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Existence of Solution of Minimax Inequalities, Equilibria in Games and Fixed Points Without Convexity and Compactness Assumptions

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Abstract This paper characterizes the existence of equilibria in minimax inequalities without assuming any form of quasiconcavity of functions and convexity or compactness of choice sets. A new condition, called "*local dominatedness property*", is shown to be necessary and further, under some mild continuity condition, sufficient for the existence of equilibrium. We then apply the basic result obtained in the paper to generalize the existing theorems on the existence of saddle points, fixed points, and coincidence points without convexity or compactness assumptions. As an application, we also characterize the existence of pure strategy Nash equilibrium in games with discontinuous and non-quasiconcave payoff functions and nonconvex and/or noncompact strategy spaces.

Keywords Minimax inequality · Saddle points · Fixed points · Coincidence points · Discontinuity · Non-quasiconcavity · Non-convexity · Non-compactness

1 Introduction

The minimax inequality [1] is one of the most important tools in nonlinear analysis and in mathematical economics. Indeed, it allows one to derive many practical and

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theoretical results in a wide variety of fields. In many situations, the Ky Fan inequality is more flexible and adaptable than other basic theorems in nonlinear analysis, such as fixed-point theorems and variational inequalities. Reference [2] remarked that it is often easier to reduce an equilibrium existence problem to a minimax inequality problem rather than to transform it into a fixed-point problem. Therefore, weakening its conditions further enlarges its domain of applicability.

The many applications of minimax inequality in different areas (such as general equilibrium theory, game theory, and optimization theory) attracted researchers to weaken the conditions on the existence of its solution. Lignola [3] relaxed the assumption on the compactness and the semicontinuity. References [4–6] weakened the condition of quasiconcavity of the function. Many other results were also obtained, such as those in [2, 5–11]. Equilibrium problems were studied in both mathematics and economics, such as those in [2, 4, 7, 8, 12–20], among which [7, 8] generalized the Ky Fan inequality to different sets. However, all the work mentioned above is assumed that the set is convex. In many practical situations, a choice set may not be convex and/or not compact so that the existing theorems cannot be applied.

In this paper, we provide characterization results on the existence of solution to the minimax inequality without any form of quasiconcavity of function or convexity and compactness of choice sets. We introduce a new condition, called *local-dominatedness property*. It is shown that the local-dominatedness condition is necessary and further, under some mild continuity condition, sufficient for the existence of a solution to a minimax inequality. We then apply the basic result to study the existence of saddle points, fixed points, and coincidence points. As an application of our basic result, we study the existence of equilibria for a noncooperative game without quasiconcavity of payoff functions and convexity or compactness of strategy spaces.

The remainder of the paper is organized as follows. In Sect. 2, we give basic terminologies used in our study. We introduce the concepts of local-dominatedness property, transfer quasiconcavity, and transfer continuity. We also provide sufficient conditions for these conditions to be true. Section 3 is dedicated to the development of existence theorems on minimax inequality for a function defined on Cartesian product of two different sets without the convexity and/or compactness assumptions. In Sect. 4, we apply our results on the minimax inequality to offers new existence theorems on saddle points without assuming convexity and/or compactness of choice sets. In Sect. 5, we provide necessary and sufficient conditions for the existence of fixed points and coincidence points. We introduce the concept of f-separability that can be used to characterize the existence of fixed point of a function without the convexity assumption. Section 6 considers the existence of coincidence points. In Sect. 7, we consider the existence of equilibria in games discontinuous and non-quasiconcavity of payoff functions and convexity and/or compactness of strategy spaces. Concluding remarks are offered in Sect. 8.

2 Notations and Definitions

Let Y be a nonempty subset of a topological space F. Denote by 2^Y be the family of all nonempty subsets of Y and $\langle Y\rangle$ the set of all finite subsets of Y. Let $S \subset Y$, denote by $\operatorname{int} S$ the relative interior of S in Y and by $\operatorname{cl} S$ the relative closure of S in Y.

A function $f: Y \to \mathbb{R}$ is upper semicontinuous on Y iff, the set $\{x \in Y, f(x) \geq c\}$ is closed for all $c \in \mathbb{R}$; f is lower semicontinuous on Y iff, $-f$ is upper semicontinuous on Y; f is continuous on Y iff, f is both upper and lower semicontinuous on Y.

Let F be a vector space and Y a convex set. A function $f: Y \to \mathbb{R}$ is quasiconcave on Y iff, for any y_1, y_2 in Y and for any $\theta \in [0, 1]$, $\min\{f(y_1), f(y_2)\} \leq f(\theta y_1 + (1-\theta)y_2)$, and f is quasiconvex on Y iff, $-f$ is quasiconcave on Y. A function $f: Y \times Y \to \mathbb{R}$ is diagonally quasiconcave in y iff, for any finite points $y^1, \dots, y^m \in Y$ and any $y \in \mathrm{co}\{y^1, \dots, y^m\}$, $\min_{1 \leq k \leq m} f(y, y^k) \leq f(y, y)$. A function $f: Y \times Y \to \mathbb{R}$ is α-diagonally quasiconcave in y iff, for any $y^1, \dots, y^m \in Y$ and $y \in \mathrm{co}\{y^1, \dots, y^m\}$, $\min_{1 \leq k \leq m} f(y, y^k) \leq \alpha$.

Definition 2.1 (α-Local-dominatedness) Let $\alpha \in \mathbb{R}$. A function $\Psi: X \times Y \to \mathbb{R}$ is said to be *α-locally dominated in y* iff, for any $A \in \langle Y \rangle$, there exists $x \in X$ such that

$$\max_{y \in A} \Psi(x, y) \leq \alpha.$$

The term *localness* reflects to choose finite subsets from Y. *Dominatedness* refers to the fact that $\Psi(x, y)$ is dominated by α. *α-local-dominatedness property* says that, given any finite set $A \subset Y$, there exists a corresponding candidate point $x \in X$ such that $\Psi(x, y)$ is dominated by α for all points in A. We will see from Theorem 3.1 below that α-local-dominatedness condition is necessary, and further, under some mild condition, sufficient for the existence of solution to the minimax inequality.

Remark 2.1 Let $H(y) = \{x \in X : \Psi(x, y) \leq \alpha\}$ for $y \in Y$. Then, the function $\Psi(x, y)$ is α-locally dominated in y if and only if the family sets $\{H(y),\ y \in Y\}$ has the finite intersection property.

Example 2.1 Consider the following function:

$$\begin{aligned} &f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \\ &(x, y) \mapsto f(x, y) = x^3 - x \times y^2. \end{aligned}$$

It is obvious that, if $x < 0$, the function $y \mapsto f(x, y)$ is not quasiconcave in y. However, it is 0-locally dominated in y. To see this, let $y_1, \dots, y_n \in \mathbb{R}$. Then there exists $x = -\max_{i=1,\dots,n}|y_i|$ such that $y_i^2 \leq x^2$ for all $i \leq n$. Thus, $-xy_i^2 \leq -x^3$, and $f(x, y_i) = x^3 - xy_i^2 \leq 0$ for all $i \leq n$.

The following Definition generalizes the γ-generalized quasiconcave in [12] and the transfer quasiconcavity in [21] to a function defined in the product of different sets.

Definition 2.2 (α-Transfer quasiconcavity) Let X be a nonempty and convex subset of a vector space E and let Y be a nonempty set. A function $\Psi: X \times Y \to \mathbb{R}$ is said to be *α-transfer quasiconcave in y* iff, for any finite subset $\{y^1, \dots, y^m\} \subset Y$, there exists a corresponding finite subset $\{x^1, \dots, x^m\} \subset X$ such that, for any subset $L \subset \{1, 2, \dots, m\}$ and any $x \in \mathrm{co}\{x^h :\ h \in L\}$, we have $\min_{h \in L} f(x, y^h) \leq \alpha$.

Remark 2.2 When $X = Y$, a sufficient condition for a function $\Psi : X \times Y \to \mathbb{R}$ to be α-transfer quasiconcave in y is that it is α-diagonally quasiconcave in y.

The following Proposition characterizes the α-local-dominatedness property if X is convex and Ψ is lower semicontinuous in x.

Proposition 2.1 *Let X be a nonempty, convex and compact subset in a topological vector space E, and let Y be a nonempty set. Suppose function $\Psi(x, y)$ is lower semicontinuous in x. Then $\Psi(x, y)$ is α-locally dominated in y if and only if it is α-transfer quasiconcave in y.*

Proof Necessity: Suppose that $\Psi(x, y)$ is α-locally dominated in y, *i.e.*, Then, for each $B \in \langle Y\rangle$, there exists $\tilde{x} \in X$ such that $\sup_{y\in B}\Psi(\tilde{x}, y) \le \alpha$. Let $\{y^1, \dots, y^m\} \in \langle Y\rangle$. Then there exists a corresponding finite subset $\{x^1, \dots, x^m\} \in \langle X\rangle$, where $x^1 = \cdots = x^m = \tilde{x}$ such that, for each $J \subset \{1, \dots, m\}$, $x \in \mathrm{co}\{x^j : j \in J\} = \{\tilde{x}\}$, $\inf_{j\in J}\Psi(x, y^j) \le \sup_{j\in J}\Psi(x, y^j) \le \alpha$, *i.e.*, $\Psi(x, y)$ is α-transfer quasiconcave in y.

Sufficiency: Suppose that $\Psi(x, y)$ is α-transfer quasiconcave in y. By Remark 2.1, $\Psi(x, y)$ is α-locally dominated in y if and only if $\{G(y) = \{x \in X : \Psi(x, y) \le \alpha\},\ y \in Y\}$ has the finite intersection property. Suppose that this family does not have the finite intersection property. Then, there exists $B = \{y^1, \dots, y^m\} \in \langle Y\rangle$ such that $\bigcap_{y\in B}G(y) = \emptyset$, *i.e.*, for each $x \in X$, there exists $y \in B$ such that $x \notin G(y)$. Let $A = \{x^1, \dots, x^m\} \in \langle X\rangle$ be the corresponding points in X such that, for each $M_k \subset \{1, 2, \dots, m\}$ and any $x \in \mathrm{co}\{x^h,\ \ h \in M_k\}$, we have $\min_{h\in M_k}\Psi(x, y^h) \le \alpha$. Let $Z = \mathrm{co}(A)$ and $L = \mathrm{span}(A) = \mathrm{span}\{x^1, \dots, x^m\}$. Since $G(y)$ is closed, we can define a continuous function $g : Z \to [0, \infty[$ by

$$g(x) = \sum_{i=1}^{m} d\bigl(x, G\bigl(y^i\bigr) \cap L\bigr),$$

where d is the Euclidean metric on L. Since $\bigcap_{y\in B} G(y) = \emptyset$, then $g(x) > 0$ for each $x \in X$. Define another continuous function $f : Z \to Z$ by

$$f(x) := \sum_{i=1}^{m} \frac{d(x, G(y^i) \cap L)}{g(x)} x^i.$$

By the Brouwer fixed-point Theorem, there exists $\overline{x} \in Z$ such that

$$\overline{x} = f(\overline{x}) = \sum_{i=1}^{m} \frac{d(\overline{x}, G(y^i) \cap L)}{g(\overline{x})} x^i. \tag{1}$$

Let $J = \{i \in \{1, \dots, m\} : (\overline{x}, G(y^i) \cap L) > 0\}$. Then, for each $i \in J$, $\overline{x} \notin G(y^i) \cap L$. Since $\overline{x} \in L$, $\overline{x} \notin G(y^i)$ for any $i \in J$. Thus, we have

$$\inf_{i\in J}\Psi\bigl(\overline{x}, y^i\bigr) > \alpha. \tag{2}$$

From (1), $\overline{x} \in \mathrm{co}\{x^i : i \in J\}$, and then by α-transfer quasiconcavity, we obtain

$$\inf_{i \in J} \Psi\left(\overline{x}, y^i\right) \leq \alpha,$$

which contradicts (2). Therefore, the function $y \mapsto \Psi(x, y)$ is α-locally dominated. □

Ansari et al. [12] defined the following concept of continuity.

Definition 2.3 (α-Transfer lower semicontinuity) Let X be a nonempty subset of a topological space, and Y be a nonempty subset. A function $f : X \times Y \to \mathbb{R}$ is said to be *α-transfer lower semicontinuous in x with respect to Y* iff, for $(x, y) \in X \times Y$, $f(x, y) > \alpha$ implies that there exist some point $y' \in Y$ and some neighborhood $\mathcal{V}(x) \subset X$ of x such that $f(z, y') > \alpha$ for all $z \in \mathcal{V}(x)$.

The α-transfer lower semicontinuity in x with respect to Y says that, if a point x in X is dominated by another point y in Y comparing to α, then there is an open set of points containing x, all of which can be dominated by a single point y'. Here, transfer lower semicontinuity in x with respect to Y refers to the fact that y may be transferred to some y' in order for the inequality to hold for all points in a neighborhood of x. The usual notion of lower semicontinuity would require that the first inequality hold at y for all points in a neighborhood of x. Thus, α-transfer lower semicontinuity is weaker than the notions of continuity used in the literature.

We have the following Proposition.

Proposition 2.2 *Any one of the following conditions is sufficient for $f(x, y)$ to be α-transfer lower semicontinuous in x with respect to Y:*

(a) *$f(x, y)$ is continuous in x;*
(b) *$f(x, y)$ is lower semicontinuous in x.*

Remark 2.3 Proposition 2.1 still hold when lower semicontinuity is weakened to transfer lower semicontinuity.

3 Minimax Inequality Without Convexity and/or Compactness

In this section, we present theorems on the existence of equilibrium in the minimax inequality for a function defined on Cartesian product of two different sets X and Y without any form of quasiconcavity of function and/or convexity and compactness of sets.

Theorem 3.1 *Let X be a nonempty and compact subset of a topological space E, and Y a nonempty set. Suppose Ψ is a real-valued function on $X \times Y$ such that $\Psi(x, y)$ is α-transfer lower semicontinuous in x with respect to Y. Then, there exists $\overline{x} \in X$ such that*

$$\Psi(\overline{x}, y) \leq \alpha, \quad \forall y \in Y \tag{3}$$

if and only if Ψ is α-locally dominated in y.

Proof Necessity: Let $\overline{x} \in X$ be a solution of the minimax inequality (3). Then, we have $\max_{y\in A}\Psi(\overline{x}, y) \le \alpha$ for any subset $A = \{y^1, \dots, y^m\} \in \langle Y\rangle$. Hence $\Psi(x, y)$ is α-locally dominated in y.

Sufficiency: If $\Psi(x, y)$ is α-transfer lower semicontinuous in x with respect to Y, then $\bigcap_{y\in Y} H(y) = \bigcap_{y\in Y} \operatorname{cl} H(y)$, where $H(y) = \{x \in X : \Psi(x, y) \le \alpha\}$. Indeed, let us consider $x \in \bigcap_{y\in Y} \operatorname{cl} H(y)$ but not in $\bigcap_{y\in Y} H(y)$. Then, there exists $y \in Y$ such that $x \notin H(y)$, i.e., $\Psi(x, y) > \alpha$. By the α-transfer lower semicontinuity of Ψ in x with respect to Y, there exist $y' \in Y$ and a neighborhood $\mathcal{V}(x)$ of x such that $\Psi(z, y') > \alpha$ for all $z \in \mathcal{V}(x)$. Thus, $x \notin \operatorname{cl} H(y')$, a contradiction. The condition that $y \mapsto \Psi(x, y)$ is α-locally dominated in y implies that $\{\operatorname{cl} H(y) :\in Y\}$ has the finite intersection property. Since $\{\operatorname{cl} H(y) : y \in Y\}$ is a compact family in the compact set X. Thus, $\emptyset \ne \bigcap_{y\in Y} H(y)$. Hence, there exists $\overline{x} \in X$ such that $\Psi(\overline{x}, y) \le \alpha$ for $y \in Y$. This completes the proof. □

Remark 3.1 The above result generalizes the existing results without assuming any form of quasiconcavity or $X = Y$. Note that, in Example 2.1, if the function f is defined on $[-1, 1] \times [-1, 1]$, then the existing results cannot be applicable since the function $f(x, y)$ is not quasiconcave in y on $[-1, 1]$ for all $x \in [-1, 1]$. However, there exists a solution since Ψ is α-locally dominated in y. The following is an example.

Example 3.1 Let

$$X = Y = \big([1/2, 1] \cup [3/2, 2]\big)^2, \quad x = (x_1, x_2), \quad y = (y_1, y_2), \quad \text{and}$$
$$F(x, y) := x_2 y_1^2 - x_1 y_2^2 - x_2 x_1^2 + x_1 x_2^2.$$

The function F is continuous over $X \times X$. For any subset $\{(y_{1,1}, y_{2,1}), \dots, (y_{1,k}, y_{2,k})\}$ of $X\times X$, let $x = (x_1, x_2) \in X$ such that $x_1 = \max_{h=1,\dots,k} y_{1,h}$ and $x_2 = \min_{h=1,\dots,k} y_{2,h}$. Then, $y_{2,h}^2 \ge x_2^2$ and $y_{1,h}^2 \le x_1^2$, for each $h = 1, \dots, k$. Thus, $-x_1 y_{2,h}^2 \le -x_1 x_2^2$, and $x_2 y_{1,h}^2 \le x_2 x_1^2$ for each $h = 1, \dots, k$. Therefore, $F(x, y_h) - F(x, x) \le 0$, $\forall h = 1, \dots, k$. The other conditions of Theorem 3.1 are obviously satisfied. Thus, the minimax inequality has a solution. Since X is not convex, the results in [1, 3–6, 9] on the existence of a solution to the Ky Fan inequality are not applicable.

When X is convex, Proposition 2.1 and Theorem 3.1 imply the following Corollary.

Corollary 3.1 *Let X be a nonempty, convex and compact subset of a topological vector space E, and Y a nonempty set. Let $\Psi : X \times Y \to \mathbb{R}$ be α-transfer lower semicontinuous in x with respect to Y. Then, the minimax inequality* (3) *has at least one solution if and only if $\Psi(x, y)$ is α-transfer quasiconcave in y.*

Theorem 3.1 can be generalized to the case where X is not compact.

Theorem 3.2 *Let X be a nonempty subset of a topological space E, and Y a nonempty set. Suppose Ψ is a real-valued function on $X \times Y$ such that*

(a) *$\Psi(x, y)$ is α-transfer lower semicontinuous in x with respect to Y*
(b) *there exists a finite subset $\{y^1, \dots, y^k\} \subset Y$ such that $\bigcap_{i=1,\dots,k} G_\alpha(y^i)$ is compact, where $G_\alpha(y) = \{x \in X : \Psi(x, y) \le \alpha\}$.*

Then, there exists $\overline{x} \in X$ such that

$$\Psi(\overline{x}, y) \le \alpha, \quad \forall y \in Y \tag{4}$$

if and only if Ψ is α-locally dominated in y.

Proof The necessity is the same as that of Theorem 3.1. We only need to prove the sufficiency. For each $y \in Y$, let $G_\alpha(y) = \{x \in X : \Psi(x, y) \le \alpha\}$. Then, by condition (a) of Theorem 3.2, we have $\bigcap_{y\in Y} G_\alpha(y) = \bigcap_{y\in Y} \operatorname{cl} G_\alpha(y)$. The condition that $y \mapsto \Psi(x, y)$ is α-locally dominated implies that $\{\operatorname{cl} G_\alpha(y) : y \in Y\}$ has the finite intersection property, and therefore $\{G_\alpha(y) \cap \bigcap_{i=1,\dots,k} G_\alpha(y^i) : y \in Y\}$ has the finite intersection property. For each $y \in Y$, the set $G_\alpha(y) \cap \bigcap_{i=1,\dots,k} G_\alpha(y^i)$ is compact. Thus, $\emptyset \ne \bigcap_{y\in Y} G_\alpha(y)$. Hence, there exists $\overline{x} \in X$ such that $\Psi(\overline{x}, y) \le \alpha$ for all $y \in Y$. This completes the proof. □

Theorem 3.3 *Let X be a nonempty subset of a topological space E, and Y a nonempty set. Let $\Psi(x, y)$ be a real-valued function on $X \times Y$. Then, the minimax inequality* (3) *has at least one solution if and only if there exists a nonempty compact subset X^0 of X such that*

(a) *$\Psi|_{X^0\times Y}(x, y)$ is α-transfer lower semicontinuous in x with respect to Y*
(b) *there exists $\bar{y} \in Y$ such that $G(\bar{y})$ is compact, where $G(y) = \{x \in X^0 : \Psi(x, y) \le \alpha\}$*
(c) *the function $y \mapsto \Psi|_{X^0\times Y}(x, y)$ is α-locally dominated in x on X^0.*

Proof Necessity: Suppose that the minimax inequality (3) has a solution $\overline{x} \in X$. Let $X^0 = \{\overline{x}\}$. Then, the set X^0 is nonempty and compact, and the restricted function $\Psi|_{X^0\times Y}$ is α-transfer lower semicontinuous in x with respect to Y.

The set $G(y) = \{x \in X^0 : \Psi(x, y) \le \alpha\}$ is compact for each $y \in X$, and for each $A \in \langle Y\rangle$, $\exists x = \overline{x} \in X^0$ such that $\max_{y\in A} \Psi(x, y) \le \max_{y\in Y} \Psi(x, y) \le \alpha$ (because $\overline{x}$ is a solution of the minimax inequality (3)).

Sufficiency: For each $y \in Y$, let $G(y) = \{x \in X : \Psi(x, y) \le \alpha\}$. Then by condition (a) of Theorem 3.3, we have $\bigcap_{y\in Y} G(y) = \bigcap_{y\in Y} \operatorname{cl} G(y)$. Condition (c) of Theorem 3.3 implies that $\{\operatorname{cl} G(y) : y \in Y\}$ has the finite intersection property, and therefore the family $\{G(y) \cap G(\bar{y}) : y \in Y\}$ has the finite intersection property. For each $y \in Y$, the set $G(y) \cap G(\bar{y})$ is compact. Thus, $\emptyset \ne \bigcap_{y\in Y} G(y) \cap G(\bar{y}) = \bigcap_{y\in Y} G(y)$. Hence, there exists $\overline{x} \in X^0$ such that for each $y \in Y$, $\Psi(\overline{x}, y) \le \alpha$. This completes the proof. □

4 Existence of Saddle Point

Saddle point is an important tool in variational problems and game theory. Much work has been dedicated to the problem of weakening its existence conditions. Almost all these results assume that a function is defined on convex sets. In this section, we present existence theorems on saddle point without any form of convexity conditions.

Consider two players, Juba and Massi with strategy sets X and Y, respectively. If Juba chooses a strategy $a \in X$ and Massi chooses a strategy $b \in Y$, the payoff is given by

$$f(a, b) := \text{gain by Massi} = \text{loss by Juba}$$

(e.g. in euro). We allow $f(a, b)$ to be negative, and, if this is the case, then player Massi can obtain a negative gain, that is, a loss of $|f(a, b)|$ euro.

Definition 4.1 A pair $(\overline{x}, \overline{y})$ in $X \times Y$ is called a saddle point of f in $X \times Y$, iff,

$$f(\overline{x}, y) \le f(\overline{x}, \overline{y}) \le f(x, \overline{y}) \quad \text{for all } x \in X \text{ and } y \in Y.$$

This definition reflects the fact that each player is individualistic.

Before we give our new results, we state two classical results on saddle point.

Theorem 4.1 (von Neumann Theorem) *Let X and Y be nonempty, compact and convex subsets in a Hausdorff locally convex vector spaces E and F, respectively, and f a real-valued function defined on $X \times Y$. Suppose that*

(a) *the function $x \mapsto f(x, y)$ is lower semicontinuous and quasiconvex on X,*
(b) *the function $y \mapsto f(x, y)$ is upper semicontinuous and quasiconcave on Y.*

Then, f has a saddle point.

Theorem 4.2 (Kneser theorem) *Let X be a nonempty and convex subset in a Hausdorff topological vector space E, and Y a nonempty, compact and convex subset of a Hausdorff topological vector space F. Let f be a real-valued function defined on $X \times Y$. Suppose that*

(a) *the function $x \mapsto f(x, y)$ is concave on X,*
(b) *the function $y \mapsto f(x, y)$ is lower semicontinuous and convex on Y.*

Then, $\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y)$.

By relaxing the convexity of function, we obtain the following Theorem.

Theorem 4.3 (Saddle point without convexity) *Let X and Y be two nonempty, compact subsets in topological spaces E and G, respectively. Let $f : X \times Y \mapsto \mathbb{R}$ be a real-valued function defined on $X \times Y$ such that $f(x, y') - f(x', y)$ is 0-transfer lower semicontinuous in (x, y) with respect to $X \times Y$. Then, the function f has a saddle point if and only if, for all $\{(a_i, b_i) : i = 1, \dots, n\} \subset X \times Y$, there exists $(x, y) \in X \times Y$ such that $f(x, b_i) \le f(a_i, y)$, for all $i = 1, \dots, n$.*

Proof Necessity: Let $(\overline{x}, \overline{y}) \in X \times Y$ be a saddle point of f. Then,

$$f(\overline{x}, y) \le f(\overline{x}, \overline{y}) \le f(x, \overline{y}) \quad \text{for all } x \in X \text{ and } y \in Y. \tag{5}$$

Suppose that there exists $A=\{(a_i,b_i),\ i=1,\dots,n\}\subset X\times Y$, such that

$$\forall(x,y)\in X\times Y, \exists i=1,\dots,n \quad \text{such that} \quad f(x,b_i)>f(a_i,y). \tag{6}$$

Let $x=\overline{x}$ and $y=\overline{y}$ in (6). Then there exists $(\tilde{a},\tilde{b})\in A$ such that

$$f(\overline{x},\tilde{b})>f(\tilde{a},\overline{y}).$$

Now choose $x=\tilde{a}$ and $y=\tilde{b}$. Inequality (5) becomes

$$f(\overline{x},\tilde{b})\le f(\overline{x},\overline{y})\le f(\tilde{a},\overline{y}).$$

Thus, we have $f(\overline{x},\tilde{b})\le f(\tilde{a},\overline{y})<f(\overline{x},\tilde{b})$, which is a contradiction. Therefore, for all $\{(a_i,b_i): i=1,\dots,n\}\subset X\times Y$, there exists $(x,y)\in X\times Y$ such that $f(x,b_i)\le f(a_i,y)$ for all $i=1,\dots,n$.

Sufficiency: Let $F: Z\times Z\mapsto\mathbb{R}$, where $Z=X\times Y$ and

$$F(z,t)=f\bigl(x,y'\bigr)-f\bigl(x',y\bigr), \quad \forall z=(x,y)\in Z \text{ and } t=\bigl(x',y'\bigr)\in Z.$$

It is easy to verify that all conditions of Theorem 3.1 are satisfied for the $F(z,t)$. Then, there exists $\overline{z}=(\overline{x},\overline{y})\in Z$ such that

$$\max_{t\in Z}F(\overline{z},t)\le 0. \tag{7}$$

Now we prove that $\overline{z}=(\overline{x},\overline{y})$ is a saddle point of the function $f(x,y)$. From (7) we get

$$\forall(x,y)\in X\times Y, \quad f(\overline{x},y)\le f(x,\overline{x}). \tag{8}$$

Letting $x=\overline{x}$ in (8), we have $\forall y\in Y$, $f(\overline{x},y)\le f(\overline{x},\overline{y})$. Letting $y=\overline{y}$ in (8), we have $\forall x\in X$, $f(\overline{x},\overline{y})\le f(x,\overline{y})$. Therefore, for all $(x,y)\in X\times Y$, we have

$$f(\overline{x},y)\le f(\overline{x},\overline{y})\le f(x,\overline{y}),$$

i.e., $(\overline{x},\overline{y})$ is a saddle point of the function $f(x,y)$. □

When X is convex, Proposition 2.1 and Theorem 4.3 imply the following Corollary.

Corollary 4.1 *Let X and Y be two nonempty, compact and convex subsets in topological spaces E and G, respectively. Let $f: X\times Y\mapsto\mathbb{R}$ be a real-valued function defined on $X\times Y$ such that $f(x,y')-f(x',y)$ is 0-transfer lower semicontinuous in (x,y) with respect to $X\times Y$. Then, the function f has a saddle point if and only if $f(x,y')-f(x',y)$ is 0-transfer quasiconcave in (x',y').*

Theorem 4.3 can be generalized by relaxing the compactness of X and Y.

Theorem 4.4 *Let X and Y be two nonempty subsets in topological spaces E and G, respectively. Let $f: X\times Y\mapsto\mathbb{R}$ such that:*

(a) *$f(x, y') - f(x', y)$ is 0-transfer lower semicontinuous in (x, y) with respect to $X \times Y$;*
(b) *there exists $\{(x_1, y_1), \ldots, (x_k, y_k)\} \subset X \times Y$ such that $\bigcap_{i=1,\ldots,k} G(x_i, y_i)$ is compact, where $G(u, v) = \{(x, y) \in X \times Y : \ f(x, v) \le f(u, y)\}$.*

Then, the function f has a saddle point if and only if, for all $\{(a_i, b_i) : i = 1, \ldots, n\} \subset X \times Y$, there exists $(x, y) \in X \times Y$ such that $f(x, b_i) \le f(a_i, y)$, for all $i = 1, \ldots, n$.

Proof The necessity part of the proof is the same as that of Theorem 4.3, and the sufficiency part of the proof is the same as that of Theorem 3.2. □

Remark 4.1 The function $f(x, y') - f(x', y)$ is 0-transfer lower semicontinuous in (x, y) with respect to $X \times Y$ if (a) $x \mapsto f(x, y)$ is lower semicontinuous function in x and (b) $y \mapsto f(x, y)$ is upper semicontinuous function in y.

5 Existence of Fixed Point

This section provides necessary and sufficient conditions for the existence of a fixed point of a function defined on a set that may not be compact or convex.

A correspondence C, defined from Y into 2^F, has a fixed point $x \in Y$ iff, $x \in C(x)$. If C is a single-valued function, then a fixed point x of C is characterized by $x = C(x)$. We start by considering the following example of a fixed-point problem:

Example 5.1 Define a function $f : X = [0, 3] \to \mathbb{R}$ by

$$f(x) := \frac{x+4}{x+1}.$$

Since $\max_{x\in[0,3]} |f'(x)| = 3$, f is a 3-Lipschitz. Moreover, since $f([0, 3]) = [\frac{7}{4}, 4] \not\subseteq [0, 3]$, all the classical fixed-point Theorems (Banach's,[1] Brouwer–Schauder–Tychonoff's,[2] Halpern–Bergman's,[3] Kakutani–Fan–Glicksberg's,[4] ... [22]) are not applicable.

Let (E, d) be a metric space. The subset $B(a, r)$ is defined by

$$B(a, r) := \{x \in E : d(x, a) < r\},$$

where $a \in X$ and $r \in \mathbb{R}^*_+$, is called open ball centered at a point a with radius r.

---

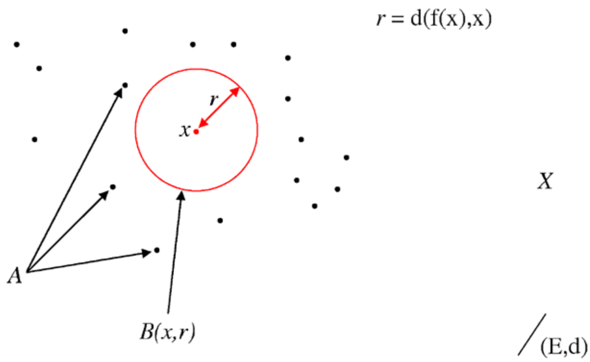
[1] Banach Fixed Point Theorem: Let (K, d) be a complete metric space and let $f : K \to K$ be a d-contraction ($d \in [0, 1[$). Then, f has a unique fixed point.

[2] Brouwer–Schauder–Tychonoff Fixed Point Theorem: Let K be a nonempty, compact and convex subset of a locally convex Hausdorff space, and let $f : K \to K$ be a continuous function. Then the set of fixed points of f is compact and nonempty.

[3] Halpern–Bergman Fixed Point Theorem: Let K be a nonempty, compact and convex subset of a locally convex Hausdorff space X, and let $C : K \rightrightarrows X$ be an inward pointing upper semicontinuous mapping with nonempty, closed and convex values. Then C has a fixed point.

[4] Kakutani–Fan–Glicksberg Fixed Point Theorem: Let K be a subset nonempty, compact and convex of a locally convex Hausdorff space, and let $C : K \rightrightarrows K$ have closed graph and nonempty and convex values. Then the set of fixed points of C is nonempty and compact.

Fig. 1 X is f-separate



Definition 5.1 Let X be a nonempty set in a metric space (E,d) and f be a function defined on X into E. The set X is called *f-separate* iff, at least one of the following conditions holds:

(a) for all $A \in \langle X \rangle$, there exists $x \in X$ such that

$$A \cap B\big(x, d\big(f(x), x\big)\big) = \emptyset;$$

(b) for all $A \in \langle f(X) \rangle$, there exists $x \in X$ such that

$$A \cap B\big(f(x), d\big(f(x), x\big)\big) = \emptyset.$$

The geometric interpretation that X being f-separate is that any finite subset A of X (or of $f(X)$) can be separated from some single point x (or $f(x)$) of X by an open ball centered in x (or in $f(x)$) with radius $r(x) = d(f(x), x)$ (see Fig. 1). By relaxing the convexity of set, we have the following theorem.

Theorem 5.1 (Fixed point without convexity assumption) *Let X be a nonempty and compact subset of a metric space (E,d) and f be a continuous function over X into E. Then, f has a fixed point if and only if X is f-separate.*

Proof Necessity: Let $\overline{x} \in X$ be a fixed point of f. Then $f(\overline{x}) = \overline{x}$, i.e., $d(f(\overline{x}), \overline{x}) = 0$. Suppose that X is not f-separate. We distinguish two cases:

(1) There exists $A = \{y_1, \dots, y_n\} \in \langle X \rangle$ such that $A \cap B(x, d(f(x), x)) \neq \emptyset$ for all $x \in X$. Then, $\forall x \in X$ there exists $y(x) \in A$ such that $y(x) \in B(x, d(f(x), x))$, i.e., $d(f(x), x) > d(x, y(x))$. Let $x = \overline{x}$ in the last inequality. Then there exists $y(\overline{x}) \in A$ such that $d(f(\overline{x}), \overline{x}) > d(\overline{x}, y(\overline{x}))$. We have $d(f(\overline{x}), \overline{x}) = 0$, and $0 > d(\overline{x}, y(\overline{x}))$, which is impossible. Thus, X is f-separate.
(2) There exists $A = \{y_1, \dots, y_n\} \in \langle f(X) \rangle$ such that, for $A \cap B(f(x), d(f(x), x)) \neq \emptyset$ for all $x \in X$. Then, $\forall x \in X$ there exists $y(x) \in A$ such that $y(x) \in B(f(x), d(f(x), x))$, i.e., $d(f(x), x) > d(f(x), y(x))$. Let $x = \overline{x}$ in the last inequality. Then there exists $y(\overline{x}) \in A$ such that $d(f(\overline{x}), \overline{x}) > d(f(\overline{x}), y(\overline{x}))$. We have $d(f(\overline{x}), \overline{x}) = 0$, and therefore $0 > d(f(\overline{x}), y(\overline{x}))$, a contradiction. Thus, for all $A \in \langle f(X) \rangle$, there exists $x \in X$ such that $A \cap B(f(x), d(f(x), x)) = \emptyset$. Hence, X is f-separate.

Sufficiency: Suppose that X is f-separate. Then, one of the following conditions holds:

(3) For all $A \in \langle X \rangle$, there exists $x \in X$ such that $A \cap B(x, d(f(x), x)) = \emptyset$. Consider the following real-valued function φ defined on $X \times X$:

$$(x, y) \mapsto \varphi(x, y) := d\big(f(x), x\big) - d(x, y).$$

The function $x \mapsto \varphi(x, y)$ is then continuous over X, $\forall y \in X$. Thus, for all $A \in \langle X \rangle$, there exists $x \in X$ such that $A \cap B(x, d(f(x), x)) = \emptyset$, and then, for all $y \in A$, $d(x, y) \geq d(f(x), x)$, i.e., $\max_{y \in A} \varphi(x, y) \leq 0$. By Theorem 3.1, there exists $\overline{x} \in X$ such that $\sup_{y \in X} \varphi(\overline{x}, y) \leq 0$. Then, for each $y \in X$, we have $d(\overline{x}, f(\overline{x})) \leq d(\overline{x}, y)$. Letting $y = \overline{x}$ in last inequality, we obtain $d(\overline{x}, f(\overline{x})) = 0$, which means $\overline{x} = f(\overline{x})$. Then, f has a fixed point.

(4) For all $C \in \langle f(X) \rangle$, there exists $x \in X$ such that $C \cap B(f(x), d(f(x), x)) = \emptyset$. Consider the following real-valued function φ defined on $X \times f(X)$:

$$(x, y) \mapsto \varphi(x, y) := d\big(f(x), x\big) - d\big(f(x), y\big).$$

The function $x \mapsto \varphi(x, y)$ is continuous over X, $\forall y \in f(X)$. Thus, for all $y \in C$, $d(f(x), y) \geq d(f(x), x)$, i.e., $\max_{y \in C} \varphi(x, y) \leq 0$. Therefore by Theorem 3.1, there exists $\overline{x} \in X$ such that $\sup_{y \in f(X)} \varphi(\overline{x}, y) \leq 0$. Then, for each $y \in f(X)$, $d(\overline{x}, f(\overline{x})) \leq d(f(\overline{x}), y)$. Letting $y = f(\overline{x})$ in last inequality, we obtain $d(\overline{x}, f(\overline{x})) = 0$, which means $\overline{x} = f(\overline{x})$. Then, f has a fixed point.

This completes the proof. □

Remark 5.1 If X is compact in a metric space, then the two conditions (a) and (b) in Definition 5.1 are equivalent.

Example 5.2 (Continued) Let us again consider Example 5.1. The function f is defined by

$$f : X = [0, 3] \to \mathbb{R}$$
$$x \mapsto f(x) = \frac{x + 4}{x + 1}.$$

The point $x = 2$ is a fixed point of f in $[0, 3]$, then the set $[0, 3]$ is f-separate.

Theorem 5.1 can be generalized by relaxing the compactness of X.

Theorem 5.2 (Fixed point without convexity or compactness) *Let X be a nonempty subset of a metric space (E, d) and f a continuous function over X into E. Suppose that there exist $\{y^1, \ldots, y^k\} \subset X$ such that $\bigcap_{i=1,\ldots,k} G(y^i)$ is compact, where*

$$G(y) = \begin{cases} \{x \in X, d(f(x), x) \leq d(x, y)\}, & \textit{if } (1) \textit{ in Definition 5.1 is satisfied}, \\ \{x \in X, d(f(x), x) \leq d(f(x), y)\}, & \textit{if } (2) \textit{ in Definition 5.1 is satisfied}. \end{cases}$$

Then, f has a fixed point if and only if X is f-separate.

Proof The necessity is the same as that of Theorem 5.1. We only need to prove the sufficiency in the case where condition (1) in Definition 5.1 is satisfied. For each $y \in Y$, let

$$G(y) = \bigl\{x \in X : d\bigl(f(x), x\bigr) \leq d(x, y)\bigr\}.$$

Since f is continuous, $G(x)$ is a closed set. Furthermore, the condition that X is f-separate implies that the family $\{G(y) : y \in X\}$ has the finite intersection property, and therefore $\{G(y) \cap \bigcap_{i=1,\dots,k} G(y^i) : y \in X\}$ has the finite intersection property. For each $y \in Y$, $G(y) \cap \bigcap_{i=1,\dots,k} G(y^i)$ is compact. Thus, $\emptyset \neq \bigcap_{y\in X} G \cap \bigcap_{i=1,\dots,k} G(y^i) = \bigcap_{y\in X} G(y)$. Hence, there exists $\overline{x} \in X$ such that each $y \in X$, $d(f(\overline{x}), \overline{x}) \leq d(\overline{x}, y)$. Then, letting $y = \overline{x}$, we obtain $f(\overline{x}) = \overline{x}$. This completes the proof. □

The following Proposition provides a sufficient condition for a set X to be f-separate.

Proposition 5.1 *Let X be a nonempty, compact and convex subset of a normed space $(E, \|.\|)$ and f a continuous function over X into E such that $X \subset f(X)$ and the function $z \mapsto \|f(z) - x\|$ is quasiconvex over X, for all $x \in X$. Then, X is f-separate and thus it has a fixed point.*

Proof Let $A = \{y_1, y_2, \dots, y_n\}$ be any set in $\langle X\rangle$. Suppose that $\forall x \in X$, $\exists y \in A$ such that

$$\|x - y\| < \bigl\|f(x) - x\bigr\|, \tag{9}$$

i.e., X is not f-separate. Consider the following multi-valued function $C : X \rightrightarrows X$:

$$x \mapsto C(x) = \Bigl\{z \in X : \min_{y\in A}\|x - y\| \geq \bigl\|f(z) - x\bigr\|\Bigr\}.$$

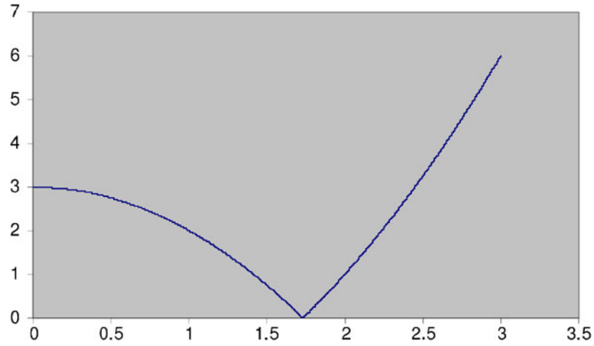
(1) The condition $X \subset f(X)$ implies that, for each $x \in X$, $C(x) \neq \emptyset$.
(2) The continuity of f and the compactness of X imply that C is upper semicontinuous over X and, for each $x \in X$, the set $C(x)$ is closed in X.
(3) The quasiconvexity of function $z \mapsto \|f(z) - x\|$ implies that $C(x)$ is convex in X, for each $x \in X$.

From (1)–(3), we conclude that the function C satisfies all the conditions of Kakutani's fixed-point Theorem in [23]. Consequently, $\exists \widetilde{x} \in X$ such that $\widetilde{x} \in C(\widetilde{x})$, i.e.,

$$\min_{y\in A}\bigl\|\widetilde{x} - y\bigr\| \geq \bigl\|f(\widetilde{x}) - \widetilde{x}\bigr\|.$$

Let $x = \widetilde{x}$ in (9). Then there exists $y(\widetilde{x}) \in A$ such that $\|\widetilde{x} - y(\widetilde{x})\| < \|f(\widetilde{x}) - \widetilde{x}\|$. Therefore, $\|f(\widetilde{x}) - \widetilde{x}\| \leq \min_{y\in A}\|\widetilde{x} - y\| \leq \|\widetilde{x} - y(\widetilde{x})\| < \|f(\widetilde{x}) - \widetilde{x}\|$, a contradiction. Thus, X is f-separate. □

Fig. 2 The graph of function $z \mapsto |z^2 - 2 - x|$, $\forall x \in [0,3]$



The following Example illustrates Proposition 5.1.

Example 5.3 Let f be the function $f : X = [0,3] \to \mathbb{R}$ defined as follows: $x \mapsto f(x) = x^2 - 2$. Then, $\max_{x\in[0,3]}|f'(x)| = 6$, and so f is a 6-Lipschitz. Moreover, since $f([0,3]) = [-2,7] \nsubseteq [0,3]$, all the classical fixed-point Theorems (Banach's, Brouwer–Schauder–Tychonoff's, Halpern–Bergman's, Kakutani–Fan–Glicksberg's, ...) are not applicable. However, since $z \mapsto |z^2 - 2 - x|$ is quasiconvex over $[0,3]$, $\forall x \in [0,3]$ (see Fig. 2), by Proposition 5.1, f has a fixed point in $[0,3]$. Indeed, $\overline{x} = (1+\sqrt{5})/2$ is such a point.

In the following Theorem, we show the existence of fixed point without the quasiconvexity of $z \mapsto \|f(z) - x\|$ or the convexity of X.

Theorem 5.3 *Let $(E, \|.\|)$ be a normed space and f a function over E into E. Suppose that there exists a compact set X in E such that*:

(a) *The restriction of f on X is continuous*;
(b) *$f(X)$ is convex in E*; *and*
(c) *$X \subset f(X)$.*

Then, *f has a fixed point.*

Proof Consider the following function:

$$\phi : X \times f(X) \to \mathbb{R}$$

defined by $(x,y) \mapsto \phi(x,y) = \|f(x) - x\| - \|x - y\|$.

- The function $x \mapsto \phi(x,y)$ is continuous over X, for each $y \in f(X)$.
- The function $y \mapsto \phi(x,y)$ is quasiconcave over $f(X)$, for each $x \in X$.

Suppose that

$$\forall x \in X, \quad \text{there exists} \quad y \in f(X), \quad \text{such that} \quad \phi(x,y) > 0. \tag{10}$$

Then, $f(X)$ can be covered by the sets

$$\theta_y = \left\{ f(x) \in f(X) : \phi(x,y) > 0 \right\}, \quad y \in f(X).$$

Since θ_y is open in $f(X)$ and $f(X)$ is compact, it can be covered by a finite number r of subsets $\{\operatorname{int}\theta_{y_1}, \dots, \operatorname{int}\theta_{y_r}\}$ of type θ_y. Consider a continuous partition of unity $\{h_i\}_{i=1,\dots,r}$ associated to this finite covering, and the following function:

$$\alpha : f(X) \to f(X), \qquad \text{such that} \qquad \alpha(y) = \sum\nolimits_{i=1}^{r} h_i(y) y_i.$$

The function α is continuous over the compact convex $f(X)$ into $f(X)$. Then, by Brouwer Fixed-Point Theorem, there exists $\tilde{y} = f(\tilde{x}) \in f(X)$ such that $\tilde{y} = f(\tilde{x}) = \sum_{i=1}^{r} h_i(\tilde{y}) y_i$. Let $J = \{i = 1, \dots, r : h_i(\tilde{y}) > 0\}$.

The quasiconcavity of $y \mapsto \phi(x, y)$ implies that

$$\min_{i \in J} \phi(\tilde{x}, y_i) \le 0. \tag{11}$$

If $i \in J$, $\tilde{y} \in \operatorname{supp}(h_i) \subset \theta_{y_i}$. Thus, $\phi(\tilde{x}, y_i) > 0$ for each $i \in J$. Therefore,

$$\min_{i \in J} \phi(\tilde{x}, y_i) > 0. \tag{12}$$

Then inequalities (11) and (12) imply $0 < \min_{i \in J} \phi(\tilde{x}, y_i) \le 0$, which is impossible. Thus, supposition in (10) is not true, *i.e.*,

$$\exists \overline{x} \in X, \text{such that } \forall y \in f(X), \qquad \text{we have } \phi(\overline{x}, y) = \| f(\overline{x}) - \overline{x} \| - \| \overline{x} - y \| \le 0.$$

Hence, $\|f(\overline{x}) - \overline{x}\| \le \|\overline{x} - y\|$ for each $y \in f(X)$. By condition (c) of Theorem, we have $X \subset f(X)$. Thus, letting $y = \overline{x}$, we obtain $\|f(\overline{x}) - \overline{x}\| = 0$ in the last inequality, which means $\overline{x}$ is a fixed point of f in X. □

We have the following Corollary.

Corollary 5.1 *Let f be a continuous function over a compact interval $X \subset \mathbb{R}$ into $\mathbb{R}$ such that $X \subset f(X)$. Then, f has a fixed point.*

Proof By Theorem 5.3, it suffices to show that $f(X)$ is convex. If $f(X)$ is reduced to a singleton, then the proof is obvious. Suppose that $f(X)$ is not reduced to a singleton. Let $y_1, y_2 \in f(X)$ and $t \in [0, 1]$. Then there exist $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Let g be a function defined as follows: $g(x) = f(x) - ty_1 - (1-t)y_2$. Thus, g is continuous over the compact interval X and $g(x_1) \times g(x_2) \le 0$. Therefore, by intermediate value Theorem, there exists $c \in [x_1, x_2] \subset X$ such that $g(c) = 0$. Hence, $ty_1 + (1-t)y_2 = f(c) \in f(X)$, i.e. $f(X)$ is convex. □

The following Examples illustrate Theorem 5.3 and Corollary 5.1.

Example 5.4 Let f be the following function:

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ &x \mapsto f(x) = x^3 + 2x^2 - 5x + 1. \end{aligned}$$

Let $X=[0,2]$. We have $\max_{x\in[0,2]}|f'(x)|=15$ so that f is a 15-Lipschitz. Moreover, since $f([0,2])=[f_{\min},5]\nsubseteq[0,2]$ where $f_{\min}\approx-2.001$, all the classical fixed-point theorems are not applicable. We have $f(0)=1$ and $f(2)-2=5$; then the intermediate value theorem cannot be applied. However, since the restriction of f on $[0,2]$ is continuous, $f([0,2])$ is convex and $f([0,2])=[f_{\min},5]\supset[0,2]$ with $f_{\min}\approx-2.001$, then, by Theorem 5.3, f has a fixed point in $[0,2]$.

Example 5.5 Let f be the following function:

$$\begin{aligned} f:\mathbb{R}&\to\mathbb{R}\\ x&\mapsto f(x)=\frac{2x^2-4}{x+2}. \end{aligned}$$

Let $X=[-1,4]$. We then have $\max_{x\in[-1,4]}|f'(x)|=\frac{17}{9}$ so that f is a $\frac{17}{9}$-Lipschitz. Moreover, since $f([-1,4])=[4\sqrt{2}-8,14/3]\nsubseteq[-1,4]$, the function $z\mapsto\|f(z)-x\|$ is not quasiconvex in z and thus all the classical fixed-point Theorems and Proposition 5.1 are not applicable.

However, since the restriction of f on $[-1,4]$ is continuous, $f([-1,4])$ is convex and $f([-1,4])=[4\sqrt{2}-8,14/3]\supset[-1,4]$, then, by Theorem 5.3, f has a fixed point in $[-1,4]$. Indeed, $\overline{x}=1+\sqrt{5}$ is such a point.

The following Theorem generalizes the existence theorems on fixed point to a multifunction mapping.

Theorem 5.4 *Let X be a nonempty and compact subset of a metric space E and C a multifunction mapping defined on X into E such that the function $x\mapsto d(x,C(x))$ is lower semicontinuous over X. Then, C has a fixed point if and only if, for each $A\in\langle X\rangle$, there exists $x\in X$ such that $d(x,C(x))\le d(x,A)$.*

Proof It is a consequence of Theorem 3.1 by defining $\Psi(x,y)=d(x,C(x))-d(x,y)$. □

Theorem 5.5 *Let X be a nonempty and compact subset of a metric space E and Y a nonempty subset of a metric space (F,d), and let C be a multifunction mapping defined on X into Y such that the function $x\mapsto d(g(x),C(x))$ is lower semicontinuous over X, where g is a continuous function defined from X into Y. Then, C has a g-fixed point (i.e., $\exists\bar{x}\in X$ such that $g(\bar{x})\in C(\bar{x})$) if and only if, for each $A\in\langle g(X)\rangle$, there exists $x\in X$ such that $d(g(x),C(x))\le d(g(x),A)$.*

Proof It is a consequence of Theorem 3.1 by defining $\Psi(x,y)=d(g(x),C(x))-d(g(x),y)$. □

Theorem 5.6 *Let X be a nonempty and compact subset of a metric space (E,d_1), and Y a nonempty subset of a metric space (F,d_2). Let C be a multifunction mapping defined on X into Y such that the function $x\mapsto d_2(g(x),C(x))$ is lower semicontinuous over X, where g is a continuous function defined from X into Y. Then, C has a g-fixed point if and only if, for each $A\in\langle X\rangle$, there exists $x\in X$ such that $d_2(g(x),C(x))\le d_1(x,A)$.*

Proof It is a consequence of Theorem 3.1 by defining $\Psi(x,y) = d_2(g(x), C(x)) - d_1(x,y)$. □

6 Existence of Coincidence Points

This section provides a necessary and sufficient conditions for the existence of coincidence points of two functions defined on a set that may not be compact or convex.

Definition 6.1 Let X be a nonempty and compact subset of a metric space (E, d_1), and let f and g be two continuous functions over X into a metric space (F, d_2). Then, f and g are said to have a coincidence point iff there exists $\overline{x} \in X$ such that $f(\overline{x}) = g(\overline{x})$.

Definition 6.2 Let X be a nonempty set in a metric space (E, d_1), and let f and g be two functions from X into a metric space (F, d_2). The set X is called *fg-separate* iff one of the following conditions holds:

(1) for all $A \in \langle X \rangle$, there exists $x \in X$ such that

$$A \cap B\big(x, d_2\big(f(x), g(x)\big)\big) = \emptyset;$$

(2) for all $A \in \langle f(X) \rangle$, there exists $x \in X$ such that

$$A \cap B\big(f(x), d_2\big(f(x), g(x)\big)\big) = \emptyset.$$

(3) for all $A \in \langle g(X) \rangle$, there exists $x \in X$ such that

$$A \cap B\big(g(x), d_2\big(f(x), g(x)\big)\big) = \emptyset.$$

Theorem 6.1 (Coincidence point without convexity assumption) *Let X be a nonempty and compact subset of a metric space (E, d_1), and let f and g be two continuous functions over X into a metric space (F, d_2). Then, f and g have a coincidence point if and only if X is fg-separate.*

Proof *Necessity*: It is the same as that of Theorem 5.1.

Sufficiency: If the first condition in Definition 6.2 is satisfied, let

$$\phi : X \times X \to \mathbb{R}, \quad (x,y) \mapsto \phi(x,y) = d_2\big(f(x), g(x)\big) - d_1(x,y).$$

If the second condition of Definition 6.2 is satisfied, let

$$\phi : X \times f(X) \to \mathbb{R}, \quad (x,y) \mapsto \phi(x,y) = d_2\big(f(x), g(x)\big) - d_2\big(f(x), y\big).$$

If the third condition of Definition 6.2 is satisfied, let

$$\phi : X \times g(X) \to \mathbb{R}, \quad (x,y) \mapsto \phi(x,y) = d_2\big(f(x), g(x)\big) - d_2\big(g(x), y\big).$$

The remaining proof of the sufficiency is the same as that in the proof of Theorem 5.1. □

Theorem 6.2 *Let E be a topological space and $(F, \|.\|)$ a normed space. Let f and g be two functions over E into F. Suppose that there exists a compact set X in E such that the restriction of f and g on X are continuous and*

$$\begin{cases} (a) f(X) \quad \text{is convex in } F; \\ (b) g(X) \subset f(X), \end{cases} \qquad \text{or} \qquad \begin{cases} (a') g(X) \quad \text{is convex in } F; \\ (b') f(X) \subset g(X). \end{cases}$$

Then, f and g has a coincidence point.

Proof If Conditions (a)–(b) are satisfied, let

$$\phi : X \times f(X) \to \mathbb{R}, \quad (x, y) \mapsto \phi(x, y) = \|f(x) - g(x)\|_F - \|g(x) - y\|_F.$$

If Conditions $1'$-$2'$ are satisfied, let

$$\phi : X \times g(X) \to \mathbb{R}, \quad (x, y) \mapsto \phi(x, y) = \|f(x) - g(x)\|_F - \|f(x) - y\|_F.$$

The remaining proof of the sufficiency is the same as that in the proof of Theorem 5.3. □

We have the following Corollary.

Corollary 6.1 *Let f and g be two continuous functions over a compact interval $X \subset \mathbb{R}$ into $\mathbb{R}$ such that $g(X) \subset f(X)$ or $f(X) \subset g(X)$. Then, f and g has a coincidence point.*

Proof See the proof of Corollary 5.1. □

7 Existence of Nash Equilibrium

As an application of our basic result on the minimax inequality, in this section we provide a result on the existence of pure strategy Nash equilibrium without assuming the convexity of strategy spaces and any form of quasiconcavity of payoff functions.

Consider the following noncooperative game in normal form:

$$G = (X_i, f_i)_{i \in I}, \tag{13}$$

where $I = \{1, \dots, n\}$ is the finite set of players, X_i is player i's strategy space which is a nonempty subset of a topological space E_i, and $f_i : X \longrightarrow \mathbb{R}$ is the payoff function of player i. Denote by $X = \prod_{i \in I} X_i$ the set of strategy profiles of the game and $f := (f_1, f_2, \dots, f_n)$ the profile of payoff functions. For each player $i \in I$, denote by $-i := \{j \in I$ such that $j \neq i\}$ the set of all players rather than player i. Furthermore, denote by $X_{-i} = \prod_{j \in -i} X_j$ the set of strategies of the players in coalition $-i$.

Definition 7.1 A strategy profile $\overline{x} \in X$ is said to be a *pure strategy Nash equilibrium* of game (13) iff,

$$\forall i \in I, \forall y_i \in X_i, \quad f_i(\overline{x}_{-i}, y_i) \le f_i(\overline{x}).$$

The aim of each player is to choose a strategy in X_i that maximizes his payoff function.

Define a function $\Psi : X \times X \to \mathbb{R}$ by

$$\Psi(x, y) := \sum_{i=1}^{n} \bigl\{ f_i(x_{-i}, y_i) - f_i(x) \bigr\}, \quad \forall (x, y) \in X \times X.$$

The following theorem generalized Theorem 1 in [21] by relaxing the convexity of strategy spaces and 0-transfer quasiconcavity of payoff function.

Theorem 7.1 (Nash equilibrium without convexity assumption) *Let $I = \{1, \dots, n\}$ be an indexed finite set, let X_i be a nonempty and compact subset of a topological space E_i. Suppose that the function $\Psi(x, y)$ is 0-transfer lower semicontinuous in x with respect to X. Then, the game $G = (X_i, f_i)_{i \in I}$ has a Nash equilibrium if and only if, for all $A \in \langle X \rangle$, there exists $x \in X$ such that, for each $i \in I$, we have*

$$f_i(y_i, x_{-i}) \le f_i(x), \quad \textit{for each } y \in A. \tag{14}$$

Proof Note that the condition (14) is equivalent to the function $y \mapsto \Psi(x, y)$ is 0-locally dominated. Then it is a straightforward consequence of Theorem 3.1 and definition of Ψ. □

Example 7.1 Suppose that in game (13), $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [1, 2] \cup [3, 4]$, $x = (x_1, x_2)$ and

$$f_1(x) = x_2 x_1^2,$$
$$f_2(x) = -x_1 x_2^2.$$

In this example, X_i is not convex, $\forall i \in I$, and the function $y_i \mapsto f_i(x_{-i}, y_i)$ is not quasiconcave for $i = 1$, so that the existing theorems on Nash equilibrium, such as in [6, 24–30] are not applicable.

However, we can show the existence of Nash equilibrium by applying Theorem 7.1. Indeed, for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$\Psi(x, y) = x_2 y_1^2 - x_1 y_2^2.$$

The function Ψ is continuous on $X \times X$. For any subset $\{({}_1y_1, {}_2y_2), \dots, ({}_ky_1, {}_ky_2)\}$ of X, let $x = (x_1, x_2) \in X$ such that $x_1 = \max_{h=1,\dots,k} {}_iy_1$ and $x_2 = \min_{h=1,\dots,k} {}_iy_2$. Then, we have

$$\begin{cases} {}_iy_2^2 \ge x_2^2, & \forall i = 1, \dots, k, \\ {}_iy_1^2 \le x_1^2. \end{cases}$$

Thus,

$$\begin{cases} -x_1\, {}_iy_2^2 \le -x_1\, x_2^2, & \forall i = 1, \dots, k, \\ x_2\, {}_iy_1^2 \le x_2\, x_1^2. \end{cases}$$

Therefore, $\Psi(x, {}_i y) \leq \Psi(x, x)$, $\forall i = 1, \dots, k$. According to Theorem 7.1, this game has a Nash equilibrium.

Theorem 7.1 can be generalized by relaxing the compactness of X.

Theorem 7.2 (Nash equilibrium without convexity and compactness of strategy sets) *Let $I = \{1, \dots, n\}$ be an indexed finite set and let X_i be a nonempty subset of a topological space E_i. Suppose that function $\Psi : X \times X \to \mathbb{R} \cup \{\infty\}$ satisfies the following conditions*:

(a) $\Psi(x, y)$ *is* 0-*transfer lower semicontinuous in x with respect to X*;
(b) *there exists $\{y^1, \dots, y^k\} \subset X$ such that $\bigcap_{i=1,\dots,k} G(y^i)$ is compact*, *where*

$$G(y) = \left\{ x \in X : \sum_{i=1}^{n} \bigl[f_i(x_{-i}, y_i) - f_i(x) \bigr] \leq 0 \right\}.$$

Then, *the game $G = (X_i, f_i)_{i \in I}$ has a Nash equilibrium if and only if*, *for each $A \in \langle X \rangle$*, *there exists $x \in X$ such that*, *for each $i \in I$*, *we have $f_i(y_i, x_{-i}) \leq f_i(x)$, $\forall y \in A$.*

8 Conclusion

In this paper, we introduced a new condition, called "local-dominatedness property," that can be used to characterize existence of equilibria in many problems which may have nonconvex and/or compact sets and have non-quasiconcave functions. We first investigated the existence of equilibrium in minimax inequalities under the local-dominatedness condition. We proved that the local-dominatedness condition is necessary, and further under mild continuity condition, sufficient for the existence of solution in minimax inequality. The basic results on the minimax inequality are then used to get new theorems on the existence of saddle points, fixed points, and coincidence points of functions. As an application of our basic result, we also characterize the existence of pure strategy Nash equilibrium in games with discontinuous and non-quasiconcave payoff functions and nonconvex and noncompact strategy spaces.

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