighborhood mechanism, then it is individually rational, Pareto efficient and group strategy-proof.*

Since all the r-neighborhood mechanisms are neighborhood mechanisms, Proposition 2 implies Theorem 1. We can enlarge the class of individually rational, efficient and group strategy-proof mechanisms to the set of neighborhood mechanism. Although Theorem 2 shows that the 1-neighborhood mechanism is the only OSP-implementable mechanism among all the r-neighborhood mechanism, the direct generalization does not hold. The 1-neighborhood mechanism is not the only OSP-implementable neighborhood mechanism when $|I| \geq 4$.

Remark 1. *We can further generalize the neighborhood mechanisms. We get a larger class of mechanisms if the range of neighborhood depends on the allocations in each step and the size of neighborhood can be non-negative integers. This set of mechanisms generalize the neighborhood mechanisms. But if earlier cycles may effect the forthcoming neighborhoods of other agents, some of these mechanisms can be bossy, and the group strategy-proofness may be violated.*

4.2 Discussion about Liu (2021)

In the neighborhood mechanisms, we define the nearest several houses of an agent i with a house h as the neighborhood of i with h. And each agent points to her most preferred house in the neighborhood. Liu (2021) introduces another class of mechanisms, where a subset of houses is available for exchange in each step. In each step, only the agents who occupy the houses in the subset are allowed to exchange with each other. All the houses are available in finitely many steps, after which the algorithm terminates. Liu (2021) use ”neighborhood” to call this subset of houses available in each step if they are next to each other.

Liu (2021) identifies the class of mechanisms called **neighborhood top trading cycles(NTTC) mechanisms**. An NTTC mechanism selects the final allocation by several steps. In each step, a set of houses is chosen exogenously. Each agent who occupies a chosen house points to the owner of her most-preferred house in this set. Agents in cycles trade and exit the step with the new houses. In each step this procedure is repeated until all chosen houses exit this step. An NTTC mechanism terminates after a step in which all the houses are selected. The selected houses are restricted to be next to each other in each step. And such a set can be path-dependently, in the sense that it can be determined according to the allocations in the previous steps. The NTTC mechanisms are individually rational, Pareto efficient and strategy-proof with single-peaked preferences. Below we introduce the definition of the NTTC mechanisms and discuss the relationship between the class of NTTC mechanisms and the class of neighborhood mechanisms.

For each NTTC mechanism, Liu (2021) defines a tree to determine the sets available for exchange along the iteration. Let e be the initial allocation. Given a subset of agents $\hat{I}$ and a subset of houses $\hat{H}$ such that $|\hat{I}| = |\hat{H}|$, a sub-allocation m is a one-to-one mapping from a $\hat{I}$ to a subset of houses $\hat{H}$, and let I_m and H_m denote the agents and houses involved in m respectively. A sub-allocation m is called to be nested in another sub-allocation $m^{'}$ if $H_m \subseteq H_{m'}$ and $I_m \subseteq I_{m'}$. The set of all sub-allocation is M. A sequence of sub-allocations $m_1 m_2 \cdots$ is called a history if $m_1 = e$ and m_{k+1} is nested in m_k for $k = 1, 2, \cdots$. If a sub-history γ is created by appending some sub-allocations to history $\gamma^{'}$, we denote $\gamma^{'} \subseteq \gamma$ and say that $\gamma^{'}$ is a sub-history of γ. For example, m_1m_2 is a sub-history of both itself and $m_1m_2m_3$. Let Γ denotes the set of all histories. Let $|\gamma|$ denotes the number of allocation s in the history γ and let $\bar{\Gamma} = \{\gamma \in \Gamma : |\gamma| = \infty\}$ be the set of all infinite histories. An available tree is a function $T : \Gamma \to 2^H \setminus \emptyset$ satisfying the condition that for any history $\bar{\gamma} \in \bar{\Gamma}$, there is a non-terminal sub-history $\gamma \subseteq \bar{\gamma}$ such that $T(\gamma^{'}) = H_m$, where m is the last sub-allocation in γ and that for any history $\bar{\gamma} \in \Gamma \setminus \bar{\Gamma}$, $T(\gamma^{'}) \subseteq H_m$, where m is the last sub-allocation in γ. An available tree T is called a neighborhood tree, if $h_i, h_j \in T(\gamma)$ and $h_i < h_k < h_j$ imply $h_k \in T(\gamma)$ for any $h_k \in H_m$ and any $\gamma \in \Gamma \setminus \bar{\Gamma}$, where m is the last sub-allocation in γ.

Fix a neighborhood tree $T : \Gamma \to 2^H \setminus \emptyset$. An NTTC mechanism operates as follows.

Step 1 Initialize with $m_1 = e$.

Step $k, k \geq 2$. Preparation sub-step: Check for each agent whether her current house is the most-preferred one among the remaining ones. If so, let the agent leave with the current house. Repeat this process until no such agent exists. If no agent remains, iteration terminates. Otherwise, denote the resulted sub-allocation by $\bar{m}_{k-1}$.

Exchange sub-step: Each agent who owns a house in the set $T(\bar{m}_1\bar{m}_2 \cdots \bar{m}_{k-1})$ points to the owner of her most preferred house in $T(\bar{m}_1\bar{m}_2 \cdots \bar{m}_{k-1})$. There exists at least one cycle(including self-cycles) and cycles don't intersect. Let agents in cycles trade and exit the step with the new houses. Repeat this process until all the houses in $T(\bar{m}_1\bar{m}_2 \cdots \bar{m}_{k-1})$ are removed in this step. Let m_k denote the allocation after all the trades in this step. If $T(m_1m_2 \cdots m_{k-1}) = H$, terminate and the matching m_k is selected. Otherwise proceed to step $k+1$.

The definition of the neighborhood tree ensures that given a neighborhood tree and a preference profile, the whole set of houses is available after a finite step and then the algorithm terminates. Liu (2021) shows that the NTTC mechanisms are individually rational, Pareto efficient and strategy-proof with single-peaked preferences. Below we discuss the relationship between the neighborhood mechanisms and the NTTC mechanisms.

In a neighborhood mechanism, the left and right size of neighborhood is determined by the identity of agents houses, but the neighborhood tree of an NTTC mechanism might be path-dependent. So a NTTC mechanism may not be in the class of neighborhood mechanisms. In the NTTC mechanisms, only selected agents are allowed to trade in each step, so they share the same range of vision in the step. In the neighborhood mechanisms, different agents may have different neighborhood in the same step. So the class of neighborhood mechanisms contains mechanisms that are not NTTC mechanisms. On conclusion, we have the following statement.

Claim 1. *The class of NTTC mechanisms and the set of neighborhood mechanisms are not comparable.*

Example 2 shows that some of neighborhood mechanisms are not NTTC mechanism, and some NTTC mechanisms are not contained in the class of neighborhood mechanisms.

Example 2. Suppose $I = \{1, 2, 3, 4\}, H = \{h_1, h_2, h_3, h_4\}$. Consider three preference profiles. In case 1, agent $1, 2, 3$ want h_4 most, while agent 4 prefers h_1. The table below illustrates the preference profile in case 1:

P_1	P_2	P_3	P_4
h_4	h_4	h_4	h_1
h_3	h_3	h_3	h_2
h_2	h_2	h_2	h_3
h_1	h_1	h_1	h_4

In case 2, agent 1 prefers h_2, agent 2 and 3 wants h_4, and agent 4 wants h_1. The following table illustrates it:

P_1	P_2	P_3	P_4
h_2	h_4	h_4	h_1
h_3	h_3	h_3	h_2
h_4	h_2	h_2	h_3
h_1	h_1	h_1	h_4

In case 3, agent 1 wants h_4, and the other agents prefer h_1. The following table illustrates it:

P_1	P_2	P_3	P_4
h_4	h_1	h_1	h_1
h_3	h_2	h_2	h_2
h_2	h_3	h_3	h_3
h_1	h_4	h_4	h_4

Now we show that the following neighborhood mechanism f is not outcome equivalent to any NTTC mechanism: agent 1,3,4 can point to any houses, and agent 2 can only point to the direct neighbor. That is, we have $l_i^j = r_i^j = 4$ for $i = 1, 3, 4, \forall j$, and $l_2^j = r_2^j = 1, \forall j$. This neighborhood mechanism selects $(4h_1, 2h_2, 3h_3, 1h_4)$ in Case 1 and $(4h_1, 1h_2, 2h_3, 3h_4)$ in Case 2. In case 1, TTC mechanism is the only NTTC mechanism selecting $(4h_1, 2h_2, 3h_3, 1h_4)$. In case 2, the only NTTC mechanism selects $(4h_1, 1h_2, 3h_3, 2h_4)$. So the neighborhood mechanism f is not in the class of NTTC mechanisms.

The following NTTC mechanism g is not outcome equivalent to any neighborhood mechanism: agent 1,2,3 trade first, and then all the agents trade. That is, the neighborhood tree T satisfies $T(e) = \{h_1, h_2, h_3\}$ and $T(\gamma) = H_m$ for all $|\gamma| > 1$, where m is the last sub-allocation in γ. The mechanism g selects $(4h_1, 2h_2, 3h_3, 1h_4)$ in Case 1 and selects $(3h_1, 2h_2, 4h_3, 1h_4)$ in Case 3. If a neighborhood mechanism selects $(4h_1, 2h_2, 3h_3, 1h_4)$ in case 1, then agent 1 and 4 can point to each other in the first step (which means $r_1^1 = l_4^4 \geq 3$), and this neighborhood mechanism selects $(4h_1, 2h_2, 3h_3, 1h_4)$ in Case 3. So the NTTC mechanism g is not in the class of neighborhood mechanisms.

Compared with the NTTC mechanisms, the neighborhood mechanisms have some appealing advantages. First, Li (2017) defines two different neighborhood trees to replicate the crawler and the dual crawler respectively, while we unify the crawler and the dual crawler with the 1-neighborhood mechanism. Besides, in the NTTC mechanisms, selected agents share the same range of vision in each step, and the neighborhood mechanisms allow different agents to have different range of vision in the same step. Moreover, the NTTC mechanisms are strategy-proof, while we show that the neighborhood mechanisms are group strategy-proof.

5 Conclusion

In this paper, we unify and generalize the TTC mechanism and the crawler with single-peaked preferences. We identify a large class of mechanisms called r-neighborhood mechanisms, and propose a more general set of mechanisms called neighborhood mechanisms which are group strategy-proof, individually rational and Pareto efficient. We show that the crawler, dual crawler and 1-neighborhood mechanism are outcome equivalent. Among all the r-neighborhood mechanisms, the 1-neighborhood mechanism is the only mechanism that can be implemented in obviously dominant strategies, while some neighborhood mechanisms other than the crawler are OSP-implementable.

As for the characterization problem, noticing that some mechanisms are strategy-proof, efficient and individually rational but they are not neighborhood mechanisms, the full characterization problem is still an open question.

A Appendix

A.1 Proof of Proposition 1

Proof. The first statement can be proved directly by the definition. Either in the N-neighborhood mechanism or in the TTC mechanism, each agent can point to the owner of the most preferred house among all the houses and does not change the target. So an agent points to and trade with the same owner as the TTC mechanism in each step. Then an agent receives the same house in the two mechanisms.

Now we prove the second statement. Given a preference profile $P = (P_1, P_2, \cdots, P_N)$, let $C(P)$ denote the matching selected by the crawler and agent i receives $C_i(P)$ in $C(P)$. Let $f(P)$ be the matching selected by the 1-neighborhood mechanism and agent i receives $f_i(P)$ in $f(P)$. We need to show that $C_i(P) = f_i(P)$ for any i. Let i_k denotes the agent matched in step k of the crawler. Let m_k denotes the number of agents who crawls in step k of the crawler. Suppose agents $i_k^1, i_k^2, \cdots, i_k^{m_k}$ crawl from houses $h_k^1, h_k^2, \cdots, h_k^{m_k}$ to $h_k^0, h_k^1, h_k^2, \cdots, h_k^{m-1}$ respectively in step k of the crawler, where $i_k^1 < i_k^2 < \cdots < i_k^{m_k}$. Then agent i_k occupies house h_k^0 at the beginning of step k of the crawler and finally receives $h_k^{m_k} = C_{i_k}(P)$.

We now show that $C_{i_1}(P) = f_{i_1}(P)$ and agent i_1 trades with agent $i_1^{m_1}$ to the house $h_1^{m_1}$ for all $m = 1, 2, \cdots, m_1$ in the 1-neighborhood mechanism. If agent i_1 prefers the house h_{i_1} most, then no agent crawls in step 1 of the crawler and $C_{i_1}(P) = f_{i_1}(P) = h_{i_1}$ holds. If agent i_1 wants to move rightwards, then agent $i_1^1, i_1^2, \cdots, i_1^{m_1}$ want to move leftwards according to the definition of the crawler. In the first step in the 1-neighborhood mechanism, agent $i_1^{m_1}$ and agent i_1 are direct neighbors and they point to each other. So agent i_1 moves from h_1^1 to h_1^2 in step 1 of the 1-neighborhood mechanism. Then agent i_1 is the direct neighbor of agent i_1^2 and trade in a cycle. By induction, This process repeat until agent i_1 trade with agent $i_1^{m_k}$ in step m_1. So agent i_1 trades with agent $i_1^{m_1}$ to the house $h_1^{m_1}$ for all $m = 1, 2, \cdots, m_1$ in the 1-neighborhood mechanism by induction. Since $h_1^{m_1}$ is the most-preferred house of agent i_1, we have $C_{i_1}(P) = f_{i_1}(P) = h_1^{m_1}$ holds.

Next we complete the proof by induction. Suppose for any $j < k$, agent i_j trades with agent i_j^n to the house h_j^n for all $n = 1, 2, \cdots, m_j$ in the 1-neighborhood mechanism and $C_{i_j}(P) = f_{i_j}(P)$ holds. We need to show that $f_{i_k}(P) = C_{i_k}(P)$ holds and agent i_k trades with $i_k^1, i_k^2, \cdots, i_k^{m_k}$ to $h_k^1, h_k^2, \cdots, h_k^{m_k}$ in some steps of the 1-neighborhood mechanism.

If $i_k > min\{i_1, i_2, \cdots, i_{k-1}\}$ holds, then agent i_k prefers the currently occupied house to the house of the right direct neighbor in any step of the crawler. So no agent crawls in step k and $h_k^0 = C_{i_k}(P)$ holds. By the induction hypothesis, for all $j < k$, if i_k crawls from a house h to another house $h^{'}$ in step j of the crawler, then agent i_k trades with i_j from h to $h^{'}$ in the 1-neighborhood mechanism. So agent i_k occupies $C_{i_k}(P)$ in some step of the 1-neighborhood mechanism. Since $f_{i_j}(P) = C_{i_j}(P)$ holds for all $j < k$, agent i_k does not trade to any house strictly better than $C_{i_j}(P)$ in the 1-neighborhood mechanism. So no agent crawls in step k and $C_{i_k}(P) = f_{i_k}(P)$ holds.

If $i_k < min\{i_1, i_2, \cdots, i_{k-1}\}$ holds, then agent i_k does not crawl in any step of the crawler, and h_k^0 is the initial endowment h_{i_k}. For any $n = 0, 1, \cdots, m_k$, we need to show that agent i_k trade with i_k^n from the house h_k^{n-1} to h_k^n in the algorithm of the 1-neighborhood mechanism. By the induction hypothesis, for any $j < k$ and $n < m_k$, if an agent i_k^n crawls from a house h to another house $h^{'}$ in step j of the crawler, then agent i_k^n trades with i_j in the 1-neighborhood mechanism. Let $i_k^0 = i_k$. Then for any $n = 0, 1, 2, \cdots, m_k$, there exist an integer t_n such that agent i_k^n occupies h_k^n at the beginning of step t_n of the 1-neighborhood mechanism. Now we show that agent i_k trades with i_k^1 from h_k^0 to h_k^1 in step $max\{t_0, t_1\}$ of the 1-neighborhood mechanism. If $t_0 = t_1$, then agent

i_k and i_k^1 point to each other in step t_0 and trade in a cycle. If $t_0 > t_1$, then we need to show that agent i_k^1 does not trade from h_k^1 to h_k^0 before step t_0 of the 1-neighborhood mechanism. Suppose not, and agent i_k^1 trades with an agent i_p, who is chosen in step p of the crawler, from h_k^1 to h_k^0 before step t_0 of the 1-neighborhood mechanism. Since agent i_k^1 does not crawls from h_k^1 to h_k^0 in the first $k-1$ steps of the crawler, we have $p \notin \{1, 2, \cdots, k-1\}$ by the induction hypothesis and thus $p > k$ holds. Meanwhile, agent i_p occupies the house h_k^0 before step t_0 implies that $i_p > i_k$ holds. The assumption that i_p trades with i_k^1 implies that i_p prefers h_k^1 to h_k^0 and thus i_p should be screened out before i_k. So we have $p < k$, a contradiction. If $t_0 < t_1$, then we need to show that agent i_k does not trade from h_k^0 to h_k^1 before step t_1. Suppose not, and agent i_k trades with an agent i_p from h_k^0 to h_k^1 before step t_1. Since each agent trades with the direct neighbor in the 1-neighborhood mechanism, the house h_{i_p} locates between h_{i_k}(which is h_k^0) and $h_{i_k^1}$. Let h^* denote the house owned by agent i_p in step k of the crawler. The assumption $i_k < min\{i_1, i_2, \cdots, i_{k-1}\}$ implies that h^* located at the right side of h_k^0. If $p > k$ holds, then agent i_k^1 cannot crawls to the left side of agent i_p before step k of the crawler. So the house h^* locates at the left side of h_k^1. But there is no remaining house between h_k^0 and h_k^1 in step k of the crawler. So $p > k$ does not hold. If $p < k$ holds, then agent i_p is removed before step k of the crawler. So $f_{i_p}(P) = C_{i_p}(P) = h^*$ and $f_{i_p}(P) R_{i_p} h_k^0$ hold. But the assumption that agent i_p trades with agent i_k to house h_k^0 implies that $h_k^0 R_{i_p} f_{i_p}(P)$ due to the single-peaked preference, a contradiction. In conclusion, agent i_k trades with i_k^1 from h_k^0 to h_k^1 in step $max\{t_0, t_1\}$ of the 1-neighborhood mechanism. Then agent i_k occupies house h_k^1 in step $max\{t_0, t_1\} + 1$ of the 1-neighborhood mechanism, and agent i_k^2 occupies h_k^2 in step t_2. For the same reason, i_k trades with i_k^2 from h_k^1 to h_k^2 in step $max\{t_0 + 1, t_1 + 1, t_2\}$ of the 1-neighborhood mechanism. By induction, we have agent i_k sequentially trades with $i_k^1, i_k^2, \cdots, i_k^{m_k}$ to $h_k^1, h_k^2, \cdots, h_k^{m_k}$ in the 1-neighborhood mechanism, where $h_k^{m_k} = C_{i_k}(P)$. Since $f_{i_j}(P) = C_{i_j}(P)$ for all $j < k$, agent i_k does not trade to any house strictly better than $C_{i_k}(P)$, and $C_{i_k}(P) = f_{i_k}(P)$ holds.

By induction, we have $C_i(P) = f_i(P)$ for any i, and the second statement is proved.

□

A.2 Proof of Theorem 1 and Proposition 2

Proof. Since Proposition 2 implies Theorem 1, we only need to prove Proposition 2. Pápai (2000) shows that a mechanism is group strategy-proof if and only if it is strategy-proof and non-bossy. We need to show that if f is a neighborhood mechanism, then f is individually rational, Pareto efficient, strategy-proof and non-bossy.

Individual rationality: The neighborhood mechanisms are individually rational, since the endowment is initially in the neighborhood. So an agents always trades to a better house, and finally receives a house weakly better than the initial one.

Pareto efficiency: We can show that the neighborhood mechanisms are efficient by induction. All the agents removed in step 1 is matched with their most preferred houses. Removing houses matched in step 1, agents removed in step 2 receive their most preferred in the remaining houses. So agents removed in step 2 cannot receive a strictly better house without hurting agents removed in step 1. Proceeding inductively, agents matched in step k receive their most preferred houses among those remaining at the beginning of step k. So the neighborhood mechanisms are efficient.

Strategy-proofness: As for strategy-proofness, we need to show that $f_i(P_i, P_{-i}) R_i f_i(P_i^{'}, P_{-i})$ holds for any $P, P^{'}$ and i. Given an agent i, a preference profile $P = (P_1, P_2, \cdots, P_N)$ and another preference $P_i^{'}$, let $h_i^k(P_i, P_{-i})$ denote the house occupied by agent i in the end of step k with the preference profile (P_i, P_{-i}) and let $H^k(P_i, P_{-i})$ denote the remaining houses at the beginning of step k with (P_i, P_{-i}). Without loss of generality, suppose the peak of P_i is to the right side of h_i. Due to the single-peakedness, the left neighbor of agent i is worse than the occupied one in any step, and thus agent i will never point to the left side of the occupied house in any step. If misreporting leads to the same house, $f_i(P_i, P_{-i}) R_i f_i(P_i^{'}, P_{-i})$ holds. Otherwise, suppose the first deviation happens in step k_0. That is, define $k_0 = min_k\{h_i^k(P_i, P_{-i}) \neq h_i^k(P_i^{'}, P_{-i})\}$. Note that the assumption that the first deviation happens at step k_0 implies that agent i is in a cycle in step $k_0 - 1$ with either (P_i, P_{-i}) or $(P_i^{'}, P_{-i})$ if $k_0 \geq 2$. So agent i points to the owner of the most preferred house in the neighborhood in step k_0 with either P_i or $P_i^{'}$. We say a house h is the (right) **boundary** of a neighborhood N_i^k, if $h \in N_i^k$ and $h^{'} \notin N_i^k$ for any $h^{'} > h$. A house h is in the **interior** of N_i^k if it is not the boundary of N_i^k. Then there are three cases of step k_0: Agent i with P_i moves to the interior of the neighborhood; Agent i with P_i moves to the boundary of the neighborhood; And agent i with P_i is not in a cycle in step k_0.

Now we show that if agent i with P_i moves to the interior of the neighborhood in step k_0, then we have $f_i(P_i, P_{-i}) R_i f_i(P_i^{'}, P_{-i})$. If truth-telling leads to a house $h_i^k(P_i, P_{-i})$ in the interior, the agent i occupies the most preferred house in the neighborhood after the trade, which is also the most preferred house among all the remaining house at the beginning of step k_0 due to the single-peakedness. So misreporting to $P_i^{'}$ cannot lead to a strictly better house.

The next case is that agent i reporting P_i moves to the boundary of the neighborhood in step k_0. In this case agent i reporting $P_i^{'}$ points to the interior of the neighborhood in step k_0. Then the peak of $P_i^{'}$ among all the remaining houses $H_i^k(P_i, P_{-i})$ is in the interior of the neighborhood. So agent i with $P_i^{'}$ will be removed with $f_i(P_i^{'}, P_{-i})$ after the next trade. Suppose the boundary house is occupied by agent $i^{'}$ at the beginning of step k_0. There is a chain from agent $i^{'}$ to agent i in step k_0 and the chain exists until agent i trades. In the following steps, agent i reporting $P_i^{'}$ points to the owner of the peak among the remaining houses, which locates between the boundary $h_i^k(P_i, P_{-i})$ and the currently occupied house. Since a cycle occurs if agent i point from the current house to either the owner of boundary or herself, the house $f_i(P_i^{'}, P_{-i})$ locates between these two houses. Then we have $f_i(P_i, P_{-i}) R_i h_i^k(P_i, P_{-i}) R_i f_i(P_i^{'}, P_{-i})$, and the deviation is not profitable.

The last case is that agent with P_i does not trade in step k_0, but agent with $P^{'}$ trades with $i^{''}$ to the house $h^{''}$ in a cycle. Then there exists a chain from agent $i^{''}$ to agent i until agent i trades, and the house $h^{''}$ is always available until a trade occurs for P_i. So if agent i with P_i points to the left side of $h^{''}$ in step k_0, she will be removed after the next trade and $f_i(P_i, P_{-i}) R_i h^{''} R_i f_i(P_i^{'}, P_{-i})$ holds. And if the agent with P_i point to the right side of $h^{''}$ in step k_0, we have $f_i(P_i^{'}, P_{-i}) = h^{''}$. The fact that $h^{''}$ is always available until the next trade of agent i with P_i implies $f_i(P_i, P_{-i}) R_i f_i(P_i^{'}, P_{-i})$.

As a result, the mechanism f is strategy-proof.

Non-bossiness: Next we show that f is non-bossy. Suppose a preference profile P and a preference $P_i^{'}$ satisfy $f_i(P_i, P_{-i}) = f_i(P_i^{'}, P_{-i})$. We need to show that $f(P_i, P_{-i}) = f(P_i^{'}, P_{-i})$. If agent i points to the same agent in each step, the same matching will be selected. Otherwise, consider the first step k where she points to different agents. Suppose agent i with P_i and $P_i^{'}$ moves to h and $h^{'}$ in the next *trade* of step k respectively. Without loss of generality, suppose agent i moves rightwards and h is closer to the occupied house, that is, either $h < h^{'}$ or $h = h^{'}$ holds. There is always a chain from the owner of h to agent i before the next trade of $P^{'}$. So the assumption that the house h is closer to the currently occupied house implies that agent i with P_i does not point to the boundary of the neighborhood in step k and will be removed after the next trade. Then $f_i(P_i, P_{-i}) = h$ holds. Agents in neighborhood mechanisms always move to the same direction, so $f_i(P_i, P_{-i}) = f_i(P_i^{'}, P_{-i}) = h$ implies $h^{'} = h$. Then agent i with either P_i or $P_i^{'}$ trades in the same cycle in the next trade and is removed after this trade in each situation. Since the preference profile of other agents are the same, the deviation does not influence the cycles in the algorithm. Then we have $f(P_i, P_{-i}) = f(P_i', P_{-i})$, and f is non-bossy.

Since f is strategy-proof and non-bossy, we conclude that f is group strategy-proof.

□

A.3 Proof of Theorem 2

Proof. We need to show that an r-neighborhood mechanism f with $|I| \geq 4$ is OSP-implementable if and only if $r = 1$ holds. When $r = 1$ holds, the 1-neighborhood mechanism f is outcome equivalent to the crawler according to Proposition 1. Bade (2019) shows that the crawler is OSP-implementable. So the 1-neighborhood mechanism is OSP-implementable.

Next we show that the r-neighborhood mechanism cannot be implemented in obviously dominate strategies if $|I| = 4$ and $r \geq 2$. Suppose an extensive-form game G together with a profile of obviously dominate strategies $(S_i(\cdot))_{i \in I}$ implements f with $I = \{1, 2, 3, 4\}$ and $H = \{a, b, c, d\}$. Agent $1(2, 3, 4$, respectively) is endowed with the house $a(b, c, d$, respectively). If $i < j$ holds, then the endowment of agent i locates at the left side of the endowment of agent j. Inspired by Li (2017) (Proof of Proposition 5), consider the following subset of the preference domain. Each agent i has two possible preferences P_i and P_i' such that,

$$P_1 : bR_1cR_1dR_1a$$
$$P_1' : dR_1'cR_1'bR_1'a,$$
$$P_2 : cR_2dR_2bR_2a$$
$$P_2' : dR_2'cR_2'bR_2'a,$$

$$P_3 : bR_3aR_3cR_3d$$
$$P_3^{'} : aR_3^{'}bR_3^{'}cR_3^{'}d,$$
$$P_4 : cR_4bR_4aR_4d$$
$$P_4^{'} : aR_4^{'}bR_4^{'}cR_4^{'}d.$$

Then we argue by contraction. Consider some history h at which agent 1 moves and two actions correspond to P_1 and $P_1^{'}$ respectively. We claim that such a history h cannot come before all such history for agent 2, 3 and 4 in the game G. That is, given such a history h for agent 1, there exist a subhistory $h^{'} \subseteq h$ and $j \in \{2, 3, 4\}$ such that agent j moves at $h^{'}$ and two actions correspond to P_j and $P_j^{'}$ respectively.

Suppose not, and suppose agent 1 with P_1 chooses the action corresponding to P_1. If agent 1 faces the other agents with $P_2^{'}$, P_3 and P_4, then agent 1 gets the house a. If agent 1 chooses the other action, and faces $P_2^{'}$, $P_3^{'}$ and P_4, then agent 1 receives house c and cP_1a holds. Thus it is not an obviously dominant strategy to choose the action corresponding to P_1. So agent 1 cannot be the first to have such a non-singleton action set.

Now consider a history h at which agent 2 moves and two actions correspond to P_2 and $P_2^{'}$. We need to show that agent 2 cannot be the first as well. Suppose agent 2 with $P_2^{'}$ chooses the action corresponding to $P_2^{'}$, and faces $P_1^{'}$, $P_3^{'}$ and P_4, then agent 2 receives the house b. But if agent 2 chooses the action corresponding P_2 and faces P_1, P_3 and $P_4^{'}$, agent 2 receives c and $cP_2^{'}b$ holds. So agent 2 cannot be the first to have such a non-singleton action set.

By symmetry, similar argument applies to agent 3 and 4 as well. Due to finiteness, there is not such a game G and a profile of obviously dominate strategies that implements f, a contradiction. So an 2-neighborhood mechanism cannot be implemented in obviously dominate strategies when there are 4 agents.

If $r \geq 3$ holds, then an r-neighborhood mechanism f is outcome equivalent to TTC mechanism when $|I| = 4$. The TTC mechanism cannot be implemented in obviously dominated strategies when there are at least 4 agents(Bade, 2019). So r-neighborhood mechanisms with $r \geq 3$ are not OSP-implementable when $|I| = 4$.

Next we show that an r-neighborhood mechanism f cannot be implemented in obviously dominate strategies with $r \geq 2$ and $|I| > 4$. Suppose not, and an extensive-form game G together with a profile of obviously dominate strategies $(S_i(\cdot))_{i \in I}$ implements f. Fix the preferences of agent i as $\bar{P}_i$ for any $i > 4$, then the game G and $(S_i(\cdot))_{i \in I}$ induce an extensive-form game $\bar{G}$ where only agent $1, 2, 3$ and 4 choose on the nodes. Then $\bar{G}$ and the profile of obviously dominate strategies $(S_i(\cdot))_{i \in \{1,2,3,4\}}$ implement f with 4 agents, a contradiction.

On conclusion, the 1-neighborhood mechanism is the only OSP-implementable mechanism with $|I| \geq 4$ in the class of r-neighborhood mechanisms.

□

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