Advanced Microeconomic Theory\textsuperscript{1}

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\textsuperscript{1}The draft of my book is for the purpose of my teaching and convenience of my students in class. Please not distribute it.
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Preface

This book is based on my lecture notes that I used to teach courses at many universities, including Texas A&M University, Shanghai University of Finance and Economics, Hong Kong University of Science and Technology, Tsinghua University, and Renmin University of China, and added nearly half the content to the notes. A Chinese version was published in 2016. This book is rich in content, and has a full range of topics, including most the typical themes in modern microeconomic theory up to the frontier such as those topics in dynamic mechanism design, auction theory and matching theory that were mainly developed in the last three decades, but not covered in the commonly used standard microeconomics textbooks. It is also an integration of my study, research, and teaching of microeconomic theory more than 30 years. The content in this book is suitable for multiple purposes. By selecting different chapters and/or sections, it can serve as the text for a graduate or advanced undergraduate course in microeconomics, a sequence course of advanced microeconomics for doctoral students, a course in advanced topics in microeconomic theory, or even a course in mathematical economics. It can be also used as an important reference book for researchers.

Economics is a discipline that is seemingly simple, but actually hard to learn, master, thoroughly understand, and truly comprehend. The reason why the economic issues are difficult to solve is that, beside the most basic objective reality that an individual (at whatever level — nation, firm, household, or a person) usually pursues their own interests under normal circumstances (so called business is business), another objective reality is that, in the vast majority of cases, information among individuals is often private, it is easy to pretend: a person said something, but it is not sure whether he tells the truth or not; even if someone stares at you, seeming to be listening attentively, but it is unknown if they really listened. This increases the difficulty of understanding and solving economic problems, and in great extent, will offset the effects of institutional arrangements. How to deal with these two most basic objective realities, what kind of economic systems, institutions, incentive mechanisms and policies should be adopted have become the core issues and themes in all areas of economics.

At the same time, economics often involves normative statement —
subjective value judgments. Different individuals have different values and different opinions. For instance, some people emphasize the efficiency of resource allocations, but some others emphasize the equality of resource allocations. Individuals often have different views on economic reforms and policies. It is easy to cause great controversy, making it difficult to understand and master economics and its logical thought.

In addition, economics is a social discipline with a particularly strong externality. It may have a large positive or negative externality. Unlike medical doctors, those who suffer or even die from poor medical skills are only a few individuals, while those who suffer from poor economic application affect all aspects of economy, society, groups and individuals. Therefore, the correct understanding, study, and master of modern economics, especially the main content of the microeconomics discussed in this book, is not only important for the theoretical creation of modern economics, but also more important for practical applications. Once faults are made and inappropriate economic policies and rules are implemented, it will affect and endanger not only individuals, but also economic development at the national level as a whole. In this way, in addition to learning economics well, when making policy suggestions, we should not decide our heads so much that we think too much about own interests. We should not only be brave in taking responsibility, but also have a sense of social conscience and responsibility.

Modern economics is a dynamic development, extremely inclusive, open discipline. It has far surpassed the neo-classical economics, and economic practice can provide rich realities for the innovative development of economic theory. From the author’s point of view, as long as rigorous inherent logical analysis (not necessarily a mathematical model) is used and rational assumptions (including bounded rationality assumptions) are adopted, such research fill in the category of modern economics. Under the framework of the discipline of modern economics, microeconomics is mainly about the theory of how individuals make decisions. It is also about the theory of how prices are determined so that it can be highly characterized by one word — pricing. It is a theory about how markets operate, and it is also a theory about how a nature market should be remedied in case of market failure. It focuses on the study of how limited resources are allocated among different uses to better meet various needs of humanity. It also constitutes the microfoundation of macroeconomics and almost all other areas of economics. It can help ones understand how to fully and properly play the decisive role of the market in resource allocation, and better play the role of the government in making market ordering and efficient and maintaining social fairness, and providing public services. It can play a dual role of "guidance" and "how to do and how best to do".

Knowing what it is and why, we should know the scope of applications of an economic theory. Otherwise, once it is widely used to guide the for-
mulation of economic policies, it will cause great negative social effects. If the training of theoretical logic and empirical quantification of modern economics is limited, and the premise of its theory is not paid attention to, blindly transplanted to the study and application of realistic problems, it will lead to serious problems, so as negate the role of modern economics and think that the basic theoretical hypothesis of the existing mainstream economics is too strong, too much attention to emphasis on mathematics and rigor is too far from reality to explain and solve practical problems. When pushed to the extreme, it will negate the basic role of modern economics in economic development and market-oriented reform.

In fact, in most cases, people who have this view do not know the preconditions, and so they do not know the scope of application of a theory and blindly use them universally. Once wrong, they blame the theory is not good, and even think that the theory is wrong. In fact, just like the theory of any discipline, any rigorous economic theory gives preconditions, it does not valid in all situations. Therefore, unless the theory itself has logical inconsistence, there is no right or wrong between them, but only which theory or model is suitable for an economic institutional environment. If there is no rigorism, how can one always draw the result of inherent logic? This is as pointed out by Professor Dani Rodrik of Harvard University. These accusations usually come from laymen or some unorthodox marginalists. Indeed, individuals who hold such opinions are often those who have limited understanding of economic theory and its methods. Therefore, whether it is to do original research or practical application research, it is necessary to learn economics well, grasp its analytical framework and methods, and pay attention to the rigor of logical analysis. This is the basic purpose of this book.

As a high-level course in microeconomics, the purpose of advanced microeconomics teaching is to reveal the rigorous inherent logic behind some common principles and concepts, and to cultivate students’ability and thinking mode to analyze economic theoretical issues in a rigorous way. It also aims to teach students how to grasp and characterize the nature of complex economic behavior and economic phenomena (i.e. modeling) in order to conduct a rigorous inherent logic economic analysis. This book systematically expounds the content of modern microeconomics from basic theory, benchmark theory, analytical framework, research methods to the latest frontier topics. Therefore, all or part of the chapters and sections can be selected for use by doctoral, postgraduate and senior undergraduate students majored in economics, finance, statistics, management, applied mathematics, and related disciplines according to the course requirements as advanced microeconomics course materials. This book can also serve

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as an important reference book for teachers and scholars engaged in economics teaching and research.

**The characteristics of this book**

While the mainstream of economics profession prior World War II focused mainly on economic thoughts, in great extent, lacking scientific rigor, but it now the main attention is seemingly given to techniques and strictness, and the profound economic thoughts behind economics are largely neglected. Many individuals are lost in mathematical models and do not know the underlying assumptions and profound thoughts and insights of these economic theories. Why can not we achieve the dialectical unity of academic thinking and thinking academics? In fact, even rigorous and original research can achieve academic thinking and thoughtful academic, many highly technical theories and methods also contain profound economic thoughts, the model reflects profound economic thoughts and a strong insight (such as the general equilibrium theory, mechanism design theory). As such, when studying economics, one should know well not only academic contents of economics but also its systems so as to master its profound thoughts and get wisdom. Although it may not be so important to understand these ideas and insights in a mature modern market system for the public to become a norm because of immersion, but it is extremely important in institutional transitional countries where the direction of economic and social transformation is not clear. Combining both, the book attempts to advocate pursuing academics with deep thoughts and for deep thoughts.

In addition, in order to enhance the understanding of the origin, development and inheritance of the various economic theories (including the economic theory expressed in advanced mathematics) discussed in this book, as well as to increase readers’ interest in learning economic theory and the comprehensiveness of knowledge, and to balance scholarship and ideas well, according to the network resources and my own understanding, I comprehensively compiled the biographies of 44 economists who made a pioneering contributions to the development of modern microeconomics.

This book also connects it with the profound philosophy and wisdom of Chinese ancient times, hoping to achieve ideological and academic thinking, and realize ideological and academic organic integration. To learn economics, we need not only to be good at economics, but also to know its way, master its profound thought and become a wise man.

**Structure of this book**

Microeconomics focuses on the study of economic problems from the analysis of individual economic behavior, and then develops various theories
and results under given or designed institutional arrangements, especially under the market system. This book also arranges thematic chapters according to this logic. Before that, however, the book also arranged an introduction to the preparatory knowledge and methods needed to learn advanced microeconomic theories.

The book is divided into seven parts. The first four parts mainly introduces the benchmark models and theories in ideal frictionless economic environments where a natural market works well, as well as analytical frameworks, methods and tools. The last three parts focuses on the market failure in term of efficiency. It mainly examines how to modify or make up the market in the presence of economic externalities, public goods, especially private information, so as to solve the problem of efficient allocations of resources. The contents of the book are as follows:

There are two chapters in Part I, which is the general introduction and preparatory knowledge of the book. Chapter 1 mainly introduces the nature and methods of modern economics, the scope of the book, the preparatory knowledge and methods of mathematics, so as to play a dual role of "what to do" and "how to do and how best to do". It begins with an overview of the nature, scope, thoughts, analytical framework, and research methods of the modern economics, especially the modern microeconomics theory, as well as the similarity between economic thoughts and profound Chinese wisdom. These contents hope to increase people’s inclusiveness to all disciplines of economics, which is not only very important to the development of various disciplines of economics, but also within disciplines, such as benchmark theories and relative realistic theories.

Chapter 2 provides almost all the commonly used mathematical analytical tools and methods used in modern economics and particularly in this book, which are rich in length and content. It can be used as the basic text or important reference book for the course of mathematical economics for studying advanced macro/micro economics. It can also serve as a manual reference for basic mathematic tolls needed to learn and study economics.

Part II consists of three chapters, which mainly discuss individual decision-making, including consumer theory, producer theory, and individual choice under uncertainty. The individual decision-making theory is the microfoundation for the establishment of many theoretical models in economics, and it occupies a central position in the way economists think about problems. At the same time, many choices are made in risk and uncertain situations. People usually need to avoid some uncertainties, for example, by purchasing insurance. So the issue of choice under risk and uncertainty is an extremely important aspect of economics.

Part III consists of four chapters. It mainly discusses game theory and market theory, including basic game theory, repeated game and reputation mechanism, cooperative game, and market theory of various market structures. Game theory has become an extremely important subdiscipline in
mainstream economics, a core field in microeconomics, and the most basic analytical tool for studying various interaction decision problems in economics. For instance, monopolistic competition, especially the discussion of oligopolistic market, requires the use of a lot of knowledge and results of game theory, so it is discussed together as applications.

Part IV consists of five chapters. It mainly discusses the benchmark market theory in the frictionless situation of perfect competition - general equilibrium theory and social welfare, including the positive theory of competitive equilibrium, the normative theory of competitive equilibrium, economic core, fair allocation, social choice theory, and general equilibrium theory under certainty. General equilibrium theory is one of the most important theories in the history of economic theory development in the past 100 years. It provides an important reference and benchmark for better studying and solving practical problems. This part will describe the nature of competitive equilibrium, and discuss how to achieve the fair allocation of resources, so as to further demonstrate the universality, optimality and rationality of the market economy system.

The above four parts mainly describe the benchmark models and theories under ideal economic environments in which the market results in efficient allocations as well as the analysis framework, methods, and tools. However, in many cases, the market is not omnipotent and often fails in term of efficiency, especially from the micro and information point of view, the market is facing many problems, leading to "market failure". Thus it is very important to analyze where the market fails and how the government, regulator, or a designer should do it. Therefore, the next three parts of this book mainly discuss how to deal with the problems that the market often fails in non-ideal situations that is more close to reality.

Part V consists of two chapters. It discusses the theories of externalities and public goods, including the typical market failures of externalities and public goods. In this part it will be shown that, in general, these "non-market goods" or "harmful goods" will lead to Pareto inefficient allocations, leading to market failure. Because of externalities and public goods, a natural market is generally not an ideal mechanism for allocating resources.

Part VI has five chapters. The most important three key words in economics are information, incentive and efficiency. Therefore, this part is devoted to discusses incentive, information and economic mechanism design, including principal-agent theory under hidden information and moral hazard, general mechanism design under complete information and incomplete information, and dynamic mechanism design. Mechanism design is not to try to change human nature, which is basically unchangeable. Thus, economic mechanism theory is to study whether and how to design a set of mechanisms (rules of game or systems) to achieve desired goals under the conditions of individual pursuing self-interests, free choice, voluntary exchange, incomplete information, and decentralized decision-making, and
to compare and judge the advantages and disadvantages of a mechanism. Whether information is symmetry and incentive is compatible are the root causes of different performances of different mechanisms.

Part VII consists of two chapters, which mainly introduce the two frontier subfields of modern microeconomic theory: auction theory and matching theory, collectively referred to as market design. Market design as a relatively new field can be regarded as the specific expansion and extension of the general mechanism design theory in Part VI, and has its wide range of applications in reality.

Teaching tips

As mentioned above, the context of this book may be taught at many different levels, which is intended for the courses of microeconomics for graduate and advanced undergraduate students, topics of advanced microeconomic theory, or a course in mathematical economics. The full-fledged and rich in content in this book also provide a wide range of choices and free play for the teaching of microeconomics at different levels and for the most important aspects that teachers consider. Teachers can flexibly select relevant chapters and sections to teach according to their teaching needs. The second chapter about the knowledge of mathematics can be also used as a teaching material or an important reference for mathematical economics.

Here are some suggestions for instructors who choose this book. (1) For microeconomics sequence I for graduates and senior undergraduates, consider the following chapters for the teaching of one semester: Chapters 3-5 on individual rational decision-making and Chapters 6-8 on game theory. (2) For microeconomics II for graduates, consider the following chapters for the teaching of one semester: Chapter 9 on market theory, Chapters 10-13 on general equilibrium Theory, Chapters 14-15 on externalities and public goods, and Chapters 16-17 on principal-agent theory. (3) For the advanced topics microeconomic theory for graduates, depending on different focuses, the instructor can choose Chapters 18-20 on the mechanism design theory, or Chapters 21-22 on market design. Of course, the selection of chapters in specific teaching needs to be based on the preference of instructors, research interests, and constraints on the course time. In addition, whether to teach advanced microeconomics I, II, or advanced topics in microeconomic theory, it is important for students to know the context of Chapter 1 on the nature, category, analytical framework, thoughts and methodologies of modern economics, and thus students should first self-study Chapter 1 or it be taught by instructors. For undergraduates to have a general understanding of the scope, ideas, analytical framework, and research methods of modern economics, they are strongly recommended for reading Chapter 1 and the introduction sections of other chapters.

Doing exercises is the most reliable and effective way to master the con-
tent of teaching materials. The exercises at the end of each chapter were
written by myself, adapted from classical textbooks and the Doctoral Qual-
ification Examination Question Bank of the Economics Department of the
world-class universities, or examples or basic conclusions of original aca-
demic papers, indicating their sources whenever possible. I would like to
express my gratitude to the anonymous authors of many exercises in this
book.

Research tips
This book can also be used as a reference book for research. Whether it is
to do original theoretical and applied research or policy-oriented research,
it is necessary to learn modern economics well, master its basic analytical
framework and research methods, and lay a solid foundation for theory
and methodology. From the book, you can learn the most advanced mi-
croeconomic theory and can engage in research. Generally speaking, study
on economics can be broadly divided into two categories: One is the study
of basic, original, and common theories and tools, most of which have no
borders and can generally apply to any region. The various theories intro-
duced in this book basically fall into this category. The second is to attempt
to solve practical or realistic issues, that is, applying the basic principles,
analytical frameworks, research methods, and analytical tools of modern
economics to study and solve the real world problems of a nation or region.
These two are dialectically unified. Do not negate the latter by the former
or negate the former by the latter. Both should be parallel and equally em-
phasised. This is just like the basic research in the natural sciences and the
technological innovation in industries. They are complementary and they
are all very important and indispensable.

In addition, it should be pointed out that modern economics, especial-
ly microeconomic theory, mainly provides two categories of theories, all
of which have strict prerequisites. The first category is to provide various
benchmark theories. Most of the theories introduced in Parts II-IV of this
book fill in the first category. The second category is to provide various rel-
atively more realistic theories. Parts V-VII are more closely related to reality
and are mainly theories that are proposed to solve practical problems. The
first category theory is mainly based on the economic environments of the
mature market economies or nations, and provides the basic theory in ideal
situations. Although it has an important role in guiding the improvement
or reform orientation, however, it deviates from reality, especially including
economic environments that are still in the process of transition. Therefore,
it is necessary to revise the benchmark theory and consider the situation-
s that have friction and are closer to reality, so as to develop the relatively
more realistic economic theory that solves specific practical problems, draw
conclusions and make predictions with intrinsic logic. Thus, these two cat-

categories of theories are a progressive relationship in the development of disciplines. The basic analytical framework, thoughts, and research methods of modern economics described in this book are not only common to the study of these two types of theories, but also can be used to better use economic theory to study real world’s economic problems. They can even lay a solid theoretical and methodological foundation for the development of economic theories suitable for studying transitional economies. This requires the special attention of researchers when using this book.

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In short, this book is a crystal of a lot of people’s wisdom in a sense. Of course, I am responsible for the deficiencies and errors of this book.

Guoqiang Tian
In my home at Texas: “Xingkong Study”
December, 2019
Part I

Preliminary Knowledge and Methods
In order for readers to grasp the content in this book more effectively, learn the modern economics, understand its profound economic thoughts and the theoretical models as well as the proofs rigorously, this part introduces the preliminary knowledge and methods of economics and mathematics.

Chapter 1 briefly introduces the nature, essence, category, thoughts and methods of modern economics discipline. We will discuss the ideas and methods of modern economics, especially those that are involved in this book. While the mainstream of economic profession prior World War I focused mainly on qualitative analysis and economic thoughts, in great extent, lacking scientific rigor and quantitative analysis, but nowadays it seems that the main attention is given to techniques and strictness, and the profound economic thoughts behind economics are largely neglected. When studying economics, one should know well not only academic contexts of economics but also its systems so as to master its profound thoughts and wisdom. Combining both, the book attempts to advocate pursuing academics with deep thoughts and for deep thoughts.

Chapter 2 introduces the basic knowledge and results of mathematics that are indispensable for studying modern economics in general and advanced microeconomic theory in particular. They are used for rigorous analysis of various economic problems and especially provide necessary mathematical knowledge and tools for models, axioms, and science of microeconomic theory to derive and prove theoretical results in microeconomics and the boundaries within which they are true.
Chapter 1

Nature of Modern Economics

In this chapter, we first introduce the nature, essence, and methodology of modern economics in general and the scope, preliminary knowledge, thoughts, and methods particularly involved in the book. We will introduce the basic terminologies, core assumptions, standard analytical framework, methodologies and techniques used in modern economics, discuss its research object of market system, its connection with ancient Chinese economic thoughts, as well as some key points one should pay attention to.

The methodologies and techniques for studying modern economics include: providing benchmark, establishing reference system, setting up studying platforms, developing analytical tools, making positive and normative analysis, mastering the basic requirements of learning economic theory, understanding the role of mathematical theory, clarifying necessary and sufficient conditions for a statement, understanding the role of mathematics and statistics in economics, and knowing well the conversion between economic and mathematical languages.

1.1 Economics and Modern Economics

1.1.1 What is economics about?

To learn well economics, one must firstly know its definition and understand its nature, connotation, scope, and concerns.

Economics is the social science that studies how to make decisions in face of resource scarcity and/or incomplete information. Specifically, it studies economic behavior and phenomena, and how rational individuals (agent, household, firm, nation, organizations, and government agencies) make trade-off choices with the limited resources.

It is actually because of the fundamental inconsistence and conflict between resource scarcity and individuals’ unlimited desires (or wants) that
economics could come into being. The core idea is that individuals, who are under the basic constraint of limited resources (limited information, capital, time, capacity and freedom) and driven by unlimited desires, must make trade-off choices in resource allocation to make the best use of limited resources to maximize the satisfaction of their needs.

Economics occupies the top position among social sciences. As a discipline of social science, it studies the problem of social choices based on logical analysis and scientific viewpoints and establishes itself via systematic exploration of the matter of choice. Such exploration not only involves the building of theory but also provides analytical tools for the test of economic data.

1.1.2 Four Basic Questions That Must Be Answered

For any economic system, regardless of whether it is the planned economy wherein the government plays a decisive role, the free economy wherein the market plays a decisive role, or the semi-market and semi-planned mixed economy wherein the state-owned economy plays a leading role, when it comes to the allocation of resources, all face the following four basic questions:

(1) What should be produced and in what quantity?
(2) How should the product be produced?
(3) For whom should it be produced and how should it be distributed?
(4) Who makes the decision on production?

These questions must be answered in all economic systems, but different economic institutional arrangements provide different answers. Whether an institutional arrangement can effectively resolve these problems depends on whether it can properly deal with the issues induced by information and incentives and results in efficient allocations of resources.

As such, two basic economic institutional arrangements have been used in the real world:

(1) The institutional arrangement of planned economies: All the four questions are answered by the government who determines most economic activities and monopolizes decision-making processes and all sectors; it makes decisions on market access, product catalog, infrastructure investment allocation, individual job assignment, product pricing, employment wage, and among others, and the risk is borne by the government.
(2) The institutional arrangement of market economies: Most economic activities are organized through free market; the decisions on what product to produce, how to produce and for whom to produce are mainly made by decentralized firms and consumers, and the risk is borne by individuals.

While almost every real-world economic system is somewhere in between these two extremes, the key is which extreme is in the dominant position. Because of pursuing personal interests and private information by individuals, the fundamental flaw of the planned economic system is that it can not effectively resolve the problems induced by information and incentives, which in turn results in inefficient allocation of resources, whereas free-market economic system provides a good solution in these respects in most situations. This is the fundamental reason why countries that once adopted planned economic system inevitably failed and why China and all the East European countries carried out market-oriented reforms and wishes to bring market to the decisive role in resource allocation.

1.1.3 What is Modern Economics?

Modern economics, which developed rapidly since the forties of the last century and was built on the basic recognition of individuals’ pursuing their self-interest, systematically studies individuals’ economic behavior and economic phenomena by adopting scientific methods for rigorous reasoning and utilizing mathematical tools – specifically, it makes historical and empirical observations of the real world, elevates the observations towards the formation of theory through rigorous logical analysis, and then again test the theory in the continuing real world – thus making itself a branch of science equipped with scientific analytical framework and research methods. This systematic inquiry not only involves the form of theory, but also provides an analytical tools for testing economic data.

Scientific economic analysis is inseparable from "three dimensions and six natures", among which "three dimensions" are "theoretical logic, practical knowledge and historical vision" and "six natures" are scientific, rigorous, realistic, targeted, forward-looking and ideological". Since social economic issues generally can not be studied by simply using real society doing experiments, we need not only theoretical analysis with internal logic inference, but also empirical quantitative analysis or test with tools such as statistics and econometrics. However, only theory and practice are not good enough, which may cause shortsightedness, because the short-term optimum is not necessarily the long-term optimum, so it also needs historical comparisons from a wide field of vision and angle of view for drawing experience and lessons. Only through the three dimensions of "theoretical logic, practical knowledge and historical vision", can we guarantee that
its conclusions or reform measures satisfy the “six natures” so that “three dimensions and six natures” are indispensable.

Indeed, all knowledge is presented as history, all science exhibits as logics, and all judgment is understood in the sense of statistics. That is why Joseph Schumpeter (see Section 2.12.2 for his biography) thought that the difference between an economic “scientist” and a general economist lies in whether they adopt the three elements when conducting economic analysis: the first element is theory for logical analysis, the second one is history for historical analysis, and the third one is statistics for empirical analysis with data.¹

For theoretical creation and practical application, it is of crucial importance to correctly understand and master the general knowledge of modern economics and the content of this book in particular. It is useful for studying and analyzing economic problems, interpreting economic phenomena and individuals’ economic behavior, setting up goals and indicating the direction of improvements. More importantly, with the help of comparative analysis from the historical perspective and quantitative analysis based on data, we can draw conclusions of inherent logic and make relatively accurate predictions through rigorous inference and analysis.

Modern economics is referred to as the “crown” of social sciences due to its extremely general analytical framework, research methods and analytical tools. Its basic ideas, analytical framework and research methodologies are powerful for studying economic problems and phenomena occurred in different countries, regions, customs and cultures, and can be applied to almost all social sciences. It can even be helpful to be good leader with strong leadership, management and work so that it is jokingly called “economics imperialism” or “omnipotent” discipline by Gary S. Becker (1930-2014, see Section 13.7.2 for his biography).

1.1.4 The Difference between Economics and Natural Science

There are three major differences between modern economics and natural science:

1. Economics studies human behavior and needs to impose some behavioral assumptions, while natural science in general does not involve the behavior of human being (of course, such distinction is not absolute; for example, biology and medicine sometimes involve human behavior. However, these involvements are not from the perspective of rationality, while
1.2. TWO CATEGORIES OF MODERN ECONOMIC THEORY

Economics considers human behavior mainly from the perspective of utilitarianism. Once individuals are involved, information is very much private (incomplete or asymmetric) and easy to disguise because their behavior is unpredictable or they reveal their information strategically, without appropriated mechanisms, making it very difficult and complicated to deal with.

(2) In the discussion and study of economic problems, positive analysis of description and normative analysis of value judgment are both needed. As people have different values and self-interests, controversies often emerge, while natural science generally makes descriptive positive analysis only and the conclusions can be verified through practice.

(3) Society cannot be simply taken for experiment or test of most conclusions in economics because policies have broad impact and large externalities, while this is not a problem for almost all branches of natural science. These three differences make the study of economics more complex and difficult, and a more detailed discussion will be carried out in the sequel.

1.2 Two Categories of Modern Economic Theory

Modern economic theory is an axiomatic way to study economic issues. Similar to mathematics, it relies on logic deductions from presupposed assumptions. It consists of assumptions/conditions, analytical frameworks and models, and conclusions (interpretation and/or prediction). These conclusions are strictly derived from the assumptions and analytical frameworks and models used, so it is an analytical method with inherent logic. This analyzing method is very helpful for clearly explaining the problem and can avoid many unnecessary complexities and disputes. Modern economics is to explain and evaluate observed economic phenomena and make predictions based on economic theory.

1.2.1 Benchmark Theory and Relatively Realistic Theory

The modern economic theory can be divided into two categories according to its function. One is benchmark economic theory that provides a benchmark or a reference system\(^2\), relatively far away from the reality and dealing with ideal situations. Parts II-IV of the book mainly provide such benchmark theories. The other is relative practical theory that aims to solve practical issues so that assumptions are closer to reality, which are usually modifications to the benchmark theory. Parts V-VII mainly provide such relative practical theories. As such, both of the two types of theories are

\(^2\)We will come back to discuss the role of the benchmark and the reference system in more detail.
very important and can be used to draw logical conclusions and make predictions. Besides, there is a progressive and complementary relationship of development and extension between these two categories. The second category of realistic theories is developed from the continuing revision of the first category of the benchmark theories, thus making the theoretical system of modern economics complete and close to the real world.

The benchmark theories are mainly built on the economic environment of mature market economies and ideal situations. Their great significance should not be underestimated, misunderstood or denied. They must not be neglected and are indispensable, and demonstrate their critical importance in at least two aspects. Firstly, though the theoretical results of this category do not exist and cannot be realized in practice, one cannot deny the fact that they play an extremely important role in guiding, orientating, and providing benchmarks. When we study and solve a problem, we need to figure out first what to do and whether it should be done and then proceed to the question of how to do. Benchmark theories answer what to do, or provide the direction and goals of improvements towards the ideal situation. Though sometimes only a relatively better result can be achieved when doing anything, we can approach closer and closer towards the best by checking the benchmark or reference system. This is why we claim that only through learning from and comparing with the best can we become better and better. Therefore, the benchmark theory provides necessary standards for judging what is better and whether it is the right direction, without which what we do may be poles apart from our goals. So, who can say that the benchmark theory is not important and deny its critical significance? Secondly, it lays down the necessary foundations for developing the other category of realistic theories.

Any theory, conclusion, or statement can only be considered relatively, otherwise there is no way to analyze or evaluate. This is true for both physics which is natural science and economics which is social science, so benchmark theories are demanded. For instance, a world with friction is relative to a world with no friction, information asymmetry is relative to information symmetry, monopoly is relative to competition, technological progress and institutional changes are relative to technological and institutional fixedness, and so on. Therefore, we must firstly develop the benchmark theory under rather ideal situations. It is like the basic laws and principles in physics that only hold under the ideal situation without friction but do not exist in reality, but still no one can deny their fundamental importance because they provide indispensable benchmarks for solving physics problems in reality. Similarly, to study real economic behavior and phenomenon that includes “friction”, we should firstly be clear about the ideal situation without “friction” and use it as a benchmark and reference system. Modern economics develops so rapidly, which is unimaginable without these economic theories with ideal conditions as the benchmark.
and reference system.

As an important part of modern economics, neoclassical economics assumes the regularity conditions of complete economic information, zero transaction cost, and convexities of consumer preference and production sets and hence falls into the first category of benchmark economic theory that provides a benchmark and reference system. Neoclassical economics considers ideal situations; though there is no artificial designed social goal, it reveals that as long as individuals are self-interested, market of free competition will naturally lead to efficient allocation of resources – which can be regarded as a rigorous statement of the “invisible hand” as proposed by Adam Smith (172-1790, see 1.17.1 for his biography) – and thus shall be set as the reference system for us to determine the direction and goals for reforms so as to improve the economic, political, and social environment, establish the competitive market system, and let market play a decisive role in the allocation of resources. The boundary conditions for market works well also tell us when the market will fail, so the government needs to play a good role.

Some considers ideal reference system is far away from the real economy, so they deny the role of neoclassical economics and then further deny the instructive role of modern economics in economic reform, which is a serious misunderstanding as they do not realize that the great gap between reality and the benchmark/reference system only shows the necessity for a nation, like China, to have market-oriented reforms and to continuously improve the efficiency in resource allocation. This kind of opinion that denies the role of benchmark theory is like the denial of physics by a junior high who has just learned several formulas of Newton’s three laws and criticized it for it is totally different in the real world. In fact, they did not grasp the role of benchmark theory. Try thinking about that without the benchmark theory in physics about free fall and uniform motion, how do we know the magnitude of frictional force so we can build a house stably and rightly? And how do we know how much frictional force should be overcome to solve problems concerning the taking-off and landing of airplanes or launching satellites? Without benchmark theory, applied physics cannot possibly be developed. The study of economics follows the same logic, so we need directions, structure, goals, and some fundamentals in mind when doing things in practice, which is especially true for reforms in transitional economies.

To have reforms for those transitional economies like China, goals must be established so it must also require benchmarks and reference systems for the reform to orientate itself. It is indeed so. For the social economic development of a country, it is needless to say more about the importance of theoretical discussion, rational thinking, and theoretical creation, but as the direction and goals of the reform should be determined in the first place, what plays the fundamental, decisive, and key role in this is the basic in-
Institutions that determine national legal systems, policies and strategies. If basic institutions of legal system, rule of law, politics, economy, society, and culture that concern a nation’s path of development and long-term stability are not determined, economic theories of the best kind may help accomplish nothing but even the opposite. In economics, there is not a kind of economic theory that is always right for all development stages but there is the fittest one for certain institutional environments.

For market-oriented reform, it is natural and necessary to set neoclassical economic theory – especially such economic theory of the first category such as general equilibrium theory that demonstrates market as the optimal economic system– as benchmark and competitive market as reference system for the orientation of the reform so that results of the reform will be continuously improved towards the better outcome.

According to the economic environment defined by these benchmarks, we need to carry out reforms of deregulation and delegation for competition neutrality, ownership neutrality, liberalization, privatization, and marketization or reforms against government monopoly of resources and control of market access. At the same time, as we are aware of that under certain circumstances market may fail, the general equilibrium theory strictly defines the applicable range of market mechanism and shows the circumstances under which market may fail so for us to know the areas in which it will be better for government agency or social plan to make rules and institutions to solve market failure; in such way, this theory plays an important role in defining the scope of market that works well.

Therefore, the study of economic problems and reforms, especially determining the direction of reforms must start from the benchmark of economics, while reforms that go against the common sense in economics will end up with nothing but failure. The benchmark and reference system strictly present the premises on which market will lead to more efficient allocation and become good market economy, and such premises show the direction of the reform. The fourth part of this book lays stress on the Arrow-Debreu general equilibrium theory (Kenneth Joseph Arrow, 1921-, see Section 10.8.2 for his biography; Gerald Debreu, 1921-2004, see Section 11.9.2 for his biography) and Lucas’s macroeconomic theory of rational expectation (referred to as neoclassical macroeconomics), the two of which are both standard theories of neoclassical economics and rigorously demonstrate that market of free competition leads to efficient allocation of resources.

Needless to say, there are various benchmarks and reference systems with different value judgements and goals that are also likely to lead to very different outcomes. For example, when students regard “pass” as their benchmark, but the result is often a failure because the test of questions is a random variable to the students. Just as Confucius and Sun Zi’s statement, those who aim at the superior get the medium, those who
aim at the medium get the inferior, and those who aim at the inferior will only lose, which illustrates the extreme importance of the choice of benchmark. Meanwhile, because many benchmark theories are basic theories under ideal conditions, though they can be instructive for the improvement or the orientation of the reform, we should also bear in mind that they are actually far from the reality and cannot be simply adopted to solve real problems in practice. That is to say, the goal does not determine the process, and a well-trained economist will not mechanically apply economic theories in the first category. In reality, there are many economists who do not analyze the dynamics in the transition, consider only developed countries but not developing countries, and neglect the objective law in special development stages. However, the target institution, especially the path and schedule for accomplishing the target institution, should not always be evaded by substituting development for reform, otherwise the composite force of theories that promote market-oriented reform will be destructed and the transitional status is likely to become permanent.

It should be pointed out that the benchmark economic environment is a rather ideal situation that must be a given factor, otherwise any question cannot be discussed if everything is changing. So, in neoclassical theories and many other economic theories, an economic environment including basic institutions and production technologies must be exogenously given. Of course, there are many theories in modern economics that are specialized in the study of institutional evolution or transition, and technological progress so they cannot be given. But even so, we still need to clarify the situation in which institution, preferences and technologies are exogenous. Since the market system under the ideal situation is the goal that we pursue, we may as well set it as given exogenous institutional arrangement so as to well investigate its desirable property. However, it does not mean that modern economics only studies situations where the institution is given. Such narrow understanding of the scope of modern economics have arisen many controversies about modern economics, thinking that only the first kind of benchmark economic theory is provided and thus taking neoclassical economics that mainly provides benchmark economic theories as the unique component of modern economics. Since neoclassical theory considers ideal situations that depart from the real world, they further think that modern economics is solidified and thus deny it, which is a great misunderstanding.

Theories in the second category is a relatively more realistic economic theories that aim to solve practical problems, which are built on presupposed assumptions closer to reality and are revisions of the benchmark theories. According to their functions, they can further be divided into two kinds: the first kind provides analytical framework, method, or tools for solving practical problems, such as game theory, mechanism design theory, principal-agent theory, auction theory, matching theory, etc.; the second
kind is to give specific policy suggestions, such as Keynes theory and rational expectation theory in macroeconomics.

Thus, modern economics consisting of the two categories of theories is a greatly inclusive and open discipline in dynamic development, which far exceeds the scope of neoclassical economics. Through weakening the assumptions of benchmark theories and standardized axiomatic formulation of descriptive theories, modern economics continuously develop the second category of economic theories, giving itself great insight, explanatory power, and predictability; meanwhile. In the author opinion, as long as the study involves rigorous logical analysis (not necessarily using mathematical models) and rationality assumption (bounded rationality assumption included), it falls in the category of modern economics.

Modern economics originated from classical economics, which was developed based on the integration of Adam Smith’s work by Thomas Robert Malthus (1766-1834, see Section 4.6.1 for his biography) and David Ricardo (1772-1823, see Section 1.17.2 for his biography), including not only benchmark theories like neoclassical marginal analysis economics established by Alfred Marshall (1842-1924, see Section 3.11.1 for his biography) and Arrow-Debreu general equilibrium theory but also many more realistic economic theories. For instance, the new institutional economics by Douglass C. North (1920-2015, see Section 5.5.1 for his biography) and mechanism design theory by Leonid Hurwicz (1917-2008, see Section 16.10.2 for his biography) both have been revolutionary development of the neoclassical theory: while neoclassical theory takes institution as given, North and Hurwicz internalized institution, taking it as changeable, shapeable, and designable, and thus formulated various institutional arrangements by complying with human nature for various environments; they both have become important components of modern economics. For another example, the development of new political economics has, to a large degree, borrowed the analytical methods and tools of the second category of economic theory.

It is important to note that because theories of the second category aim to provide analytical framework, methods, and tools for solving practical problems and give specific policy suggestions mostly based on mature modern market system, its application should be handled with caution so as to avoid troubles. In fact, for no matter original theory or theory that provides analytical tools, and no matter the first category of theory that provides benchmarks or reference systems or the second category of theory that aims to solve practical economic problems, every rigorous economic theory in modern economics has self-consistent inherent logic and thus must have boundaries and scopes within which they are applicable. Therefore, much mathematics is often needed, which incurs a common criticism on modern economics’ overemphasis on details and increasing involve-
1.2. TWO CATEGORIES OF MODERN ECONOMIC THEORY

ment of mathematics, statistics, and models, making economic questions even more obscure and difficult to understand.

As a matter of fact, the reason why modern economics uses a lot of mathematics and statistics is that one should not take real society as an experiment when studying economic and social problems as well as giving economic policies, otherwise, the cost is likely too high. That is to say, a theory once is adopted, without knowing boundary conditions, it may result in great negative externality. Though policy makers and the public do not need to know details or premises of rigorous theoretical analysis, economists who propose policy suggestions must know. If the premise is not considered in the policy suggestion and application, it may result in serious problems and even disastrous results, so mathematics is needed to rigorously identify the boundary conditions and applicable scope, at the same time, it also provides basis for system innovation. Meanwhile, the application of a theory or formulation of a policy will often than not need tools of statistics and econometrics for quantitative analysis or empirical test.

In addition, in most cases, as real society cannot be simply used for experiment, the larger perspective of history is needed for vertical and horizontal comparisons. What is more, many unnecessary disputes can also be avoided in the discussion of questions. Hurwicz believed that the biggest problem of traditional economic theories was arbitrary explanation of concepts, while the greatest significance of the axiomatic method lies in its definite and clear-cut formulation of the theory, providing a commensurable research paradigm and analytical framework for discussion and criticism.

Thus, as the basic theoretical foundation for market economic system, modern economics lays much on the introduction of research methodology and analytical framework of natural sciences to study social economic behavior and phenomena, stress on the inherent logic from assumption, derivation, and conclusion, stress on mathematics and mathematical models as basic analytical tools, and stress on empirical research based on mathematical statistics and econometrics as well as lab experimental research, field research, computer simulation research, showing strong colors of practicality, empiricism, and natural science. Modern economics is very different from other humanities and social sciences with distinct ideologies and values.

1.2.2 Three Roles of Economic Theory

Economic theory has at least three roles.

The first role is to provide various benchmarks and reference systems to set up desired goals to catch up with or create so as to point a direction for improving. Through reform, transformation and innovation guided by theory, the economy in the real world is impelled increasingly closer to the
ideal state.

The second role is that it can be used to learn and understand the real economic world, and to explain economic phenomena and economic behavior so as to solve real problems, which is the major content of modern economics.

The third role is that it can be used to make logically inherent inferences and predictions. Practice is the sole criteria for testing truth but not the sole criteria for predicting truth. In many cases, problems may still arise if only historic examination and existing data are used for economic prediction, so theoretic analysis with inherent logic is needed. Through the logical analysis of economic theory, we can make logically inherent inferences and predictions on the possible outcomes under given economic environments, behavior of economic agents and economic institutional arrangements. This will guide us to solve economic problems in reality in a better way. As long as the preassumptions in a theoretic model are roughly met, we can obtain scientific conclusions and make basically correct predictions and inferences accordingly, so we may know the outcomes. For instance, the theoretic inference that planned economy is unfeasible proposed in 1920s by Friedrich Hayek (1899-1992, see Section 2.12.1 for his biography) has this kind of insight. A good theory can deduce the logically inherent result without social experiments, which can solve the problem that economics cannot make experiments on real society to a great extent. What we need to do is to check whether the assumptions made on economic environments and behavior are reasonable (experimental economics that is popular in recent years is mainly engaged in fundamental theoretical research such as testing individual behavioral assumptions). For example, we are not allowed to issue currency recklessly just for the sake of studying the relationship between inflation and unemployment. Like astronomers and biologists, most of the time economists can only utilize existing data and phenomena to test and revise the theory.

Of course, as indicated above, we should not exaggerate the role of economic theory and expect it to solve key and fundamental problems. What is fundamental, key, and decisive is the basic constitution and institution that determine the nation’s fundamental trend. If the basic system that concerns the direction of the country and long-term prosperity in politics, economy, society, and culture is not determined, the best kind of economic theory cannot help too much and may even lead to the opposite of our wish.

1.2.3 Microeconomic Theory

A notable feature of microeconomic theory is to set up theoretical hypothesis or modeling for economic activities of self-interested individuals, especially in market economy, and conduct rigorous analysis and examine how
the market works on such basis.

The whole microeconomics runs through a main theme – price or pricing: which factors affect pricing? whether enterprises have pricing power? how to have pricing advantages? and how to make optimal pricing? Therefore, we need to study the demand, supply, characteristics and functions of market and the pricing in all kinds of markets and various economic environments so as to maximize individuals’ interests. As a result, microeconomics is also called price theory.

Microeconomics is the core of economics and the theoretical foundation of all branches of modern economics. It enables us to use simplified assumptions for in-depth analysis of complex world so as to find clues from the complex, making the complex problems become relatively simple. It can help us to extract the most useful information from things unrelated and think of various issues using the method of economics so as to make explanations and predictions that conform to reality. Other branches, such as macroeconomics, finance, applied econometrics, etc., all rely on the support of microeconomic theory.

1.3 Modern Economics and Market System

A main purpose of modern economics is to study the objective laws of market and individuals’ (such as consumers and firms) behavior in the market. Specifically, it studies how to realize harmony among self-interested individuals in the market, how the market allocates social resources, and how to achieve economic stability and sustainable growth, etc. Therefore, for the purpose of better learning modern economics, one should have have a general understanding of the function and advantage of modern market mechanism.

1.3.1 Market and Market Mechanism

Here we will have a brief introduction of the operation and basic functions of market and how market coordinates individuals’ economic activities without needing excessive participation or intervention of the government.

Market: Market is a trade mode where buyer and seller conduct voluntary exchanges. It refers not only to the place where buyer and seller conduct exchanges but also to any form of trading activities, such as auction and bargaining mechanisms.

When learning microeconomics, it is important to keep in mind that any transaction in the market has both buyers and sellers. For a buyer of any good, there is a corresponding seller. The final outcome of the market process is determined by the rivalry of relative forces of sellers and
buyers in the market. There are three forms of competition for such rivalry: consumer-producer competition, consumer-consumer competition, and producer-producer competition. In the discussion of this book, readers will find that the bargaining position of consumers and producers in the market is limited by these three sources of competition in economic transactions. Competition in any form is like a disciplinary mechanism that guides the market process and has different impacts on different markets.

**Market mechanism**: Market mechanism or price mechanism is an economic institution in which individuals make decentralized decision guided by price, which is usually a narrow definition of market institution. In fact, a nature market mechanism or market system is the set of all systems and mechanisms closely related with market (including the system of market laws and regulations). As a form of economic organization featuring decentralized decision-making, voluntary cooperation and voluntary exchange of products and services, it is one of the greatest inventions in human history and by far the most successful means for human beings to solve their economic problems. The establishment of market mechanism is not a conscious, purposeful human design, but a natural process of evolution. In the opinion of Hayek, market order is a spontaneous extended order of economy which evolves through long-term choices and trials and errors. The emergence, development and further extension of modern economics are mainly based on the study of market system. At first glance, the operation of market is amazing and beyond comprehension. In the market system, decisions on resource allocation are independently made by producers and consumers who pursue their own interests under the guidance of market price without the imposition of any command or order. Market system unknowingly solves the four basic questions that no economic system can evade: what to produce, how to produce, for whom to produce, and who makes the decision.

Under market system, firms and individuals make the decision on voluntary exchange and cooperation. Consumers seek the maximal satisfaction on their demand while firms pursue profits. In order to maximize profits, firms must have meticulous plans for the most efficient use of resources. That is to say, for resources with similar usage or quality, they will choose the ones of the lowest possible cost. The best use of things originally has different meanings from the standpoints of firms and economy, respectively, but price links them up, which, as a result, harmonizes the interest of firms and that of the whole society and leads to efficient allocation of resources. The price level reflects the supply and demand of resources in economy and the degree of scarcity of resources. For example, in the case of inadequate timber supply and ample steel supply in economy, timber will be expensive while steel will be inexpensive; to reduce expenses and make more profits, firms will try to use more steel and less timber. In doing so, firms do not take the interests of society into consideration, but
the outcome is totally in line with social interests, which is precisely the underlying role of price of resource that does the trick. Resource price coordinates the interest of firms and that of the whole society and solves the problem of how to produce. Price system also guides firms to make production decisions in the interest of society. What to produce and who has the final say? The consumer has the final say. Firms only need consider to produce the product that has a higher price. Yet in the market system, the price level exactly reflects the social needs. For instance, poor harvests and the corresponding rising grain price will encourage farmers to produce more grain. As such, profit-pursuing producers “come to the rescue” under guidance, and the problem of what to produce is solved. Moreover, the market system also addresses the problem of how to distribute products among consumers. If a consumer really needs a shirt, he or she will offer a higher price than others. Profit-pursuing producers will for sure sell the shirt to the consumer who offers the highest price. Thus, the problem of for whom to produce is addressed. All these decisions are made by producers and consumers in a decentralized manner – thus, the problem of who makes the decision is also solved.

As such, market mechanism easily coordinates the seemingly incompatible individual interest and public interest. As early as two hundred years ago, Adam Smith, Father of Modern Economics, saw the harmony and wonder of market mechanism in his masterpiece *The Wealth of Nations* (Adam Smith, 1776). He regarded the competitive market mechanism as an “invisible hand”. Under the guidance of the invisible hand, individuals pursuing their own interests unintentionally head for a common goal and thus achieve the maximization of social welfare:

“... every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it ... he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.”

Smith closely studied how market system combines self-interestedness of individuals with social interests and the division and cooperation of labor. The core of Smith’s thoughts is that if division of labor and exchange of goods are totally voluntary, then exchange will only occur when we realize the result of exchange is mutually beneficial to both parties of exchange, otherwise no one will exchange. As long as there are benefits, individuals driven by self-interests will voluntarily cooperate. External pressure is not the necessary condition for cooperation. Even if there are language barriers, as long as mutual benefits exist, exchange can still be carried on as normal. At usual times, market mechanism works so perfectly that individuals cannot feel its existence. With the metaphor of “invisible hand”,
Smith pointed out the importance of voluntary cooperation and voluntary exchange in economic activities of the market. However, the thought on the welfare brought about by market system for the people, either at the times of Smith or today, was not and still have not been fully recognized by all. The Arrow-Debreu general equilibrium theory to be discussed in the book had formal statement of Smith’s "invisible hand", which rigourously demonstrates how market of free competition could lead to the maximization of social welfare and has proved the optimality of market in allocating resources.

1.3.2 Three Functions of Price

As discussed above, the normal operation of market system is realized via the price mechanism. As analyzed by the Nobel laureate in economics, Milton Friedman (1912-2006, see Section 4.6.2 for his biography), price performs three functions in organizing a rapidly changing economic activities involving hundreds of millions of individuals:

1. Transmitting information: price transmits production and consumption information in the most efficient way;
2. Providing incentive: price provides incentives for individuals to carry out consumption and production in an optimal way;
3. Determining income distribution: endowment of resources, price, and the efficiency of economic activities determine the income distribution.

In fact, as early as the Han dynasty of China, Sima Qian (a Chinese historian of the Han dynasty who is considered the father of Chinese historiography) had already noticed and summarized the law of commodity price fluctuation, saying that all commodities "will return to being inexpensive when it becomes expensive to the extreme and will become expensive when it comes inexpensive to the extreme", so for the sake of becoming rich, individuals shall make good use of this law to "pursue interests along with opportunity".

Function 1 of Price: Transmitting Information

Price guides the decision-making of participants and transmits the information of changes in supply and demand. When the demand for a certain commodity increases, sellers will notice the increase of sales and thus place more orders with wholesalers who will place more orders with manufacturers so the price will go up, and the manufacturers will then invest more factors of production to produce this commodity. In this way, the message of increasing demand for this commodity is received by all related parties.
The price system transmits information in a highly efficient way and it only transmits information to those who need it. Meanwhile, the price system not only transmits information but also produces a certain incentive to ensure the smooth transmission of information so that information will not be detained in the hands of those who do not need it. Those who transmit information are internally motivated to look for those who are in need of information while those who need information are internally motivated to acquire information. For example, ready-to-wear apparel manufacturers are always hoping to get the best kind of cloth and constantly looking for new suppliers. Meanwhile, cotton cloth manufacturers are also always reaching out to clients to attract them with good quality and cheap price of their products by various means of publicity. Those who are not involved in such activities will surely be uninterested and indifferent to prices and supply and demand of cotton cloth. The general mechanism design theory as discussed in Chapter 18 of this book will demonstrate that competitive market mechanism is the most efficient in the use of information for it requires the least information and so lowest transaction cost. In 1970s, Hurwicz et al. already proved that for the neoclassical economic environment of pure exchange, no other economic mechanisms can achieve efficient resource allocation using less information than does the competitive market mechanism.

**Function 2 of Price: Providing Incentive**

Price can also provide incentive so that individuals will react to the changes of supply and demand. When the demand of a commodity decreases, an economic society should provide certain incentive so that manufacturers of the commodity will increase production. One of the advantages of the market price system is that prices not only transmit information but also provides incentive for individuals to react to the information voluntarily for the sake of their own interest, so consumers are motivated to consume in an optimal way while producers are motivated to conduct production in the most efficient way. The incentive function of price is closely related to the third function of price: determining income distribution. As long as the increased gain brought by increased production (i.e., marginal revenue) exceed the increased cost (i.e., marginal cost), producers will continue to increase production until the two are even and maximal profits are realized.

**Function 3 of Price: Determining Income Distribution**

In a market economy, an individual’s income depends upon the resource endowment he owns (e.g., assets, labor) and the outcomes of economic activities he’s engaged in. When it comes to income distribution, one always wants to separate price’s function of income distribution out from the other
functions, in the hope of more equal income distribution without affecting
the other two functions of transmitting information and providing incentive. However, the three functions are closely related and all indispensable.
Once price no longer influences income, its functions of transmitting informa-
tion and providing incentive will no longer exist either. If one’s income
does not depend upon the price of labor or commodities he offers to oth-
ers, why would he bother to acquire the information of price and market
demand and supply and react to such information? If one gets the same
income no matter how he works, then who would like to do a good job? If
no benefits are given for innovation and inventions, then who are willing to
make efforts in this regard? If price has no impact on income distribution,
it will also lose the other two functions.

1.3.3 The Superiority of Market System

Modern market system is a sophisticated and delicate economic mechanism that has emerged, gradually taken shape, and been constantly improved
in the long-term evolution of human society. The fundamental and deci-
sive role of market mechanism in resource allocation is the key to market
economy’s capacity for optimal resource allocation. The optimality here
has the same meaning as the Pareto optimality (efficiency) proposed by
Vilfredo Pareto (1848–1923, see his biography in Section 11.9.1) which will
be discussed in Chapter 11 in more detail. It means that, under existing
resource constraints, there is no such a scheme of resource allocation that
can make some participants better off without hurting the welfare of any
others. Even though Pareto optimality fails to consider the issue of social
fairness and justice in term of equity, it provides a basic criterion of whether
the resource is wasted or not in terms of social benefit for an economic sys-
tem and evaluates social economic effects in terms of feasibility. According
to it, if an allocation is not efficient, there is room for improvement.

Two fundamental theorems of welfare economics, which we will dis-
cuss in Part IV on general equilibrium theory, provide a rigourous for-
mal expression of Adam Smith’s assessment. As the formal statement of
his ‘invisible hand’, the theorems prove that free competitive market can
maximize social welfare and achieve the market optimality in terms of re-
source allocation. The First Fundamental Theorem of Welfare Economics
proves that when individuals pursue their own self-interest, and if eco-
nomic agents have unlimited or locally non-satiable desire, the competitive
market system can achieve Pareto-efficient allocation for economic environ-
ments with private divisible goods, complete information, and no external-
ity. The Second Fundamental Theorem of Welfare Economics proves that for the neo-classical economic environments, any Pareto-
efficient allocation can be achieved by reallocating initial endowments and
competitive market equilibrium without the need to introduce other eco-
nomic systems to replace the market mechanism. Precise statements and rigorous proofs of these theorems will be given in Chapter 11.

The economic core theorem and the limit theorem on the core in Chapter 12, from another perspective, prove that market system can benefit the social stability and is optimal and unique in terms of resource allocation, being the result of natural selection with objective inherent logic in economic activities. Moreover, for competitive market mechanism to lead to optimal allocation of resources, the modern market economic system can well solve the problem of social stability and orderliness. According to the basic connotation of the economic core theorem, when the allocation of social resources has the core property, there will not be any coalition (i.e., a group of agents) that are unsatisfied with the allocation and want to improve their welfare by controlling and utilizing their own resources. In this sense, there will be no powers or groups that constitute threat to society, and hence society will be relatively stable. As the economic core theorem reveals, when a market achieves competitive equilibrium, equilibrium allocation caused by market equilibrium will have the core property under some regularity conditions, such as monotonicity, continuity, and convexity (diminishing marginal rate of substitution) of preference. The limit theorem on the core tells us that, under the greatest reality of individuals’ pursuit of self-interest, as long as individuals are given economic freedom (i.e. the freedom to cooperate and exchange voluntarily) and perfect competition, the outcome will be identical to that of competitive market equilibrium without giving any institutional arrangements in advance. Thus, market system is not an invention but an inherent economic rule and spontaneous order, which is objective as the law of nature. The policy implication of this conclusion is that the market should let to make the decision when competitive market mechanism is able to obtain optimal allocation. Only under the circumstance where the competitive market is incapable will other mechanisms be designed to make up for market failures.

Moreover, the competitive market mechanism is not only optimal and unique in social stability maintenance and efficient resource allocation but also efficient in the transmission of information. In the 1970s, Hurwicz et al. proved that, for neoclassical pure exchange economies, there is no other economic mechanism which can lead to efficient allocation of resources using less information than the competitive market mechanism. In 1982, Jordan further proved that, in pure exchange economies, market mechanism is the only one which achieves the efficient allocation using the least information. Tian (2006) further proved that this conclusion is true not only in pure exchange economies but also for economies with production, and meanwhile market mechanism is unique. Hence, here comes an important inference: in no matter command planned economy, state-owned economy, or mixed economy, information needed to realize efficient allocation of
resources is surely more than that in a competitive market mechanism, so those economic systems are not informational efficient, meaning that they require more information to achieve the optimal allocation of resources. This conclusion provides an important theoretical explanation for the issue such as why China needs market-oriented economic reform and the privatization of state-owned economy. The uniqueness result of information efficiency will be discussed in Chapter 18.

Even though the market mechanism cannot perfectly solve the problem of social equity featured by the large gap between the rich and the poor, as long as the government can try the best to provide fair opportunities and equal value of resources for all individuals and let the market play its role instead of filling in for the market, fair and efficient resource allocation could be reached by the market, as the fairness theorem in Chapter 12 enlightens us. The above statements and proofs about the optimality, uniqueness, and fairness of the modern competitive free market system in resource allocation and its contribution to social stability are all important contents of the general equilibrium theory.

Joseph Schumpeter also discussed the optimality of the market mechanism from the perspective of how interactions (dynamic game) between competition and monopoly leads to innovation-driven growth. His innovation theory told us that the valuable competition was not merely price competition, but more importantly, the competition in new commodities, new technologies, new markets, new supply sources, and new combinations. Therefore, the root of long-term vitality of market economy was innovation and creativity, which stemmed from entrepreneurship and entrepreneurs’ constant, creative destruction of the market equilibrium, referred to as ‘creative destruction’ by him. It basically depends on profit-pursuing entrepreneurs and private economy to cultivate the soil for innovation and to encourage and protect innovation.

In fact, competition and monopoly, like supply and demand, can form an astonishing unity of opposites through the power of market, thus revealing the beauty and power of the market system. Market competition and enterprise innovation are inseparable. If there is no competition, there will be no motivation of innovation either, like what happens in the state-owned enterprises of state monopoly. Indeed, competition results in profit decline. The fiercer the competition becomes, the faster the corporate profits decline. This provides enterprises with strong incentives to innovate for the sake of survival. Innovating enterprises may gain a monopoly position that implies monopoly profits, which will attract more enterprises to participate in the competition. Thus, it forms a repeated cycle of “competition → innovation → monopoly profit → competition”, in which market competition tends to achieve equilibrium but innovation breaks the equilibrium. Market constantly goes through such games to inspire enterprises to continuously pursue innovation. Through this dynamic game, market
maintains its vitality and further leads to economic development and social welfare. So, in order to encourage innovation, the government should enforce the law of intellectual property rights protection.

While people may realize the importance of entrepreneurs and entrepreneurship, but not all fully recognize the fundamental importance of the institutional basis behind the creation of entrepreneurship and entrepreneurship, and innovation and development need institutional support. This is because entrepreneurs and entrepreneurship do not appear by chance, entrepreneurship is derivative and superficial, and it must be built on a basic meta-institution, all of which require a good institutional environment as a prerequisite. As such, Baumol (1990) extended Schumpeter’s innovation theory and argued that innovation and entrepreneurship depend on institutional choice, and are therefore endogenous variables. If the rule of game that affects the choice of entrepreneurial behaviors is abnormal or destructive, then innovation and entrepreneurship cannot reach its full potential. Indeed, so far, there are three industrial revolutions in the world: the industrial revolution in Britain, the second industrial revolution led by the United States and Germany, and the third industrial revolution taking artificial intelligence manufacturing as the core. The occurrence and development of each industrial revolution are closely related to the institutional innovation, which provides support for the smooth development of the industrial revolution.

For instance, in order to encourage competition and form externalities of technological innovation, the anti-monopoly law should be enacted. Also, the protection for intellectual property rights will not last forever but will be limited within a certain number of years so that they will not become fixed or perpetual oligopoly or monopoly. Thus, technological innovations operate on the basis of institutional innovations. These two kinds of innovations are like an action-reaction pair. A good institution can reduce the transaction costs of innovation, create conditions for cooperation, provide incentives for innovation, and facilitate the internalization of the benefits of innovation. One goal of constructing a technological innovation system is to promote the interaction and cooperation among innovative elements.

Innovation means to break rules and regulations, which inevitably contains high risks. High-tech innovation in particular means high risks and extremely low possibility of success for venture capital investment, but once it succeeds, it will bring considerable returns and then attract more investment. However, it is impossible for state-owned enterprises to take such high risks due to the fact that it naturally lacks the incentive mechanism to take risks. On the contrary, private enterprises dare the most to take the risk out of strong incentive to pursue self-interest, so they are the most creative and innovative entities. Therefore, entrepreneurial innovation (not fundamental scientific research) mainly takes place in private business. Ac-
tually, Sima Qian, in the history of Chinese thoughts, had also affirmed that competition and survival of the fittest were a natural tendency. He believed that one did not rely on certain trades to get rich, while wealth was not exclusively occupied by certain people; a capable person would accumulate wealth, whereas those incapable would lose their properties.

What should be noted is, with the emergence of innovation in financial technology, the deviation between the real economic situation and the ideal state will be decreased. The innovation will push the real market economy toward the ideal state of market economy described by Adam Smith, Hayek, Kenneth Joseph Arrow, Gerard Debrue, and Ronald H. Coase (1910-2013, see Section 14.6.2 for his biography). According to no matter the role of competitive market as the ‘invisible hand’ described by Adam Smith, or the general equilibrium in the perfectly competitive market by Arrow and Debrue and the theory with the zero transaction cost in the perfectly competitive market by Coase which will be discussed with details in this book, or the statement that competition benefits innovation from Schumpeter, the market can all be proved to be the optimal. The basic conclusion of these theories is that the perfectly competitive market leads to efficient allocation of resources and social welfare maximization. Of course, the perfectly competitive market merely provides a reference system or an ultimate goal, which means that the more competitive the market is, the better; the more symmetric the information is, the better. Such a perfectly competitive market system, however, does not exist in reality because the communication cost and transaction cost, including the important financing cost, cannot be zero (although it can be closer and closer).

With internet as the medium for finance, the transaction cost is becoming smaller and smaller. Due to the innovation and development of financial technology, to some extent, the perfectly competitive market, as the first kind of economic theory, is not just an ideal state but tends to approach the reality increasingly closer with the disruptive innovation of financial technology. Financial technology will greatly reduce the cost of information communication in reality so as to make market economic activities closer to the ideal state of perfect competition and thus more efficient.

1.4 Governance Boundaries of Government, Market, and Society

The theoretical conclusion about market optimality actually relies on an implicit assumption of the critical importance of fundamental institution. That is to say, there should be a mature governance structure to regulate government, market, and society.
1.4. **GOVERNANCE BOUNDARIES OF GOVERNMENT, MARKET, AND SOCIETY**

1.4.1 **Three Dimensions of State Governance**

State governance involves three dimensions: **government, market, and society**. Market mechanism may give one the wrong impression that, in a market economy, one seems to be allowed to do whatever he likes to pursue self-interest, which, however, is actually not true. There is no completely laissez-faire market economy that is totally independent of the government in the world. A well-functioned market requires proper and effective integration of government, market, and society, which form a three-dimensional structure of state governance. A completely laissez-faire market without governance and regulation is not omnipotent. As we shall discuss in Parts V-VII of this book, market often fails in circumstances such as monopoly, unfair income distribution, polarization of rich and poor, externality, unemployment, inadequate supply of public goods, and information asymmetry, thus resulting in inefficient allocation of resources and various social problems.

Economic development and governance are the inherent dialectical relationship and shall be correctly understood. Economic development mainly focuses on the improvement of a nation’s hard power while governance stresses the construction of soft power. Of course, governance should be all-dimensional from different aspects including governance systems of government and market, social equity and justice, culture, values, etc. How the relationship between government and market and between government and society is handled often determines the effect of state governance. If they cannot be well balanced, there may be a series of serious problems and crises including excessive gap between the rich and the poor, unequal opportunities, etc., preventing an inclusive market economy and a tolerant and harmonious society from coming into being. In this way, in the logic of governance, there is good kind and bad kind of governance that will lead to good or bad market economy and good or bad social norms.

Therefore, governance should not be simply taken as being equivalent to rule, control, or regulation, or regarded as the opposition of development, making it difficult to attend to both governance and development simultaneously. To have an efficient market and harmonious society, it should build a limited government that is capable, accountable, effective and welfare-enhancing, leading towards the desired governance featuring the principle of the rule of law.

1.4.2 **Good Market Economy Only Comes from Good State Governance**

Market economy can be classified into **“good market economy”** and **“bad market economy”**. Whether it is good or bad depends on the system of state governance and whether the governance boundaries among govern-
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ment, market, and society are clearly and appropriately defined. For a good market system, the government allows the market to fully play its role, and in case of market failures, the government can play an appropriated role, which does not mean that the government should directly intervene in economic activities but asks the government to enact proper rules or institutions to make the market efficient or resolve the problem of market failures so as to achieve incentive compatible outcome so that individual interest and social interest are consistent. One of the most successful examples of institutional design is the enactment of the basic constitution of the U.S. when it was just founded, which made the U.S. had become the most powerful nation in the world through only more than 100 years.

A good, tolerant, and efficient modern market economy should protect the private interests of individuals to the best through institutions or laws, and meanwhile limit and counterbalance the government and its public powers as much as possible; in this sense, it is a contractual and rule-of-law economy constrained by an agreement in commodity exchange, rule of market operation, and reputation. Under the constraint of individuals’ pursuit of self-interests, resources, and information asymmetries in an economic society, to build a strong state with enriching people, first of all, individuals shall be endowed with private rights, the core of which includes the basic right to survival, freedom of choice for one to pursue happiness, and private property rights. Through their participation in full competition, voluntary cooperation, and voluntary exchange of market mechanism and their pursuit of self-interest, efficient allocation of resources and maximization of social welfare can be realized. Thus, modern market economy is established upon the basis of the rule of law that works in two ways: first, it is of fundamental importance to restricts arbitrary government intervention in market economic activities; second, it further supports and promotes the market in ways including definition and protection of property rights, enforcement of contracts and laws, maintenance of fair market competition, etc., so as to let market play the fundamental and decisive role in resource allocation and give full play to the three basic functions of price, namely transmitting information, providing incentives and determining income distribution. What is more, a good market requires good social norms, for one’s pursuit of self-interest should be on the premise of respect for others’ pursuit of self-interest, and self-interest and fair competition should run parallel to one another with no contradiction. The spirit of compromise and respect for other’s standard of values judgment are the premise for normal proceeding of exchanges.

On the other hand, in a bad market economy, for lack of adequate ruling and governance capacity in the economic and social transformation, the government is only unable to provide necessary and enough public goods and services to make up for market failures, but the government’s excessive economic activities lead public powers are not effectively counterbalanced,
property rights of state-owned enterprises are not clearly defined, and the
government is involved in numerous rent-seeking and corruption so that
equity and justice in social and economic areas are greatly impaired. This
breeds the so-called “State Capture”, which refers to the phenomenon that
by providing personal interests for government officials, economic agents
interfere in decisions on laws, rules and regulations and thus, without go-
ing through fair competition, they convert their personal interests to the
basis of rule of games of the whole market economy which become a great
deal of policy arrangements that produce high monopoly profits for specif-
ic individuals and dignitaries at the expense of enormous social costs and
the decrease of government credibility. As a result, inefficient balance in
public choice continues in the long run. The behavior of scrambling for so-
cial and governmental resources by means of unfair rent-seeking instead of
fair competition will not only cause market failures but more importantly
gradually results in bad social norms in the long run, bringing about distor-
tion of social resource allocation and values, moral decline, absence of
good faith, “false, big, and empty” in words and deeds, frivolity of soci-
ety, and increased factors of instability, which finally result in enormous
explicit and implicit transaction costs. Some sociologists call such social s-
tatus “social corruption”, meaning that social cells of the social organism
are dead and suffering functional failures.

Therefore, in the three-dimensional framework of government, market,
and society, government as an institutional arrangement with strong pos-
itive and negative externalities plays a vital role. It can make the market
efficient, become the impetus for economic development, help construct
a harmonious society, and realize sustainable development. On the other
hand, it may also make the market inefficient, lead to various social con-
tradictions, become tremendous resistance for the benign development of
society and economy and exert bad social impacts. Almost all countries in
the world adopt market economy, but a majority of them did not achieve
sound and rapid development. Among many reasons, the most fundamen-
tal one is the lack of reasonable and clear defined governance boundaries
between government, market, and society so there is over-playing, under-
playing and mis-playing of government role. Only when the governmen-
t appropriately tightens its omnipresent “visible hand” and the functions
and governance scope of the government are appropriately and reasonably
defined can it be expected to reasonably defined governance boundary be-
tween government, market and society.

1.4.3 The Principle in Defining the Boundaries of Governance A-
mong Government, Market and Society

How to reasonably define the governance boundaries among government,
market, and society? The answer is to let the market do whatever it can
do well while the government does not participate in economic activities directly (however, it is necessary for the government to maintain market order and guarantee strict implementation of contracts and rules); as for those that the market cannot do, or when it is not appropriate for the market to be involved considering other factors such as national security, the government can then directly participate in economic activities. That is to say, when considering the construction of a harmonious society and benign development of economy, or when transforming government functions and innovating the management mode, consideration shall be given to the boundaries of market, government, and society. For instance, the government should exit from competitive sectors. Only in case of market failure should the government play its role in solving problems by itself or along with the market. However, the basic guideline is that the government should not directly intervene in economic activities but enact proper rules and institutions to solve problems of market failures. Because of individuals’ pursuit of self-interest and private information are at the heart of many economic activities, direct intervention in economic activities (e.g. large amounts of state-owned enterprises and arbitrary restriction of market access and interference with commodity prices) often would not generate desired outcomes. In this respect, mechanism design theory can play an important role in making the market more efficient and solving the problem of market failures. In Hurwicz’s opinion, "law-making by the U.S. Congress or other legislative bodies equals to designing new mechanisms".

Under modern market economy, the basic and sole functions of government can be summarized as “maintenance” and “service”, that is, making the fundamental rules to ensure national security, social stability, and economic order; and providing public goods and services. Just as Hayek pointed out, government has two basic functions: firstly, the government must be responsible for law enforcement and defense against enemies; secondly, the government must provide services that market is unable to provide or unable to fully provide. Meanwhile, he also stated that “it is indeed most important that we keep clearly apart these altogether different tasks of government and do not confer upon it in its service functions the authority which we concede to it in the enforcement of the law and defence against enemies.”\(^3\) This requires that in addition to undertaking necessary functions, the government should separate its powers to market and society. Abraham Lincoln gave a clear and incisive defined functions of the government:

“The legitimate object of government, is to do for a community of people, whatever they need to have done, but can not do, at all, or can not, so well do, for themselves—in their separate, and individual capacities. In all that the people can

Meanwhile, good-inclusive-efficient modern market economy and state governance mode need an independent autonomous civil society with a strong ability of interest coordination as an auxiliary informal institutional arrangement. Otherwise, the explicit and implicit transaction costs of economic activities would be tremendous and it is hard to establish the most basic relationship of trust in society.

In summary, a reasonable and clear defined governance boundary among government, market, and society is a prerequisite for establishing a good-inclusive-efficient market economic system and achieving benign development featured by efficiency, fairness, and harmony. Of course, the transition to an efficient modern market system is often a long process. Due to various constraints, governance boundaries of government, market, and society cannot be clearly defined in one leap, but a series of transitional institutional arrangements are often needed. However, with the deepening of transition, some transitional institutional arrangements may decline in efficiency, and may even degenerate into invalid institutional arrangements or negative institutional arrangements. If governance boundaries of government, market, and society cannot be timely and appropriately clarified while some temporary, transitional institutional arrangements (such as government-led economic development) are fixed as permanent and ultimate institutional arrangements, it is then impossible to achieve an efficient market and a harmonious society. With the development of modern economics, its analytical framework and research methods play an irreplaceable role in the study of how to reasonably and clearly define governance boundaries among government, market, and society and how to conduct comprehensive governance.

1.5 Three Institutional Arrangements for Comprehensive Governance

As discussed above, in order to establish a well-functioning and efficient modern market system, it is necessary to coordinate and integrate the relationship of the three basic coordination mechanisms, namely, government, market, and society, so as to regulate and guide individuals’ economic behavior and conduct comprehensive governance. Government, market, and society correspond exactly to the three basic elements of governance, incentive, and social norms in an economy. Mandatory public governance and formal institutional arrangement like incentive market mechanism as two elements of comprehensive governance which are overlapping, through long-term accumulation, may help guide and mould normative informal

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4 Abraham Lincoln’s Quotes on Government (1854).
institutional arrangements, enhance the predictability and certainty of social and economic activities, and greatly reduce transaction costs. The informal institutional arrangement mentioned here is social norms and culture. Just like enterprises, the first-class enterprise does branding, the second-class enterprise develops technologies, and the third-class enterprise produces, in which it is corporate culture that plays the key role. Similarly, it is the position of the government that plays the key role in the establishment of good or bad social norms and market economy.

To put it in another way, the three basic institutional arrangements for comprehensive governance are employed to “enlighten with reason, guide with interests, and persuade with emotions”, the three of which are realized and implemented mainly by government, market, and society, respectively, on the level of state governance. To “enlighten with reason” is to work through the stimulus of legal principles and reasons; to “guide with interests” is to link up economic activities with revenue through the stimulus of rewards and penalties, thus becoming the incentive mechanism; to “persuade with emotions” is to work through the stimulus of emotions and shared beliefs, as sometimes relationships, friendships, and emotions, especially shared beliefs and concepts, can help solve big problems, which as a kind of social culture will greatly reduce the transaction cost.

1.5.1 Regulatory Governance

Regulatory governance, as the basic institutional arrangement and management rule, is enforcement. The basic criterion for whether such rules and regulations shall be formulated is whether it is easy for clear definition (according to the possibility of information transparency and symmetry) and whether the costs of information acquisition, supervision, and enforcement are high. If a regulation is too costly in terms of supervision, it will not be feasible for enforcement. Protection of property rights, contract implementation, and proper supervision all call for relevant rules, which thus requires a third party to oversee the enforcement of the rules. The third party is then a government agency. In order to maintain market order, the role of government is inevitable. As government is also an economic agent, it become both the referee and the player, so as it will have huge influence. This requires procedures and rules to constrict the behavior of the government. Regulations on other economic agents and market should be quite the opposite. Due to information asymmetry, regulations in this regard should not be detailed so as not to interfere too much with the freedom of choice of economic agents.

Institutional arrangements that “enlighten with reason” and deter people like sticks are similar to the thoughts of Legalism in ancient China. However, one big problem of the Legalism was it aimed to govern economic agents only but exerted no restriction on the government. So it is
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rule by law but not rule of law. Besides, this kind of institutional arrangement only considered the cruelty in individuals’ fight for powers and profits but neglected the influence of individuals’ emotion upon behavior. The employment of the heavy hand of Legalism without considering other institutional arrangements will often lead to coercive powers and hinders the formation of modern market economy.

1.5.2 Incentive Mechanism

Incentive mechanism like market mechanism is inducing, which is of the widest application and also a main concern of this book. Due to information asymmetry and high cost of information acquisition, and in addition, human nature, especially the nature of self-interest, can hardly be changed. Therefore, specific operation rules need to mobilize individuals’ enthusiasm through incentive mechanisms such as market to realize incentive compatibility so as to make individuals who even pursue their own interests to the things the mechanism designer wants to do. Reputation and integrity under the market incentive mechanism are also one kind of punishment incentive mechanism. Integrity is essential in doing business, which does not mean that business owners perform integrity out of their own will, but means they have no other choice because otherwise, they will be forced out from the market. Besides, integrity can save economic costs and lower transaction costs.

Institutional arrangements that “guide with interests” are similar to the thoughts of Taoism in ancient China. Individuals are all self-interested and have limited ideological realm, the fact of which should always be fully recognized and respected. It is not difficult to stop “talking” about interests, but in reality, it is hard not to “value” interests. Taoism believed that “the law of nature is beneficial but not harmful” and emphasized compliance with natural tendency and governance by non-interference; however, they neglected two necessary conditions for governing by non-interference, namely, the basic institution and the role of government.

1.5.3 Social Norms

Social norms are informal institutional arrangements that are implemented by neither forces nor incentives. Solving problems with mandatory laws and inducing incentives in the long term will become a kind of social norms, beliefs, and culture that requires neither enforcement nor incentive, such as corporate culture, folk custom, religious faith, ideology, and concepts. This is the way of the minimum transition cost. Especially when people shared the same concepts, problems will be much easier to be resolved and the working efficiency will be greatly improved. Otherwise, even if one problem is solved through mandatory command, inducing incentive
mechanism, or relations, there will always be new problems to be appear in the same way, which will cause large implementation cost. Chapter 7 of this book will discuss the important role of social norms in stimulating voluntary cooperation among individuals.

Even so, social norms that preach morality and “persuade with emotions” are still largely limited by the reality of individuals’ ideological height. Relying on improvement of humanity, social norms lack the power of constraint and have limited scope of governance. This kind of institutional arrangements are similar to the thought of Confucianism in ancient China. The thought of “rule of morality” in Confucianism over-emphasized ethical relations among people but deliberately overlooked the relation of economic interests. It was successful in managing a family but biased in the governance of a nation. Benevolence and morality can be dominant, or at least considerably important in a family or a small group but may not be very effective in the face of general people. Benevolence is highly personal, the effect of which will weaken as with the enlargement of the realm. Those who rely on others’ benevolence to acquire necessities of life cannot be satisfied in most cases. As there are differences between beggars and ordinary people, though people may all need help from others, they cannot count on mere benevolence for everything. Therefore, the experience of managing a family cannot be simply popularized to all economic and social activities, otherwise it may results in many problems and even disastrous outcomes. Especially under the environment of modern market economy and the circumstance of limited ideological height of people, reliance on only internal ethical norms and absence of external laws and incentive mechanisms will make market economy slide towards a bad situation.

1.5.4 Complementarity and Hierarchy Structure of the Three Institutional Arrangements

These three institutional arrangements all have two sides, positive and negative. They also have different functions, ranges of application, and limitations. As emphasis on only one of them will have serious negative consequences, it is required for the three of them to play their own roles and complement each other. Actually, if making friends on interests, the friendship will fall apart when the interests are exhausted; if making friends with influence, the friendship will fall apart when the influence fails; if making friends with powers, the friendship will fall apart when their powers are lost; if making friends out of love, one will be hurt when the love ends; only when one makes friends with true heart can the friendship last forever. Making friends with true heart is the best but very difficult, for even couples may not be able to do that.

It should be pointed out that, even though the book mainly discusses the problem of economic incentives, the regulatory governance (or in-
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...stitution) is still the most basic and fundamental in the three kinds of arrangements for it establishes the most basic institutional environment, has strong positive and negative externalities, and determines whether the role of government is appropriate or not, thereby determining the effect of incentive mechanism design and the formation of good or bad social norms. In addition, for the formulation of institutional arrangements of both regulatory governance and incentive mechanism, the principle should not mean to and also basically can not change the self-interested nature of human beings. Instead, it should make use of individuals’ unchangeable self-interestedness to guide them to do something beneficial to the society. The design of an institution should conform to the self-interested nature of individuals rather than trying to change it. Individuals’ self-interestedness cannot be simply judged as good or evil, but the key lies in what kinds of institutions are used and towards what direction it is guided. Different institutional arrangements will result in individuals’ different responses to incentives and different trade-off choices, thereby leading to very different consequences. As Deng Xiaoping put forward, “good institutions can make bad people unable to have wilful acts, while bad institutions can make good people unable to do good things and even go to the opposite side.”

Putting it in plain language, bad institutions can turn people into demons, but good institutions can even turn demons into people. Therefore, the use of “emotion, reason, and interest” should be synthesized and vary with individuals, cause, place, and time to analyze and solve problems case by case. The criterion for deciding which to use is determined by the importance of regulations, the degree of information symmetry, and the cost of supervision and law enforcement. All in all, the three institutional arrangements all have their boundary conditions. To "enlighten with reason" should depend on the availability of information symmetry and difficulty of legal supervision. The law will be meaningless if its cost of supervision and execution is too high to enforce it.

To sum up this and the previous sections, a well-functioning market needs government, market, and society all in the right place so that the three-dimensional structure of state governance can be effectively interconnected and integrated. Defining the boundaries between government and market and between government and society involves two levels. The first level is defining the boundaries. We should firstly know the appropriated boundaries among them. The prerequisite for efficient market and normal society is to build a limited government that is capable, accountable, effective and welfare-enhancing, so the reasonable position of government is of vital importance. The principle here is that the market should be allowed to do what it can do, while the government should do what the

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market can not do or can not do well, and so the function of government can be generalized as maintenance and service. The second level is the identification of priorities. So, which is the key? The answer is institution. After knowing the boundaries of the three, we then need to sort them out. Who is the one to sort them out? The answer is government. Government, market, and society respectively correspond to the three kinds of basic arrangements, namely, governance, incentive, and social norm. Then, who is the one to regulate the position of government? It must be the rule of law that mainly restricts the government, but not the rule by law that mainly restricts people. Regulatory governance (or institution) is the most important and foundational arrangement, which lays down the most basic institutional environment, has great positive or negative externalities, determines whether the position of government is appropriate, and hence determines the effect of the incentive mechanism design and the formation of good or bad social norms. However, is government willing to limit its power? Normally speaking, of course not. Therefore, the power needs further partition and separate the responsibilities and power of administrative department, law making department, and judicial department.

Therefore, the underlying governance system is of determinant importance. Only when the governance boundaries between government and market and between government and society are reasonably defined through comprehensive governance by the three dimensions of institution, the rule of law, and civil society that can regulate, restrict, and supervise governmental power, can problems of efficiency, social equity and justice be resolved, phenomenon of corruption and bribery be eradicated, and healthy relations among government, market, society, enterprises, and individuals be built, which means relations of benign interaction between all of them. When benign interaction is realized, the government can then strengthen the efficiency of the market by continuous enactment and enforcement of laws so as to truly promote the long-term peace and stability of a nation.

1.6 Ancient Chinese thoughts on Market

Many basic concepts of market economy and conclusions of economics, including the idea that commodity price is determined by market and the invisible hand of Adam Smith, have been stated in a remarkably profound way thousands of years ago. Most of fundamental ideas, core assumptions and basic conclusions of modern economics, such as behavior assumption of pursuing self-interest, economic freedom, governance by the invisible hand, social division of labor, the intrinsic relationship between national prosperity and individual wealth and between development and stability, and the relationship between government and market had all been discussed by ancient Chinese philosophers. Some examples are given as fol-
As early as over 3,000 years ago, JIANG Shang (also known as JIANG Ziya and JIANG Taigong, an ancient Chinese strategist and adviser) believed that “averting risks and pursuing interests” is the innate nature of human beings, that is to say, “All people resent death and enjoy life, welcome virtue and chase profits.” He put forward the people-centered thought of dialectical unity between wealth of the people and stability and strength of the state and the fundamental law of national governance by stating that “the state is not the property of one man but of all people. The man who shares interests with all men will win the state”, and provided the fundamental strategy of state governance, that is, the government should take the common interests, risks, welfare and livelihood of the populace as its own, so as to obtain an incentive-compatible result that the populace shares the same interests and risks with the government. JIANG Shang also gave an incisive answer to the relationship and priority of wealth of the state and wealth of the people, that is, “A kingly state makes its people rich, a ruling state makes lower-rank officials wealthy, a barely surviving state makes higher-rank officials and dignitary affluent, and an unprincipled, ill-governed state only makes itself prosperous.” His advice was accepted by King Wen of Zhou Dynasty then, who ordered to open the granary to help the poor and reduce taxes to enrich the people. The Western Zhou thus became a growing power.

Over 2,600 years ago, GUAN Zhong (a Legalist chancellor and reformer of the State of Qi in ancient China) had deep insights on many economic thoughts. The core of his economic thought was “the theory of self-interest”. In Guan Zi: Jinzang (On Maintaining Restraint), he explained social economic activities with individuals pursuing interests very vividly and profoundly: “All men pursue interests and avert harm. When doing business, merchants hasten on the way day and night and make light of traveling from afar because, for them, interests are on their way. When fishermen go fishing in the sea, though the sea is hundreds of meters deep, they sail against the current for hundreds of miles day and night because, for them, interests are in water. So as long as there is interest, people would climb the mountain regardless of its height and go to sea regardless of its depth. Therefore, if those who know well how to govern the origin of interest, people will naturally admire the state and settle. The governor does not need to push them to go or lead them to come. Without being bothered or disturbed, people will get rich in a natural course. It’s like a bird incubating eggs, the process of which is invisible and silent but the result is noticeable when it’s done.” This was basically a very vivid demonstration of Adam Smith’s “invisible hand”, only more than 2,000 years earlier than the latter. In his book Guan Zi, GUAN Zhong presented the law of demand by stating that “The devaluation comes from the excess, while the value from the scarcity”, and also drew the basic conclusion that individuals’ wealth leads to national stability, security, prosperity and power by saying that “Only at times of plenty will people observe the etiquette. Only when they are well-clad and fed will they have a sense
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of honor and shame.” He further pointed out that “State governance must start with enriching the people. When the people become well-off, the state will be easy to govern. If they are in poverty, the state will be hard to govern.” ··· “Usually, an orderly state is rolling in prosperity while a disorderly one is deep in poverty. So a king versed in ruling a state must give priority to making people wealthy over governance itself.” Besides, comprehensive governance is another essential point in Guan Zhong’s thought of state management. For example, with respect to vassal kings, Guan Zhong suggested “restraining them with interest, associating with them with trust, admonishing them with military power” so that vassal kings “wo not dare to defy the king and will accept his interest, trust his benevolence, and fear his military force.” It does not require much effort to find out that there are certain corresponding relations between “restraining them with interest, associating with them with trust, admonishing them with military power” mentioned here and the three institutional arrangements we have discussed above.

More than 2400 years ago, SUN Tzu (a military general, strategist and philosopher in ancient China)’s book entitled The Art of War is on military strategies and tactics, but its principles and ideas are very similar to the market behavior and decision-making of firms. The first chapter “Laying Plans” of the book coincide, to a large extent, with the basic analytical framework of modern economics and can be fully adopted in the context of accomplishing an endeavor. It can be the rule of accomplishing big goals, making right decisions, and winning competitions in governing a nation and managing an enterprise or organization. Meanwhile, he also gave the basic conclusion of information economics: it is possible to achieve the optimal outcome (“the best is first best”) only under complete information; under information asymmetry, we can at most obtain suboptimal outcome (“the best is second best”). Hence we have the well-known saying: “If you know the enemy and yourself, you will not endanger yourself in a hundred battles. If you know yourself but not the enemy, for every victory gained you will also suffer a defeat. If you know neither the enemy nor yourself, you will succumb in every battle.”

In the same period, a more remarkable fact was that LAO Tzu (a famous philosopher of ancient China, the founder of Taoism) presented the supreme law of comprehensive governance: “Govern the state with fairness, use tactics of surprise in war, and win the world by non-intervention.” (Chapter 57, Tao Te Ching) This is the essential way of governing a state or administering an organization, which can be abstracted in common parlance as being righteous in deed, flexible in practice and minimal in intervention so for the government to realize governing by non-intervention. Lao Tzu considered “Tao” as the invisible inner law of nature, while “Te” (meaning the inherent character, integrity, virtue) as the concrete embodiment of “Tao”. He deemed that the governance of state and people should follow the Way of Heaven (referring to the objective laws governing nature
or the manifestations of heavenly will), the Virtue of Earth and the Principle of Non-intervention by saying that “Man follows the Earth; Earth follows the Heaven; Heaven follows the Tao; The Tao follows what it is.” (Chapter 25, Tao Te Ching) In addition, he also pointed out that “Difficult tasks alwaysstem (and should be tackled) from easy parts, and great undertakings always start with small beginnings.” (Chapter 63, Tao Te Ching) That is to say, for whatever we do, success lies in details. All these statements mentioned above demonstrate that Lao Tzu’s thought of non-intervention does not mean “doing nothing” as is commonly regarded, which is a wrong interpretation of Lao Tzu’s real intention. The non-intervention discussed by Lao Tzu is a relative concept, which requires non-intervention in major aspects but action and care in specific aspects. In other words, we should never lose sight of the general goal and begin by tackling small practical problems at hand to take action.

Over 2,300 years ago, SHANG Yang (an important Chinese statesman) of the State of Qin used the example of hare to expound the utmost importance of establishing private property rights and how the clear well-defined property rights can “determine ownership and set disputes” and help establish the market order. He came to this conclusion 2,300 years earlier than Coase. In Shang Jun Shu (The Book of Lord Shang), Shang Yang wrote that “when a hare is running, one hundred men chase after it; this is not because the hare can be divided into one hundred shares, but its ownership is not determined yet. On the other side, hare traders fill the market, but thieves dare not take one because the ownership has been determined. For this reason, when ownership is not determined, sages like Yao, Shun, Yu, and Tang would also chase after it; when ownership is definite, even a greedy thief wo not dare to take it.” The hare is chased because people are motivated to strive for its ownership, and even sages would do the same. On the contrary, ownership of a captured hare on market is determined, so others cannot take it as they will.

About 2,100 years ago, SIMA Qian (a Chinese historian of the Han dynasty who is considered the father of Chinese historiography) made an incredible statement in his work Shiji: Huozhi Liezhuan (Records of the Grand Historian: Biographies of Merchants), “Jostling and joyous, the whole world comes after interest; racing and rioting, after interest the whole world goes,” which succeeded Guan Zhong’s thought of self-interest; besides, he also demonstrated the economic thought of achieving social welfare through social division of labor based on self-interest, which is similar to that of Adam Smith. SIMA Qian investigated the development of social and economic life and realized the importance of social division of labor. He wrote that “All these goods are what people want, which are the necessities for life.” Thus, “People rely on the farmer for food, the planter for wood, the craftsman for utensil, and the businessman for circulation.” Moreover, he believed that the whole social economy composed of agriculture, forestry, industry, and commerce should develop in a natural course without the constraint of administrative orders.
Also in the Biographies of Merchants, SIMA Qian continued to write that “Are there any orders and instructions to mobilize or assembly the people periodically? Each man makes his efforts to satisfy his own needs based on his own abilities. People seek for purchasing the cheap and selling the expensive. With diligence and commitment, each man delights in his own business, like water flowing downwards ceaselessly. People gather together on their own initiative and produce various goods without any orders. Is not it the proof of compliance with the law of nature?”

In addition, SIMA Qian’s thought is also sparkling with wisdom in the philosophy of government governance, the importance of economic freedom, and the order of priority of several basic institutional arrangements. Sima Qian gave a very incisive conclusion in the Biographies of Merchants, “The master way is to follow the natural course and not to intervene, to guide with interests comes second, to teach with moral comes third, to rule by regulations comes next, and the last (worse) is that government competes for interests with people.”

Confucius affirmed that the pursuit of personal material interests on the premise of social ethics is justifiable. He said, “It is a shame for one to be poor and obscure when the moral way is holding its sway in a state; it is a shame for one to be rich and honored when chaos reigns in a state.” (The Analects: Tai Bo) By saying so, Confucius encouraged people to pursue decent material wealth. The Analects also recorded Confucius’s compliments on his disciple Zigong (Duanmu Ci), who was a merchant. In The Analects: Xianjin, it recorded that “Confucius said, ‘Hui, who is nearly attained to perfect virtue, is often in want. Ci, who does not acquiesce in his destiny and has taken to trading, is often accurate in judgments.’” Here, Confucius compared Yan Hui, his favorite disciple, with Zigong: the former was almost perfect in morality but often lived in poverty, which did not seem to be the right way of living, while the latter who did not follow the arrangement of destiny and went into business turned out excellent in predicting the market.

These ancient Chinese economic thoughts are extremely profound. What Adam Smith discussed had already been addressed by ancient Chinese philosophers much earlier. Yet as those ancient Chinese statements were just summaries of experience, they did not form rigorous scientific subjects or disciplines, they did not provide boundary conditions and scopes for conclusions, or make logically inherent analyses. As a result, little is known to the outside world.

In remaining parts of this chapter, we will have a rough discussion on core assumptions, key points, analytical frameworks, and research methodology of modern economics to help you understand the rigorous analysis of the content covered by this book.
1.7 A Cornerstone Assumption in Economics

Every subject of social sciences imposes assumptions on individual behavior as the logical starting point of its theoretical system, so here we will expound this in more detail. As discussed above, the essential distinction of social science and natural science is that the former studies individuals’ behavior and needs to make some basic assumptions on individuals’ behavior whereas the latter studies the natural environments and objects. Economics is a subject for it not only studies and explains economic phenomena and does positive analyses but also studies individual behavior so as to make better predictions and value judgments.

1.7.1 Selfishness, Self-love, and Self-interest

When talking about individuals’ behavior, three words are often involved: self-love, selfish, and self-interest, the three of which are related with and also different from each other. Self-love means one’s esteem and affection for oneself, which can be positive as it encourages individuals to keep themselves pure on the one hand and can also be negative for it may lead to massive ego and even self-harm or self-deceit. Self-love can also generate self-interestedness and even selfishness.

Selfish means benefiting oneself on the premise of harming others. Being selfish makes one greedy; being greedy makes one ambitious; being ambitious makes one vain and arrogant; being vain makes individuals lose themselves, while being arrogant may even make individuals ruthless and offensive.

Self-interest means benefiting oneself at the price of benefiting others. So, self-interest makes one reasonable and rational but selfishness only breeds greed. In other words, it is altruism for the sake of self-interest. To pursue and obtain interest, individuals have to choose altruistic behavior. This is the self-interest assumption adopted in the study of economics. When self-love and self-interest complement one another, individuals will have clear self-knowledge; when self-love is related with selfishness, individuals will go through moral decline. So, when individuals are dominated by self-love, it does not necessarily mean they will disregard others because self-love may also be combined with self-interest.

1.7.2 Practical Rationality of Self-interested Behavior

In this book, especially in the proof that market economic mechanism leads to optimal resource allocation, a key assumption is that individual behavior is driven by self-interest in normal situations, which is actually the most basic assumption in economics. This is not only an assumption but also the biggest reality as well as the cornerstone of the modern economics.
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This assumption also applies to the handling of relations among nations, units, households, and individuals, being an objective reality or constraint that must be considered when studying and solving political, social, and economic problems. For example, when dealing with the relation between two nations, as a citizen, one must protect the interests of his own nation and speak and act from the standpoint of the nation, and may be sentenced to penalties if he divulges state secrets; when dealing with the relation between enterprises, as an employee, one must protect the interests of his organization and if he leaks firm secrets to competitors, he may also be sentenced according to the consequence. The self-interest assumption is often questioned by this saying: why are there families if individuals are rationally self-interested and pursue personal interests? In fact, when the question comes to family, individuals act in the interest of their own families. That is to say, under normal circumstances, individuals care about their own families instead of others’. The discussion on relation between individuals also follows the same reasoning. In practice, many have misunderstandings of this assumption and interpret it in a simple and narrow sense as an assumption about individual persons in every case.

It is necessary to assume that individuals are self-interested because it conforms to the basic reality, and more importantly, the risk is minimal: even if self-interest assumption is wrong, it will not lead to serious consequences; on the contrary, if one adopts altruistic behavior assumption, once it proves wrong, the consequences will be much more serious than the former situation. In fact, the rules of game adopted under the self-interest assumption also apply to altruistic individuals in most cases, and institutional arrangements or game rules and individuals’ trade-off under altruistic behavior assumption are much simpler. However, once altruistic behavior assumption proves wrong, the consequences will be much more severe than those of incorrect self-interest assumption and may even be disastrous. Actually the assumption is also very important for making right judgment on individual behavior in daily life. Just imagine the costs incurred if a selfish, cunning person is mistaken for a simple, selfless, and honest person and even trusted with important responsibilities, you will understand the serious consequences of the wrong assumption of altruistic behavior.

Acknowledging that individuals are self-interested shows a realistic and responsible attitude towards solving social and economic problems. This is why we need disciplines of the laws of the state to prevent opportunists from taking advantage of loopholes in the institutions under altruism assumption. On the contrary, if altruism is used as the premise to solve social and economic problems, the consequence can be disastrous. For example, in the organization of production, if we deny individuals’ self-interest and only motivate individuals by emphasizing contribution to the nation and group, the result would be that everyone wants to take advantage of the
institutions to get an equal share and wishes to benefit from others’ contribution, and consequently how can a nation can be rich?

1.7.3 Self-interest Vas Altruism

It should be specially noted that although the self-interest assumption holds true in most cases, it also has its application boundaries. Under abnormal circumstances like natural or man-made disasters, war, earthquake, and others’ crisis, individuals will often demonstrate their altruistic and selfless character, make sacrifices to fight for the nation, and help others in crisis. This is another form of rationality, i.e., altruism, with which individuals are willing to sacrifice their lives (even animals have the instinct), otherwise extreme individualism or egoism may arise. For example, when a nation was invaded by another nation, many citizens are willing to sacrifice their lives for protecting their nation. However, under normal and peaceful environment, when engaged in economic activities, individuals often pursue their own interests. All these demonstrate that self-interest and altruism are not at all contrary to each other but only natural responses to different situations and environments.

Thus, we can see self-interest and altruism as relative terms. In fact, such duality can also be seen in animals. For example, when the wild goats were chased to the edge of a cliff, the old goats would sacrifice themselves by making the first jump, so that the younger goats or goatlings could jump on them and have a chance of running away. Adam Smith not only wrote the foundational work of The Wealth of Nations, but also wrote The Theory of Moral Sentiments to argue that individuals should have sympathy and a sense of justice. These two works complement each other in the academic thought system of Adam Smith. It is indeed so. Under the reality of individual’s self-love and self-interest, morality should be a kind of balance, an equilibrium outcome, and a convention realized through social division of labor and cooperation. Under the guidance of proper institutions, individuals voluntarily divide the work and cooperate so as to build a harmonious, civilized, stable, and orderly society. It is against human nature to regard self-interest as an opposition of moral, or is biased and wrong to regard self-interest as being equal to selfishness. On the contrary, the organic combination of moral ethics and self-interest can actually promote social civilization and individual’s decency. The biggest advantage of modern market relies on its utilization of the power of self-interest to counteract the weakness of benevolence so that those obscure hard-workers can also be satisfied. Therefore, we should not neglect the role of benevolence and moral in the formation of market system. Social progress cannot be relied on those who always want to hurt the interest of others.

All in all, self-interested individuals can be benevolent, altruistic, and moral. “Self-interest” should not be “at others’ cost”. There are limitation-
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s and boundaries for self-interest and altruism, but selfishness that benefits oneself at others’ cost is the origin of evil and greed. Rationally self-interested behavior will conform to social norms as a necessary constraint. We agree to educate individuals to pursue personal interest without violating public order and to protect public interest built upon the basis of individual rationality, but disagree with the economic idealism that bases policies on the ignorance of personal interest and violation of reasonable personal interest in the name of defending public interest. In short, the self-interested behavior under constraints of laws and regulations must be distinguished from selfishness in violation of laws and regulations that hurt others’ interests, the former of which should be protected while the latter should be opposed.

Even with self-interested behavior, there is a difference in the extent. Under ideal situations, the less self-interested individuals are, the better. However, it is also impossible to eliminate self-interest completely. We can safely say that self-interest is the logical starting point for economics. If all human beings are unselfish and always considerate to others, then economics involving human behavior will turn out useless so that industrial engineering or input-output analysis might be enough. It is out of the consideration that self-interest is an objective reality and individuals tend to pursue their own interest when involving in economic activities that China carries out reform and opening-up and makes the transition from planned economy to market economy.

1.8 Key Points in Modern Economics

Economists usually base on some of the following basic assumptions, constraints, axioms or principles when studying economic issues:

(1) Scarcity of resources;
(2) Information asymmetry and decentralized decision-making:
   individuals prefer decentralized decision-making;
(3) Economic freedom: voluntary cooperation and voluntary exchange;
(4) Decision-making under constraints;
(5) Incentive compatibility: the system or economic mechanism should solve the problem of conflict interests among individuals or economic units, i.e., induce individuals’ incentive to do what you want them to do;
(6) Well-defined property rights;
(7) Equity in opportunity;
(8) Allocative efficiency of resources.
Relaxing any of these assumptions and constraints may result in different conclusions. The consideration and application of these assumptions, constraints and principles are also useful for people to deal with daily affairs. Although they seem to be simple, it is not easy to thoroughly understand and skillfully use them in reality. In the following, we briefly discuss these key points, conditions, axioms, and principles, respectively.

1.8.1 Scarcity and Limitation of Resources

Economics stems from the fact that resources are limited in the world (at least the mass of earth is finite). As long as an individual is self-interested and his material desires are infinite (the more he has, the better), it is impossible to realize distribution according to wants, and the problem of how to use limited resources to meet the wants has to be addressed. Hence we need economics.

1.8.2 Information Asymmetry and Decentralized Decision-making

In addition to the biggest objective reality of self-interested nature of individuals, another fundamental objective reality is that in most cases, information is private or asymmetric among economic agents so that the effect of institutional arrangements adopted may be inadvertently neutralized. These are why economic problems are difficult to solve. For example, a person’s words could be high-sounding, but it is hard to tell whether they truly mean it; listeners seems to concentrate, but you do not know if the message is well-received by them. This is what we mean when we say that a person may “say one thing but means another”, “one’s inner world is unpredictable”, and “human bing is the most difficult to deal with”. The fundamental reason for such phenomena is private information. Private information together with the self-interested nature of individuals can often lead to conflicts of interests among economic agents. If there is no proper governance system to reconcile, in order to obtain a variety of limited resources, it may cause ones to speak and do things “false, big, empty” void prevailing, slogans and reality are often disconnected, likely become the norm. This is why social sciences, especially economics, are more complex and difficult to study than natural sciences. Also due to information asymmetry, centralized decision-making is often inefficient, while decentralized decision-making, such as the use of market mechanism, is required to solve economic problems.

Only when complete information is acquired can the outcome be the first best. However, private information is often hard to obtain, so incentive mechanism is needed to induce truthful information. Information acquisition incurs costs; as such, we can only obtain second-best outcome in most situations. This is a basic conclusion in the principal-agent theory,
optimal contract theory, and optimal mechanism design theory that will be discussed in Part VI of this book. In most circumstances, information is asymmetric, the best we can do is the second-best. Without reasonable institutional arrangements, there will be incentive distortion where inducing information inevitably incurs costs; therefore, the outcome cannot be first-best. Thus, it is particularly important to make information symmetry, without which many misunderstandings may arise. By communicating with others, you let others understand you (signaling) and also get to know others (screening) so as to make information more symmetry, clear up misunderstandings and reach consensus, which is the fundamental premise of obtaining desired outcomes.

Excessive intervention of government in economic activities and the over-playing of government role will lead to low efficiency, which, in root, is caused by information asymmetry. There are many problems in information acquisition and discrimination of the government. If decision-makers are able to have all related information, centralized decision-making featuring direct control would not be problematic, and it would be a simple question of optimal decision-making. However, it is impossible for decision-makers to have all related information at hand. That is why individuals prefer decentralized decision-making. That is also why economists call for the design of various incentive mechanisms, a decentralized decision-making method featuring indirect control, should be used to stimulate individuals to do as decision-makers desire, or to achieve the goals decision-makers aspire for. We will focus on the issue of information and incentive in Parts VI and VII.

It is worth mentioning that centralized decision-making also has its advantages in some aspects, especially in the decision-making regarding quick major changes. For instance, centralized decision-making is more efficient when a nation, unit or an enterprise is laying down the vision, orientation, and strategies or making big decisions. However, such major changes might bring about huge success or severe mistakes. For example, the decision of adopting the reform and opening-up policy has led to rapid development of the Chinese economy. In contrast, the decision of “Cultural Revolution” almost pushed the Chinese economy to the verge of collapse. One solution to this problem is to give public opinion full respect and select great leaders.

1.8.3 Economic Freedom and Voluntary Exchange

Because of economic agents’ pursuit of their own interests and information asymmetry, institutional arrangements of the mandatory “stick” style are often not effective. So, we need to give individuals more freedom of economic choice, which is the most important right among the three private rights (right to survival, freedom of choice for one to pursue happiness, and
private property right). Thus, we should mobilize economic agents with economic freedom based on voluntary cooperation and exchange through inducing incentive mechanism such as market. Therefore, the freedom of economic choice ("deregulation") plays a vital role in market mechanism with decentralized decision-making ("decentralization"), being a prerequisite for the normal operation of market mechanism and also a fundamental precondition to ensure efficient allocation of resources under competitive market mechanism.

In fact, the Economic Core Theorem to be discussed in Chapter 12 reveals that once full economic freedom is given and free competition, voluntary cooperation and exchange are allowed, even without any institutional arrangement in advance, the outcome of resource allocation driven by the self-interested behavior of individuals will be theoretically consistent with equilibrium allocation of a perfectly competitive market. The essence of the Economic Core Theorem can be summarized as follows: under the rationality assumption, as long as economic freedom and competition are given even if institutional arrangement is not considered, the economic core obtained will be a competitive market equilibrium.

China’s reform and opening up over the past 40 years have proved this theme in practice. When analyzing the reasons for China’s remarkable economic achievements, despite any other crucial issues, the critical key is to give individuals more freedom of economic choice. Reform practice from rural to urban areas indicates that wherever there are looser policies and a greater degree of economic freedom provided for producers and consumers, there will be higher levels of economic efficiency. So-called China’s miraculous economic growth actually stems from the government’s delegation of powers to the market, while its imperfect market today has actually been a result of excessive government intervention and inadequate or inappropriate government regulation and institutional arrangements.

1.8.4 Constraints and Feasible Choices

Doing things under constraints is one of the most fundamental principles in economics, as the saying goes that ones have to bow under the eaves. Everything has its objective constraints, i.e., individuals make trade-off choice under existing constraints. Individuals’ choice is determined by objective constraints and subjective preferences. Constraints include material constraint, information constraint, and incentive constraint, which make it difficult for economic agents to achieve their goals. In economics, one embodiment of the basic idea of constraints is the budget constraint line of consumer theory as we will discuss in Chapter 3, which states that an individual’s budget is constrained by the prices of the commodities and his income. For an enterprise, constraints include available technologies and prices of inputs, under which, the goal of maximum profit requires firms to
determine the quantity of production, technologies to be adopted, quantity of each input, even pricing for products, response to competitor’s decision-making, etc. The development of a person or even a nation is faced with various constraints, including political, social, cultural, environmental, and resource constraints. If we do not make constraint conditions clear, it is hard to have things done.

When introducing a reform measure or an institutional arrangement, it is a must to consider feasibility and meet the objective constraints; meanwhile, the implementation risk is expected to be reduced to the minimum so that social, political and economic turmoil will not be caused. Therefore, feasibility is a necessary condition to judge whether a reform measure or institutional arrangement is conducive to economic development and the smooth transformation of the economic system. In a nation’s economic transformation, to have an institutional arrangement feasible, it must conform to the institutional environment of the specific stage of the country’s development.

Participation constraint is very important when considering incentive mechanism design, which means that an economic agent can benefit, or at least shall not suffer losses, from economic activities, otherwise he will not participate, or even oppose the rules or policies to be implemented. Individuals who pursue the maximization of self-interest will not automatically accept an institutional arrangement but will make a choice between acceptance and refusal. Only under an institutional arrangement where the individual’s benefit is not less than his reservation benefit (or he does not accept the arrangement) will the individual be willing to work, product, trade, distribute, and consume. If a reform measure or an institutional arrangement does not meet the participation constraint, individuals may give up. If everyone is reluctant to accept the reform measure or institutional arrangement, it cannot be successfully implemented. Mandatory reform may arouse opposition and cause social instability so that development will not be possible. Therefore, participation constraint is closely related to social stability and is a basic judgment of social stability in development.

1.8.5 Incentive and Incentive Compatibility

Incentive is one of the core concepts in economics. Each individual has his own self-interest; to obtain interests from some activity, one must also pay the cost. By the comparison between benefits and costs, individuals may be willing (have incentive) to get something done or well done, or be reluctant or unwilling to get it done or well done, and thus will have rational incentive response to the rules of game. However, it often leads to incentive-incompatible conflicts of interests among individuals or between individuals and society and brings about chaos. The thought of incentive compatibility has been discussed by Adam Smith in *The Theory of Moral*
Sentiments, as he stated that “...in the great chess-board of human society, every single piece has a principle of motion of its own, altogether different from that which the legislature might choose to impress upon it. If those two principles coincide and act in the same direction, the game of human society will go on easily and harmoniously, and is very likely to be happy and successful. If they are opposite or different, the game will go on miserably, and the society must be at all times in the highest degree of disorder.” The reason is that under given institutional arrangements or rules of game, individuals will make optimal choices according to their own interests, but such choice will not automatically satisfy interests or goals of others and society, and meanwhile information asymmetry makes it difficult to implement social optimum by command.

A good institutional arrangement or rule is able to guide self-interested individuals to act subjectively for themselves but objectively for others, making individuals’ social and economic behavior beneficial to the nation and the individuals as well as to themselves. This is the core content of modern economics.

Everything an individual does involves interests and costs (benefits and costs), making incentive an ubiquitous issue to deal with in daily work and life. As long as the benefits and costs are not equal, there will be different incentive reactions. To maximize profits, an enterprise has the incentive to use resources in the most efficient way and provide incentives to guide employees to make the greatest effort. Outside the enterprise, changes of profits provide an incentive for resource holders to change their ways of using the resource; inside the enterprise, incentive influences the way of using the resource and the effort employees put in work. To make management efficient, one must be clear about the role of incentive in organizations and how to construct incentive to guide subordinates to exert the greatest effort in work.

Since the interests of individuals, society, and economic organizations cannot be completely the same, how can self-interest, mutual benefits, and social interests be organically combined? It then requires incentive compatibility, with which the reform measures and institutional arrangements adopted can greatly mobilize individuals’ enthusiasm for production and work. Therefore, to implement a goal of one’s own or of the society, proper rule of game shall be given, under which, when individuals involved pursue self-interest, the goal can also be achieved. That is to say, it unifies the self-interest of individuals and mutual benefits among individuals so that when pursuing self-interest, each individual can help attain the goal intended by the society or other individuals. We will focus on the issue of how to achieve incentive compatibility in Parts VI-NII of the book.

*Adam Smith: The Theory of Moral Sentiments.*
1.8.6 Property Rights as a Incentive Scheme

Property rights are an important component of market economy. In the previous section about the ancient Chinese thoughts on market, we mentioned that over 2,300 years ago, Shang Yang of the State of Qin used the example of hare to expound the utmost importance of establishing private property rights and how the clear definition of property rights can “determine ownership and set disputes” and help establish the market order.

Property rights include ownership rights, using rights, and decision rights of property. A clear definition of property rights will help clearly define the attribution of profits, thus providing incentives for property owners to consume and produce in the most efficient way, to provide quality products and good service, to build reputation and credibility, and to maintain their own commodities, housing, and equipment. If property rights are not clearly defined, the enterprises’ enthusiasm will be harmed, giving rise to incentive distortion and moral risk. For example, unclear property rights in state-owned enterprises will surely cause low efficiency, bring about ubiquitous corruption like rent-seeking or interest transfer, squeeze private economy, be detrimental to innovation, and lead to unfair competition. In the market mechanism, incentives are given to individuals mainly in forms of property possession and profit acquisition. The Coase Theorem to be discussed in Chapter 14 of the book is a benchmark theorem in property rights theory, which claims that when there is neither transaction cost nor income effect, as long as property rights are clearly defined, an efficient allocation of resources can be achieved through voluntary coordination and cooperation.

1.8.7 Equity in Outcome and in Opportunity

“Equity in outcome” is a goal that an ideal society wants to achieve. However, for human society with self-interested behavior, this “outcome equity” often brings low efficiency. So, in what sense can equity be consistent with economic efficiency? The answer is that if ones use “equity in opportunity” as the value judgment standard, fairness and efficiency can be consistent. “Equity in opportunity” means that there should not be any barrier to hinder individuals from pursuing their goals in their capacity, and there should be a starting point of equal competition for every individual. The Fairness Theorem to be introduced in Chapter 12 tells us that as long as individuals’ initial endowments are of equal value, through the operation of competitive market, allocation of resources that is not only efficient but also equitable can be achieved even if individuals pursue self-interest. A concept similar to “equity in opportunity” is “individual equality” (also known as, “All men are created equal”), which means that, although individuals are born with different values, genders, physical conditions, cul-
tural backgrounds, capacities, ways of life, “individual equality” requires due respect for such individual differences.

As individuals have different preferences, the seemingly equal distribution of milk and bread may not satisfy everyone. Therefore, in addition to equal allocation that defines equity as absolute equalitarianism, the concept of equity in other senses is also used in the discussion of economic issues. For instance, equitable allocation to be introduced in Chapter 12 considers both subjective and objective factors, which means that everyone is satisfied with their own shares.

\subsection{1.8.8 Allocative Efficiency of Resources}

Whether resources are efficiently allocated is a basic criterion to evaluate the quality of an economic system. In economics, efficient allocation of resources usually refers to Pareto efficiency or optimality, which means that there does not exist other feasible allocation such that at least someone is better off without hurting any others. As such, it requires not only efficient consumption and production but also production of products that can best meet needs of consumers.

It may be remarked that, when it comes to economic efficiency, we should distinguish between three types of efficiency: firm’s production efficiency, industrial production efficiency, and allocation efficiency of an economy. By saying that a firm’s production is efficient, we mean that with a given input, output is maximum, or, with a given output, its inputs are minimum. Industry is the aggregation of all firms’ production for a commodity and the efficiency of which can be similarly defined. It should be noted that the efficiency of a firm does not imply the efficiency of an industry. The reason is that if the production materials of firms with outdated technologies are given to those firms with advanced technologies, there will be more outputs for the whole industry. At the same time, even if production of the whole industry is efficient, the allocation of resources may not be (Pareto) efficient.

The concept of Pareto efficiency in allocating resources is applicable to any economic institution. It just provides a basic criterion of value judgment for an economic institution from the viewpoint of social benefit and makes evaluation on the economic effect from the perspective of feasibility. It can be applied to planned economy, market economy, or mixed economy. The First Theorem of Fundamental Welfare Economics to be introduced in Chapter 11 proves that when individuals pursue their own self-interest, a perfectly competitive market will lead to efficient allocation of resources.
1.9 Understanding Modern Economics Properly

A proper and profound understanding of modern economics can help individuals correctly use basic principles and analytical methods of economics to study various economic issues under different economic environments, behavioral assumptions, and institutional arrangements. The different schools and theories of modern economics per se show the speciality, universality, and generality of the analytical framework and methodologies of modern economics. Under different economic environments, different assumptions and specific models will surely be required. Only in this way will the theory developed be able to explain different economic phenomena and individuals’ economic behavior, and more importantly, make logically inherent analyses, draw conclusions of inherent logic, or make scientific predictions and reasoning under various economic environments that are close to the theoretical assumptions. However, because of the complexity of economic environments, for the sake of semantic and logic clarity, modern economics uses various rigorous mathematical tools to build economic models so as to develop economic theory. Rigorous mathematical tools are often difficult to master, which results in frequent misunderstandings and criticisms on modern economics. In addition to the common misunderstandings of benchmark theories as discussed before, there are misunderstandings about modern economics concerning the following several aspects.

1.9.1 How to Regard the Scientific Nature of Modern Economics

One of the main misunderstandings is that economics is not science and includes many seemingly conflicting or contradicting theories. Criticisms are often heard that there exist too many different economic theories coming from different economic schools in modern economics, making it difficult to figure out which is right and which is wrong. People who hold such opinion fail to understand that all economic theories do not break away from the two basic categories of theory as discussed before. It is precisely because of the complexity of objective reality, different economic, political, social and cultural environments in different countries and regions, different thoughts and preferences of people, and different economic goals that people pursue, that different theoretical economic models and economic institutional arrangements need to be developed accordingly.

It is easy to understand that different economic theories or models should be developed for different economic, social, and political environments, but difficult for many to comprehend why different economic theories are developed under the same economic environment. Thus, some individuals, while disavowing modern economics and its scientific characteristics, scorch economists by saying that 100 economists will have 101 opinions. In fact, it is who they do not realize that it’s like that we need maps for differ-
tent purposes such as traffic, travel, military, etc., though there is only one earth; under the same given economic environment, we will need to develop different economic theories and provide different economic institutional arrangements for different goals.

The fact that different opinions of economists will arise for the same problem just shows the precision and perfection of modern economics because when premise and environment changes, the conclusion should change accordingly, which is especially true in the case of the second category of economic theory that aims to solve practical problems. Besides, as different individuals will also have different subjective judgements of values, there is few universally correct conclusion that satisfies everyone and is feasible in all situations, otherwise we would not need case-by-case analysis according to time and place and adjust to changing circumstances. It is similar to the philosophical thoughts of using military forces in war and using medicine to treat diseases: medicines vary as diseases vary; when considering and solving economic problems, case-by-case analysis shall be done according to time, place, individual, and occasion. The difference is that as economics has great externalities as we’ve discussed before, while a charlatan may result in serious problems (even death) to patients with wrong medicines, the wrong prescription of economic policy will influence a larger group of people and even a nation.

Though there are different economic theories and models, either benchmark economic theory that provides benchmark or reference system or the second category of economic theory that aims to solve practical problems is definitely not different “Economics”. The basic analytical framework and methodologies of modern economics, just like those of mathematics, physics, chemistry, engineering, etc., are not bounded by regions or nations. The basic principles, methodologies, and analytical framework can be used to study a variety of economic issues under any economic environment and institution and to study the economic behavior and phenomena in a specific area and time period. The analytical framework and methodologies to be introduced later can be used to study and conduct comparative analysis on almost every economic phenomenon and issue. In fact, this is exactly in which the power and glamour of the analytical framework of modern economics lies: its essence and core require that the economic, political and social environment conditions at a specific time and place must be considered and clearly defined when doing research. Modern economics can be used to study economic issues and phenomena under human behavior as manifested in different nations and regions, customs and cultures. Its basic analytical framework and methodologies can also be used to study other social phenomena and human decision making. It is proven that because of the universality and generality of the analytical framework and methodologies of modern economics, in the past few decades, various analytical methods and theories have been extended to other disciplines like
political science, sociology, and humanities.

1.9.2 How to Regard the Mathematical Nature of Modern Economics

Besides the criticism that the assumptions of economic theory do not conform with reality and economics is not science, another common criticism is that economics pays too much attention to details and involves an increasing amount of mathematics, statistics, and models, making questions even more obscure and incomprehensible. However, the reason that economics uses much mathematics and statistics is for the sake of rigor and quantifiability of empirical studies. Though decision-makers in the administration and the general public do not need to understand details or premises of the rigorous theoretical analysis, it is a must for economists who put forward policy suggestions to know. As economic theory will generate great externalities once adopted, blind application without considering the premise may bring about serious problems and even disastrous consequences. That is why mathematics should be employed to rigorously define the boundary condition of a theory. Meanwhile, the application of a theory or enactment of a policy often requires tools of statistics and econometrics to carry out quantitative analysis or empirical test. Also, because the real economic society is too complicated, the use of mathematical models in economic theories can help abstract and depict the real economic world for people to deeply understand and comprehend problems to be solved in reality. We will further discuss the role of mathematics in modern economics later in this chapter.

1.9.3 How to Regard Economic Theory

Each theory or model in economics that aims to solve practical problems adopts the axiomatic analysis, which is composed of a set of presupposed assumptions on economic environments, behavior patterns, and institutions and conclusions based on these assumptions. Considering the complexity of economic environments and the diversity of individual preferences in reality, the more general the presupposed assumptions of a theory are, the more powerful the theory is. If the presupposed assumptions of a theory are too restrictive, the theory will lack generality and thus be less useful in reality. As economics is used to serve the society and the government, the theory must be of some breadth. So, a necessary requirement for a good theory is its generality: the more general it is, the larger explanatory power it will have and the more useful it will be. The general equilibrium theory that studies competitive market is such a theory. It proves that the existence of competitive equilibrium and leads to efficient resource alloca-
tion under the condition of very general preference relations and production technologies.

Even so, theories of social sciences, especially those of modern economics, like all theorems in mathematics, have their boundary conditions. As discussed before, because of the great externalities of economic theories, when discussing or applying an economic theory, we need to pay attention to its presupposed assumptions and application scope, for conclusions of any economic theory are not absolute but only hold true when the assumptions are satisfied. Whether this point is recognized when discussing economic issues is a basic judgment of whether an economist is well-trained. As economic issues are closely related to daily life, even ordinary people can give certain opinions about economic problems such as inflation, business climate, balance or imbalance of supply and demand, unemployment, stock market, and housing market. For this reason, many people do not regard economics as a science. Indeed, “economics” would not be a science if it did not take into account constraints or base on accurate data and rigorous theoretical logical analysis. A well-trained economist always discusses issues based on certain economic theories and is fully aware of the boundary conditions for the relationship among economic variables and the inherent logic of the conclusions. It is very important to fully understand the boundary conditions of economic theories, otherwise one will not be able to distinguish between the theory and the reality but instead be liable to go to two extremes: either simply applying the theory to reality regardless of constraints in reality, or totally denying the value of modern economic theory.

The first extreme viewpoint overestimates the role of theory and abuses theory. For example, some ones disregard the realistic objective constraints facing a nation, blindly or mechanically apply the two categories of theories in economics to solve the nation’s problem, and indiscriminately copy models to study the problem, thinking that the inclusion of mathematical models will make good papers and good theories. Conclusions and suggestions obtained by simply copying economic theories or models, in the case of no matter benchmark economic theory or the second category of economic theory that is more close to reality, for general application to the reality of nation without taking consideration of various constraints and boundary conditions created by the nation’s real situation and economic institutional environment will often lead to serious problems. In fact, no matter how general the theory and behavioral assumption are, they have application boundaries and limitations and thus should not be used indiscriminately. Especially for those theories developed based on an ideal state far from reality and mainly for purposes of establishing reference system, benchmark, and goal, we should not apply them directly; otherwise, we might get wrong conclusions. Without the sense of social responsibility or good training in economics, a person may overestimate the role of theo-
ry and blindly apply or misuse economic theory in real economy without taking the premise into account, which could bring severe consequences, affect social and economic development, and lead to huge negative externalities. For instance, the conclusion of first welfare theorem that competitive market leads to efficient allocation of resources is subject to a series of preconditions and its abuse will give rise to severe policy errors and serious damage to the real economy.

Another extreme viewpoint totally denies the role of theory. People holding such point of view underestimate and even deny the practical significance of modern economics, including its behavioral assumption, analytical framework, basic principles, and research methodology. In fact, just as the benchmark theory of modern economics, there is no discipline in the world all of whose assumptions and principles coincide with reality perfectly (like concepts of free fall without air resistance and fluid motion without friction in physics). We should not deny the scientific feature and usefulness of a discipline, and modern economics as well, just based on this. We learn modern economics to acquire not only its basic principles and usefulness but also the way of thinking, asking, and solving questions. As discussed before, the value of benchmark theories lies not in direct explanation of the reality but in provision of study platform and reference system for developing new theories to explain the real world. With these methods, economists can be enlightened on how to solve economic problems in reality. In addition, as discussed in the previous section, a theory applicable to one nation or region may not be applicable to another nation or region due to different environments. Instead of applying the theory mechanically and indiscriminately, we need to modify the original theory to develop new theories according to a given economic environment and individual behavior pattern.

Surely, there is another extreme viewpoint, who do not think that market failures may occur under any circumstance and that market has externalities. They thus deny the practical significance of economic theory, believing that modern economics is strongly hypothetical while these assumptions are not needed because market does not have boundaries.

Sometimes it is heard that a theory or conclusion was toppled. As not all the conditions of a theory are not in line with the reality, some think that the theory is incorrect and has been toppled. Generally speaking, the logic is not scientific or even wrong. Assumptions, even those in the second category of theories that aim to solve practical problems, cannot fully coincide with reality or cover every possible case. A theory may be applicable to the economic environment of one place but inapplicable to that of another nation or region. But, as long as there is no inherent logic error, we cannot say that the theory is wrong and needs to be overturned. We may only state that it is not applicable at a certain place or a certain time.

Besides, another common mistake is trying to draw a general theoret-
1.9. UNDERSTANDING MODERN ECONOMICS PROPERLY

ical conclusion just based on some specific examples, which is a mistake in methodology. Of course, here we do not deny the unique role of history, culture, and thought of each country in the establishment of their own discourse of economics.

1.9.4 How to Regard Experiments in Economics

Another criticism about economics is that it is not an experimental science at all and thus negate the scientific feature of economics. Such viewpoint is a misunderstanding. First of all, as with the rapid development of experimental economics in recent years, economics is using more and more lab experiment, field experiment, and computer simulation experiment to study various economic problems. Experimental economics also tests individual’s behavior and rationality of behavioral assumption through experiments and thus becomes an important tool to examine whether an economic theory fits the objective reality. Theorists have also obtained important information from experiments so as to promote the advancement of theories (There are discussions of many economists on how to understand experiments in economics on the website of Al Roth). What is more, experiments in economics have already walked out of laboratory toward field experiments (See relevant discussion by John List).

Indeed, from the empirical perspective, in real economic activities, experiments in economics are of indispensable advantage to verify policies and systems, especially for the need of institutional transition. After constant exploration by early scholars and the systematic synthesis of methodologies and tools of experiments in economics by Vernon Smith, winner of the 2002 Nobel Memorial Prize in Economic Sciences, experimental economics as an important empirical tool has received increasing attention in market mechanism design. When external environment changes rapidly and new technologies spring up like mushrooms, reform becomes an inevitable choice, but meanwhile, strategic risk and social cost of various policy suggestions and proposals have to be considered with caution. Therefore, it naturally becomes a difficult point and key in the institutional reform to find a way to comprehensively and thoroughly examine in advance problems that might arise out of new proposals on institution. For instance, in the earlier days of reform and open-up, China has taken various measures including “special economic zone policy”, “pioneering pilot scheme”, “typical example as the lead”, etc. Economic experiments are consistent with such measures in the guiding thought of lowering risk and cost of the reform, but there are still major differences in methodology between economic experiments and pilot experiments for experience accumulation. In comparison with the pilot method, firstly, the economic experiment is liable to focus on a single research question so that each experiment examines effects of only one policy and characteristics of only one mechanis-
Secondly, the economic experiment employs rather normal techniques and tools. There are multiple influential factors in real economic activities, while the methodology of economic experiments is able to control factors irrelevant to the research question so as to focus on the effect of one specific factor on certain economic phenomenon. Moreover, in comparison with the pilot method, it is less costly to conduct economic experiments. Besides, study on the relation between genetics and economic behavior is meaningful. Once the relation between the two is figured out, it will lay down the foundation for economics to become a discipline of science just as natural science.

Surely, we must also admit that quite a few economic theories, like general equilibrium theory for comprehensive analysis, cannot easily be tested by social experiments in case of policy mistakes that will bring about huge risks to economy and society. This is the greatest difference from natural sciences, as in natural sciences, natural phenomena and objects can be studied through experiments and theories can be tested and further developed in the laboratory. Astronomy might be the only exception, but it involves no individual behavior. Once that is involved, things will become more complicated. Moreover, extreme accuracy can be attained in the application of theories in natural sciences. For instance, in the construction of buildings or bridges and manufacturing of missiles or nuclear weapons, accuracy of any extent is attainable for all the parameters are controllable and the interrelationship between variables can be experimented. However, in economics, many factors affecting economic phenomena are uncontrollable.

Economists are often criticized for inaccurate economic forecasts. We can explain this from two perspectives. From the subjective perspective, it is due to the quality whether economists have systematic and rigorous training in economics. If not, they may be unable to figure out main causes of problems or make correct logical analyses and inferences when they discuss and try to solve economic problems and thus will prescribe wrong medicines (if such medicine exists) to economic problems. From the objective perspective, some economic factors that influence economic results may suddenly and uncontrollably change, thus making the prediction inaccurate, even though they are made by well-trained economists with good economic intuition and insight. An economic issue involves not only human behavior which might complicate the matter but also many other uncontrollable factors. Although an economist might be wise and insightful, his predictions are liable to deviation because these uncontrollable factors that will influence economic results may change. For instance, even for a good economist with sound judgment, his economic forecasts might become quite inaccurate once sudden changes take place in the economic, political or social environment. Some people may question that no matter how economics develops and what the reason is, inaccurate prediction is
the norm while accurate prediction is merely luck. It is true in principle, as economic fluctuation is random variable, making it impossible to make accurate prediction. However, since the probability for an event to happen and the quality of economists vary, a well-trained economist can better judge the probability of an event and be more likely to be accurate in their prediction, influenced by the subjective factor of inaccurate prediction as mentioned above.

How can the problem that economic theory cannot experiment on society under many circumstances be compensated? The answer is logically inherent analysis, based on which inherent conclusions and inferences can be drawn and comparisons and empirical data test can be done through horizontal and vertical perspectives of history. In this way, when conducting economic analysis or giving policy suggestions, according to the three dimensions of economic analysis as discussed before, firstly, there should be theoretical analysis of inherent logic to define applicable boundary conditions and scope. Meanwhile, tools of statistics, econometrics or experimental economics shall be utilized for empirical quantitative analysis or test, and the great perspective of history should be taken for vertical and horizontal comparative analysis. So, when conducting economic analysis or giving policy suggestions, there should be not only theoretical analysis of inherent logic, historical comparative analysis from the great perspective, but also empirical quantitative analysis based on data and statistics, the three of which are all indispensable. The three-in-one study methodology compensates to a large extent the problem that many economic theories cannot or cannot easily experiment on society.

The method of logically inherent analysis of economics is to fully understand and characterize first relevant circumstances (economic environment, situation, and status quo) for a problem to be solved to ascertain what the problem is and its causes, apply proper economic theories accordingly, draw conclusions, and make accurate forecasts and inferences. As long as the status quo accords with causes (economic environments and behavioral assumptions) presupposed in the economic model, logically inherent conclusions can be drawn according to economic theories, and thus solutions can be obtained or certain institutional arrangement can be suggested for different circumstances (varying with time, place, individual, and case). The method of logically inherent analysis can help make academic inferences on possible results under the circumstance where real economic and social environment, behavior pattern of economic agents, and economic institutional arrangement are given and thus provide guidance for solving real economic problems. In other words, once we make clear the problem and its causes and apply proper economic theory accordingly (like the prescription), if such theory exists, we can then use the right remedy of comprehensive treatment and draw logically inherent conclusions so as to make accurate predictions and inferences. Otherwise, severe consequences
might arise.

It is true that the result of an economic theory may not be tested by experiments on society under many circumstances, and data does not provide the sole basis for analysis. Practice is only the sole criterion for testing truth but not for predicting truth. So, it is the logically inherent analysis and historic comparison that should be relied upon, and thus theory becomes so important. Like a doctor prescribing for his patient or a mechanic repairing a car, the hardest thing is diagnosis of disease or cause of failure. The criterion for a good doctor lies in whether he can accurately find the true cause of the problem. Once the cause is identified and there is a remedy, it is relatively easier to prescribe for the patient, unless he is a complete charlatan. For economic problems, the prescription is economic theories. Once we truly understand the characteristics of economic environment, thoroughly investigate the situation, and reasonably assume individual’s behavior, we will likely increase the likelihood of success.

1.10 Basic Analytical Framework of Modern Economics

There are basic laws to follow for doing anything. The way that modern economics studies and solves problems is similar to how people deal with personal, household, economic, political and social affairs. As is known to all, in order to do something well and associate with others, the first thing is to understand national conditions and customs, i.e., to know the real environment and the conduct and personality of the persons to interact with; on such basis, one determines the way of dealing with them and affairs accordingly and makes incentive response after trading off advantages and disadvantages to obtain the best outcome; last, one makes value judgement on the choice and evaluates the rule of game that is used. The basic analytical framework and research methods of modern economics exactly follow this mode to study economic phenomena, human behavior, and how people weigh trade-offs and make decisions. Of course, a major difference between these two is the rigorous reasoning of modern economics which uses formal models to identify the logical relationship between presupposed assumptions and conclusions. Such analytical framework is of great generality and consistency.

A standard academic work needs to spell out first of all the problems to be studied and resolved or the economic phenomena to be explained. That is to say, economists should first identify research objectives and their significance, provide readers with information on the overview and progress of the issues under study through literature review, and illustrate the work’s innovation on technical analyses or theoretical conclusions. Then, they discuss how to address the issues raised and draw conclusions.
Although economic issues under study may be quite different, the basic analytical framework used is basically the same, which is an axiomatic approach to study economic problems. The analytical framework for a standard economic theory in modern economics consists of the following five parts or steps/components: (1) Specifying economic environments; (2) Making behavioral assumptions; (3) Setting institutional arrangements; (4) Determining equilibrium outcome; and (5) Making evaluations. Any economics paper written with clarity and logical consistency is basically made up of these five parts, especially the former four parts, no matter what the conclusion is and whether the author realizes it or not. So to speak, writing an economics paper is innovative writing with logical structure and analysis in such steps. Once you understand these components, you will grasp the basic writing pattern of academic economics papers and find it easier to learn modern economics. These five steps are also helpful for understanding economic theory and its proof, and finding research topics and doing research.

Before discussing the five components in detail, we should define first the term “institution”. Institution is usually defined as a set of rules related with social, political and economic activities that dominate and restrict the behavior of various social classes (Schultz, 1968; Ruttan, 1980; North, 1990). When people consider an issue, they always treat some factors as exogenously given variables or parameters while others as endogenous variables or dependent variables. These endogenous variables depend on the exogenous variables, and thus are functions of those exogenous variables. In line with the classification method of Davis-North (1971, pp 6-7) and the issue to be studied, institution can be divided into two categories: institutional environment and institutional arrangement. Institutional environment is the set of a series of basic economic, political, social and legal rules that form the basis for formulating production, exchange and distribution rules. Of these rules, the basic rules and policies that govern economic activities and property and contract rights constitute the economic institutional environment. Institutional arrangement is the set of rules that dominate the potential cooperation and competition among economic participants. It can be interpreted as the generally known rules of the game, with different rules of game leading to different incentive reactions of individuals. In the long run, institutional environment and institutional arrangement will affect each other and change. Yet in most cases, as Davis-North clearly pointed out, people usually regard economic institutional environment as an exogenously given variable, while consider economic institutional arrangement (such as market system) as being exogenous or endogenous, depending on the issues to be studied and discussed.
1.10.1 Specifying Economic Environment

The primary component of the analytical framework of modern economics is to specify the economic environment where the issue or object to be studied lies. An economic environment is usually composed of economic agents, their characteristics, the institutional environment of economic society, the informational structure and so on, which are treated as exogenous variables and parameters. They are the embodiment of the basic idea on constraints.

How can we specify economic environment? It can be divided into two levels: (1) objective and realistic description of economic environment and (2) concise and acute characterization of the essential features. The former is science and the latter is art, the two of which should be combined and balanced. The more clear and accurate the description of economic environment is, the greater the likelihood of obtaining correct theoretical conclusions. Also, the more refined and acute the characterization of economic environment is, the easier it is to argue and understand the theoretical conclusions. Only by combining these two levels together can we capture the essence of issues under study, as specifically discussed below:

**Description of economic environment**: The first step in every economic theory of modern economics is to give an approximately objective description of economic environment where the issue or object to be studied lies. A reasonable, useful economic theory should properly describe the specific economic environment. Though different nations and areas have different economic environments, which usually lead to different conclusions, the basic analytical framework and methodologies utilized are the same. A basic common point of studying economic issues is to describe the economic environment. The more clear and accurate the description of economic environment is, the greater the likelihood of obtaining correct theoretical conclusions.

**Characterization of economic environment**: When describing the economic environment, a question equally important as clear and accurate description of the economic environment is how to concisely and acutely characterize economic environment in order to capture the essence of a problem. Because most facts and phenomena in reality may be significantly irrelevant to the economic issue to be analyzed and solved, completely objective description of the economic environment is not at all helpful but may even confuse us with trivial details. If we exactly depict all aspects, it surely is very accurate and truthful description of economic environment, but this kind of simple listing only presents numerous confusing facts without capturing key points and the essence of the problem. In order to avoid trivial aspects and focus on the most critical and central issues, we need to characterize economic environment specifically according to the demands of the issue to be studied. For example, when discussing consumer behav-
ior in Chapter 3, we simply describe consumers as a composition of preference relation, consumption space, and initial endowment regardless of gender, age, or wealth of consumers. When discussing the production theory in Chapter 4, a producer can be characterized as the production possibilities set. When studying transitional economy, we cannot simply mimic and apply conclusions derived under mature market economic environment but need to characterize basic features of transitional economies, though the basic analytical framework and methodologies of modern economics still can be used.

We often hear some people criticize modern economics as useless because it uses a few simple assumptions to plainly summarize complex situations. In fact, this is also the basic research methodology of physics. In the study of the relationship between two physical variables, both theoretical research and experimental operation will fix other variables that will influence the object of study. In many cases, clarifying every aspect (especially unrelated aspect) is not necessary and may even lead to the loss of focus. It is like drawing maps for different goals and purposes as we mentioned before: people need a tourist map for traveling, a traffic map for driving, and a military map for war. These maps only describe certain characteristics of a region, so they are not the whole picture of the real world. Why do people need tourist map, traffic map and military map? The reason is that they are for different purposes. If we depict the entire real world into a map, although it completely describes the objective reality, how can it be a useful map?

Therefore, economics is not only a science but also the art of abstracting and depicting real economic environment. Economics uses concise and profound characterization of economic environment to describe causes of problems, conduct analysis of inherent logic, and thus obtain logical conclusions and inferences. A good economist should be able to accurately grasp the most essential characteristics of the current economic situation in his/her study. Only when we truly make clear the causes and current situation can we solve the specific problems with proper remedies (economic theory adopted). Of course, to do this, one needs to have the basic training in economics.

How to describe the economic environment often leads to very different theoretical results, even the results of creating a school. Because the economic environment is very complex, in many cases, economics cannot only carry out descriptive analysis like natural science, but also need to refine and characterize the behavior of economic environment in an abstract way to find out the most important characteristics, which often makes economists have a certain degree of subjective judgment. Different subjective judgments will lead to different specifications of economic environment, which will in turn lead to different economic theories, economic schools or theoretical results. For example, there are many schools in
macroeconomics: Keynesian school, post Keynesian school, rational expectation school (or neoclassicism school), monetarism school, supply school and new institutional school. In fact, the antagonism between these schools is not as great as that of non economists or media thought. The schools have many things in common: the same basic analytical framework, the same research method (using economic model and market equilibrium to analyze market) and the same object (studying the interaction and change law of macroeconomic variables under the arrangement of market system). They all believe in market system. It is believed that in terms of the long-term or general trend of economic operation, there will be an optimal market equilibrium. The differences among these theories are mainly caused by the differences in describing the economic environment, especially whether the impact or interference to the economic system comes from the demand side or the supply side, whether the information about economic fluctuation is sufficient or insufficient, and whether the time effect of the interference is caused by the difference many many assumptions, such as lag or instant.

1.10.2 Making Behavioral Assumptions

The second basic component of the analytical framework of modern economics is to make assumptions on the behavior mode of economic agents. This is the key difference between economics and natural science. The assumption is of fatal importance and is the foundation of economics. Whether an economic theory is convincing and has practical value and whether an institutional arrangement or economic policy is conducive to sustainable and rapid economic development mainly depends on whether the individual behavior assumed truly reflects the behavior of most individuals and whether the institutional arrangement and individuals’ behavior are incentive-compatible, i.e., whether individuals’ reaction to incentive is also beneficial to others or society.

In general, under a given environment and a rule of game, individuals will make trade-offs choice according to their behavioral disposition. Thus, when deciding the rules of game, policies, regulations or institutional arrangements, we need to take into account the behavioral pattern of participants and make correct judgments. Like dealing with different individuals in daily life, we need to know whether they are selfless and honest or not. Different rules of game should be instituted when faced with different participants. When facing an honest person who tends to tell the truth, the way to deal with him or the game rules imposed on him may often be comparatively simple. When facing totally selfless individuals, the rules to deal with him can be even simpler, for one does not need to take precautions or invest much energy (in designing the rules of game) to deal with him, and the rules may seem not so important. On the contrary, when facing a cunning and dishonest person, the way to deal with him will be
quite different and takes much energy so that the rules will be much more complicated. As such, making right judgments on individuals’ behavior is a very important step for the study of how individuals react to incentives and make trade-off choices. When studying economic problems such as economic choice, interaction between economic variables and how they change, it is also important to determine the behavioral pattern of economic agents.

As mentioned above, under normal circumstances, a reasonable and realistic assumption about individuals’ behavior used by economists is self-interest assumption, or the stronger rationality assumption, i.e., economic agents pursue the maximization of benefits. Bounded rationality is to make the best choice according to the knowledge and information an agent has, which belongs to the category of rationality assumption. In the consumer theory to be discussed later, we assume that consumer pursues utility/satisfaction maximization; in the producer theory, we assume that producer pursues the profit maximization; in game theory, there are various equilibrium solution concepts describing the behavior of economic agents, which are given based on different behavioral assumptions. Any economic agent, in his contact with others, implicitly assumes others’ behavior.

The assumption of (bounded) rationality is largely reasonable in some sense. From a practical point of view, as mentioned before, there are three basic kinds of institutional arrangements: mandatory regulation (for situations with small operation cost and relatively easy information symmetry), incentive mechanism (for cases of information asymmetry), and social norms (composed of ideology, ideal, morals, customs, etc., that are norms of self-discipline). If individuals are all selfless and of very high ideological levels, there will be no need for the rigid “stick-style” mandatory regulation that “enlightens with reason” or the flexible market system that “guides with interest”.

### 1.10.3 Setting Economic Institutional Arrangements

The third basic component of the analytical framework of modern economics is to set up economic institutional arrangements, which are usually referred to as institutions or rules of game. Different strategies or rules of game should be taken for different situations, different environments, and individuals with different goals and behavioral patterns. When the situation or environment changes, the strategies or rules of game will also change accordingly. When an economic environment is given, agents need to decide the economic rules of game, which is called economic institutional arrangement. Determination of institutional arrangement is important for doing anything. Economics studies and gives various economic institutional arrangements or economic mechanisms according to different economic environments and behavioral assumptions. Depending on the issue
under discussion, an economic institutional arrangement could be exogeniously given (in which case it will degenerate into the institutional environment such as the perfectly competitive market under which we study consumer or producer problem) or endogenously determined.

As discussed before, there are three basic kinds of institutional arrangements to guide individuals’ behavior: mandatory regulatory governance or government intervention, institutional norm of incentive mechanism, and didactic social norm. The three means play different roles and also have respective applicable ranges and limitations. Didactic social norm relies on the improvement of humanity and lacks constraining force; mandatory regulatory governance or government intervention incurs high information costs, and over intervention will hurt individual freedom; compared with the other two means, incentive mechanism is the most effective one. This is why economists pay much attention to institutions.

Therefore, for the enaction of an institutional arrangement of no matter regulatory governance or incentive mechanism, the purpose is not to change the self-interested nature of individuals, but to make use of such unchangeable self-interestedness so as to guide individuals to do things that will objectively benefit society. Mechanism design should follow the nature of individuals but not try to change such nature. There is no good or evil in the discussion of individuals’ self-interestedness, and the key lies in how to guide it with institutions. Different institutional arrangements will induce different incentive reactions and trade-off choices and thus may lead to very different results.

Any theory of economics involves economic institutional arrangements. Standard modern economics mainly focuses on market system and studies how individuals make trade-off decisions in a market system (such as the consumer theory, producer theory, and general equilibrium theory) and under what economic environments will market equilibrium exist. It also makes value judgments on the result of resource allocation under different market structures (the criterion is based on whether resource allocation is efficient and fair). In these studies, market system is normally assumed to be exogenously given. By so doing we can simplify the problem so as to focus on the study of individuals’ economic behavior and how individuals make trade-offs choices.

Of course, as the exogeneity assumption of institutional arrangements is not entirely reasonable in many cases, different economic institutional arrangements should be given depending on different economic environments and individuals’ behavioral patterns. As shall be discussed in Parts V-VI of the book, there will be market failures (i.e., inefficient allocation of resources and non-existence of market equilibrium) in many situations, so we will need to find an alternative mechanism or a better economic mechanism. In that case, we need to treat institutional arrangement to be endogenously determined by the economic environment and individual
behavior. Thus, economists should give a variety of alternative economic institutional arrangements for different purposes.

When studying the economic behavior and choice issues of a specific economic organization, economic institutional arrangements should especially be endogenously determined. New institutional economics, transition economics, modern theory of the firm, especially the economic mechanism design theory, information economics, optimal contract theory, auction theory and matching theory that have developed in the last few decades, study and give various economic institutional arrangements for a wide range from the state to individuals according to different economic environments and behavioral assumptions. Part VI will give elaborate discussions on the issue of incentive design in economic institutional arrangement.

1.10.4 Determining Equilibrium

The fourth basic component of the analytical framework of modern economics is to make trade-off choices and determine the “best” outcome. Given the economic environment, institutional arrangement (rules of game) and other constraints that should be obeyed, individuals will react to incentives based on their own behavior, weigh and choose an outcome from available and feasible outcomes. Such an outcome is called equilibrium. In fact, the concept of equilibrium is not hard to understand. It means that among various feasible and available choices, the one being finally chosen is called the equilibrium. Those who are self-interested will choose the best one for themselves; those who are altruistic may choose an outcome that is favorable to others. Thus, the so-called equilibrium, which refers to a state without deviation incentives for all economic agents, is a static concept.

The equilibrium defined above may be the most general definition in economics. It embraces the equilibria in textbooks that are reached by independent decisions under the drive of self-interested motivation and all kinds of technology or budget constraints. For instance, under market system, for the producer, a profit-maximizing production plan under the constraint of production technology is called equilibrium production plan; for the consumer, a utility-maximizing consumption set under the budget constraint is called consumption equilibrium. When producers, consumers, and their interactions reach a state where there is no incentive for deviations, a competitive market equilibrium for each commodity is obtained.

It should be noted that equilibrium is a relative concept. The equilibrium outcome depends on economic environments, participants’ behavior patterns (whether in terms of rationality assumption, bounded rationality assumption, or other behavioral assumptions), and the rules of game by which individuals react to incentives; it is the “best” choice relative to these factors. Note that due to bounded rationality, it may not be the opti-
mal choice in objective reality, but is the “best” one chosen by individuals according to their preferences and information and knowledge in hand.

1.10.5 Making Evaluations

The fifth basic component of the analytical framework of modern economics is to make evaluations and value judgments on equilibrium outcome and institutional arrangements undertaken. After making their choices, individuals usually hope to evaluate the equilibrium that then arises and compare it with the ideal outcome (for instance, efficient allocation, fair allocation, incentive compatibility, informational efficiency, etc.), so as to make further assessments and value judgments on economic institutional arrangement—whether the economic institutional arrangement adopted has led to certain “optimal” outcome; and test whether the theoretical result is consistent with the empirical reality and whether it can provide correct predictions or practical significance. Finally, they evaluate the economic institution and rules adopted to find out whether there is room for improvement. In short, in order to achieve better results for doing something, after finishing it, we should evaluate the effects, whether it is worth continuing efforts, and whether there is a possibility for improvement. Thus, we need to make evaluations and value judgments on equilibrium outcome under an institutional arrangement and trade-off choice so as to find out the institutions that are best suited to the development of a nation.

When making evaluations on an economic mechanism or institutional arrangement, one of the most important criteria adopted in modern economics is whether the institutional arrangement is in line with the principle of efficiency. Surely, as economic environment and individuals’ behavioral patterns in reality keep changing and science and technology are also continuously developing, the exact Pareto optimality may never be truly realized. Just like Newton’s three laws of motion, free fall, and fluid flow without friction in physics, it only is an ideal state and provides the direction of improvement on economic efficiency. As long as the improvement of economic efficiency is desired, individuals will constantly pursue to approach this goal as much as possible. With such an ideal standard as Pareto optimality, we then have the benchmark to compare, measure, and evaluate various economic institutional arrangements in the real world and see how far they are from this ideal goal so as to learn the room of improvement on economic efficiency and make resource allocation approach the Pareto optimality as close as possible.

Nonetheless, Pareto optimality is not the sole criterion, and there is another value judgment called equity or fairness. Market system achieves efficient allocation of resources, but it also faces many problems, such as social injustice caused by huge wealth gap. There is a variety of definitions of equity and fairness. The equitable allocation to be introduced in
Chapter 12 takes into account both objective equity and subjective factors, and more importantly, it can achieve equitable and efficient outcomes at the same time. This is the basic conclusion of the Fairness Theorem to be introduced in Chapter 12. Another important criterion for evaluating an economic institutional arrangement is incentive compatibility.

In summary, the five components discussed above constitute the analytical framework underlying almost all standard economic theories and models, no matter how much mathematics is used, or whether the institutional arrangement is exogenously given or endogenously determined. In the study of economic issues, one should define first the economic environment, and then examine how the self-interested behavior of individuals affects each other under exogenously given or endogenously determined mechanisms. Economists usually take “equilibrium”, “efficiency”, “information” and “incentive compatibility” as key aspects of consideration to observe the effects of different mechanisms on individual behavior and economic organizations, explain how individual behavior achieves equilibrium, and evaluate and compare the equilibrium. Using such a basic analytical framework in the analysis of economic issues is not only compatible in methodology but may also lead to surprising (but logically consistent) conclusions.

1.11 Basic Research Methodology in Modern Economics

We have discussed the five components of the basic analytical framework of modern economics: specifying economic environment, making behavioral assumptions, setting institutional arrangements, determining equilibrium, and making evaluations. Broadly speaking, any economic theory consists of these five aspects. Discussion on the five components naturally leads to the question of how to combine them appropriately, gradually deepen the study of various economic phenomena, and develop new economic theories. This is what we will discuss in this section: the basic research methodology and key points, which include setting up benchmarks, establishing reference systems, building studying platforms, developing analytical tools, constructing rigorous models, and conducting positive and normative analyses.

The research methodology of modern economics is to firstly provide basic studying platforms for all levels and aspects and then establish benchmarks and reference system so as to present the criterion to evaluate equilibrium outcomes and institutional arrangements. Building studying platform, setting benchmark, and establishing reference system are of great
importance to the construction and development of any discipline, and econometrics is no exception.

1.11.1 Setting up Benchmark

Evaluation or judgement of anything can only be relative but not absolute, so there should be a benchmark, which also applies to the discussion of economic issues. In economics, benchmark refers to a relatively ideal state or simple economic environment. As discussed in the first section of this chapter, to study a realistic economic issue and develop a new theory, we usually need to consider first it under a relative ideal economic environment to develop a simple result or theory. Then, we discuss the result in a non-ideal economic environment, which is closer to reality, develop a more general theory, and compare it with that developed under the benchmark situation.

In this sense, benchmark is relative to non-ideal economic environments and new theories to be developed that are closer to reality. For instance, complete information environment is the benchmark for the study of incomplete information. When studying economic issues under private information, we need to fully understand first the situation of complete information (highly unrealistic though it is). Only when we are clear about the situation of complete information can we study well economic issues taking place under the circumstance of private information. So is the case with theoretical research in economics. We start from the ideal state or simple scenarios before considering more realistic or general scenarios; we learn from others’ research results before innovating on the existing theories. New theories are always developed on the basis of prior research findings and results. It is like that Newton’s mechanics makes possible Einstein’s theory of relativity, while the theory of relativity makes possible the non-conservation of parity put forward by Chen-Ning Franklin Yang and Tsung-Dao Lee.

1.11.2 Establishing Reference System

Reference system refers to economic models and theories generated in an ideal situation, such as the general equilibrium theory, that is, a perfectly competitive market will lead to efficient resource allocation. Establishing reference system is of great importance to the construction and development of a new discipline, including economics. Although economic theories set as the reference system may include many assumptions that do not accord with reality, at least they can help in: (1) simplifying the issue and capturing its characteristics directly; (2) establishing a comparative measurement criterion that is conductive to the assessment of the gap between the existing system and the ideal system, understanding the reality, and
establishing the direction of improvement; (3) studying how to reform or further carry out theoretical innovation, which can be used as a reference system for further analysis.

Although an economic theory as a reference system might have many unrealistic assumptions, they are very useful and can serve as the reference for further analysis. This is similar to our practice of setting role models in life. The importance of the reference system does not lie in whether it accurately describes the real world or not, but in establishing the measurement for a better understanding of the reality. Like a mirror, it helps reveal the gap between theoretical models or realistic economic institutions and the ideal state. It is essentially important in the sense that it points out the direction of efforts and adjustments and the extent of adjustments. If a person has no goal and is unaware of the gap and rough direction of making efforts, how can he make improvements? Not to mention achieving his goal.

General equilibrium theory is such a reference system. As we know that a perfectly competitive market will lead to efficient allocations of resources, although there is no such market in reality, if we make efforts toward that direction, efficiency will be enhanced. That is why we have institutional arrangements such as the anti-trust law to protect market competition. By virtue of the reference system with perfect competition and complete information as the benchmark, we can study what outcomes can be obtained from economic institutional arrangements that are closer to the reality (such as some monopolistic or transitional economic institutional arrangements) where assumptions in the general equilibrium theory are not valid (incomplete information, imperfect competition, externalities), and compare them with the results obtained from general equilibrium theory in the ideal state. In so doing, we will know whether an economic institutional arrangement (be it theoretical or realistic) is efficient in allocating resources and utilizing information, and how far away the economic institutional arrangement adopted in reality is from the ideal situation. Based on this, we can make policy suggestions accordingly. In this sense, general equilibrium theory also serves as a reference theory that gives a criterion for evaluating institutional arrangements and the corresponding economic policies in reality.

Just like a knife, however sharp, will not work if its holder does not cut in the right direction; a person, however smart, might go nowhere if he is unclear of his goal and direction. Therefore, we should have lofty ideals, which let us know the direction and goals of making efforts. Even if we cannot get there eventually, but, only learning from the best and comparing with the best can we do better and even better, being inspired to continuously approach the ideal.
1.11.3 Building Studying Platforms

A studying platform in modern economics consists of some basic economic theories, models or methods which provide the basis for deeper analysis. The methodology of modern economics is very similar to that of physics, i.e., simplifying the issue first to capture the core essence of the issue. In case that many factors breed an economic phenomenon, we need to make clear the impact of every factor. This can be done by studying the effect of one factor at a time while assuming other factors remain unchanged. The theoretical foundation of modern economics is modern microeconomics, and the most fundamental theory in microeconomics is individual choice theory to be discussed later – consumer theory and producer theory, which are the basic studying platform and cornerstone of modern economics. This is why almost all textbooks of modern economics start from the discussion of consumer theory and firm theory. They provide the fundamental theories explaining how individuals make choices as consumers and firms and establish the studying platform for further study of individual choice.

Generally speaking, the equilibrium choice of an individual depends not only on one’s own choice, but also on others’ choices. In order to study individual choice, we need to clearly understand what is the most important factors for an individual to make his choice in the absence of influence by other agents. The consumer theory and producer theory are developed through this approach. It is assumed that economic agents are in the institutional arrangement of a perfectly competitive market. Therefore, every agent will take price as given, and individual choice will not be affected by others’ choices. The optimal choice is determined by subjective factors (such as the pursuit of utility or profit maximization) and objective factors (such as budget line or production constraints).

This methodology that involves simplification and idealization of the question sets a basic platform for deepening research. It is like the approach in physics: in order to study a undertaken problem, we go to the essence first, start with the simplest situation that excludes frictions from consideration, and then gradually deepen the research and consider the more general and complicated cases. In microeconomics, theories of market structures like monopoly, oligopoly and monopolistic competition are generated from more general cases, where producers can influence each other. To study the choice issue under the more general situation where economic agents can influence each other’s decision-making, economists have developed a very powerful analytical tool – game theory.

General equilibrium theory is a more sophisticated studying platform based on consumer theory and producer theory. Consumer theory and producer theory provide a fundamental platform for studying individual choice problems, while general equilibrium theory provides a fundamental platform for analyzing how to reach market equilibrium through
interactions of individuals and all markets. Mechanism design theory provides an even higher-level platform for studying, designing and comparing various institutional arrangements and economic mechanisms. It can be used to study and prove the optimality and uniqueness of perfectly competitive market mechanism in resource allocation and information requirement, and more importantly, in case of market failure, it offers methods of how to design alternative mechanisms. Under some regularity conditions, the institutional arrangement of perfectly competitive market will not only lead to efficient allocation of resources, but also prove the most efficient in terms of information requirement (mechanism operation cost, transaction cost) as it requires the least amount of information. Under other circumstances of market failure, we need to design a variety of alternative mechanisms for different economic environments. Furthermore, the studying platform also creates conditions for providing reference systems for evaluating various kinds of economic institutional arrangements. In other words, it provides a criterion for measuring the gap between reality and the ideal state.

1.11.4 Developing Analytical Tools

For the research of economic phenomena and economic behavior, we also need various analytical tools besides the analytical framework, benchmark, reference system, and studying platform. Modern economics requires not only qualitative analysis but also quantitative analysis to find the boundary conditions for each theory to be true so that the theory will not be misused. Thus, a series of powerful analytical tools should be provided, which are usually given as mathematical models or even diagrams. The power of these tools lies in that their ability to help us deeply analyze the intricate economic phenomena and economic behavior through a simple and clear diagram or mathematical structure. Examples include the demand-supply curve, game theory, principal-agent theory for studying information asymmetry, Paul A. Samuelson’s (1915-2009, see Section 3.11.2 for his biography) overlapping generations model, dynamic optimality theory, etc. Of course, there are exceptions which are not expressed with analytical tools. For instance, the Coase’s theory is established and demonstrated through words and basic logical deduction only.

1.11.5 Constructing Rigorous Analytical Models

Logical and rigorous theoretical analysis is needed when we explain economic phenomena or economic behavior and make conclusions or economic inferences. As mentioned above, each theory holds true under certain conditions. Modern economics requires not only qualitative analysis but also quantitative analysis to define the boundary condition for a theory to
be true so that the theory wo not be misused, just like medication and pharma-
cology where we have to be clear about the application and functions of medicines. We need to establish rigorous analytical models identifying clearly the conditions under which a theory holds true. Lack of related mathematical knowledge will make it difficult to have an accurate understanding of the connotation of a definition, or to have discussions on related issues, not to mention defining boundary conditions or constraints. Therefore, it is not surprising that mathematics and mathematical statistics are used as basic analytical tools, and they are also among the most important research methods in modern economics.

1.11.6 Positive Analysis Vas Normative Analysis

As per the research method, economic analysis can be divided into two cat-
egories, one being positive or descriptive analysis, the other being norma-
tive or value judgement analysis. Another major difference between eco-
nomics and natural science is that the latter essentially only makes positive analysis while the former involves both positive and normative analysis.

Positive analysis explains how economy operates. It only gives facts and provides explanations (thus verifiable), but does not make value judgement of economic phenomena or offers means of revision. For example, an important task of modern economics is to describe, compare and analyze such phenomena as production, consumption, unemployment and price, and to predict possible outcomes of different policies. Consumer theory, producer theory, and game theory are typical examples of positive approach.

Normative analysis makes value judgments on economic phenomena. It not only explains how economy operates, but also attempts to find out means of revision. Therefore, it always involves value judgments and opinions of the economists and is thus not checkable through facts. For instance, some may lay more emphasis on economic benefits while others focus on equity or social justice. With the differences between the two methods in mind, we can avoid many disputes when discussing economic issues. Economic mechanism design theory is a typical example of normative approach.

Positive analysis is the foundation for normative analysis, while norma-
tive analysis is the extension of positive analysis. In this sense, the foremost task of economics is to make positive analysis, and then normative analysis follows. General equilibrium theory includes both the former (such as the existence, stability and uniqueness of competitive equilibrium) and the latter (the First and Second Fundamental Theorems of Welfare Economics).
1.12 Practical Role of the Analytic Framework and Methodologies

The most basic analytic framework and research methods of modern economics that have been discussed are of practical uses. Even though these analytical framework and methodologies seem simple, it is actually not easy to comprehend and use them in your life, study, and research. However, once the idea is mastered, you will get lifelong benefits. They can make you wise, profound, and able to think scientifically. They can help you to study the highbrow pure economic theories and also provide advices to solve practical issues in your life and work.

First of all, from the aspect of studying modern economics, if analytic framework and methodologies are mastered, you will not be confused by abstract models and abstruse mathematics. Because no matter how profound mathematics, how many formulae, and how complicated economic models are used in an economic theory, it still uses the above analytical framework and methodologies. If the basic analytical framework and methodologies are grasped and born in your mind as the main thread, you will not lose the way or the focus and will basically understand its general idea. Therefore, you can temporarily put aside incomprehensible technical details and get the framework and conclusion straightened out first; only then we manage to comprehend the details. That is to say, you should grasp first the main point and general idea, know its objectives, train of thought as well as conclusions, and then come to the details. Moreover, once you master the basic analytic framework and methodologies, you can have a correct idea of modern economics and cannot be misled, affecting your interests in studying modern economics. The reason why people often criticize modern economics and its methodologies is that their judgements are mostly not based on methods of scientific analysis and may even only rely on their subjective assumptions. If you do not understand the basic analytical framework and methodologies, it is possible that their judgments will mislead you, make you lose the right way of studying modern economics, and even overlook and resist the study of modern economics.

Second, in the aspect of modern economic research, once you understand and comprehend the basic analytical framework and methodologies, you will be more qualified to do the research. For many people who want to do economic research, even though they have understood the modern economics quite well and have read quite a number of related papers, they still find it hard to do research. They do not know how to do research, or their research findings are not significant or widely recognized. Actually, it would not be so difficult to do research on economics as long as they understand the basic analytic framework and methodologies and have basic mathematic knowledge and the ability of logical analysis. Doing research,
to some extent, is step-by-step writing with inherent logic according to the five components of the basic analytical framework. The basic analytic framework and methodologies can help you improve your research and innovative abilities.

For example, if you are interested in a specific economic issue or phenomenon, or you want to put forward a new theory with stronger explanatory power to guide the resolution of practical economic issues, you will need to reasonably and precisely describe and depict the economic environment and economic agent's behavior, employ existing analytical tools or develop new ones to build a model as simple as possible, and then do the deduction and proof. If you only want to extend and improve the original theory, you should analyze whether original assumptions about economic environment and behavior and models fit the reality and whether the assumptions can be relaxed to derive new or more general conclusions. It might be easier to do the work of extensions and improvements. Of course, you can also modify the specification of economic environment or other components so as to get a different or even more important conclusion. It is how the school of macroeconomics and many theories under information asymmetry were developed. If you are going to criticize an economic theory, you should criticize which parts of its analytic framework are unreasonable, illogical, or unrealistic and in what way rather than criticizing economics and its methodologies as a whole.

Finally, understanding modern economics, its methodologies, and analytical framework may help us have better thinking and deal with everyday concerns and other individuals in a better way. It can make you more thoughtful, insightful, and capable in work. It is often heard that modern economics is highbrow and metaphysical since it involves so much mathematics and so much difficulty to learn; what will it be used for? Actually, the basic approach of dealing with others and things in daily life is similar to the basic framework in economic analysis. For example, when you are in a new place preparing to do something or teaming up with others, the first thing you need to do is acquainting yourself with the local environment and situations, which is similar to "specifying economic environment" in the framework; then, you need to know the local culture and custom, the behavioral pattern and personality of your counterpart, etc., similar to "making behavioral assumptions"; taken all the information together, you can decide your rules of dealing with people, which is similar to "setting economic institutional arrangements"; the next step is to choose the optimal scheme by trading off among feasible options, which is similar to the "determining equilibrium"; the last step is to summarize and reflect your decisions, actions, and your ways of dealing with things to see whether they are the most effective approach, whether they achieve the best outcome, whether they are fair and reasonable, whether individuals’ enthusiasm is aroused, whether people have reactions to the incentive, and whether the
intended goal of incentive compatibility is achieved, which is similar to “making evaluations”. Besides, when the environment and condition are changed, or the subject you are working with is changed, the rule should also be changed accordingly. If you can do according to the five aspects and adjust the rules with changes of the condition, better results can surely come about. This may be one of the best ways to deal with daily life and work. Moreover, some conclusions of economic theories can also help you to think and solve problems.

1.13 Basic Requirements for Mastering Modern Economics

There are three basic requirements for learning and mastering modern economic theory:

1. The basic requirement is to master the basic concepts and definitions, which is a reflection of logical thinking and clear mind. This is the prerequisite not only for discussion and analysis of questions and logically inherent analysis, but also for a good command of economics. Otherwise, different definitions of terms may cause great confusions and thus lead to unnecessary disputes.

2. One should be able to clearly state all theorems or propositions and also be clear about the basic conclusions and their conditions. Otherwise, a small misunderstanding an economic theory in application to the analysis of issues may lead to serious problems. Just like there is a scope of application for medicines, any theory or institution also has its proper applicable arrange. If we go beyond that arrange, problems are liable to arise, bringing about tremendous negative externalities. In many cases, social or economic problems occur just because economists misuse some theories without a good understanding of their applicable conditions. In this sense, an eligible economist is like a qualified medicine doctor who needs to know well the properties and efficacy of different kinds of medicines when prescribing for his patients.

3. One has to grasp how the basic theorems or propositions are proved (ideas and processes). A good economist, like a good doctor, should know what the matter is and why it is so, and also have medicine properties and pathology both in mind. Then, he can gain a deeper understanding and a better command of the theories he has learned.

If you meet these requirements, it would be easy to refresh your memory even if you forget the proof of some conclusions. Economics relies mainly upon the analysis of inherent logic, which is also the power of modern economic theory. That is why it is important to accurately grasp the theories and their application scopes.
1.14  Distinguishing Sufficient and Necessary Conditions

When discussing economic issues, it is important to distinguish between sufficient conditions and necessary conditions, which can help people think clearly and avoid unnecessary debates. A necessary condition is a condition that is indispensable in order for an assessment to be true. A sufficient condition is a condition that guarantees the assessment to be true. For instance, some people often negate market economy based on the examples of poor countries, which adopt market economy but remains poor, so they argue that a nation should not embark on the path of market economy. These people do not realize the difference between sufficient and necessary conditions: the adoption of market economy is a necessary condition rather than a sufficient condition for a nation to become prosperous and strong. In other words, if a country wants to be prosperous, it must adopt market economy. This is because one cannot find any wealthy nation in the world that is not a market economy. However, market mechanism is just a necessary but not sufficient condition for prosperity. One must also admit that market mechanism does not necessarily lead to national prosperity. To be sufficient, other supplementary systems are needed such as the rule of law, strong capacity of nation, suitable political system, etc.

Indeed, as discussed before, there is a distinction between good market economy and bad market economy, the reason of which is that although (based on observation of reality) market mechanism is indispensable for national prosperity, there are many other factors that also affect the prosperity of a nation, such as the degree of government intervention, political system, law, religion, culture, and social structure, thus making market mechanism be labeled as good or bad.

1.15  The Role of Mathematics and Statistics in Modern Economics

Mathematics and statistics are of extreme significance for people to have a good knowledge of the nature and to manage daily affairs. As the well-known statistician, C. Radhakrishna Rao pointed out that mathematics is a kind of logic to deduce results on a given premise, while statistics is a rational method acquired through experience and a kind of logic to verify the premise with a given result. Rao believes that “All knowledge is, in final analysis, history. All sciences are, in the abstract sense, mathematics. All judgements are, in their rationale, statistics.”

1.15. THE ROLE OF MATHEMATICS AND STATISTICS IN MODERN ECONOMICS

depicts the significance of mathematics and statistics and their respective connotations.

Mathematics and statistics are also important in modern economics. Almost every field in modern economics more or less uses mathematics, statistics and econometrics. The width and depth of mathematics involved may even exceed those in physical science. The reasons for this include: the fact that modern economics is increasingly becoming a science, the use of mathematical analytical tools, and the more complex and influential social systems. Thus, when considering and studying economic problems, it is required to have a rigorous theoretical model for internal logic analysis and determine the arrange for a theoretical conclusion to be true, and then have empirical test for the given results by means of econometric analysis. Hence, it is not surprising that mathematics and mathematical statistics are used as the basic analytical tools and also become the most important analytical tools in modern economics. Those who study and conduct research on modern economics must have a good knowledge of mathematics and mathematical statistics.

Modern economics mainly adopts mathematical language to make assumptions on economic environments and individual behavior patterns, uses mathematical expressions to illustrate logical relations between economic variables and economic rules, builds mathematical models to study economic issues, and finally follows the logic of mathematical language to deduce conclusions. Without the related mathematical knowledge, it is hard to grasp the essence of concepts and discuss related issues, let alone conduct research and figure out the necessary boundary or constraints when giving conclusions. Therefore, it is of necessity to master sufficient mathematical knowledge if you want to learn modern economics well, engage in research on modern economics, and become a good economist.

People with little knowledge of mathematics are unlikely able to master the basic theories and analytical tools of modern economics or understand advanced economic textbooks or papers. They may find excuses such as it is important to produce economic thoughts, or mathematics is far away from practical economic issues. No one could deny the importance of economic thoughts for they are the output of research. Without the tool of mathematics, how could we rigorously figure out boundary conditions and the applicable scope of economic thoughts or conclusions? Without knowledge of the conditions and scope, how could we defend against abuse or misuse of economic thoughts or conclusions? How many people could develop such profound economic thoughts without using mathematical models as Adam Smith and Ronald H. Coase did? Even so, economists have never stopped studying what conditions are required for their conclusions to be true. Without strict arguments, the thoughts or conclusions could not be widely recognized. It is indeed so. As mentioned above, the
economic thoughts presented by philosophers in ancient China like JIANG Shang, LAO Tzu, SUN Tzu, GUAN Zhong and SIMA Qian are extremely profound, and some ideas of Adam Smith had already been mentioned by these philosophers long ago. However, their thoughts have never been known to the outside world because they were just observation or conclusions of experience that did not form a scientific system nor involved logically inherent analyses using scientific methods.

There is another misunderstanding that research of economic issues with mathematics is remote from reality. Actually, most of the mathematical knowledge is developed on the basis of practical demands. People who have basic knowledge of physics, physical science history or history of mathematical thought will know that both primary and advanced mathematics are originated from the demands of scientific development and reality. As such, why can not mathematics be used to study practical economic issues? The foundation of mathematics and modern economics is very important for one to be a good economist. If you know mathematics well and master the fundamental analytical framework and research methodology of modern economics, you could learn modern economics more easily and greatly enhance study efficiency.

The functions of mathematics in the theoretical analysis of economics are as follows: (1) It makes the language more precise and concise and the statement of assumptions more clear, which can reduce many unnecessary disputes resulting from ambiguous definitions. (2) It makes the analytical logic more rigorous and makes it possible to clearly state the boundary, application scope and conditions for a conclusion to hold true, and point out the direction of theoretical innovation beyond the limitations of the original theory. Otherwise, the abuse of a theory may occur. For example, when discussing the issue of property rights, people would quote Coase theorem, thinking that as long as the transaction cost is zero, there will be efficient allocation of resources. Up to now, there are still many people (including Coase himself when giving his assessments) who do not know that this conclusion is normally not true if the utility (payment) function is not quasi-linear. (3) Mathematics can help obtain the results that cannot be easily attained through intuition. For example, from intuition, according to the law of supply and demand, competitive markets will achieve market equilibrium through the adjustment of market prices by the “invisible hand” as long as the supply and the demand are not equal. Yet this conclusion is not always true. Scarf (1960) gave counterexamples of market instability. (4) It helps improve and extend the existing economic theory. Examples in this respect are plenty in the study of economic theory. For instance, economic mechanism design theory is an improvement and extension of general equilibrium theory.

Qualitative theoretical analysis and quantitative empirical analysis are both needed for studying economic problems. Statistics and econometrics
play an important role in these analyses. Statistics focuses more on data collection, description, sorting and providing statistical methods, whereas econometrics identifies economic structure by economic theory, focuses more on testing economic theory, evaluating economic policy, making economic forecast and identifying the causal relationship between economic variables. In order to better evaluate economic models and make more accurate predictions, theoretical econometricians have been continuously developing increasingly powerful econometric tools.

It is, however, noteworthy that economics is not mathematics. Mathematics in economics is used as a tool to consider or study economic problems. Economists just employ mathematics to express their opinions and theories more rigorously and concisely and to analyze the interdependent relationship among economic variables. With the metrication of economics and the precision of various presupposed assumptions, economics has become a social science with a rigorous system.

However, a person with good knowledge of mathematics does not necessarily become a good economist. It also requires a deep understanding of the analytical framework and research methodologies of modern economics and a good intuition and insight of real economic environments and economic issues. The study of economics not only calls for the understanding of terms, concepts and results from the perspective of mathematics (including geometry), but more importantly, even when those are given by mathematical language or geometric figures, we should try the best to get clear their economic meanings and the underlying economic thoughts. Thus we should avoid being confused by the mathematical formulas or symbols in the study of economics. So we say that, to become a good economist, one needs to pursue academics with deep thoughts and for deep thoughts.

1.16 Conversion between Economic and Mathematical Languages

The product of economic research is economic inferences and conclusions. A standard economics paper usually consists of three parts: (1) it raises questions, states the significance, and identifies the research objective; (2) it establishes economic models and rigorously expresses and proves the inferences; (3) it uses non-technical language to explain the conclusions and provides policy suggestions. That is to say, an economic conclusion is usually obtained through the following three stages: non-mathematical language stage $\rightarrow$ mathematical language stage $\rightarrow$ non-mathematical language stage. The first stage proposes economic ideas, concepts or conjectures, which may stem from economic intuition or historical and foreign experience. As they have not been proved by theories yet, they can be re-
The first stage is very important because it is the origin of theoretical research and creation. The second stage verifies whether the proposed economic ideas or conjectures hold true or not. The verification requires economists to give formal and rigorous proofs through axiomatic analysis by introducing economic models and analytical tools, and if possible, to test them with empirical data. The conclusions and inferences obtained are usually expressed in mathematical language or technical terms, which may not be understandable to non-experts. Therefore, they may not be adopted by the public, government officials or policy makers. Thus, these conclusions expressed by technical language can be regarded as intermediate products of general production.

Economic studies should serve the real economic world. Therefore, the third stage is to express the conclusions and inferences by non-expert language rather than expert language, making them more understandable to the general public. Policy implications and profound meanings of conclusions and insightful inferences conveyed through non-expert language will be final products of economics. It is notable that in both the first and the third stages, economic ideas and conclusions are presented by common, non-technical and non-mathematical language, but the third phase is a kind of enhancement of the first phase. As a matter of fact, the three-stage form of common language - technical language - common language is a normal research method widely adopted by many disciplines.

1.17 Biographies

1.17.1 Adam Smith

Adam Smith (1723-1790), the primary founder of economics, is well-known as the father of modern economics. Adam Smith finished his study of Latin, Greek, mathematics, and ethics in the University of Glasgow in Britain. After that, he worked in the University of Glasgow as a professor in Logic and Moral Philosophy, and even once as the honorary position of Lord Rector. The Wealth of Nations, published in 1776, is Smith’s most influential work and also a great contribution to the establishment of economics as an independent discipline. This book even is the most influential work among all publications in the field of economics. His main academic thought was affected by Bernard de Mandeville (1670-1731), Francis Hutcheson (1694-1764), David Hume (1711-1776), J. Vanderlint (year of birth unknown, died in 1740), George Berkeley (1685-1753), and so on.

Smith suggested that the economic development of human society was the outcome of spontaneous action of tens of millions of individuals whose behavior followed the power of instinct and was driven by their self-interested
nature. Smith regarded this power as the “invisible hand”, which is also his idea of allowing the rule of market to play the decisive role in the organization of economic society. *The Wealth of Nations* denied the attention to land in physiography but valued labor as the most important and believed that the division of labor could increase the efficiency of production.

Thomas Robert Malthus and David Ricardo focused on summarizing Smith’s theories into a theory known as the classical economics in the twentieth century (where modern economics thus originated). Malthus further extended Smith’s theories to the problem of surplus of population. Ricardo, on the other hand, put forward the iron law of wages, suggesting that the surplus of population may lead to the consequence that even workers’ livelihood cannot be guaranteed. Smith assumed that the increase in wages would accompany the increase in production, which seems more correct from today’s perspective. Theories involved in his book not only set up the division of labor theory, but also pioneered in areas including monetary theory, theory of value, theory of distribution, capital accumulation theory, theory of taxation, and so on. In addition, Marx’s labor theory of value built upon the basis of Ricardo’s political economy also received indirect influence from Smith’s theory.

Before the foundational work *The Wealth of Nations*, Smith also wrote *The Theory of Moral Sentiments* (first published in 1759). In this book, he mainly argued that people should have sympathy and sense of justice, which may happen under unusual circumstances (for example, when people are in trouble or when a nation is invaded). He tried hard to prove how self-interested individuals controlled their emotions and behavior, especially their selfish sentiment and behavior, so that incentive compatibility between social interest and self-interest could be achieved. The system of economic theory built by Smith in *The Wealth of Nations* is based on his arguments in *The Theory of Moral Sentiments*.

Smith worked on the two works at the same time and revised them repeatedly until death. They became two organic and complementary components in his academic thinking system. *The Theory of Moral Sentiments* states the problem of morality, while *The Wealth of Nations* states the problem of economic development. Smith regarded *The Wealth of Nations* as the continuation of his thought in *The Theory of Moral Sentiments*. The two books differ in mood, scope of discussion, structure arrangement, and emphasis; for example, the control for self-interested behavior in *The Theory of Moral Sentiments* relied on the sympathy and sense of justice, while in *The Wealth of Nations*, it relied on competitive mechanism. Nonetheless, the discussions about motivation of self-interested behavior were the same in essence. It can be suggested that individuals’ sympathy, sense of justice, and pursuit of self-interest are just different reactions to different circumstances (unusual or usual). In *The Theory of Moral Sentiments*, Smith saw “sympathy” as the core of judgement, but when it is regarded as the moti-
ivement of individual’s behavior, it will turn out totally different result.

1.17.2 David Ricardo

David Ricardo (1772-1823), the representative of the Classical Political Economy, together with Thomas Robert Malthus (1766-1834), integrated Smith’s theory into classical economics. Ricardo was born to a Jewish family and his father was a stock broker. He went to a business college in the Netherlands at 12 and worked on stock exchange with his father at 14. He was engaged in stock exchange independently in 1793, and by the time he was 25, he had already owned a wealth of two million pounds. After that, he began to study mathematics and physics. When he first read Adam Smith’s *The Wealth of Nations* in 1799, he began to study economic issues. At the age of 37, he published his first paper in economics and then went very smooth in this field. During the 14 years of his short academic life, he left numerous of works, papers, notes, letters, and speeches. Among those, the most famous one is the *Principles of Political Economy and Taxation* published in 1817. Ricardo was rather self-conceited. He said that his point of view was different from Smith and Malthus who enjoyed great prestige then, and, in Britain, there would be less than 25 people who could understand his book. In 1819, Ricardo was elected as a member of parliament.

Starting with Bentham’s version of utilitarianism, Ricardo established a theoretical system based on the labor theory of value and centered on the theory of distribution. He insisted in the principle stating that the value of a commodity was decided by the labor cost in it. He also criticized Smith’s theory of value by putting forward that labor that could decide value was socially necessary labor. In addition, labor that could decide the value of a commodity were not only direct living labor but also labor in factors of production. Ricardo suggested that all value was produced by labor and was distributed among three classes: wages that were decided by the value of essential means of subsistence of workers; profit was the surplus when wages were deducted; and land rent was the surplus when wages and profit were deducted.

Based on the the labor theory of value, Ricardo established the theory of comparative advantage. In *On the Principles of Political Economy and Taxation*, he clearly stated that “The value of a commodity, or the quantity of any other commodity for which it will exchange, depends on the relative quantity of labour which is necessary for its production”. He further stated that “the exchangeable value of these commodities, or the rule which determines how much of one shall be given in exchange for another, depends almost exclusively on the comparative quantity of labour expended on each”. The profit of each party in international trade is also totally related to the exchangeable value of all commodities in the international market, namely, the relative price level.
Ricardo regarded the free flow of factors of production, such as capital and labor, among regions and industries within a country as the fundamental reason of equalized rate of profit. The flow of factors between nations, however, would inevitably be interrupted by force or even totally stopped due to various reasons. Ricardo concluded that, it was the immobility of factors between countries that decided “the same rule which regulates the relative value of commodities in one country, does not regulate the relative value of the commodities exchanged between two or more countries”. Since there are numerous reasons for different relative prices of one commodity in different countries, there is room of profit for all the participating countries in the international trade. The premise of it, however, is that each country knows its advantage compared with others; that is to say, they are sure about their own comparative advantages.

1.18 Exercises

Exercise 1.1 (Economics and three dimensions of scientific economic analysis)
Answer the following questions:

1. What is the definition of Economics?
2. What are the two great objective realities facing the study of economic problems?
3. What is modern economics?
4. What is "three dimensions and six natures"?
5. Why does scientific economic analysis need "three dimensions and six natures"?

Exercise 1.2 (Differences between economics and natural science) Answer the following questions:

1. What are the main differences between economics and natural science?
2. Why these differences make economic research more complex and difficult?

Exercise 1.3 (Two basic categories of economic theory) Answer following questions:

1. Which two categories of economic theories can be classified according to their functions?
CHAPTER 1. NATURE OF MODERN ECONOMICS

2. Describe the connotation and function of each category, as well as the relationship between these two categories.

3. How should we correctly regard and deal with the interaction between these two kinds of economic theory?

Exercise 1.4 (The fundamental functions of economic theory) Answer the following questions:

1. What are the three main functions of economic theory?

2. Why is there not a best kind of economic theory that is always right and fits every development stage but a kind that fits certain institutional environment the best?

3. What are two common misunderstandings of economic theories?

Exercise 1.5 (Market and market mechanism) Answer the following questions:

1. From the perspectives of information and incentive, what are the advantages of the market economic system compared with the planned economic system?

2. Under the condition of market economy, what are three basic functions of price?

3. What is the superiority of market system?

4. What are the three development stages an economy will go through? How can the efficiency-driven and even innovation-motivated development be realized? What is the basic economic institution behind this?

Exercise 1.6 (The dialectical relationship between competition and monopoly) Answer the following questions:

1. Why do people want to introduce competitive mechanism in the view of social resource allocation, but enterprises want monopoly? Please state the dialectical relationship between competition and monopoly.

2. What does the Innovation Theory of Schumpeter tell us? Please state the importance of innovation-driven development.

3. Why do the innovation and entrepreneurship depend on institutional choice, and are therefore endogenous variables?

Exercise 1.7 (The boundaries among government, market and society) Answer the following questions:
1. Why is it necessary to reasonably define the boundaries between government and market and between government and society?

2. How should the boundaries between government and market and between government and society be roughly defined?

3. Why does a well-governed nation need to reasonably define not only the boundaries between government and market but also those between government and society?

**Exercise 1.8 (The three basic institutional arrangements for state governance)**

Answer the following questions:

1. What are three basic institutional arrangements for state governance?

2. State the range of application and limitation of each of the three basic institutional arrangements. Which one is most basic and important?

**Exercise 1.9 (The logic of development and governance)**

Answer the following questions:

1. How shall we correctly understand the logic of development and governance and dialectical relationship between the two?

2. Explain the achievements and limitations of the economic reform in China according to this framework.

**Exercise 1.10 (Ancient Chinese Economic Thought)**

Answer the following questions:

1. Give five examples to state the thought of market economy in ancient China.

2. Why could not those numerous deep economic thoughts in ancient China form a scientific economic theory?

**Exercise 1.11 (The cornerstone assumptions of modern economics)**

Answer the following questions:

1. State the relationship and distinction between self-love, selfishness, and self-interestedness?

2. Why does economics use the self-interest assumption as the most basic, important, and central assumption?

3. How shall we regard self-interestedness and altruism?

**Exercise 1.12 (Key points in modern economics)**

Answer the following questions:
1. What are the key points of modern economics? Please state each of them generally.

2. State the meanings, advantages, and limitations of centralized and decentralized decision-making.

3. Why are economic freedom and competition extremely important to economic development?

4. Why doing things under constraints is one of the most fundamental principles in economics?

5. What is the relationship between incentive and information?

6. Why are clearly defined property rights important to efficient allocation of resources?

7. Discuss the differences between equity in outcome and equity in opportunity. Which one does not conflict with efficiency?

**Exercise 1.13 (Proper understanding of modern economics)** Answer the following questions:

1. How shall we regard the scientific nature of modern economics?

2. How shall we regard the mathematical nature of modern economics?

3. How shall we regard the economic theory correctly?

4. How shall we regard the critics that economics cannot be tested through experiment?

**Exercise 1.14 (Basic analytical framework of modern economics)** Answer the following questions:

1. What are the components that constitute the basic analytical framework of a standard modern economic theory?

2. Why do different economic environments need different economic theories?

3. Why are different economic theories needed even for the same economic reality or environment under many circumstances?

4. Why should evaluation be included in the analytical framework?

5. Taking Coase Theorem as an example, expound on its economic analytical framework.
6. What are practical usages of the basic analytical framework and re-
search methodologies of modern economics?

**Exercise 1.15 (Benchmarks and reference system)** Answer the following ques-
tions:

1. What are definitions of benchmark and reference system?
2. Why are setting up benchmarks and establishing reference system the
premise of discussing economic problems?
3. What are the typical examples of reference systems?

**Exercise 1.16 (Methodologies)** Answer the following questions:

1. When considering the problem of economic reform, why is it impor-
tant to distinguish necessary condition from sufficient condition?
2. Why does it need both positive and normative analysis when dis-
cussing economic problems?
3. How shall we regard the role of mathematics and statistics in modern
economics?
4. How shall we complete the conversion between economic and math-
ematical languages?

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Chapter 2

Preliminary Knowledge and Methods of Mathematics

This chapter provides a lengthy and self-contained basic mathematical knowledge and results that are indispensable tools of modern economics in general and advanced microeconomic theory in particular, including basic knowledge and results of topology, linear algebra, mathematical analysis, fixed point theory, static optimization, dynamic optimization and probability theory. We will use relevant results. The readers with more mathematics training will also find this chapter useful as important reference course for most of the principal mathematical theorems that arise in modern economics.

We begin with some of those knowledge that will often encounter.

2.1 Basic Set Theory

This section introduces some basic concepts and results of set theory.

2.1.1 Set

A set $S$ is a collection of elements. According to the number of elements, a set can be a **finite set**, for example, $S = \{1, 3, 5, 7, 9\}$; an **infinite countable set**, for example, $S = \mathcal{N}$, where $\mathcal{N}$ is the set of all natural numbers; or an **infinite uncountable set**, for example, $S = \mathcal{R}$, where $\mathcal{R}$ is the set of all real numbers. A countable set can be finite or infinite. A set can also be described with some properties, for example, $S = \{1, 3, 5, 7, 9\} = \{x : x < 10, x \in \mathcal{N}, \frac{x}{2} \notin \mathcal{N}\}$. The **empty set** $\emptyset$ is a set consisting of no elements.

A subset $T$ of a set $S$ is also a set, and any element in $T$ belongs to $S$, denoted by $T \subseteq S$. If $T$ is a subset of $S$ and $S$ has at least one element that does not belong to $T$, then $T$ is a **proper subset** of $S$. If $T$ and $S$ are subsets of each other, then these two sets are equal, that is, $T = S$. 
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The union of two sets $T$ and $S$ is denoted by $T \cup S = \{x: x \in T \text{ or } x \in S\}$; the intersection of two sets $T$ and $S$ is denoted by $T \cap S = \{x: x \in T \text{ and } x \in S\}$.

The complement set of $S$ in the universal set $U$ is denoted by $S^c = \{x: x \in U, x \notin S\}$. The complement set of the universal set $U$ is the empty set, and the complement of the empty set is the universal set. The difference between sets $S$ and $T$ is denoted by $S \setminus T$ or $S - T$, which is defined as $S \setminus T = S - T = \{x: x \in S, x \notin T\}$.

The complement of the union or the intersection of any number of sets satisfies De Morgan’s law:

$$(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c;$$

$$(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c.$$ The product of sets $T$ and $S$ is denoted by $S \times T = \{(s,t): s \in S, t \in T\}$. The $n$-dimensional real space is defined as $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

2.1.2 Mapping

In order to discuss the number of elements of a set, we first introduce the concept of mapping or function.

Definition 2.1.1 (Mapping) Given sets $A$ and $B$, if for each element $x$ in $A$, there always exists an element $y$ in $B$ related to it, then this relation is called mapping or function, denoted by $f: A \to B$. The set $A$ is called domain of $f$, and the set $B$ is called range.

The image of $f$ is the set of points in the range into which some points in the domain is mapped, i.e., $I = \{y: y = f(x), \text{ for some } x \in A\}$.

The following definition gives the types of mapping.

Definition 2.1.2 Given sets $A$ and $B$, and a mapping $f: A \to B$, we call $f$ a surjection if $\{y \in B: f(x) = y, x \in A\} \equiv f(A) = B$; an injection if $f(x) \neq f(x')$ holds for all $x \neq x'$; a bijection if $f$ is both a surjection and an injection.

If there is a bijection $f$ between sets $A$ and $B$, then $A$ and $B$ are equivalent, denoted by $A \sim B$. Next we will discuss the number of elements of a set.

Definition 2.1.3 Let $J_n = \{1, 2, \cdots, n\}$ be the set of the first $n$ positive integers, and $J$ the set of all positive integers.
2.2. **BASIC LINEAR ALGEBRA**

(1) A set \( A \) is **finite**, if there exists a certain \( n \) such that \( A \sim J_n \);

(2) A set \( A \) is **countable**, if \( A \sim J \);

(3) A set \( A \) is **countable**, if it is either a finite set or a countable set;

(4) A set \( A \) is **uncountable**, if it is neither finite nor countable.

Both sets of natural and rational numbers are countable sets, but the real number set is not. The following conclusion shows that the set of all binary numbers is not countable.

**Theorem 2.1.1** Suppose that \( A \) is a set consisting of all sequences made up of 0 and 1. Then it is uncountable.

The proof is referred to the proof of Theorem 2.14 in Rudin (1976)'s Principles of Mathematical Analysis.

Since real numbers can be represented in binary, the set of real numbers is equivalent to the set \( A \) above, which is uncountable. Any interval \((a, b)\) is equivalent to the real space since \( f(x) = \frac{y}{1-|y|} \) with

\[
y = \frac{x - \frac{(a + b)}{2}}{\frac{(b - a)}{2}}
\]

is a bijection of them, thus any real interval is uncountable.

### 2.2 Basic Linear Algebra

#### 2.2.1 Matrix and Vector

We use \( \mathbb{R}^n \) to represent a set of all \( n \)-tuple real numbers. The elements of it are called **points** or **vectors**. \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \) represents a column vector, and \( x_i \) is the \( i \)th component of the vector \( x \). \( x' = (x_1, \cdots, x_n) \), a row vector, is defined as the **transposition** of \( x \). If not specially specified, vectors refer to column vectors in general.

The inequality signs \( \geq, \geq \) and \( > \) about vectors are defined as follows. Let \( a, b \in \mathbb{R}^n \), then \( a \geq b \) represents that \( a_s \geq b_s \) for all \( s = 1, \cdots, n \); \( a \geq b \) represents that \( a \geq b \) but \( a \neq b \); \( a > b \) represents that for all \( s = 1, \cdots, n \), \( a_s > b_s \).

In economics, it is usually required to solve the system of linear equations, which now can be easily expressed and solved by linear algebra.
We consider a system of \(m\) linear equations with \(n\) variables \((x_1, x_2, \ldots, x_n)\):

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\
    \cdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m
\end{align*}
\]

where the letter with double subscript, \(a_{ij}\), denotes the coefficient appearing in \(i\)th equation and attached to the \(j\)th variable \(x_j\), and \(d_j\) denotes the constant term on the right side of the \(j\)th equation.

It can be expressed more succinctly by the following matrix form:

\[
Ax = d,
\]

where \(A\), \(x\), \(d\) are respectively:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \cdots & \cdots & \cdots & \cdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

\[x = \begin{bmatrix}
    x_1 \\
    x_2 \\
    \cdots \\
    x_n
\end{bmatrix}
\]

\[d = \begin{bmatrix}
    d_1 \\
    d_2 \\
    \cdots \\
    d_m
\end{bmatrix}
\]

\(A\) is called the **coefficient matrix** of a \(m \times n\) system of equations, which consists of \(m\) rows and \(n\) columns; \(x\) is called a **variable vector**, and \(d\) is a **constant vector**. An \(n\)-dimensional vector can be viewed as a special \(n \times 1\) matrix.

As a shorthand device, the array in matrix \(A\) can be written more simple as

\[
A = [a_{ij}]_{m \times n} \quad (i = 1, 2, \cdots, m; \ j = 1, 2, \cdots, n).
\]

### 2.2.2 Matrix Operations

Here we give a brief introduction to some common matrix operations.

**Equality of Two Matrices:** \(A = B\) if and only if \(a_{ij} = b_{ij}\) holds for all \(i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n\).
2.2. BASIC LINEAR ALGEBRA

Addition and Subtraction of Matrices: $A \pm B = [a_{ij}] \pm [b_{ij}] = [a_{ij} \pm b_{ij}]$.

Note that addition and subtraction make sense only if the dimensions of matrices are the same.

Scalar Multiplication of matrices: $\lambda A = \lambda[a_{ij}] = [\lambda a_{ij}]$.

Matrix Multiplication: Given two matrices $A_{m \times n}$ and $B_{p \times q}$, matrix multiplication requires a compatibility condition: the number of columns in matrix $A$ is the same as the number of rows in matrix $B$, that is, $n = p$. If the compatibility condition is satisfied, the dimension of the product of $AB$ is $m \times q$. $AB$ is defined as:

$$AB = C$$

with $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{l=1}^{n} a_{il}b_{lj}$. Obviously, the matrix product of $AB$ is not necessarily equal to $BA$.

Identity Matrix

An identity matrix is a square matrix with the same number of rows and columns, which is assumed to be $n$.

An identity matrix of order $n$, denoted by $I_n$, is a square matrix with ones in its principal diagonal and zeros everywhere else.

It has the following properties:

Property 1:

$$I_mA_{m \times n} = A_{m \times n}I_n = A_{m \times n}$$

Property 2:

$$A_{m \times n}I_nB_{n \times p} = (A_{m \times n}I_n)B_{n \times p} = A_{m \times n}B_{n \times p}$$

Property 3:

$$(I_n)^k = I_n$$

This matrix is analogous to 1 in real space.

Null Matrix

A null or zero matrix (not necessarily square matrix) –denoted by 0, plays the role of the number 0.

A $m \times n$ null matrix is simply a matrix whose elements are all zero.

Null matrices obey the following rules of operation:

$$A_{m \times n} + 0_{m \times n} = A_{m \times n}$$;

$$A_{m \times n}0_{n \times p} = 0_{m \times p}$$;

$$0_{q \times m}A_{m \times n} = 0_{q \times n}.$$
Remark 2.2.1  

(1) $CD = CE$, $C \neq 0$ does not imply that $D = E$, e.g.,

$$
C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}, \quad
D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad
E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}.
$$

(2) Even if $A$ and $B \neq 0$, we still have $AB = 0$, e.g.,

$$
A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad
B = \begin{bmatrix} -2 & 4 \\ -1 & -2 \end{bmatrix}.
$$

2.2.3 Linear Dependence of Vectors

One of the most important properties among vectors is the linear dependence.

**Definition 2.2.1 (Linear Dependence)** A set of vectors $v^1, \cdots, v^n$ is linearly dependent, if and only if there exists a vector $v^i$ which is a linear combination of the others, namely, $v^i = \sum_{j \neq i} \alpha_j v^j$.

**Example 2.2.1** The following three vectors

$$
v^1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}
$$

are linearly dependent, since

$$
3v^1 - 2v^2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = v^3
$$

or

$$
3v^1 - 2v^2 - v^3 = 0
$$

where $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a zero vector.

2.2.4 Transpose and Inverse of Matrix

We first discuss the transpose of a matrix. The transpose of a $m \times n$ matrix $A$ is a matrix which is obtained by interchanging the rows and columns of the matrix $A$. Formally, we have

**Definition 2.2.2 (Transpose of Matrix)** $B = [b_{ij}]_{n \times m}$ is said to be the transpose of the matrix $A = [a_{ij}]_{m \times n}$, denoted by $A'$ or $A^T$, if $a_{ji} = b_{ij}$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$. 


Definition 2.2.3 We have the following types of matrices about the transpose of a matrix:

The matrix $A$ is said to be **symmetric** if $A' = A$.

The matrix $A$ is said to be **antisymmetric** if $A' = -A$.

The matrix $A$ is said to be **orthogonal** if $A'A = I$.

Properties of Transpose:

a) $(A')' = A$;

b) $(A + B)' = A' + B'$;

c) $(\alpha A)' = \alpha A'$, where $\alpha$ is a real number;

d) $(AB)' = B'A'$.

Next we discuss the **inverse** of a square matrix. The inverse of matrix $A$, denoted by $A^{-1}$, should satisfies

$$AA^{-1} = A^{-1}A = I.$$ 

While the transpose of matrix always exists, the inverse does not necessarily exist.

Remark 2.2.2 The following statements are true:

1). Not every square matrix has an inverse, that is, squareness is a necessary but not sufficient condition for the existence of an inverse. If a square matrix $A$ has an inverse, $A$ is said to be **nonsingular**. If $A$ has no inverse, it is said to be a **singular** matrix.

2). If $A$ is nonsingular, then $A$ and $A^{-1}$ are inverse of each other, i.e., $(A^{-1})^{-1} = A$.

3). If $A$ is $n \times n$, then $A^{-1}$ is also $n \times n$.

4). The inverse of $A$ is unique.

5). $AA^{-1} = I$ implies $A^{-1}A = I$.

6). Suppose that $A$ and $B$ are nonsingular matrices with dimension $n \times n$.

   (a) $(AB)^{-1} = B^{-1}A^{-1}$

   (b) $(A')^{-1} = (A^{-1})'$
CHAPTER 2. PRELIMINARY KNOWLEDGE AND METHODS OF MATHEMATICS

2.2.5 Solving a Linear System

We shall now discuss the inverse matrix and how to solve the system of linear equations. Consider a system of $n$ equations with $n$ unknowns:

$$Ax = d.$$  

If $A$ is nonsingular, then multiplying both sides by $A^{-1}$ gives:

$$A^{-1}Ax = A^{-1}d.$$  

Therefore, $x = A^{-1}d$ is the unique solution of the linear system $Ax = d$, where $A^{-1}$ is unique.

Before applying the method of inverse matrix to solve linear systems, we need to determine first whether matrix is nonsingular. Secondly, we shall solve it by the Cramer’s rule.

There are two ways to test the nonsingularity of a square matrix $A$. One is to see whether the row or column vectors of a matrix are linearly independent or not; the other is to see whether the determinant of a square matrix is equal to zero.

An $n \times n$ square matrix $A$ could be written as a set of vectors in terms of rows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix}$$

where $v_i' = [a_{i1}, a_{i2}, \cdots, a_{in}]$, $i = 1, 2, \cdots, n$. Whether a square matrix $A$ is nonsingular is determined by whether the vectors $v_i'$, $i = 1, 2, \cdots, n$ are linearly independent.

**Determinant of a Matrix**

The determinant of an $n$th order square matrix $A = (a_{ij})$, denoted by $|A|$ or $\text{det}(A)$, is a uniquely defined scalar associated with that matrix. Determinants are defined only for square matrices. Before giving the definition of an $n$th order square matrix $A$, we give the definition of $A$ with 2th and 3th orders, respectively.

For a $2 \times 2$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

its determinant is defined as follows:

$$|A| = a_{11}a_{22} - a_{12}a_{21}.$$
In view of the dimension of matrix $A$, $|A|$ as defined in the above is called a \textbf{second-order determinant}.

For a $3 \times 3$ matrix $A$, its \textbf{third-order determinant} is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

From the definition of the determinant for matrix $A$ with 2th or 3th order, it is defined as the sum of all possible products in which each product consists of elements in different row and column. This is true also for a general $n$th order square matrix $A$. Then, for a $2 \times 2$ matrix $A$, its determinants is defined as:

$$|A| = \sum_{(a_1, \cdots, a_n)} (-1)^{I(a_1, \cdots, a_n)} a_{1a_1} a_{2a_2} \cdots a_{na_n},$$

where $(a_1, \cdots, a_n)$ is the permutation of $(1, \cdots, n)$, and $I(a_1, \cdots, a_n)$ is the number of inversion times when reordering $(1, \cdots, n)$.

For example, $(2, 1, 3)$ is reordered once by $(1, 2, 3)$, and $(2, 3, 1)$ is reordered twice by $(1, 2, 3)$.

There is a simple method to calculate the determinant of a matrix, that is, the \textbf{Laplace expansion}:

$$|A| = \sum_{k=1}^{n} (-1)^{l+k} a_{lk} \times \det(M_{lk}), \text{ for any } l \in \{1, \cdots, n\},$$

where $M_{lk}$ is the $n-1$th order square matrix that results from $A$ by deleting the $l$-th row and the $k$-th column, called the \textbf{minor of} $a_{lk}$.

The Laplace expansion of an $n$th-order determinant will reduce the problem to one of evaluating $n$ minors, each of which is of the $(n-1)$th order, and the repeated application of the process will methodically lead to lower and lower orders of determinants, eventually culminating in the basic second-order determinants. Then the value of the original determinant can be easily calculated.

Even though one can expand $|A|$ by any row or any column, as the numerical calculation is concerned, a row or column with largest number of 0’s or 1’s is always preferable for this purpose, because a 0 times its cofactor is simply 0.

\textbf{Basic Properties of Determinants}

1. The determinant of a matrix $A$ has the same value as that of its transpose $A'$, i.e., $|A| = |A'|$. Thus, row independence is equivalent to column independence.
2. The multiplication of any one row (or column) by a scalar \( k \) will change the value of the determinant \( k \)-fold.

3. The interchange of any two rows (columns) will alter the sign but not the numerical value of the determinant.

4. A scale of any row (column) is added to any other row (column) does not change the value or the sign of the determinant.

5. If two rows (or columns) are proportional, i.e., they are linearly dependent, then the determinant will vanish.

6. \( \det(AB) = \det(A)\det(B) \).

7. \( \det(A^{-1}) = \frac{1}{\det(A)} \). Hence, a necessary condition for the existence of \( A^{-1} \) is that \( \det(A) \neq 0 \).

Using these properties, we can simplify the matrix (e.g. obtain as many zero elements as possible), the Laplace expansion of a determinant will become a much more manageable task.

Next is a formula for solving the inverse of a nonsingular square matrix. Let \( A^{-1} = (d_{ij}) \), then

\[
d_{ij} = \frac{1}{\det(A)} (-1)^{i+j} \det(M_{ij}).
\]

The Cramer’s Rule given below summarizes how to solve a linear system. For a system of linear equations:

\[
Ax = d,
\]

where

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix},
\]

\[
d' = (d_1, \cdots, d_n),
\]

\[
x' = (x_1, \cdots, x_n).
\]

The solution is:

\[
x_j = \frac{\det(A_j)}{\det(A)},
\]

where:

\[
A_j = \begin{bmatrix}
a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn}
\end{bmatrix}
\]

which is obtained by replacing the \( j \)th column of \( |A| \) with constant terms \( d_1, \cdots, d_n \). This result is the statement of Cramer’s rule.
2.2.6 Quadratic Form and Matrix

A function \( q \) with \( n \) variables is called a **quadratic form** if it has the following expression:

\[
q(x_1, x_2, \cdots, x_n) = a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
+ a_{22}x_2^2 + \cdots + 2a_{2n}x_2x_n \\
\vdots \\
+ a_{nn}x_n^2.
\]

Let \( a_{ij} = a_{ji}, i < j \), and then \( q(x_1, x_2, \cdots, x_n) \) can be written as

\[
q(x_1, x_2, \cdots, x_n) = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\
+ a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\
\vdots \\
+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j \\
= x'Ax,
\]

where

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

is called a matrix of quadratic form. Since \( a_{ij} = a_{ji} \), \( A \) is an \( n \)th-order symmetric square matrix.

**Definition 2.2.4** For a matrix of quadratic form \( A \), the quadratic form \( q(u_1, u_2, \cdots, u_n) = u'Au \) is said to be

(a) positive definite (PD) if \( q(u) > 0 \) for all \( u \neq 0 \);

(b) positive semidefinite (PSD) if \( q(u) \geq 0 \) for all \( u \neq 0 \);

(c) negative definite (ND) if \( q(u) < 0 \) for all \( u \neq 0 \);

(d) negative semidefinite (NSD) if \( q(u) \leq 0 \) for all \( u \neq 0 \).

Otherwise \( q \) is called indefinite (ID).

Sometimes, we say that a matrix \( D \) is, for instance, positive definite if the corresponding quadratic form \( q(u) = u'Du \) is positive definite.

A necessary and sufficient condition for a matrix of quadratic form \( A \) to be positive definite is that all its minors are positive. That is,
A necessary and sufficient condition for a quadratic form $A$ to be negative definite is that its minors are negative and positive alternately, that is,

$$|A_1| = A_{11} > 0;$$

$$|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0;$$

$$\ldots$$

$$|A_n| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0.$$  

2.2.7 Eigenvalues, Eigenvectors and Traces

If a square matrix $A$ and a real number $\lambda$ satisfies the equation $Ax = \lambda x$, then $\lambda$ is called the eigenvalue of $A$, and the vector $x$ is called the eigenvector of $A$ belonging to the eigenvalue $\lambda$.

Eigenvalues and some properties of matrix, such as positive or negative definiteness, have close connections. The following theorem characterizes the relation between the eigenvalues and positive (or negative) definiteness.

**Theorem 2.2.1** A Matrix of quadratic form $A$ is

- positive definite, if and only if eigenvalues $\lambda_i > 0$ for all $i = 1, 2, \ldots, n$;
- negative definite, if and only if eigenvalues $\lambda_i < 0$ for all $i = 1, 2, \ldots, n$;
- positive semi-definite, if and only if eigenvalues $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, n$;
- negative semi-definite, if and only if eigenvalues $\lambda_i \leq 0$ for all $i = 1, 2, \ldots, n$;
- indefinite, if at least one eigenvalue is positive and at least one eigenvalue is negative.

For a symmetric matrix $A$, there is a convenient decomposition method. Matrix $A$ is **diagonalizable** if there exists a non-singular matrix $P$ and a diagonal matrix $D$ such that

$$P^{-1}AP = D.$$  

Matrix $U$ is orthogonal matrix if $U' = U^{-1}$. 
2.2. BASIC LINEAR ALGEBRA

Theorem 2.2.2 (The Spectral Theorem for Symmetric Matrices) Suppose that $A$ is a symmetric matrix of order $n$ and $\lambda_1, \cdots, \lambda_n$ are its eigenvalues. Then there exists an orthogonal matrix $U$ such that

$$U^{-1}AU = \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{bmatrix}$$

or

$$A = U \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{bmatrix} U'.$$

Usually, $U$ is the orthogonal matrix formed by eigenvectors. It has the property $U'U = I$. “Orthogonal” means that for any column $u$ of the matrix $U$, $u'u = 1$.

The power operation of symmetric matrix has a convenient form:

$$A^k = U \begin{bmatrix} \lambda_1^k \\ & \ddots \\ 0 & \lambda_n^k \end{bmatrix} U'.$$

If the eigenvalues of $A$ are nonzero real numbers, then the inverse of $A$ can be reformulated as follows:

$$A^{-1} = U \begin{bmatrix} \lambda_1^{-1} & 0 \\ & \ddots \\ 0 & \lambda_n^{-1} \end{bmatrix} U'.$$

Another common concept about square matrix is the trace. The trace of an $n$th-order $A$ is $tr(A) = \sum_{i}^n a_{ii}$. It also has following properties:

1. $tr(A) = \lambda_1 + \cdots + \lambda_n$;
2. If $A$ and $B$ have the same dimension, then $tr(A + B) = tr(A) + tr(B)$;
3. If $a$ is a real number, $tr(aA) = a \cdot tr(A)$;
4. $tr(AB) = tr(BA)$, if $AB$ is a square matrix;
5. $tr(A') = tr(A)$;
6. $tr(A'A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$. 
2.3 Basic Topology

Topology that is a branch of mathematics studies the basic properties of topological spaces and various kinds of mathematical structures defined on them. This branch is originated from the study of the set of points on real axis, manifolds, metric spaces, and functional analysis.

There are two branches of topology. One focusing on using analysis method is called general topology, point-set topology, or analytic topology. If further subdivided, point-set topology also has a branch: differential topology. The other branch emphasizes on the use of algebraic method, called algebraic topology. However, these branches tend to be unified. Topology is widely applied in functional analysis, Lie Group, differential geometry, differential equations and many other branches of mathematics.

Here we give a very brief introduction to the basic knowledge of point-set topology, and apply them to establish some important conclusions about the properties of sets and continuous mappings between sets.

2.3.1 Topological Space

Definition 2.3.1 Suppose that $X$ is a nonempty set. We call a family of subsets of $X$, denoted by $\tau$, a topology of $X$ if

1. both $X$ and the empty set belong to $\tau$;
2. the union of any number of members in $\tau$ is in $\tau$;
3. the intersection of a finite number of members in $\tau$ is in $\tau$.

The set $X$ together with its topology $\tau$ is called a topological space, denoted by $(X, \tau)$; members in $\tau$ are called the open sets of this topological space.

Example: Examples of topological spaces:

1. **(Discrete Topology)** Suppose that $X$ is a nonempty set. $\tau = 2^X$ gives a discrete topology.
2. **(Trivial Topology)** Suppose that $X$ is a nonempty set. $\tau = \{X, \varnothing\}$ gives a trivial topology.
3. **(Euclidean Topology)** Suppose that $R$ is the set of all real numbers. $\tau$ defined as a collection of open sets gives the Euclidean topology (see the following definition).
4. **(Quotient Topology)** Suppose that $X$ is a nonempty set. By a given equivalence relation $R$, we partition $X$ into disjoint subsets, all of which make up a new collection, denoted by $X/R$. We specify the subset $U$ of $X/R$ as an open set, then $X/R$ gives a quotient topology if and only if the union of any elements of $U$ is an open set belonging to $X$. 
2.3. BASIC TOPOLOGY

Although the study object of topology can be an arbitrary type of sets, for the convenience of understanding and application, in the following, we mainly introduce some commonly used topological spaces, especially the metric spaces in the finite dimensional real space.

2.3.2 Metric Space

We first illustrate the definitions of metric and metric space. Metric is a measure of distance. A metric space \((X, d)\) is composed of a set \(X\) and the metric \(d\) defined on the elements of \(X\). The metric space may be of finite or infinite dimensions, depending on the topology structure defined on \(X\).

Metric should satisfy three basic assumptions. For any \(p, q, r \in X\), we have

1. \(d(p, q) > 0\) if and only if \(p \neq q\);
2. \(d(p, q) = d(q, p)\);
3. \(d(p, q) \leq d(p, r) + d(r, q)\).

Remark 2.3.1 If the metrics on the same set are different, then the corresponding metric spaces are different. For example,

1. Metric Space 1: \((X = \mathbb{R}^n, d_1)\), \(\forall x^1, x^2 \in X, d_1(x^1, x^2) = \sqrt{\sum_i (x^1_i - x^2_i)^2}\), and it is called the n-dimensional Euclidean space.
2. Metric Space 2: \((X = \mathbb{R}^n, d_2)\), \(\forall x^1, x^2 \in X, d_2(x^1, x^2) = \sum_i |x^1_i - x^2_i|\).
3. Metric Space 3: \((X = \mathbb{R}^n, d_3)\), \(\forall x^1, x^2 \in X, d_3(x^1, x^2) = \max\{|x^1_1 - x^2_1|, \cdots, |x^1_n - x^2_n|\}\).

Although the remaining discussions in this section are also true for general metric spaces, we mainly focus on Euclidean spaces for convenience of statement.

2.3.3 Open Sets, Closed Sets and Compact Sets

With the concept of metric, we can define clearly the proximity between points. In an \(n\)-dimensional Euclidean space, given \(x^0 \in \mathbb{R}^n\), the set of all points of distances less than \(\epsilon\) from \(x^0\) is called an open ball with radius \(\epsilon\) and center \(x^0\), denoted by \(B_\epsilon(x^0)\). A related concept is closed ball, which is given by the set of all points of distances less than or equal to \(\epsilon\), denoted by \(B^*_\epsilon(x^0)\).

Next, we give the definition of closed sets and compact sets.
Definition 2.3.2 The set \( S \subseteq \mathbb{R}^n \) is an open set if for any \( x \in S \), there always exists an \( \epsilon > 0 \) such that \( B_{\epsilon}(x) \subseteq S \).

Based on the definition of open sets, the following theorem gives some basic properties of open sets:

Theorem 2.3.1 (Open Sets in \( \mathbb{R}^n \)) In terms of open sets, the following conclusions are true.

1. The empty set \( \emptyset \) is an open set.
2. The universal space \( \mathbb{R}^n \) is an open set.
3. The union of open sets is an open set.
4. The intersection of a finite number of open sets is an open set.

**Proof.** (1) Since \( \emptyset \) has no elements, the proposition “for each points in \( \emptyset \), there is an \( \epsilon, \cdots \)”, satisfies the definition of an empty set.

(2) For any point in \( \mathbb{R}^n \) and any \( \epsilon > 0 \), according to the definition of an open ball, the set \( B_{\epsilon}(x) \) consist of points in \( \mathbb{R}^n \). Hence, \( B_{\epsilon}(x) \subseteq \mathbb{R}^n \), and then \( \mathbb{R}^n \) is open.

(3) For all \( i \in I \), let \( S_i \) be an open set. We need to show \( \bigcup_{i \in I} S_i \) is an open set. Suppose that \( x \in \bigcup_{i \in I} S_i \). Then for some \( i' \in I \), we have \( x \in S_{i'} \). Since \( S_{i'} \) is open, we have \( B_{\epsilon}(x) \subseteq S_{i'} \) for an \( \epsilon > 0 \). It then follows that \( B_{\epsilon}(x) \subseteq \bigcup_{i \in I} S_i \), and thus \( \bigcup_{i \in I} S_i \) is open.

(4) Suppose that \( B = \bigcap_{k=1}^{n} B_k \). If \( B = \emptyset \), it is clear that \( B \) is an open set. If \( B \neq \emptyset \), for any \( x \in B \), obviously, we have: for any \( k \in \{1, \cdots, n\} \), \( x \in B_k \). Since \( B_k \) is an open set, there must exist an \( \epsilon_k > 0 \) such that \( B_{\epsilon_k}(x) \subseteq B_k \). Let \( \epsilon = \min\{\epsilon_1, \cdots, \epsilon_n\} \). Then for any \( k \in \{1, \cdots, n\} \), \( B_{\epsilon}(x) \subseteq B_k \), so \( B_{\epsilon}(x) \subseteq B \). Hence, \( B \) is an open set.

The following theorem shows the relationship between open sets and open balls.

Theorem 2.3.2 (Each open set is a union of open balls) Suppose that \( S \) is an open set. Then for each \( x \in S \), there exists an \( \epsilon_x > 0 \) such that \( B_{\epsilon_x}(x) \subseteq S \), and also \( S = \bigcup_{x \in S} B_{\epsilon_x}(x) \).

**Proof.** Suppose that \( S \subseteq \mathbb{R}^n \) is an open set, then it follows from the definition of open sets that for any \( x \in S \), there exists an \( \epsilon_x > 0 \) such that \( B_{\epsilon_x}(x) \subseteq S \). We now need to show that \( x' \in S \) implies \( x' \in \bigcup_{x \in S} B_{\epsilon_x}(x) \), and \( x' \in \bigcup_{x \in S} B_{\epsilon_x}(x) \) implies \( x' \in S \).

If \( x' \in S \), then it follows from the definition of open balls with centre \( x' \) that \( x' \in B_{\epsilon_{x'}}(x). \) But \( x' \) belongs to any union containing this open ball. Hence, we have \( x' \in \bigcup_{x \in S} B_{\epsilon_x}(x) \).
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If \( x' \in \bigcup_{x \in S} B_{x'}(x) \), then \( x' \in B_{x'}(x) \). Since \( B_{x'}(x) \subseteq S \), it follows that \( x \in S \).

We now discuss closed sets and first give the definition of closed sets based on the definition of open sets.

**Definition 2.3.3 (Closed Sets in \( \mathbb{R}^n \))** If the complement of \( S \), that is \( S^c \), is an open set, then \( S \) is a closed set.

We also have some conclusions about the basic properties of closed sets.

**Theorem 2.3.3 (Closed Sets in \( \mathbb{R}^n \))** In terms of closed sets, the following conclusions are true.

1. The empty set \( \emptyset \) is a closed set.
2. The universal space \( \mathbb{R}^n \) is a closed set.
3. The intersection of any closed sets is a closed set.
4. The union of a finite number of closed sets is a closed set.

**Proof.**

1. Since \( \emptyset = \{\mathbb{R}^n\}^c \), and \( \mathbb{R}^n \) is an open set, it follows from the definition of closed sets that \( \emptyset \) is a closed set.

2. Since \( \{\mathbb{R}^n\}^c = \emptyset \), and \( \emptyset \) is an open set, it follows from the definition of closed sets that \( \mathbb{R}^n \) is a closed set.

3. Suppose that for all \( i \in I \), \( S_i \) is a closed set in \( \mathbb{R}^n \). Then we need to show that \( \bigcap_{i \in I} S_i \) is closed. Since \( S_i \) is closed, its complement \( S_i^c \) is an open set. The union \( \bigcup_{i \in I} S_i^c \) is also open. It follows from the De Morgan’s laws that \( \bigcap_{i \in I} S_i^c = \bigcup_{i \in I} S_i \) holds. Since \( \bigcup_{i \in I} S_i^c \) is open, and then its complement \( \bigcap_{i \in I} S_i \) is closed.

4. Let \( C_1 \) and \( C_2 \) be closed sets and denote \( C = C_1 \cup C_2 \). Since \( C_1 \) and \( C_2 \) are closed, \( C_k^c = B_k, k = 1, 2 \), are open. It follows from the properties of open sets above that \( B_1 \cap B_2 \) is an open set, and thus \( C = (B_1 \cap B_2)^c \) is a closed set.

Next we discuss the concept of point sets related to open and closed sets.

**Definition 2.3.4** For set \( S \) in \( \mathbb{R}^n \), a point \( x \in S \) is called **limit point (or cluster point or accumulation point)** if for any \( \epsilon > 0 \), \( B_\epsilon(x) \cap S \neq \emptyset \), i.e., every neighbourhood of \( x \) contains at least one point of \( S \) different from \( x \) itself. The collection of all limit points of set \( S \) is denoted by \( \partial S \). For set \( S \), a point \( x \in S \) is called an **interior point** of \( S \), if there is an \( \epsilon > 0 \) such that \( B_\epsilon(x) \subseteq S \).

Now, we can redefine the open set as follows: a set is open if every element in the set is an interior point. Similarly, closed sets can also be defined as follows: a set is called a closed set if all limit points of the set belong to itself. In addition, for any set \( S \) in a metric space, the smallest
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closed set containing $S$ is called the closure of the set, denoted by $\bar{S} = S \cup \partial S$ or $\text{cl} S$. Obviously, if $S$ is closed, $\bar{S} = S$.

Next, we discuss a class of special but widely used closed sets, namely compact sets. We first introduce the concept of bounded sets.

Definition 2.3.5 (Bounded Set) If a set $S$ in $\mathbb{R}^n$ is contained in a ball (open or closed ball) with radius $\epsilon$, then $S$ is called bounded. In other words, if for some $x \in \mathbb{R}^n$, there is an $\epsilon > 0$ such that $S \subseteq B_\epsilon(x)$, then $S$ is bounded.

Definition 2.3.6 (Compact Set) If a set $S \subseteq \mathbb{R}^n$ is closed and bounded, then it is compact.

Compact set is a crucial concept in mathematical analysis, but the definition of compact sets given above is only for finite dimensional spaces. The definition of compact sets for infinite dimensional spaces is based on the concept of open cover. Whether a set is in finite or infinite dimension, there is another way to define compact sets, that is, a set is compact if each open cover of a set has a finite subcover.

We first introduce the concept of open covering.

Definition 2.3.7 (Open Cover) For a set $S$ and a collection of open sets $\{G_\alpha\}$ in metric space $X$, if $S \subseteq \bigcup_\alpha G_\alpha$, then $\{G_\alpha\}$ is called an open cover of $S$; if the index set $\{\alpha\}$ is finite, it is called a finite open cover.

Next, we discuss an important feature of compact set. The following Heine-Borel theorem, is also known as the finite covering theorem, proved that the above two ways of definition are consistent for the compact sets in finite dimensional spaces.

Theorem 2.3.4 (Heine-Borel Theorem or Finite Covering Theorem) For a set $S \subseteq \mathbb{R}^n$, the following two arguments are consistent:

1. $S$ is a bounded closed set;
2. Any open cover of $S$ has a finite subcover $\{G_\alpha\}$. That is, for $\{G_\alpha\}$, there is a finite set $\{1, \cdots , n\} \subseteq \{\alpha\}$ such that $S \subseteq \bigcup_{i=1}^n G_i$.

The proof is referred to the proof of Theorem 2.41 in Rudin’s Principles of Mathematical Analysis.

2.3.4 Connectedness of Set

Now we introduce the concept and properties of connected sets.

Definition 2.3.8 (Connected Set) For a set $S$ in metric spaces, if there do not exist two sets $A$ and $B$ such that $A \cap B = B \cap A = \emptyset$, and $S \subseteq A \cup B$, then $S$ is called a connected set.
The following theorem illustrates the characteristic of connected sets.

**Theorem 2.3.5** The set $S \subseteq \mathbb{R}^1$ is connected if and only if it satisfies the following property: for any $x, y \in S$, if $x < z < y$, then $z \in S$.

The proof is referred to the proof of the theorem 2.47 in Rudin’s *Principles of Mathematical Analysis*. Obviously, the whole real space is connected, and intervals in real space, such as $(a, b)$ and $[a, b]$, are all connected sets.

### 2.3.5 Sequences and Convergence

We first give some concepts of the sequences and convergence.

**Definition 2.3.9 (Sequence in $\mathbb{R}^n$)** Let $Z$ be the set of positive integers. A sequence in $\mathbb{R}^n$ is a function which maps $Z$ into $\mathbb{R}^n$, represented by $\{x^k\}_{k \in Z}$, and for each $k \in Z$, $x^k \in \mathbb{R}^n$.

For all sufficiently large $k$, if each element of sequence $\{x^k\}$ can arbitrarily approach a point in $\mathbb{R}^n$, then we call that the sequence converges to this point. Formally, we have the following definition:

**Definition 2.3.10 (Convergent Sequence)** If for each $\epsilon > 0$, there is a $\bar{k}$ such that for all $k \in Z$ larger than $\bar{k}$, $x^k \in B_\epsilon(x)$, then we call that the sequence $\{x^k\}_{k \in Z}$ converges to $x \in \mathbb{R}^n$.

Like subsets of a set, we have the concept of subsequences of a sequence.

**Definition 2.3.11 (Subsequence)** If $J$ is an infinite subset of $Z$, then $\{x^k\}_{k \in J}$ is called a subsequence of $\{x^k\}_{k \in Z}$ in $\mathbb{R}^n$.

**Definition 2.3.12 (Bounded Sequences)** If for $M \in \mathbb{R}$ and any $k \in Z$, $\|x^k\| \leq M$, then the sequence $\{x^k\}_{k \in Z}$ in $\mathbb{R}^n$ is bounded.

The following is a property of the subsequence of a bounded sequence.

**Theorem 2.3.6 (Bounded Sequences)** Each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

### 2.3.6 Convex Set and Convexity

Convex set is an important kind of set, which is widely used in economics. For example, sets of budget constraints are generally convex sets and have a strong economic meaning.

We first define the convex sets.

**Definition 2.3.13** If for any two elements $x^1, x^2 \in S$ and any $t \in [0, 1]$, we have $tx^1 + (1-t)x^2 \in S$, then the set $S \subseteq \mathbb{R}^n$ is a convex set.
If \( z = tx^1 + (1-t)x^2, t \in (0, 1), \) then point \( z \) is called the weighted average or convex combination of \( x^1 \) and \( x^2 \). If \( z = \sum_{l=1}^{k} \alpha^l x^l, x^l \in S, \alpha^l \in [0,1], l \in \{1, \cdots, k\}, \sum_l \alpha^l = 1, \) then \( z \) is also a convex combination of \( \{x^l\} \).

We have the following theorems about convex sets.

**Theorem 2.3.7**  If both sets \( S \) and \( T \) are convex, then their intersection \( T \cap S \) is also convex.

Any set can be convexified, i.e., it has a convex hull, denoted by \( \text{co}S \).

**Definition 2.3.14**  The **convex hull** of a set \( S \subseteq \mathbb{R}^n \) is the smallest convex set containing \( S \), denoted by \( \text{co}S \).

The following theorem illustrates how to convexify a set.

**Theorem 2.3.8**  For a set \( S \subseteq \mathbb{R}^n \), its convex hull is

\[
\text{co}S = \left\{ y \in \mathbb{R}^n, \; y = \sum_{l=1}^{k} \alpha^l x^l, x^l \in S, \alpha^l \in [0,1], \forall l \in \{1, \cdots, k\}, \sum_l \alpha^l = 1 \right\},
\]

that is, the convex hull of \( S \) is formed by convex combinations of all finite points in \( S \).

The points of convex hull are made up of convex combinations of finite points. The following Caratheodory theorem simplifies the way of convexification in finite dimensional real space.

**Theorem 2.3.9 (Caratheodory Theorem)**  If the set is in a finite dimensional real space, namely \( S \subseteq \mathbb{R}^n \), the points of its convex hull \( \text{co}S \) can be written as the convex combination of at most \( n + 1 \) points in \( S \).

The following theorems show that the convex hull of a compact set is a compact set.

**Theorem 2.3.10**  If \( S \subseteq \mathbb{R}^n \) is a compact set, then its convex hull \( \text{co}S \) is also a compact set.

See A3.1 of Kreps (2013) for the proof of the above three theorems.

Every point in a convex hull is a convex combination of finite points in a set, but it doesn’t mean that it must be a convex combination formed by other points. If a point is not a convex combination formed by other points, we define such a point as the extreme point. For compact sets, the structure of convex hull will be more simplified. The following Krein-Milman theorem characterizes the convex hull of compact sets.

**Theorem 2.3.11 (Krein-Milman Theorem)**  If a set \( S \) is a compact set of the finite dimensional real space, and \( EX(S) \) is the set of the extreme points of set \( S \), then \( \text{co}S = \text{co} EX(S) \), which means the convex hull of a compact set is composed of finite convex combinations of all the extreme points.
2.4 Single-Valued Function

2.4.1 The continuity of functions

The continuity of functions can be defined in any topological space. However, in order to state conveniently and introduce the commonly used results, here, without loss of generality, suppose $X \subseteq \mathbb{R}^n$.

Definition 2.4.1 (Continuity) For a function $f : X \rightarrow \mathbb{R}$ and $x_0 \in X$, if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

we call that $f$ is continuous at $x_0$; or equivalently, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $x \in X$ with $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| < \varepsilon;$$

or equivalently, the upper contour set of $f$ at $x_0$

$$U(x_0) \equiv \{x' \in X : f(x') \geq f(x_0)\}$$

and its lower contour set

$$L(x_0) \equiv \{x' \in X : f(x') \leq f(x_0)\}$$

are both the closed subsets of $X$.

If $f$ is continuous at any $x \in X$, we say the function $f : X \rightarrow \mathbb{R}$ is continuous on $X$.

Although the three definitions of continuity are all equivalent, the third definition is easier to verify. The idea of continuity is very intuitive. If we draw the function, the curve has no disconnected point.

The function is continuous, and then the change of $f(x)$ is small when $x$ changes slightly.

The following theorem illustrates the relationship between the continuity of functions and the open sets.

Theorem 2.4.1 (Continuity and Inverse Image) Let $D$ be a subset of $\mathbb{R}^m$, then the following conditions are equivalent.

1. $f : D \rightarrow \mathbb{R}^n$ is continuous.
2. For each open ball $B$ in $\mathbb{R}^n$, $f^{-1}(B)$ is also open in $D$.
3. For each open set $S$ in $\mathbb{R}^n$, $f^{-1}(S)$ is also open in $D$.

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose that (1) holds and $B$ is an open ball in $\mathbb{R}^n$. Picking any $x \in f^{-1}(B)$, we have $f(x) \in B$. Since $B$ is open in $\mathbb{R}^n$, then there
is an \(\varepsilon > 0\) such that \(B_\varepsilon(f(x)) \subseteq B\), and it follows from the continuity of \(f\) that there is a \(\delta > 0\) such that \(f(B_\delta(x) \cap D) \subseteq B_\varepsilon(f(x)) \subseteq B\). Hence, \(B_\delta(x) \cap D \subseteq f^{-1}(B)\). Since \(x \in f^{-1}(B)\) is arbitrary, it can be seen that \(f^{-1}(B)\) is open in \(D\), thus (2) is established.

(2) \(\Rightarrow\) (3). Suppose that (2) holds and \(S\) is open in \(\mathbb{R}^n\). Then \(S\) can be written as a union of open balls \(B_i(i \in I)\) such that \(S = \bigcup_{i \in I} B_i\). Hence, \(f^{-1}(S) = f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)\). It follows from (2) that each set \(f^{-1}(B_i)\) is open in \(D\), and then \(f^{-1}(S)\) is the union of open sets in \(D\). Therefore \(f^{-1}(S)\) is also open in \(D\). Since \(S\) is an arbitrary open set in \(\mathbb{R}^n\), (3) is established.

(3) \(\Rightarrow\) (1). Suppose that (3) holds. Take \(x \in D\) and \(\varepsilon > 0\). Then, since \(B_\varepsilon(f(x))\) is open in \(\mathbb{R}^n\), it follows from (3) that \(f^{-1}(B_\varepsilon(f(x)))\) is open in \(D\). Since \(x \in f^{-1}(B_\varepsilon(f(x)))\), there is a \(\delta > 0\) such that \(B_\delta(x) \cap D \subseteq f^{-1}(B_\varepsilon(f(x)))\), which means that \(f(B_\delta(x) \cap D) \subseteq B_\varepsilon(f(x))\). Therefore, \(f\) is continuous at \(x\). Since \(x\) is arbitrary, (1) is established. 

We have the following conclusion for a continuous function whose domain is a compact set.

**Theorem 2.4.2 (The continuous image of a compact set is a compact set)** Suppose that \(f : D \subseteq \mathbb{R}^m \to \mathbb{R}^n\) is a continuous function. If \(S \subseteq D\) is a compact set in \(D\), then its image \(f(S) \subseteq \mathbb{R}^n\) is compact in \(\mathbb{R}^n\).

### 2.4.2 Upper Semi-continuity and Lower Semi-continuity

The upper semi-continuity and lower semi-continuity of functions are weaker than continuity. Suppose that \(X\) is an arbitrary topological space.

**Definition 2.4.2** A function \(f : X \to \mathbb{R}\) is said to be upper semi-continuous at point \(x_0 \in X\) if we have

\[
\limsup_{x \to x_0} f(x) \leq f(x_0);
\]

or equivalently, for any \(\epsilon > 0\), there is a \(\delta > 0\) such that for any \(x \in X\) with \(|x - x_0| < \delta\), we have

\[
f(x) < f(x_0) + \epsilon;
\]

or equivalently, the upper contour set \(U(x_0)\) of \(f\) is a closed set of \(X\).

A function \(f : X \to \mathbb{R}\) is said to be upper semi-continuous on \(X\) if \(f\) is upper semi-continuous at every point \(x \in X\).

**Definition 2.4.3** A function \(f : X \to \mathbb{R}\) is said to be lower semi-continuous on \(X\) if \(-f\) is upper semi-continuous.

It is clear that a function \(f : X \to \mathbb{R}\) is continuous on \(X\) if and only if it is both upper and lower semi-continuous.
2.4. Transfer Upper and Lower Continuity

A weaker concept of continuity is transfer continuity. It is used to completely characterize the extreme values of functions or preferences (see a series of papers by Tian (1992, 1993, 1994), Tian & Zhou (1995) and Zhou & Tian (1992)).

**Definition 2.4.4** A function \( f : X \rightarrow \mathbb{R} \) is said to be transfer (weakly) upper continuous on \( X \), if for any points \( x, y \in X \), \( f(y) < f(x) \) means that there exists a point \( x' \in X \) and a neighbourhood \( N(y) \) of \( y \) such that \( f(z) < f(x') \) (\( f(z) \leq f(x') \)) for all \( z \in N(y) \).

**Definition 2.4.5** A function \( f : X \rightarrow \mathbb{R} \) is said to be transfer (weakly) lower continuous on \( X \), if \( -f \) is transfer (weakly) upper continuous on \( X \).

**Remark 2.4.1** It is clear that the upper (lower) semi-continuity of a function implies the transfer upper (lower) continuity (let \( x' = x \)); while the transfer upper (lower) continuity implies the transfer weakly upper (lower) continuity of the function, and the converse may not be true. We will then prove that a function \( f \) has the maximal (minimal) value on the compact set \( X \) if and only if \( f \) is transfer weakly upper continuous on \( X \), and the set of maximal (minimal) points of \( f \) is compact if and only if \( f \) is transfer upper (lower) continuous on \( X \).

2.4.4 Differentiation and Partial Differentiation of Functions

The differentiability of a function in one-dimensional real space measures the sensitivity to the change of function’s value with respect to a change in an independent variable. Let \( X \) be a subset of \( \mathbb{R} \).

**Definition 2.4.6 (Derivative)** The derivative of \( f : X \rightarrow \mathbb{R} \) at point \( x_0 \in X \) is defined as

\[
f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},
\]

where \( \Delta x = x - x_0 \).

Obviously, if a function has derivative at a point, then it must be continuous, but it may not be true for the converse. We may use the derivatives to find the limit of a continuous function of which the numerator and denominator approach to zero (or infinity), that is, we have the following L’Hôpital rule:

**Theorem 2.4.3 (L’Hôpital Rule)** Suppose that \( f(x) \) and \( g(x) \) are differentiable on an open interval \( I \) except possibly at \( c \). If \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \) or
\[ \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm \infty, \quad g'(x) \neq 0 \quad \text{for all } x \in I \text{ with } x \neq c, \] and
\[ \lim_{x \to c} \frac{f(x)}{g(x)} \text{ exists. Then} \]
\[ \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}. \]

Higher order derivatives and partial derivatives are widely used in economics.

**Definition 2.4.7 (Higher Order Derivative)** The \( n \)th order derivative of \( f : X \to \mathbb{R} \) at \( x_0 \in X \) is defined as
\[ f^{[n]}(x_0) = \lim_{\Delta x \to 0} \frac{f^{[n-1]}(x_0 + \Delta x) - f^{[n-1]}(x_0)}{\Delta x}. \]

In a multidimensional real space \( X \subseteq \mathbb{R}^n \), we can define the concept of partial differentiation of a function \( f : X \to \mathbb{R}, f(x_1, \ldots, x_n) \), to measure the degree of change of the function value with respect to a change of one of \( n \) independent variables, with the others being held constant.

**Definition 2.4.8 (Partial Derivative)** The partial derivative of \( f : X \to \mathbb{R}, X \subseteq \mathbb{R}^n \) with respect to \( x_i \) at \( x_0 = (x_0^1, \ldots, x_0^n) \in X \) is defined as
\[ \frac{\partial f(x^0)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_0^1, \ldots, x_0^i + \Delta x_i, \ldots, x_0^n) - f(x_0)}{\Delta x_i}. \]

We characterize the degree of change of a multidimensional function in different directions in the way of matrix, which is called gradient vector.

**Definition 2.4.9 (Gradient Vector)** Let \( f \) be a function defined on \( \mathbb{R}^n \) that has partial derivatives. Define the gradient of \( f \) as a vector
\[ Df(x) = \left[ \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right]. \]

Suppose that \( f \) has second-order partial derivative. We define the Hessian matrix of \( f \) at \( x \) as an \( n \times n \) matrix \( D^2 f(x) \):
\[ D^2 f(x) = \left[ \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right]. \]

If all the second-order partial derivatives are continuous, then
\[ \frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}, \]
and thus the above matrix is a symmetric matrix.
2.4. SINGLE-VALUED FUNCTION

2.4.5 Mean Value Theorem and Taylor Expansion

Theorem 2.4.4 (Féram lemma) Let $X$ be a subset of $\mathbb{R}$. Suppose that

(i) $f : X \to \mathbb{R}$ is well-defined in a neighborhood $N(x_0)$ of $x_0$, and $f(x) \leq f(x_0)$ or $f(x) \geq f(x_0)$ in this neighborhood;

(ii) $f(x)$ is derivable at point $x_0$.

Then we have $f'(x_0) = 0$.

Theorem 2.4.5 (Rolle Theorem) Suppose that $f$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $f(a) = f(b)$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

From Roll Theorem, we can have the well-known and useful Lagrange’s Theorem or the Mean-Value Theorem.

Theorem 2.4.6 (The Mean-Value Theorem or the Lagrange Formula) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The above mean value theorem is also true for multivariate $x$. If function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, there is $z \in \mathbb{R}^n$ such that

$$f(y) = f(x) + Df(z)(y - x).$$

Proof. Let $g(x) = f(x) - \frac{f(b) - f(a)}{b - a} x$. Then, $g$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $g(a) = g(b)$. Thus, by Rolle or Féramat’s Theorem, there exists a point $c \in (a, b)$ such that $g'(c) = 0$, and therefore $f'(c) = \frac{f(b) - f(a)}{b - a}$.

An variation of the above mean-value theorem is in form of integral calculus:

Theorem 2.4.7 (Mean-Value Theorem of Integral Calculus) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then there exists a number $c \in (a, b)$ such that

$$\int_a^b f(x) dx = f'(c)(b - a).$$

The second variation of the mean-value theorem is the generalized mean-value theorem:
Theorem 2.4.8 (Cauchy’s Theorem or the Generalized Mean -Value Theorem)
If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists a point \( c \in (a, b) \) such that
\[
(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).
\]
Taylor’s expansion is a useful method for solving approximation.
Consider a continuously differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \), \( x, y \in \mathbb{R}^n \), by the mean-value theorem, we know that there exist \( z, w \in (x, y) \) such that the following two equations hold:
\[
f(y) = f(x) + Df(z)(y - x),
\]
\[
f(y) = f(x) + Df(x)(y - x) + \frac{1}{2!}(y - x)'D^2f(w)(y - x),
\]
where \((y - x)'\) is the transpose of the vector \((y - x)\).

Generally, we have the following theorem:

Theorem 2.4.9 (Taylor’s Theorem)
Given any function \( f(x) \), if there exists \((n+1)\)th order derivative at \( x_0 \), then the function can be expanded at \( x_0 \):
\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + R_n
\]
\[
\equiv P_n + R_n,
\]
where \( P_n \) represents the \( n \)-th order polynomial, and \( R_n \) is the Lagrange’s remainder:
\[
R_n = \frac{f^{(n+1)}(P)}{(n + 1)!}(x - x_0)^{n+1},
\]
where \( P \) is a point between \( x \) and \( x_0 \), and \( n! \) is the factorial of \( n \):
\[
n! \equiv n(n - 1)(n - 2) \cdots (3)(2)(1).
\]
We have the following approximation of function by Taylor’s expansion. If \( y \) approximates \( x \), then
\[
f(y) \approx f(x) + Df(x)(y - x),
\]
\[
f(y) \approx f(x) + Df(x)(y - x) + \frac{1}{2}(y - x)'D^2f(x)(y - x).
\]

2.4.6 Homogeneous Functions and Euler’s Theorem

Definition 2.4.10 Let \( X = \mathbb{R}^n \). A function \( f : X \to \mathbb{R} \) is said to be homogeneous of degree \( k \) if for any \( t \), \( f(tx) = t^k f(x) \).
An important result concerning homogeneous function is the Euler’s theorem.

Theorem 2.4.10 (Euler’s Theorem) A function \( f : \mathbb{R}^n \to \mathbb{R} \) is homogeneous of degree \( k \) if and only if
\[
kf(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i.
\]
2.4.7 Implicit Function Theorem

If a variable \( y \) is clearly expressed as a function of \( x \), we call \( y = f(x_1, x_2, \cdots, x_n) \) an explicit function. In many cases, \( y \) is not an explicit function, and the relationship between \( y \) and \( x_1, \cdots, x_n \) is expressed by the following equation:

\[
F(y, x_1, x_2, \cdots, x_n) = 0.
\]

For a domain \( D \), if for each vector \( x \in D \), there is a unique determined value \( y \) satisfying the above equation, then \( y \) is an implicit function of \( x \), denoted by \( y = f(x_1, x_2, \cdots, x_n) \). Thus, the question is how to determine whether there is a unique value \( y \) satisfying this equation for every \( x \) in a certain domain. The following implicit function theorem indicates that under certain conditions the implicit function \( y = f(x_1, x_2, \cdots, x_n) \) determined by \( F(y, x_1, x_2, \cdots, x_n) = 0 \) not only exists but also is differentiable.

Theorem 2.4.11 (Implicit Function Theorem) Let \( X = \mathbb{R}^n \). Suppose that a function \( F(y, x_1, x_2, \cdots, x_n) = 0 \) satisfies the following four conditions:

(a) \( F_y, F_{x_1}, F_{x_2}, \cdots, F_{x_n} \) are continuous in the domain \( X \) containing \( (y_0, x_1^0, x_2^0, \cdots, x_n^0) \);

(b) \( F(y, x_1, x_2, \cdots, x_n) \) has continuous partial derivatives with respect to \( x \) and \( y \) in the domain \( X \);

(c) \( F(y_0, x_1^0, x_2^0, \cdots, x_n^0) = 0 \);

(d) The partial derivative \( F_y \) of \( F(y, x_1, x_2, \cdots, x_n) \) with respect to \( y \) at \( (y_0, x_1^0, x_2^0, \cdots, x_n^0) \) is not equal to zero.

Then:

1. In a neighbourhood \( N(x^0) \) of \( (x_1^0, x_2^0, \cdots, x_n^0) \), the function \( y = f(x_1, x_2, \cdots, x_n) \) of \( (x_1, x_2, \cdots, x_n) \) can be defined implicitly, which satisfies \( F(y(x_1, \cdots, x_n), x_1, x_2, \cdots, x_n) = 0 \) and \( y_0 = f(x_1^0, x_2^0, \cdots, x_n^0) \).

2. \( y = f(x_1, x_2, \cdots, x_n) \) is continuous in \( N(x^0) \).

3. \( y = f(x_1, x_2, \cdots, x_n) \) has continuous partial derivatives in \( N(x^0) \), which is given by:

\[
\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}, \quad i = 1, \cdots, n.
\]

2.4.8 Concave and Convex Function

Next we discuss the concavity and convexity of functions. Concave functions, convex functions and quasi-concave functions are common functions in economics and have strong economic significance. They hold a special position in optimization problems.

Let \( X \subseteq \mathbb{R}^n \) be a convex set.
Definition 2.4.11  For a function $f : X \to \mathbb{R}$, if for any $x, x' \in X$ and any $t \in [0, 1]$, we have
\[ f(tx + (1-t)x') \geq tf(x) + (1-t)f(x'), \]
then $f$ is said to be concave on $X$.

If for all $x \neq x' \in X$ and $0 < t < 1$, we have
\[ f(tx + (1-t)x') > tf(x) + (1-t)f(x'), \]
then $f$ is said to be strictly concave on $X$.

Definition 2.4.12  If $-f$ is (strictly) concave on $X$, then $f : X \to \mathbb{R}$ is called a (strictly) convex function on $X$.

Remark 2.4.2  We have the following results:

1. A linear function is both convex and concave.
2. The sum of two concave (convex) functions is still concave (convex).
3. The sum of a concave (convex) function and a strictly concave (convex) function is strictly concave (convex).

The statement that the function $f : X \to \mathbb{R}$ is concave on $X$ is equivalent to the statement that for any $x_1, \cdots, x_m \in X$ and any $t_i \in [0, 1]$, we have
\[ f(t_1x_1 + t_2x_2 + \cdots + t_mx_m) \geq t_1f(x_1) + \cdots + t_mf(x_m). \]

This formula is also called Jensen’s inequality. If $t_i$ is regarded as the probability of $x_i$, when $f : X \to \mathbb{R}$ is concave on $X$, Jensen’s inequality implies that the expectation of function value with respect to a random variable is not greater than the function value with respect to the expectation of the random variable, i.e.,
\[ f(E(X)) \geq E(f(X)). \]

When a function is twice differentiable, its convexity or concavity can be determined by whether the second-order partial derivative matrix is positive (negative) definite.

Remark 2.4.3  A function $f$ defined on $X$ has a continuous second-order partial derivative, then it is a concave (convex) function if and only if its Hessian matrix $D^2f(x)$ is negative (positive) semi-definite on $X$. It is strictly concave (convex) if and only if its Hessian matrix $D^2f(x)$ is negative (positive) definite on $X$. 
Remark 2.4.4 The strict concavity of the function $f(x)$ can be determined by testing whether the principal minors of the Hessian matrix change signs alternately, namely,

$$
\begin{vmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{vmatrix} > 0,
$$

$$
\begin{vmatrix}
  f_{11} & f_{12} & f_{13} \\
  f_{21} & f_{22} & f_{23} \\
  f_{31} & f_{32} & f_{33}
\end{vmatrix} < 0,
$$

and so on, where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. This algebraic condition is very useful for testing second-order conditions of optimality.

2.4.9 Quasi-concave and Quasi-convex Function

Quasi-concave functions are frequently used in economics. The concept of quasi-concavity is weaker than that of concavity.

Let $X \subseteq \mathbb{R}^n$ be a convex set.

Definition 2.4.13 For a function $f : X \to \mathbb{R}$, if for any $x, x' \in X$ and any $t \in [0, 1]$, we have

$$
f(tx + (1-t)x') \geq \min\{f(x), f(x')\},
$$

then $f$ is called quasi-concave on $X$.

If for any $x, x' \in X$ and any $t \in [0, 1]$, we have

$$
f(tx + (1-t)x') > \min\{f(x), f(x')\},
$$

then $f$ is strictly quasi-concave on $X$.

If $-f$ is (strictly) quasi-concave on $X$, then the function $f : X \to \mathbb{R}$ (strictly) quasi-convex on $X$.

Remark 2.4.5 The following facts are clear:

(1) If a function $f$ is (strictly) concave (convex), then it is (strictly) quasi-concave (convex);

(2) The function $f$ is (strictly) quasi-concave if and only if $-f$ is (strictly) quasi-convex;

(3) An arbitrary (strictly) monotone function defined on an interval of one-dimensional real number space is both (strictly) quasi-concave and (strictly) quasi-convex;

(4) The sum of two quasi-concave (convex) functions is generally not a quasi-concave (convex) function.
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The following theorem correlates the quasi-concavity of a function to the convexity of upper contour set.

**Theorem 2.4.12 (Quasi-concavity and Upper Contour Sets)** \( f : X \to \mathcal{R} \) is a quasi-concave function if and only if for any \( y \in \mathcal{R} \), the upper contour set \( S(y) \equiv \{ x \in X : f(x) \geq y \} \) is a convex set.

**Proof.** Necessity: Let \( x^1 \) and \( x^2 \) be two points of \( S(y) \) (if \( S(y) \) is the empty set, then we can complete the proof immediately since the empty set is convex). We need to show: if \( f \) is quasi-concave, all points in the form of \( x^t \equiv t x^1 + (1 - t) x^2, t \in [0, 1] \) is in \( S(y) \).

Since \( x^1 \in S(y) \) and \( x^2 \in S(y) \), by the definition of upper contour set, we have \( f(x^1) \geq y \) and \( f(x^2) \geq y \).

Now, we consider any \( x^t \). Since \( f \) is quasi-concave, then:

\[
\begin{align*}
f(x^t) & \geq \min[f(x^1), f(x^2)] \\
& \geq y.
\end{align*}
\]

Thus, \( f(x^t) \geq y \), and then \( x^t \in S(y) \). Therefore, \( S(y) \) must be a convex set.

Sufficiency: we need to show: if for all \( y \in \mathcal{R} \), \( S(y) \) is a convex set, then \( f(x) \) is a quasi-concave function. Let \( x^1 \) and \( x^2 \) be two arbitrary points in \( X \). Without loss of generality, suppose \( f(x^1) \geq f(x^2) \). Since for all \( y \in \mathcal{R} \), \( S(y) \) is a convex set, then \( S(f(x^2)) \) must be convex. It is also clear that \( x^2 \in S(f(x^2)) \), and since \( f(x^1) \geq f(x^2) \), we have \( x^1 \in S(f(x^2)) \). Hence, for any convex combination of \( x^1 \) and \( x^2 \), we must have \( x^t \in S(f(x^2)) \). It follows from the definition of \( S(f(x^2)) \) that \( f(x^t) \geq f(x^2) \). Consequently, we must have

\[
f(x^t) \geq \min[f(x^1), f(x^2)].
\]

Therefore, \( f(x) \) is quasi-concave. \( \square \)

The following theorem characterizes the properties of quasi-concave functions, that is, quasi-concavity is robust to monotonic transformation.

**Theorem 2.4.13** Suppose that the function \( f : X \to \mathcal{R} \) is quasi-concave on \( X \), and \( h : \mathcal{R} \to \mathcal{R} \) is a monotonically non-decreasing function. Then the composite function \( h(f(x)) \) is also quasi-concave. If \( f \) is strictly quasi-concave and \( h \) is strictly increasing, then the composite function is strictly quasi-concave.

When a function \( f \) defined on a convex set \( X \) has continuous second order partial derivatives, the bordered Hessian determinant is defined as follows:

\[
|B| = \begin{vmatrix}
0 & f_1 & f_2 & \cdots & f_n \\
1 & f_{11} & f_{12} & \cdots & f_{1n} \\
f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_n & f_{n1} & f_{n2} & \cdots & f_{nn}
\end{vmatrix}
\]
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The principal minors of the bordered Hessian determinant $B$ are as follows:

$$|B_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix},$$

$$|B_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix},$$

$$\vdots,$$

$$|B_n| = |B|.$$

Then the necessary condition for $f : X \to \mathbb{R}$ to be a quasi-concave function is

$$|B_1| \leq 0, \ |B_2| \geq 0, \ |B_3| \leq 0, \ \cdots, \ (-1)^n|B_n| \geq 0.$$

The sufficient condition for $f : X \to \mathbb{R}$ to be a strictly quasi-concave function is:

$$|B_1| < 0, \ |B_2| > 0, \ |B_3| < 0, \ \cdots, \ (-1)^n|B_n| > 0.$$

The necessary condition for $f : X \to \mathbb{R}$ to be a quasi-convex function is:

$$|B_1| \leq 0, \ |B_2| \leq 0, \ \cdots, \ |B_n| \leq 0.$$

The sufficient condition for $f : X \to \mathbb{R}$ to be a strictly quasi-convex function is:

$$|B_1| < 0, \ |B_2| < 0, \ \cdots, \ |B_n| < 0.$$

2.4.10 Separating and Supporting Hyperplane Theorems

Separating hyperplane theorem also has crucial applications in economics. First recall that if $X \subseteq \mathbb{R}^n$ is a compact set, then it is bounded and closed. $X$ is convex set, if for any $x, x' \in X$ and any $0 \leq t \leq 1$, $tx + (1 - t)x' \in X$. The convex set implies that the connections between any two points in the set belong to this set.

**Theorem 2.4.14 (Separating Hyperplane Theorem)** Suppose that $A, B \subseteq \mathbb{R}^m$ are convex and $A \cap B = \emptyset$. Then, there is a vector $p \in \mathbb{R}^m, p \neq 0$ and $c \in \mathbb{R}$ such that

$$px \leq c \leq py \ \forall x \in A, \forall y \in B.$$

Moreover, suppose that $B \subseteq \mathbb{R}^m$ is convex and closed, $A \subseteq \mathbb{R}^m$ is convex and compact, and $A \cap B = \emptyset$. Then, there is a vector $p \in \mathbb{R}^m, p \neq 0$ and $c \in \mathbb{R}$ such that $A, B$ are strictly separated, namely

$$px < c < py \ \forall x \in A, \forall y \in B.$$

**Theorem 2.4.15 (Supporting Hyperplane Theorem)** Suppose that $A \subseteq \mathbb{R}^m$ are convex and $y \in \mathbb{R}^m$ is not an interior of $A$ (i.e., $y \notin \text{int}A$). Then, there is a vector $p \in \mathbb{R}^m$ with $p \neq 0$ such that

$$px \leq py \ \forall x \in A.$$
Unlike the Separating Hyperplane Theorem, the Supporting Hyperplane Theorem above does not need assume that the intersection of two sets $A$ and $\{y\}$ is a empty set.

**Definition 2.4.14** Let $C \subseteq \mathbb{R}^m$. $C$ is called a cone if for any $x \in C$ and $\lambda \in \mathbb{R}$, we have $\lambda x \in C$.

**Proposition 2.4.1** A cone $C$ is convex if and only if $x, y \in C$ implies $x + y \in C$.

**Proposition 2.4.2** Let $C \subseteq \mathbb{R}^m$ be a closed and convex cone, and $K \subseteq \mathbb{R}^m$ be a compact and convex cone. Then $C \cap K \neq \emptyset$ if and only if for any $p \in C$, there is $z \in K$ such that $p \cdot z \leq 0$.

## 2.5 Multi-Valued Function

Set-valued mapping refers to the situation where the image of a mapping may not a single point but a set.

### 2.5.1 Point-to-Set Mappings

Suppose that $X$ and $Y$ are two subsets of a topological vector space (such as the Euclidean space).

**Point-to-set mapping** is also called correspondence or multi-valued function. A correspondence $F$ maps points $x$ in a domain $X$ into sets in $Y$ (for example, maps point $x$ in $X \subseteq \mathbb{R}^n$ into the range $Y \subseteq \mathbb{R}^m$), denoted by $F : X \rightarrow 2^Y$. One also uses $F : X \rightarrow Y$ or $F : X \rightarrow Y$ to denote the multi-valued mapping $F : X \rightarrow 2^Y$.

**Definition 2.5.1** Let $F : X \rightarrow 2^Y$ be a correspondence.

1. If $F(x)$ is non-empty for every $x \in X$, then the correspondence $F$ is said to be **non-empty valued**;
2. If $F(x)$ is a convex set for every $x \in X$, then the correspondence $F$ is said to be **convex valued**;
3. If $F(x)$ is a closed set for every $x \in X$, then the correspondence $F$ is said to be **closed valued**;
4. If $F(x)$ is compact for every $x \in X$, then the correspondence $F$ is said to be **compact valued**;
5. If $F(x)$ is open for every $x \in X$, then the correspondence $F$ is said to have **open upper sections**;
6. If the preimage $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open for every $y \in Y$, then the correspondence $F$ is said to have **open lower sections**.
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Definition 2.5.2 Let \( F : X \to 2^X \) be a correspondence from \( X \) to \( X \) itself.

(1) \( F \) is said to be **FS-convex** if for any \( x_1, \ldots, x_m \in X \) and its convex combination \( x_\lambda = \sum_{i=1}^{m} \lambda_i x_i \),\(^1\) we have
\[
x_\lambda \in \bigcup_{i=1}^{m} F(x_i).
\]

(2) \( F \) is said to be **SS-convex** if for any \( x \in X, x \notin \text{co} F(x) \).\(^2\)

Remark 2.5.1 It is easy to verify that correspondence \( P : X \to 2^X \) is SS-convex if and only if correspondence \( G : X \to 2^X \) defined by \( G^{-1}(x) = X \setminus P(x) \) is FS-convex.

Specially, for function \( f : X \to \mathbb{R} \), define upper contour set
\[
U(x) = \{ y \in X : f(y) \geq f(x) \}, \ \forall x \in X,
\]
strict upper contour set
\[
U_s(x) = \{ y \in X : f(y) > f(x) \}, \ \forall x \in X,
\]
lower contour set
\[
L(x) = \{ y \in X : f(y) \leq f(x) \}, \ \forall x \in X,
\]
and strict lower contour set
\[
L_s(x) = \{ y \in X : f(y) < f(x) \}, \ \forall x \in X.
\]

The following equivalence results are used later.

Proposition 2.5.1 The following arguments are equivalent:

(1) \( f : X \to \mathbb{R} \) is quasi-concave;
(2) \( U : X \to 2^X \) is a convex-valued correspondence;
(3) \( U_s : X \to 2^X \) is a convex-valued correspondence;
(4) \( U_s : X \to 2^X \) is SS-convex;
(5) \( U : X \to 2^X \) is FS-convex.

Proof. It is clear that (1) implies (2), (2) implies (3), (3) implies (4), and (5) implies (1). We just need to show that (4) implies (5). Suppose not, there is a finite set \( \{x_1, x_2, \ldots, x_m\} \subset X \) and certain convex combination \( x_\lambda = \sum_{j=1}^{m} \lambda_j x_j \), such that \( x_\lambda \notin \bigcup_{j=1}^{m} U(x_j) \). Thus, for all \( j \), we have \( x_\lambda \in L_s(x_j) \), that is, \( x_j \in U_s(x_\lambda) \) and hence \( x_\lambda \notin \text{co} U_s(x_\lambda) \), a contradiction. \( \square \)

---

\(^1\)The concept of FS-convex is introduced by Fan (1984) & Sonnenschein (1971), so it is called FS-convex.

\(^2\)The concept of SS-convex is introduced by Shafer & Sonnenschein (1975), so it is called SS-convex.
2.5.2 Upper Hemi-continuous and Lower Hemi-continuous Correspondence

Intuitively, a correspondence is continuous if a small change in $x$ only leads to a small change in the set $F(x)$. Unfortunately, giving a formal definition of continuity for correspondences is not so simple. Figure 2.1 shows a continuous correspondence.

The notions of hemi-continuity are usually defined in terms of sequences (see Debreu (1959) and Mask-Collell et al. (1995)). Although they are relatively easy to verify, they depend on the assumption that a correspondence is compact-valued. The following definitions are more formal (see Border, 1985).

**Definition 2.5.3** For a correspondence $F : X \to 2^Y$ and a point $x$, $F$ is said to be upper hemi-continuous at $x$ if for each open set $U$ containing $F(x)$, there is an open set $N(x)$ containing $x$ such that $F(x') \subseteq U$ for all $x' \in N(x)$.

$F$ is said to be upper hemi-continuous on $X$ if the correspondence $F$ is upper hemi-continuous at every $x \in X$, or equivalently, for every open subset $V$ of $Y$, $\{x \in X : F(x) \subseteq V\}$ is always an open subset of $X$.

**Remark 2.5.2** Upper hemi-continuity captures the idea that $F(x)$ should not “suddenly contain new points” when passing through a point $x$, in other words, $F(x)$ does not jump if $x$ changes slightly. That is, if one starts at a point $x$ and moves a little to $x'$, upper hemi-continuity at $x$ implies that there is no point in $F(x')$ that is not close to some points in $F(x)$.

**Definition 2.5.4** For a correspondence $F : X \to 2^Y$ and a point $x$, correspondence $F$ is said to be lower hemi-continuous at $x$ if for every open
set \( V \), \( F(x) \cap V \neq \emptyset \), there exists a neighborhood \( N(x) \) of \( x \) such that \( F(x') \cap V \neq \emptyset \) for all \( x' \in N(x) \).

If \( F \) is lower hemicontinuous at every \( x \), or equivalently, the set \( \{ x \in X : F(x) \cap V \neq \emptyset \} \) is open in \( X \) for every open set \( V \) of \( Y \), then \( F \) is said to be lower hemicontinuous on \( X \).

**Remark 2.5.3** Lower hemicontinuity captures the idea that any element in \( F(x) \) can be “approached” from all directions, in other words, \( F(x) \) does not suddenly become much smaller if one changes \( x \) slightly. That is, if one starts at \( x \) and \( y \in F(x) \), lower hemicontinuity at \( x \) implies that if one moves a little from \( x \) to \( x' \), there is some \( y' \in F(x') \) that is close to \( y \).

Combining the concepts of upper and lower hemicontinuity, we can define the continuity of a correspondence.

**Definition 2.5.5** A correspondence \( F : X \to 2^Y \) is said to be **continuous at** \( x \in X \) if it is both upper hemicontinuous and lower hemicontinuous at \( x \in X \). The correspondence \( F : X \to 2^Y \) is said to be **continuous on** \( X \) if it is both upper hemicontinuous and lower hemicontinuous on \( X \).

Figure 2.2 shows the correspondence that is upper hemicontinuous, but not lower hemicontinuous. To see why it is upper hemicontinuous, imagine an open interval \( U \) that encompasses \( F(x) \). Now consider moving a little to the left of \( x \) to a point \( x' \). Clearly \( F(x') = \{ \hat{y} \} \) is in the interval. Similarly, if we move to a point \( x' \) a little to the right of \( x \), then \( F(x) \) will be inside the interval so long as \( x' \) is sufficiently close to \( x \). So it is upper hemicontinuous. On the other hand, it is not lower hemicontinuous. To see this, consider the point \( y \in F(x) \), and let \( U \) be a very small interval around \( y \) that does not include \( \hat{y} \). If we take any open set \( N(x) \) containing \( x \), then it will contain some point \( x' \) to the left of \( x \). But then \( F(x') = \{ \hat{y} \} \) will contain no points near \( y \), i.e., it will not intersect \( U \). Thus, the correspondence is not lower hemicontinuous.

Figure 2.3 shows the correspondence that is lower hemicontinuous, but not upper hemicontinuous. To see why it is lower hemicontinuous: For any \( 0 < x' < x \), note that \( F(x') = \{ \hat{y} \} \). Let \( x_n = x' - 1/n \), \( y_n = \hat{y} \). Then \( x_n > 0 \) for sufficiently large \( n \), \( x_n \to x' \), \( y_n \to \hat{y} \), and \( y_n \in F(x_n) = \{ \hat{y} \} \). So it is lower hemicontinuous. It is clearly lower hemicontinuous for \( x_i > x \). Thus, it is lower hemicontinuous on \( X \). On the other hand, it is not upper hemicontinuous. If we start at \( x \) by noting that \( F(x) = \{ \hat{y} \} \), and make a small move to the right to a point \( x' \), then \( F(x') \) suddenly contains many points that are not close to \( \hat{y} \). So this correspondence fails to be upper hemicontinuous.

**Remark 2.5.4** As it turns out, the notions of upper and hemicontinuous correspondence reduce to the standard notion of continuity for a function.
Figure 2.2: The correspondence is upper hemi-continuous, but not lower hemi-continuous

Figure 2.3: The correspondence is lower hemi-continuous, but not upper hemi-continuous

if \( F(\cdot) \) is a single-valued correspondence, i.e., a function. That is, \( F(\cdot) \) is a single-valued upper (or lower) hemi-continuous correspondence if and only if it is a continuous function.

**Remark 2.5.5** Based on the following two facts, both notions of hemi-continuity can be characterized by sequences.

(a) If a correspondence \( F : X \rightarrow 2^Y \) is compact-valued, then it is upper hemi-continuous if and only if for any \( \{x_k\} \) and \( \{y_k\} \), where \( x_k \rightarrow x \), \( y_k \in F(x_k) \), there exists a converging subsequence \( \{y_{k_m}\} \), such that \( y_{k_m} \rightarrow y \) and \( y \in F(x) \).

(b) A correspondence \( F : X \rightarrow 2^Y \) is said to be lower hemi-continuous at \( x \) if and only if for any \( \{x_k\} \) and \( y \in F(x) \), where \( x_k \rightarrow x \), there is a sequence \( \{y_k\} \) such that \( y_k \rightarrow y \) and \( y_k \in F(x_k) \).
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2.5.3 The Open and Closed Graphs of Correspondence

**Definition 2.5.6** A correspondence $F : X \to 2^Y$ is said to be **closed** at $x$ if for any $\{x_k\}$ and $\{y_k\}$, where $x_k \to x$ and $y_k \to y$, $y_k \in F(x_k)$, we have $y \in F(x)$. $F$ is said to be **closed** or has a closed graph if $F$ is closed for all $x \in X$, or equivalently graph $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed.

Regarding the relationship between upper hemi-continuity and closed graph, we have the following results.

**Proposition 2.5.2** Let $F : X \to 2^Y$ be a correspondence.

(i) Suppose that $Y$ is compact and $F : X \to 2^Y$ is closed-valued. If $F$ has a closed graph, it is upper hemi-continuous.

(ii) Suppose that $X$ and $Y$ are closed and $F : X \to 2^Y$ is closed-valued. If $F$ is upper hemi-continuous, then it has a closed graph.

Because of fact (i), a correspondence with a closed graph is sometimes used to define a hemi-continuous correspondence in the literature. But one should keep in mind that they are not the same in general. For example, let $F : \mathbb{R}_+ \to 2^\mathbb{R}$ be defined by

$$F(x) = \begin{cases} \left\{ \frac{1}{x} \right\}, & \text{if } x > 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

The correspondence is closed but not upper hemi-continuous. Also, define $F : \mathbb{R}_+ \to 2^\mathbb{R}$ by $F(x) = (0, 1)$. Then $F$ is upper hemi-continuous but not closed.

**Definition 2.5.7** A correspondence $F : X \to 2^Y$ is said to be **open** if its graph $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is open.

**Proposition 2.5.3** Let $F : X \to 2^Y$ be a correspondence. Then,

(1) if a correspondence $F : X \to 2^Y$ has an open graph, then it has open upper and lower sections.

(2) If a correspondence $F : X \to 2^Y$ has open lower sections, then it must be lower hemi-continuous.
2.5.4 Transfer Closed-valued Correspondence

The concepts of transfer closed, transfer open, transfer convex and others for the multivalued mapping (correspondence) introduced in Tian (1992, 1993) and Zhou and Tian (1992) weaken the conditions for establishing some basic mathematical theorems in nonlinear analysis and the existence of equilibrium solution of optimization problems. They can be used to obtain many characterization results, such as the existence of the maximal element of preference relations and the existence of Nash equilibrium. These conclusions are given in the corresponding chapters of this book.

Denote by \( \text{int} D \) and \( \text{cl} D \) the set of interior points and the closure of set \( D \), respectively.

**Definition 2.5.8** A correspondence \( G : X \to 2^Y \) is said to be **transfer closed-valued** on \( X \) if for any \( x \in X \), \( y \not\in G(x) \) implies that there is an \( x' \in X \) such that \( y \not\in \text{cl} G(x') \).

**Definition 2.5.9** A correspondence \( P : X \to 2^Y \) is said to have **transfer open upper sections** on \( X \) if for any \( x \in X \) and \( y \in Y \), \( y \in P(x) \) implies that there is a point \( x' \in X \) such that \( y \in \text{int} P(x') \).

**Remark 2.5.6** If a correspondence is closed-valued, then it is a transfer closed-valued (it is obtained by letting \( x' = x \)); if a correspondence has open upper sections, then it has the transfer open upper sections (let \( x' = x \)). Also, the correspondence \( P : X \to 2^Y \) has transfer open upper sections in \( X \) if and only if \( G : X \to 2^Y \) defined by \( G(x) = Y \setminus P(x) \) is transfer closed-valued in \( X \).

**Remark 2.5.7** For any function \( f : X \to \mathbb{R} \), the correspondence \( G : X \to 2^Y \) defined by

\[
G(x) = \{ y \in X : f(y) \geq f(x) \}, \forall x \in X
\]

is transfer closed-valued if and only if \( f \) is transfer upper continuous on \( X \).

The following proposition significantly weakens the various continuity conditions involved when proving many optimization problems.

**Proposition 2.5.4 (Tian (1992))** Let \( X \) and \( Y \) be two topological spaces, \( G : X \to 2^Y \) be a correspondence from point to set. Then

\[
\bigcap_{x \in X} \text{cl} G(x) = \bigcap_{x \in X} G(x)
\]

if and only if \( G \) is transfer closed-valued on \( X \).
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**Proof.** Sufficiency: We need to show

\[ \bigcap_{x \in X} \text{cl} G(x) = \bigcap_{x \in X} G(x). \]

It is clear that

\[ \bigcap_{x \in X} G(x) \subseteq \bigcap_{x \in X} \text{cl} G(x), \]

thus we just need to show that

\[ \bigcap_{x \in X} \text{cl} G(x) \subseteq \bigcap_{x \in X} G(x). \]

Suppose not, then there is a \( y \) such that \( y \in \bigcap_{x \in X} \text{cl} G(x) \), but \( y \notin \bigcap_{x \in X} G(x) \). Hence, there is a \( z \in X \) such that \( y \notin G(z) \). Note that \( G \) is transfer closed-valued on \( X \), then there exists a \( z' \in X \) such that \( y \notin \text{cl} G(z') \), thus \( y \notin \bigcap_{x \in X} \text{cl} G(x) \), a contradiction.

Necessity: Suppose that

\[ \bigcap_{x \in X} \text{cl} G(x) = \bigcap_{x \in X} G(x). \]

If \( y \notin G(x) \), then

\[ y \notin \bigcap_{x \in X} \text{cl} G(x) = \bigcap_{x \in X} G(x), \]

and thus \( y \notin \text{cl} G(x') \) for some \( x' \in X \). Hence, \( G \) is a transfer closed-valued correspondence on \( X \).

Similarly, we can define the transfer convexity.

**Definition 2.5.10 (Transfer FS-convex)** Let \( X \) be a topological space and \( Z \) be a convex subset of \( X \). A correspondence \( G : X \to 2^Z \) is said to be transfer FS-convex on \( X \) if for any finite set \( \{x_1, x_2, \cdots, x_n\} \subseteq X \), there is a corresponding finite set \( \{y_1, y_2, \cdots, y_n\} \subseteq Z \) such that for any subset \( \{y_{i_1}, y_{i_2}, \cdots, y_{i_s}\} (1 \leq s \leq n) \), we have

\[ \text{co } \{y_{i_1}, y_{i_2}, \cdots, y_{i_s}\} \subseteq \bigcup_{r=1}^{s} G(y_{i_r}). \]

**Definition 2.5.11 (Transfer SS-convex)** Let \( X \) be a topological space, and \( Z \) be a convex subset of the topological space. A correspondence \( P : Z \to 2^X \) is said to be transfer SS-convex on \( X \) if for any finite set \( \{y_1, y_2, \cdots, y_n\} \subseteq X \), there always exists a finite set \( \{y_1, y_2, \cdots, y_n\} \subseteq Z \) such that for any subset \( \{y_{i_1}, y_{i_2}, \cdots, y_{i_s}\} (1 \leq s \leq n) \) and any \( y_{i_0} \in \text{co } \{y_{i_1}, y_{i_2}, \cdots, y_{i_s}\} \), we have \( x_{i_r} \notin P(y_{i_0}) \).
Remark 2.5.8 Unlike FS-convex and SS-Convex, when defining the transfer FS-convex and transfer SS-convex, we do not assume that correspondences are mapping from itself to itself. It is clear that when $X = Z$ and picking $y_i = x_i$, FS-convex implies transfer FS-convex and SS-convex implies transfer SS-convex. Similarly, it is not difficult to verify that the correspondence $P : X \rightarrow 2^Z$ is transfer SS-convex if and only if $G : X \rightarrow 2^X$ defined by $G^{-1}(x) = Z \setminus P(x)$ is transfer FS-convex.

2.6 Static Optimization

The optimization problem is the core issue in economics. Rationality is a key assumption about individual decision-makers in economics. Individuals pursue maximizing their personal interests, and the basis analysis is to solve optimization problems. This section introduces various methods for solving static optimization problems.

2.6.1 Unconstrained Optimization

The optimization problem discusses whether a function can reach the maximum or minimum on a given set. Let $X$ be an arbitrary topological space. First, we give the following concepts:

Definition 2.6.1 (Local Optimum) If $f(x^*) \geq f(x)$ ($f(x^*) > f(x)$) for all $x$ in some neighbourhoods of $x^*$, then the function is said to have local maximum (unique local maximum) at point $x^*$.

If $f(\tilde{x}) \leq f(x)$ ($f(\tilde{x}) < f(x)$) for all $x \neq \tilde{x}$ in some neighbourhoods of $\tilde{x}$, then the function is said to have local minimum (unique local minimum) at $\tilde{x}$.

Definition 2.6.2 (Global Optimum) If $f(x^*) \geq f(x)$ ($f(x^*) > f(x)$) for all $x$ in the domain of the function, then the function is said to have global (unique) maximum at $x^*$; if $f(x^*) \leq f(x)$ ($f(x^*) < f(x)$) for all $x$ in the domain of the function, then the function is said to have global (unique) minimum at $x^*$.

A classical conclusion about global optimization is the so-called Weierstrass theorem.

Theorem 2.6.1 (Weierstrass Theorem) Any upper (lower) semi-continuous function must reach its maximum (minimum) on a compact set, and the set of maximal points is compact.

Transfer continuity can be used to generalize the Weierstrass Theorem by providing sufficient and necessary conditions for a function $f$ to reach global maximum (minimum) on a compact set $X$, sufficient and necessary

**Theorem 2.6.2 (Tian-Zhou Theorem I)** Suppose that $X$ is a compact set in an arbitrary topological space. The function $f : X \to \mathbb{R}$ has a maximum (minimum) on $X$ if and only if $f$ is transfer weakly upper (lower) continuous on $X$.

**Proof.** Since $f$ is transfer weakly upper continuous on $X$ if and only if $-f$ is transfer weakly lower continuous, we just need to show the case that the function has a maximal point.

Sufficiency: We prove it by contradiction. Suppose that $f$ does not have a maximum on $X$. Then for each $y \in X$, there is $x \in X$ such that $f(x) > f(y)$. It follows from the transfer weak upper continuity of $f$ that there is a $x' \in X$ and a neighbourhood $\mathcal{N}(y)$ of $y$ such that $f(x') \geq f(y)$ for all $y' \in \mathcal{N}(y)$. Hence, we have $X = \bigcup_{y \in X} \mathcal{N}(y)$. Since $X$ is compact, there is a finite number of points $\{y_1, y_2, \ldots, y_n\}$ such that $X = \bigcup_{i=1}^{n} \mathcal{N}(y_i)$. Let $x'_i$ be the corresponding points such that $f(x'_i) \geq f(y')$ for all $y' \in \mathcal{N}(y_i)$. $f$ must have the maximum in the finite subset $\{x'_1, x'_2, \ldots, x'_n\}$. Without loss of generality, suppose $x'_1$ satisfying $f(x'_1) \geq f(x'_i)$ for $\forall i = 1, 2, \ldots , n$. It follows from the previous assumption that $f$ has no maximum on $X$, that is, $x'_1$ is not the maximal point of $f$ on $X$. Thus there exists $x \in X$ such that $f(x) > f(x'_1)$. However, since $X = \bigcup_{i=1}^{n} \mathcal{N}(y_i)$, there is $j$ such that $x \in \mathcal{N}(y_j)$ and then $f(x'_j) \geq f(x)$. Thus $f(x) > f(x'_1) \geq f(x'_j) \geq f(x)$, a contradiction. Therefore, $f$ has a maximum on $X$.

Necessity: Clearly. Let $x'$ be a maximal point of $f$, then $f(x') \geq f(y')$ holds for all $y' \in X$.

In many cases, when proving the existence of competitive equilibrium and the existence of equilibrium in a game, we not only need to prove the existence of optimal results, but also prove that the set of the optimal results is compact.

**Theorem 2.6.3 (Tian-Zhou Theorem II)** Suppose that $X$ is a compact set in an arbitrary topological space, and $f : X \to \mathbb{R}$ is a function. The set of maximal (minimal) points of $f$ on $X$ is nonempty and compact if and only if $f$ is transfer upper (lower) continuous on $X$.

**Proof.** We only need to prove the case with a set of maximal points.

Necessity: Suppose that the set of maximal points of $f$ on $X$ is nonempty and compact. If $f(y) < f(x)$ for any $x, y \in X$, then $y$ cannot be a maximal point of $f$ on $X$. It follows from the compactness of the set of maximal points that there is a neighbourhood $\mathcal{N}(y)$ of $y$ that does not contain any maximal points of $f$ on $X$. Let $x'$ be a maximal point of $f$ on $X$,
then \( f(z) < f(x') \) for all \( z \in N(y) \). Thus, \( f \) is transfer upper continuous on \( X \).

Sufficiency: First note that \( G : X \rightarrow 2^Y \) defined by

\[
G(x) = \{ y \in X : f(y) \geq f(x) \}, \forall x \in X
\]

is a transfer closed-valued correspondence if and only if \( f \) is transfer upper continuous on \( X \). Since \( f \) is transfer upper continuous on \( X \), according to Proposition 2.5.4, we have \( \bigcap_{x \in X} cl \, G(x) = \bigcap_{x \in X} G(x) \), and hence the set of maximal points is closed.

Since \( f \) has a maximal point on any finite subset \( \{x_1, x_2, \cdots, x_m\} \subseteq X \), let \( f(x_1) \geq f(x_i) \) hold for all \( \forall i = 1, \cdots, m \). Then we have \( x_1 \in G(x_i) \) for all \( i = 1, \cdots, m \), and thus

\[
\emptyset \neq \bigcap_{i=1}^m G(x_i) \subseteq \bigcap_{i=1}^m cl \, G(x_i),
\]

namely, the class of sets \( \{cl \, G(x) : x \in X\} \) has the property of finite intersection on \( X \). Since \( \{cl \, G(x) : x \in X\} \) is a collection of closed sets in compact set \( X \), \( \emptyset \neq \bigcap_{x \in X} cl \, G(x) = \bigcap_{x \in X} G(x) \). It implies that there is \( x^* \in X \) such that \( f(x^*) \geq f(x) \) for all \( x \in X \). Since the set of maximal points \( \bigcap_{x \in X} cl \, G(x) \) is a closed subset of the compact set \( X \), it is also compact. \( \square \)

In order to easily determine whether a function has an extreme point, the following gives the method of finding extreme values by differential method. We first give the necessary conditions for interior extreme points without constraints, and then give the sufficient conditions.

Generally, there are two necessary conditions for the interior extreme point, that is, the first and second order necessary conditions.

**Theorem 2.6.4 (The first-order necessary condition for interior extreme points)**

Suppose that \( X \subseteq \mathbb{R}^n \). If a differentiable function \( f(x) \) reaches a local maximum or minimum at an interior point \( x^* \in X \), then \( x^* \) is the solution to the following system of simultaneous equations:

\[
\frac{\partial f(x^*)}{\partial x_1} = 0,
\]

\[
\frac{\partial f(x^*)}{\partial x_2} = 0,
\]

\[
\vdots
\]

\[
\frac{\partial f(x^*)}{\partial x_n} = 0.
\]
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**Proof.** Suppose that \( f(x) \) reaches the local extreme value at an interior point \( x^* \), then we need to prove that \( Df(x^*) = 0 \). Although this proof is not the simplest one, it will be very useful when considering the second order condition.

Choose any vector \( z \in \mathbb{R}^n \), and then construct a familiar univariate function of any scalar \( t \):

\[
g(t) = f(x^* + t z)
\]

First, for \( t \neq 0 \), \( x^* + tz \) gives a vector that is different from \( x^* \). For \( t = 0 \), \( x^* + t z \) is equal to \( x^* \), thus \( g(0) \) is exactly the value of \( f \) at \( x^* \). According to the assumption that \( f \) attains an extremum at \( x^* \), \( g(t) \) must reach a local extreme at \( t = 0 \). It follows from Fermat Theorem given by Proposition 2.4.4 that \( g'(0) = 0 \). Taking the derivative of \( g(t) \) by the Chain Rule gives:

\[
g'(t) = \sum_{i=1}^{n} \frac{\partial f(x^* + t z)}{\partial x_i} z_i.
\]

When \( t = 0 \) and using \( g'(0) = 0 \), we have

\[
g'(0) = \sum_{i=1}^{n} \frac{\partial f(x^*)}{\partial x_i} z_i = Df(x^*)z = 0.
\]

Since the above equation holds for any vector \( z \), including the unit vector, it means that each partial derivative of \( f \) must equal to zero, namely

\[
Df(x^*) = 0.
\]

**Theorem 2.6.5 (The second-order necessary conditions for interior extreme points)**

Suppose that \( f(x) \) is twice continuously differentiable on \( X \subseteq \mathbb{R}^n \).

1. If \( f(x) \) reaches a local maximum at the interior point \( x^* \), then the Hessian matrix \( H(x^*) \) is negative semi-definite.
2. If \( f(x) \) reaches a local minimum at the interior point \( \tilde{x} \), then \( H(\tilde{x}) \) is positive semi-definite.

**Proof.** Let \( g(t) = f(x + tz) \), \( z \in \mathbb{R}^n \) and \( x \) be a stationary point of \( f \). If \( f \) attains a stationary point at \( x \), then \( g \) gets a stationary point at \( t = 0 \). Further, for any \( t \), we have

\[
g'(t) = \sum_{i=1}^{n} \frac{\partial f(x + t z)}{\partial x_i} z_i.
\]

We have the second order derivatives:

\[
g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f(x + t z)}{\partial x_i \partial x_j} z_i z_j.
\]
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Now suppose that \( f \) reaches maximum at \( x = x^* \). Since \( g''(0) \leq 0 \), then the value of \( g''(t) \) at \( x^* \) and \( t = 0 \) is

\[
g''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f(x^*)}{\partial x_i \partial x_j} z_i z_j \leq 0,
\]

or \( z^T H(x) z \leq 0 \). Since \( z \) is arbitrary, it implies that \( H(x^*) \) is negative semi-definite. Similarly, if \( f \) is minimized at \( x = \tilde{x} \), then \( g''(0) = 0 \) and \( H(\tilde{x}) \) is positive semi-definite.

Then the sufficient conditions for optimization will be discussed.

**Theorem 2.6.6 (The sufficient conditions for interior extreme points)** Suppose that \( f(x) \) is twice continuously differentiable on \( X \subseteq \mathbb{R}^n \). Then we have:

1. If \( f_i(x^*) = 0 \) and \( (-1)^i D_i(x^*) > 0, i = 1, \cdots, n \), then \( f(x) \) has a local maximum at \( x^* \).
2. If \( f_i(\tilde{x}) = 0 \) and \( D_i(\tilde{x}) > 0, i = 1, \cdots, n \), then \( f(x) \) has a local minimum at \( \tilde{x} \).

The local optimum is in general not equal to the global optimum, but under certain conditions, these two are consistent with each other.

**Theorem 2.6.7 (Local and Global Optimum)** Suppose that \( f \) is a concave and twice continuously differentiable function on \( X \subseteq \mathbb{R}^n \), and \( x^* \) is an interior point of \( X \). Then the following three statements are equivalent:

1. \( Df(x^*) = 0 \).
2. \( f \) has a local maximum at \( x^* \).
3. \( f \) has a global maximum at \( x^* \).

**Proof.** It is clear that (3) \(\Rightarrow\) (2), and it follows from the previous theorem that (2) \(\Rightarrow\) (1). Thus, we just need to prove that (1) \(\Rightarrow\) (3).

Suppose that \( Df(x^*) = 0 \), then \( f \) is concave implies that for all \( x \) in the domain, we have:

\[
f(x) \leq f(x^*) + Df(x^*)(x - x^*).
\]

These two formulas mean that for all \( x \), we must have

\[
f(x) \leq f(x^*).
\]

Therefore, \( f \) reaches a global maximum at \( x^* \). \(\square\)

**Theorem 2.6.8 (Strict Concavity/Convexity and Uniqueness of Global Optimum)** Let \( X \) be a topological vector space.
2.6. STATIC OPTIMIZATION

(1) If a strictly concave function $f$ defined on $X$ reaches a local maximum value at $x^*$, then $x^*$ is the unique global maximum point.

(2) If a strictly convex function $f$ reaches a local minimum value at $\tilde{x}$, then $\tilde{x}$ is the unique global minimum point.

**Proof.** Proof by contradiction. If $x^*$ is a global maximum point of function $f$ but not unique, then there is a point $x' \neq x^*$ such that $f(x') = f(x^*)$. Suppose that $x^t = tx' + (1-t)x^*$. Then strict concavity requires that for all $t \in (0,1)$,

$$f(x^t) > tf(x') + (1-t)f(x^*).$$

Since $f(x') = f(x^*)$,

$$f(x^t) > tf(x') + (1-t)f(x^*) = f(x').$$

This contradicts the assumption that $x'$ is a global maximum point of $f$. Therefore, the global maximum point of a strictly concave function is unique. The proof of part (2) is similar, and hence omitted.

**Theorem 2.6.9 (The sufficient condition for the uniqueness of global optimum)**

Suppose that $f(x)$ is twice continuously differentiable on $X \subseteq \mathbb{R}^n$. We have:

(1) If $f(x)$ is strictly concave and $f_i(x^*) = 0$, $i = 1, \ldots, n$, then $x^*$ is a unique global maximum point of $f(x)$.

(2) If $f(x)$ is strictly convex and $f_i(\tilde{x}) = 0$, $i = 1, \ldots, n$, then $\tilde{x}$ is a unique global minimum point of $f(x)$.

2.6.2 Optimization with Equality Constraints

Equality-Constrained Optimization

An optimization problem with equality-constraints has the following form: Suppose that a function of $n$ variables defined on $X \subseteq \mathbb{R}^n$ with $m$ constraints, where $m < n$. The optimization problem is:

$$\max_{x_1, \ldots, x_n} f(x_1, \ldots, x_n)$$

s.t. $g^1(x_1, \ldots, x_n) = 0,$

$g^2(x_1, \ldots, x_n) = 0,$

$$\vdots$$

$g^m(x_1, \ldots, x_n) = 0.$

The most important conclusion of the equality-constrained optimization problem is the Lagrange theorem, which gives a necessary condition for a point to be the solution of the optimization problem.
The Lagrange function of the above equality-constrained problem is defined as:

\[ L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g^j(x), \quad (2.6.1) \]

where \( \lambda_1, \cdots, \lambda_m \) are called Lagrange multipliers.

The following Lagrange theorem presents how to solve optimization problems under equality constraints.

**Theorem 2.6.10 (First-Order Necessary Condition for Interior Extremum)**

Suppose that \( f(x) \) and \( g^j(x), j = 1, \cdots, m, \) are continuously differentiable functions defined on \( X \subseteq \mathbb{R}^n, x^* \) is an interior point of \( X \) and an extreme point (maximal or minimal point) of \( f \) —— here \( f \) is subject to the constraint of \( g^j(x^*) = 0, \) where \( j = 1, \cdots, m. \) If the gradient \( Dg^j(x^*) = 0, j = 1, \cdots, m, \) are linearly independent, then there is a unique \( \lambda^*_j, j = 1, \cdots, m, \) such that:

\[
\frac{\partial L(x^*, \lambda^*)}{\partial x_i} = \frac{\partial f(x^*)}{\partial x_i} + \sum_{i=1}^{m} \lambda^*_j \frac{\partial g^j(x^*)}{\partial x_i} = 0, \quad i = 1, \cdots, n.
\]

The following proposition gives the sufficient conditions for interior extreme values with equality constraints.

**Proposition 2.6.1 (Second-Order Necessary Condition for Interior Extremum)**

Suppose that \( f \) and \( g^1, \cdots, g^m \) are twice continuously differentiable functions, and \( x^* \) satisfies the necessary conditions of Theorem 2.6.10. Let the bordered Hessian determinant

\[
|\bar{H}_r| = \det \begin{pmatrix}
0 & \cdots & 0 & \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^1}{\partial x_r} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{\partial g^m}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_r} \\
\frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_r} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\frac{\partial g^1}{\partial x_r} & \cdots & \frac{\partial g^m}{\partial x_r} & \frac{\partial^2 L}{\partial x_r \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_r \partial x_r} \\
\end{pmatrix}, \quad r = m+1, 2, \cdots, n
\]

take value at \( x^*. \) Thus

1. If \((-1)^{r-m+1} |\bar{H}_r(x^*)| > 0, r = m + 1, \cdots, n, \) then \( x^* \) is the local maximum of the optimization problem.
2. If \( |\bar{H}_r(x^*)| < 0, r = m + 1, \cdots, n, \) then \( x^* \) is the local minimum of the optimization problem.
Specially, when there is only one equality constraint, that is, \(m = 1\), the bordered Hessian determinant \(|\bar{H}|\) becomes:

\[
|\bar{H}| = \begin{vmatrix}
0 & g_1 & g_2 & \cdots & g_n \\
g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\
g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_n & \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn}
\end{vmatrix}
\]

where \(\mathcal{L}_{ij} = f_{ij} - \lambda g_{ij}\). The first-order condition is

\[
\lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_n}{g_n}
\]

The principal minors of the bordered Hessian are

\[
|\bar{H}_2| = \begin{vmatrix}
0 & g_1 & g_2 \\
g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\
g_2 & \mathcal{L}_{21} & \mathcal{L}_{22}
\end{vmatrix},
|\bar{H}_3| = \begin{vmatrix}
0 & g_1 & g_2 & g_3 \\
g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\
g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\
g_3 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33}
\end{vmatrix}, \ldots
\]

It leads to the following two conclusions.

**Conditions for Minimum with Equality Constraints**

1. \(\mathcal{L}_\lambda = \mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_n = 0\) [First-order Necessary Condition];

2. \(|\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0, \ldots, (-1)^n|\bar{H}_n| > 0\).

**Conditions for Maximum with Equality Constraints**

1. \(\mathcal{L}_\lambda = \mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_n = 0\) [First-order Necessary Condition];

2. \(|\bar{H}_2| < 0, |\bar{H}_3| < 0, |\bar{H}_4| < 0, \ldots, |\bar{H}_n| < 0\).

Note that when the constraint function \(g\) is linear, \(g(x) = a_1x_1 + \cdots + a_nx_n = c\), all the twice partial derivatives of \(g\) are equal to zero, so the bordered determinant \(|B|\) and the bordered Hessian determinant have the following relations:

\[
|B| = \lambda^2 |\bar{H}|
\]

Thus, the sequential principal minors of the bordered determinant have the same signs. As such, as long as the objective function is strictly quasi-concave, the first order necessary condition is also a sufficient condition to have the maximum value.
2.6.3 Optimization with Inequality Constraints

Consider an optimization problem with inequality constraints:

\[ \text{max } f(x) \]
\[ \text{s.t. } g_i(x) \leq d_i, \quad i = 1, 2, \cdots, k. \]

If a point \( x \) makes all constraints held with equality, \( Dg_1(x), Dg_2(x), \cdots, Dg_k(x) \) are linearly independent, then \( x \) is said to satisfy the \textit{constrained qualification}. Here the symbol \( D \) represents the partial differential operator.

\textbf{Theorem 2.6.11 (Kuhn-Tucker Theorem)} Suppose that \( x \) solves the inequality-constrained optimization problem and satisfies the constrained qualification condition. Then, there is a set of Kuhn-Tucker multipliers \( (\lambda_i \geq 0, \ i = 1, \cdots, k) \) such that

\[ Df(x) = \sum_{i=1}^{k} \lambda_i Dg_i(x). \]

Moreover, we have the complementary slackness conditions:

\[ \lambda_i \geq 0, \quad \text{for all } i = 1, 2, \cdots, k. \]
\[ \lambda_i = 0, \quad \text{if } g_i(x) < d_i. \]

Comparing the Kuhn-Tucker theorem with Lagrange multipliers in the equality-constrained optimization problem, we see that the major difference is that the signs of the Kuhn-Tucker multipliers are nonnegative while the signs of the Lagrange multipliers can be positive or negative. This additional information can be useful in various occasions.

The Kuhn-Tucker theorem only provides a necessary condition for a maximum. The following theorem states conditions that guarantee that the above first-order conditions be sufficient.

\textbf{Theorem 2.6.12 (Kuhn-Tucker Sufficiency)} Suppose that \( f \) is concave and \( g_i, i = 1, \cdots, k \), are convex. If \( x \) satisfies the Kuhn-Tucker first-order conditions, then \( x \) is a global solution to the constrained optimization problem.

We can weaken the conditions in the above theorem when there is only one constraint. Let \( C = \{ x \in \mathbb{R}^n : g(x) \leq d \} \). We have the following propositions.

\textbf{Proposition 2.6.2} Suppose that \( f \) is quasi-concave and the set \( C \) is convex (this is true if \( g \) is quasi-convex). If \( x \) satisfies the Kuhn-Tucker first-order conditions, then \( x \) is a global solution to the constrained optimization problem.

Sometimes we require \( x \) to be nonnegative. Suppose that we have the optimization problem:

\[ \text{max } f(x) \]
\[ \text{s.t. } g_i(x) \leq d_i, \quad i = 1, 2, \cdots, k, \]
\[ x \geq 0. \]
Then the Lagrange function in this case is given by
\[ L(x, \lambda) = f(x) + \sum_{l=1}^{k} \lambda_l [d_l - g_l(x)] + \sum_{j=1}^{n} \mu_j x_j, \]
where \( \mu_1, \ldots, \mu_n \) are the multipliers associated with constraints \( x_j \geq 0 \).

The first-order conditions are
\[ \frac{L(x, \lambda)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} - \sum_{l=1}^{k} \lambda_l \frac{\partial g_l(x)}{\partial x_i} + \mu_i = 0, \quad i = 1, 2, \ldots, n. \]
\[
\lambda_l \geq 0, \quad l = 1, 2, \ldots, k. \\
\lambda_l = 0, \quad \text{if } g_l(x) < d_l. \\
\mu_i \geq 0, \quad i = 1, 2, \ldots, n. \\
\mu_i = 0, \quad \text{if } x_i > 0.
\]

Eliminating \( \mu_i \), we can equivalently write the above first-order conditions with nonnegative choice variables as
\[ \frac{L(x, \lambda)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} - \sum_{l=1}^{k} \lambda_l \frac{\partial g_l(x)}{\partial x_i} \leq 0, \quad \text{with equality if } x_i > 0, \quad i = 1, 2, \ldots, n, \]
or in matrix notation,
\[ Df - \lambda Dg \leq 0, \]
\[ x[Df - \lambda Dg] = 0, \]
where we have written the product of two vectors \( x \) and \( y \) as the inner production, i.e., \( xy = \sum_{i=1}^{n} x_i y_i \). Thus, if we are at an interior optimum, we have
\[ Df(x) = \lambda Dg. \]

### 2.6.4 The Envelope Theorem

**The Envelope Theorem without Constraints**

Consider the following maximization problem:
\[ M(a) = \max_x f(x, a). \]

The function \( M(a) \) gives the maximum of the objective function as a function of parameter \( a \).

Let \( x(a) \) be the value of \( x \) that solves the maximization problem. Then we can also write \( M(a) = f(x(a), a) \). It is often of interest to know how \( M(a) \) changes as \( a \) changes. The envelope theorem tells us the answer:
\[ \frac{dM(a)}{da} = \left. \frac{\partial f(x, a)}{\partial a} \right|_{x=x(a)}. \]
The conclusion is particularly useful. This expression says that the derivative of \( M \) with respect to \( a \) is given by the partial derivative of \( f \) with respect to \( a \), holding \( x \) fixed at the optimal choice. This is the meaning of the vertical bar to the right of the derivative. The proof of envelope theorem is a relatively straightforward calculation.

**The Envelope Theorem with Constrain**

Now consider a more general parameterized constrained maximization problem of the form:

\[
M(a) = \max_{x_1,x_2} g(x_1, x_2, a)
\]

s.t. \( h(x_1, x_2, a) = 0 \).

The Lagrangian for this problem is

\[
\mathcal{L} = g(x_1, x_2, a) - \lambda h(x_1, x_2, a),
\]

and the first-order conditions for interior points are

\[
\begin{align*}
\frac{\partial g}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} &= 0, \\
\frac{\partial g}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} &= 0, \\
h(x_1, x_2, a) &= 0.
\end{align*}
\]

These conditions determine the optimal choice functions \((x_1(a), x_2(a))\), which in turn determine the maximum value function

\[
M(a) \equiv g(x_1(a), x_2(a), a).
\]

The envelope theorem gives us a formula for the derivative of the value function with respect to a parameter in the maximization problem. Specifically, the formula is

\[
\frac{dM(a)}{da} = \left. \frac{\partial \mathcal{L}(x, a)}{\partial a} \right|_{x = x(a)}
= \left. \frac{\partial g(x_1, x_2, a)}{\partial a} \right|_{x_i = x_i(a)} - \lambda \left. \frac{\partial h(x_1, x_2, a)}{\partial a} \right|_{x_i = x_i(a)}
\]

As before, special attention should be paid to the interpretation of these partial derivatives: they are the derivatives of \( g \) and \( h \) with respect to \( a \), holding \( x_1 \) and \( x_2 \) fixed at their optimal values.
2.6.5 Maximum Theorems

In optimization problems, we often need to check if an optimal solution is continuous in parameters, say, to check the continuity of the demand function. We can apply the so-called **maximum theorem** to these problems.

**Berge’s Maximum Theorem**

**Theorem 2.6.13 (Berge’s Maximum Theorem)** Let $A$ and $X$ be two topological spaces. Suppose that $f(x, a) : A \times X \rightarrow \mathbb{R}$ is a continuous function, and the constraint set $F : A \rightarrow 2^X$ is a continuous correspondence with non-empty compact values. Then, the maximum value function (also called marginal function)

$$M(a) = \max_{x \in F(a)} f(x, a)$$

is a continuous function, and the maximum correspondence

$$\mu(a) = \arg \max_{x \in F(a)} f(x, a)$$

is upper hemi-continuous.

**Walker’s Maximum Theorem**

In many cases of optimization problems, the preference of an economic agent may not be represented by a utility function. Walker (1979) generalized the Berge’s maximum theorem to the case of maximal element under the open preference relation. The Walker’s maximum theorem allows the preference relations and constraint sets to vary with parameters.

**Theorem 2.6.14 (Walker’s Maximum Theorem)** Let $A$ and $Y$ be two topological spaces. Suppose that $U : Y \times A \rightarrow 2^Y$ is a correspondence with an open graph. The constraint set $F : A \rightarrow 2^Y$ is a continuous and non-empty compact-valued correspondence. Define the maximum correspondence $\mu : A \rightarrow 2^Y$ as

$$\mu(a) := \{y \in F(a) : U(y, a) \cap F(a) = \emptyset\},$$

$\mu$ is a compact-valued upper semi-continuous correspondence.

**Tian-Zhou Maximum Theorem**

Both Berge’s and Walker’s maximum theorems depend on the continuity (or open graph) of the constraint correspondence and the objective function (preference correspondence).

Tian and Zhou (1995) relaxed these assumptions, generalized and characterized the Berge’s and Walker’s maximum theorems. Here is a description of the generalized Berge’s maximum theorem. We first give the following definition of transfer continuity.
Definition 2.6.3 Let $A$ and $Y$ be two topological spaces, and $F : A \to 2^Y$ be a correspondence. A function $u : A \times Y \to \mathcal{R} \cup \{\infty\}$ is said to be \textbf{quasi-transfer upper continuous in $(a, y)$ with respect to $F$} if, for every $(a, y) \in A \times Y$ with $y \in F(a)$, $u(a, z) > u(a, y)$ for some $z \in F(a)$ implies that there is a neighbourhood $N(a, y)$ of $(a, y)$ such that for any $(a', y') \in N(a, y)$ with $y' \in F(a')$, there is a $z' \in F(a')$ such that $u(a', z') > u(a', y')$.

The following definition is a natural generalization of transfer upper continuity.

Definition 2.6.4 Let $A$ and $Y$ be two topological spaces, and $F : A \to 2^Y$ be a correspondence. A function $u : A \times Y \to \mathcal{R} \cup \{\infty\}$ is said to be \textbf{transfer upper continuous on $F$} if, for every $(a, y) \in A \times Y$ with $y \in F(a)$, $u(a, z) > u(a, y)$ for some $z \in F(a)$ implies that there is a point $z' \in Y$ and a neighbourhood $N(y)$ of $y$ such that for any $y' \in N(y)$ with $y' \in F(a)$, we have $u(a, z') > u(a, y')$ and $z' \in F(a)$.

Theorem 2.6.15 (Tian-Zhou Maximum Theorem) Let $A$ and $Y$ be two topological spaces, and $u : A \times Y \to \mathcal{R} \cup \{\infty\}$ be a function. Suppose that $F : A \to 2^Y$ is a compact and closed valued correspondence. Then the maximum correspondence $\mu : A \to 2^Y$ is a nonempty, compact-valued and closed correspondence if and only if $u$ is transfer upper continuous in $y$ on $F$, and quasi-transfer upper continuous in $(a, y)$ with respect to $F$. Moreover, if $F$ is upper hemi-continuous, then the correspondence of extreme value $\mu$ is also upper hemi-continuous.

This theorem relaxes the upper semi-continuity of the objective function and the constraint correspondence in the Berge’s maximum theorem.

2.6.6 Continuous Selection Theorems

The continuous selection theorem is a powerful tool to prove the existence of equilibrium, and it is closely related to the fixed point theorem to be introduced below. The basic conclusion of a continuous selection theorem is that if a correspondence is lower hemi-continuous with non-empty convex values, there is a continuous function so that for all points in the domain, the function value is a subset of the correspondence.

Definition 2.6.5 Let $X \subseteq \mathcal{R}^n$, $Y \subseteq \mathcal{R}^m$ and $F : X \to 2^Y$ be a correspondence. If for any $x \in X$, we have $f(x) \in F(x)$, then the single valued function $f : X \to Y$ is said to be a \textbf{selection} corresponding to $F$.

Theorem 2.6.16 (Michael(1956)) Let $X \subseteq \mathcal{R}^n$ be compact. Suppose that $F : X \to 2^{\mathcal{R}^m}$ is a lower hemi-continuous correspondence with closed and convex values, then $F$ has a continuous selection, that is, there exists a single-valued continuous function $f : X \to \mathcal{R}^m$ such that $f(x) \in F(x)$ for all $x \in X$. 

For the infinite dimension space, we have the following Browder Theorem.

**Theorem 2.6.17 (Browder, (1968))** Let $X$ be a Hausdorff compact space and $Y$ be a locally convex topological vector space. Suppose that $F : X \rightarrow 2^Y$ is a correspondence with open lower sections and convex values. Then $F$ has a continuous selection, that is, there is a single-valued continuous function $f : X \rightarrow Y$ such that $f(x) \in F(x)$ for all $x \in X$.

Since the open lower section of a correspondence implies the lower hemi-continuity of the correspondence (see Proposition 2.5.3), we then have the following result.

**Corollary 2.6.1 (Yannelis-Prabhakar (1983))** Let $X \subseteq \mathbb{R}^n$. Suppose that $F : X \rightarrow 2^{\mathbb{R}^n}$ is a correspondence with open lower sections and convex values. Then $F$ has a continuous selection, that is, there is a single-valued continuous function $f : X \rightarrow Y$ such that $f(x) \in F(x)$ for all $x \in X$.

### 2.6.7 Fixed Point Theorems

The fixed point theorem plays a crucial role in proving the existence of equilibrium. It is the most commonly used method for determining whether there is a solution of equilibrium equations. John von Neumann (1903-1957, see Section 5.8.1 for his biography) was the first to propose the results that are essentially the fixed point thereom in the two papers published in 1928 and 1937, respectively.

**Definition 2.6.6** Let $X$ be a topological space and $f : X \rightarrow X$ be a single-valued function from $X$ to itself. If there is a point $x^* \in X$ such that $f(x^*) = x^*$, then $x^*$ is called a **fixed point** of function $f$.

**Definition 2.6.7** Let $X$ be a topological space and $F : X \rightarrow 2^X$ is a correspondence from $X$ to itself. If there is a point $x^* \in X$ such that $x^* \in F(x^*)$, then $x^*$ is called a **fixed point** of correspondence $f$.

There are some important fixed point theorems which are widely used in economics.

**Brouwer’s Fixed Theorem**

The Brouwer’s fixed point theorem is one of the most fundamental and important fixed point theorems.

**Theorem 2.6.18 (Brouwer’s Fixed Theorem)** Let $X$ be a non-empty, compact, and convex subset of $\mathbb{R}^m$. If a function $f : X \rightarrow X$ is continuous on $X$, then $f$ has a fixed point, i.e., there is a point $x^* \in X$ such that $f(x^*) = x^*$ (See Figure 2.4).
Example 2.6.1 \( f : [0, 1] \rightarrow [0, 1] \) is continuous, then \( f \) has a fixed point \( x \). To see this, let \( g(x) = f(x) - x \). Then, we have

\[
\begin{align*}
g(0) &= f(0) \geq 0 \\
g(1) &= f(1) - 1 \leq 0.
\end{align*}
\]

From the mean-value theorem, there is a point \( x^* \in [0, 1] \) such that \( g(x^*) = f(x^*) - x^* = 0 \).

Kakutani’s Fixed Point Theorem

In applications, the mapping is often a correspondence, so Brouwer’s fixed point theorem cannot be used directly and the Kakutani’s fixed point theorem is commonly used.

Theorem 2.6.19 (Kakutani’s Fixed Point Theorem (1941)) Let \( X \subseteq \mathbb{R}^m \) be a non-empty, compact, and convex subset. If a correspondence \( F : X \rightarrow 2^X \) is an upper hemi-continuous correspondence with non-empty compact and convex values on \( X \), then \( F \) has a fixed point, i.e., there is a point \( x^* \in X \) such that \( x^* \in F(x^*) \).

Browder’s Fixed Point Theorem

It follows from Theorem 2.6.16 that we have the following Browder’s Fixed Point Theorem.

Theorem 2.6.20 (Browder (1968)) Let \( X \subseteq \mathbb{R}^n \) be a compact and convex subset. Suppose that a correspondence \( F : X \rightarrow 2^{\mathbb{R}^m} \) is convex-valued with open lower sections, then \( F \) has a fixed point, i.e., there is a point \( x^* \in X \) such that \( x^* \in F(x^*) \).
Michael’s Fixed Point Theorem

It follows from Theorem 2.6.17 that Michael’s Fixed Point Theorem is given as follows.

**Theorem 2.6.21 (Michael (1956))** Let \( X \subseteq \mathbb{R}^n \) be a compact and convex subset. Suppose that \( F : X \rightarrow 2^{\mathbb{R}^m} \) is a lower hemi-continuous correspondence with closed and convex values, then \( F \) has a fixed point, i.e., there is a point \( x^* \in X \) such that \( x^* \in F(x^*) \).

Tarsky’s Fixed Point Theorem

Tarsky’s fixed point theorem is a very different type of fixed point theorems. It does not require the function to have any kind of continuity, but only requires that the function be monotonic and non-decreasing, and be defined on the domain composed of intervals. It is becoming more and more important in the application of economics, especially in the game with a monotonic payment function.

**Theorem 2.6.22 (Tarsky’s Fixed Point Theorem (1955))** Let \( [0,1]^n \) be the \( n \) times product of interval \([0,1]\). If \( f : [0,1]^n \rightarrow [0,1]^n \) is a non-decreasing function, then \( f \) has a fixed point, i.e., there is a point \( x^* \in X \) such that \( f(x^*) = x^* \).

Contraction Mapping Theorem

In many dynamic economic models, we not only need to prove the existence of equilibrium, but also prove the uniqueness of equilibrium. The contraction mapping principle is an important tool to solve this problem. It is also the most basic and simplest theorem of existence in functional analysis. Many of the existence theorems in mathematical analysis are its special cases. Its basic conclusion is that a contraction mapping from a complete metric space to itself has a unique fixed point.

**Definition 2.6.8** Let \((X,d)\) be a complete metric space, and \( f : X \rightarrow X \) be a single-valued function from \( X \) to itself. If for any point \( x, x' \in X \), there is \( \alpha \in (0,1) \) such that \( d(f(x), f(x')) < \alpha d(x, x') \), then \( f \) is a **contraction mapping**.

**Theorem 2.6.23 (Banach Contraction Mapping Theorem)** Suppose that \( f : X \rightarrow X \) is a contraction mapping from a complete metric space \( X \) to itself, then \( f \) has a unique fixed point on \( X \).

The Characterization of the Existence of Fixed Point

All the above fixed-point theorems are only sufficient conditions for the existence of fixed points. Recently, Tian (2017) introduced a series of con-
cepts of recursive transfer continuity, and gave a sufficient and necessary condition for the existence of fixed points.

We first introduce the concept of diagonal transfer continuity introduced by Baye-Tian-Zhou (1993).

**Definition 2.6.9** A function $\varphi : X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is said to be **diagonally transfer continuous in $y$** if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z \in X$ and a neighborhood $V_y \subset X$ of $y$ such that $\varphi(z, y') > \varphi(y', y')$ for all $y' \in V_y$.

Now we define the concept of recursively diagonally transfer continuity.

**Definition 2.6.10** (Recursive Diagonal Transfer Continuity) A function $\varphi : X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is said to be **recursively diagonally transfer continuous** if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z_0 \in X$ (possibly $z_0 = y$) and a neighborhood $V_y$ of $y$ such that $\varphi(z_m, y') > \varphi(y', y')$ for all $y' \in V_y$ and for any finite subset $\{z^1,\ldots,z^m\} \subseteq X$ with $z^m = z$ and $\varphi(z, z^{m-1}) > \varphi(z^{m-1}, z^{m-1}), \varphi(z^{m-1}, z^{m-2}) > \varphi(z^{m-2}, z^{m-2}), \ldots, \varphi(z^1, z^0) > \varphi(z^0, z^0)$ for $m \geq 1$.

**Theorem 2.6.24** (Tian’s Fixed Point Theorem(2017)) Let $X$ be a nonempty and compact subset of a metric space $(E, d)$ and $f : X \rightarrow X$ be a function. Then, $f$ has a fixed point if and only if the function $\varphi : X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$, defined by $\varphi(x, y) = -d(x, f(y))$, is recursively diagonally transfer continuous in $y$.

2.6.8 Variation Inequality

Ky-Fan minimax inequality is one of the most important results in nonlinear analysis. It is equivalent to many important mathematical theorems in certain sense, such as KKM lemma, Sperner lemma, Brouwer’s fixed point theorem and Kakutani’s fixed point theorem (can be derived from each other). In many disciplines, such as variation inequalities, mathematical programming, partial differential equations and economic models, it can be used to prove the existence of equilibrium solutions.

**Theorem 2.6.25** (Ky-Fan minimax inequality) Let $X \subseteq \mathbb{R}^m$ be a nonempty, convex and compact set, and let $\phi : X \times X \rightarrow \mathbb{R}$ be a function that satisfies the following conditions:

1. for all $x \in X$, $\phi(x, x) \leq 0$;
2. $\phi$ is lower semi-continuous in $y$;
3. $\phi$ is quasi-concave in $x$.

Then there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq 0$ holds for all $x \in X$. 
2.6. STATIC OPTIMIZATION

Ky-Fan inequality has been generalized in various forms in mathematical literature. Tian (2017) fully characterized the existence of solutions to Ky-Fan inequalities, and gave the sufficient and necessary conditions for the existence of Ky-Fan inequalities.

**Definition 2.6.11** Let $X$ be a topological space. A function $\phi : X \times X \to \mathbb{R} \cup \{\pm \infty\}$ is said to be $\gamma$-recursively transfer lower semicontinuous in $y$ if, whenever $\phi(x, y) > \gamma$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood $V_y$ of $y$ such that $\phi(z, V_y) > \gamma$ for any sequence of points $\{z^1, \ldots, z^{m-1}, z\}$ with $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, $\ldots$, $\phi(z^1, z^0) > \gamma$, $m = 1, 2, \ldots$. Here $\phi(z, V_y) > \gamma$ means that $\phi(z, y') > \gamma$ for all $y' \in V_y$.

**Theorem 2.6.26 (Tian, 2017)** Let $X$ be a compact subset in a topological space, $\gamma \in \mathbb{R}$, and $\phi : X \times X \to \mathbb{R} \cup \{\pm \infty\}$ be a function satisfying $\phi(x, x) \leq \gamma$ for all $x \in X$. Then there is a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$ if and only if $\phi$ is $\gamma$-recursively diagonally transfer lower hemi-continuous in $y$.

### 2.6.9 FKKM Theorems

The Knaster-Kuratowski-Mazurkiewicz (KKM) lemma is quite basic and is in some ways more useful than Brouwer’s fixed point theorem.

**Theorem 2.6.27 (KKM Theorem)** Let $X \subseteq \mathbb{R}^m$ be a convex set. Suppose that $F : X \to 2^X$ is a correspondence such that

1. $F(x)$ is closed for all $x \in X$;
2. $F$ is FS-convex, i.e., for any $x_1, \ldots, x_m \in X$ and its convex combination $x_\lambda = \sum_{i=1}^m \lambda_i x_i$, we have $x_\lambda \in \bigcup_{i=1}^m F(x_i)$,

then

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$  

The following is a generalized version of KKM lemma by Ky Fan (1984).

**Theorem 2.6.28 (FKKM Theorem)** Suppose that $X \subseteq \mathbb{R}^m$ is a convex set, $\emptyset \neq X \subseteq Y$, and $F : X \to 2^Y$ is a correspondence such that

1. $F(x)$ is closed for all $x \in X$;
2. $F(x_0)$ is compact for some $x_0 \in X$;
(3) \( F \) is \( FS \)-convex, i.e., for any \( x_1, \ldots, x_m \in X \) and its convex combination \( x_\lambda = \sum_{i=1}^{m} \lambda_i x_i \), we have
\[
x_\lambda \in \bigcup_{i=1}^{m} F(x_i).
\]
Then
\[
\bigcap_{x \in X} F(x) \neq \emptyset.
\]

This theorem has many generalizations. Tian (2017) also gave the sufficient and necessary conditions for establishing the FKKM theorem:

**Theorem 2.6.29 (Tian, 2017)** Let \( X \) be a nonempty compact set in a topological space \( T \), and \( F : X \to 2^X \) be a correspondence satisfying \( x \in F(x), x \in X \). Then \( \bigcap_{x \in X} F(x) \neq \emptyset \) if and only if the correspondence \( \phi : X \times X \to \mathbb{R} \cup \{ \pm \infty \} \) defined by
\[
\phi(x, y) = \begin{cases} 
\gamma, & \text{if } (x, y) \in G, \\
+\infty, & \text{otherwise}
\end{cases}
\]
is \( \gamma \)-recursively transfer semi-continuous with respect to \( y \), where \( \gamma \in \mathbb{R} \) and \( G = \{(x, y) \in X \times Y : y \in F(x)\} \).

### 2.7 Dynamic Optimization

We generally encounter various constraints when making optimal decisions, and the constrained optimization problems in the last section are all among different variables in the same period. However, individuals often need to make decisions in the dynamic environment, and the early decisions will affect the decisions in the later periods. Dynamic optimization, dynamic programming, or optimal control provides analytical frameworks and tools for solving optimization problems in dynamic environments. In this section, we discuss the calculus of variation, Hamilton equation and the basic results of dynamic programming. We focus mainly on continuous cases of dynamic optimization problems defined on \( X \subseteq \mathbb{R} \).

#### 2.7.1 Calculus of Variation

A general dynamic optimization problem has the following form:

\[
\max \int_{t_0}^{t_1} F(t, x(t), x'(t)) dt \quad (2.7.4)
\]
\[
s.t. \quad x(t_0) = x_0, x(t_1) = x_1. \quad (2.7.5)
\]
2.7. DYNAMIC OPTIMIZATION

The above optimization problem is to choose a function \( x(t) \) subject to the constraints in (2.7.5) to maximize the objective function (2.7.4). Calculus of variation is a common method to solve such problems. Let \( x^*(t) \) be the solution to the above optimization problem, and the necessary condition is that the solution must satisfy the Euler equation:

\[
F_x[t, x^*(t), x'^*(t)] = \frac{dF_x[t, x^*(t), x'^*(t)]}{dt}, \quad t \in [t_0, t_1].
\] (2.7.6)

Next, we will derive the Euler equation of dynamic optimization.

We say the function satisfying the constraint (2.7.5) is admissible. Let \( x(t) \) be admissible and let \( h(t) = x(t) - x^*(t) \) be the difference between \( x(t) \) and the optimal selection. We then have \( h(t_0) = h(t_1) = 0 \).

For any constant \( a \), \( y(t) = x^*(t) + ah(t) \) is also admissible. In this way, the dynamic optimization problem can be transformed into solving under what conditions \( a = 0 \) is the optimal choice under dynamic optimization.

\[
g(a) = \int_{t_0}^{t_1} F[t, y(t), y'(t)] dt = \int_{t_0}^{t_1} F[t, x^*(t) + ah(t), x'^*(t) + ah'(t)] dt.
\] (2.7.7)

The first-order condition of optimization is obtained by differentiating (2.7.7) with respect to \( a \) and then is set to 0:

\[
g'(0) = \int_{t_0}^{t_1} F_x[t, x^*, x'^*(t)]h(t) + F_{x'}[t, x^*, x'^*(t)]h'(t) dt = 0.
\] (2.7.8)

Using integration by parts on the second part of the right side of the equation (2.7.8) gets:

\[
\int_{t_0}^{t_1} \left\{ F_x[t, x^*, x'^*(t)] - \frac{dF_x[t, x^*(t), x'^*(t)]}{dt} \right\} h(t) dt = 0.
\] (2.7.9)

If equation (2.7.9) holds for any continuous function \( h(t) \) that satisfies the constraint \( h(t_0) = h(t_1) = 0 \), the Euler equation (2.7.6) also holds (see Kamien & Schwartz (1991)).

**Example 2.7.1 (Kamien & Schwartz (1991))** Suppose that an enterprise receives an order, requiring \( B \) units of products delivered at time \( T \). Assume that the production capacity of the enterprise is limited, and the unit cost of production is proportional to the output. In addition, completed products need to be stocked and the inventory cost per unit is a constant. Business
managers need to consider production problems from now (time 0) to delivery date (time T). Suppose that at time \( t \in [0, T] \), the inventory of the enterprise is \( x(t) \), and the change of inventory depends on the production of the enterprise, that is,
\[
\dot{x}(t) = y(t),
\]
where \( y(t) \) is the productivity at time \( t \). At \( t \), the cost of the enterprise is \( c_1 \dot{x}(t) + c_2 x(t) \) or \( c_1 u(t) + c_2 x(t) \), where \( c_1 u(t) \) is the unit cost of production when yield is \( u(t) \), and \( c_2 \) is the unit cost of inventory. The goal of the enterprise is to minimize costs (including both production costs and inventory costs), therefore the dynamic optimization problem is
\[
\min \int_0^T [c_1 \dot{x}^2(t) + c_2 x(t)] dt \tag{2.7.10}
\]
subject to:
\[
x(0) = 0, x(T) = B, \dot{x}(t) \geq 0.
\]
In expression (2.7.10), \( u(t) \) is called a control variable, and \( x(t) \) is called a state variable. Using the calculus of variation to solve the optimization problem, we have
\[
F[t, x(t), \dot{x}(t)] = c_1 \dot{x}^2(t) + c_2 x(t).
\]
The Euler equation is:
\[
c_2 = 2c_1 \dddot{x}(t).
\]
With the constraint conditions: \( x^*(0) = 0, x^*(T) = B \), we solve the above Euler equation and get:
\[
x^*(t) = \frac{c_2}{4c_1} t(t - T) + Bt/T, \quad t \in [0, T].
\]
Integrating Euler equation (2.7.6) gives:
\[
F_x = F_{x'} + F_{x''} \dot{x}' + F_{x'''} \dot{x}'''.
\] \tag{2.7.11}
Now we introduce the Hamilton equations to avoid taking second-order derivatives. Let \( p(t) = F_{x'}[t, x(t), \dot{x}(t)] \), and Hamilton equation is:
\[
H(t, x, p) = -F(t, x, \dot{x}) + px'.
\] \tag{2.7.12}
In equation (2.7.12), \( p(t) \) can be regarded as the shadow price. The total differential of equation (2.7.12) is:
\[
dH = -F_t dt - F_x dx - F_{x'} dx' + p dx' + x' dp = -F_t dt - F_x dx + x' dp.
\]
The partial derivatives of equation (2.7.12) with respect to \( x \) and \( p \), respectively, are:
\[ \frac{\partial H}{\partial x} = -F_x; \]
\[ \frac{\partial H}{\partial p} = x'. \]

Since \(-F_x = -(dF_x'/dt) = -p'\), we have two Euler equations under first-order conditions:

\[ \frac{\partial H}{\partial x} = -p'; \]
\[ \frac{\partial H}{\partial p} = x'. \]

Thus, the Euler equations are only the necessary conditions for solving dynamic optimization, and the sufficient conditions involve the second-order conditions. After deriving the first-order conditions by the calculus of variation, it is clear that the second-order condition is:

\[ g''(0) = \int_{t_0}^{t_1} \left[ F_{xx} h^2 + 2F_{xx'} hh' + F_{xx'} (h')^2 \right] dt \leq 0. \]

It is easy to verify that if the objective function \( F \) is concave in \( x \) and \( x' \), then the second-order condition is satisfied.

Denote \( F = F(t, x, x') \), \( F^* = F(t, x^*, x'^*) \), and let \( h(t) = x(t) - x^*(t) \), then we have \( h'(t) = x'(t) - x'^*(t) \), hence

\[ \int_{t_0}^{t_1} (F - F^*) dt \leq \int_{t_0}^{t_1} [(x - x^*) F^*_x + (x' - x'^*) F^*_x'] dt \\
   = \int_{t_0}^{t_1} (h F^*_x + h' F^*_x') dt \\
   = \int_{t_0}^{t_1} h(F^*_x - dF^*_x/dt) dt = 0. \]

It can be proved (see Kamiem & Schwartz (1991, p.43)) that the first-order condition, namely, the Euler equation will be met as long as \( F_{xx'} \leq 0 \), and thus the dynamic maximization problem is solved. As for the dynamic minimization, the first-order condition is also a sufficient condition if the second-order condition satisfies \( F_{xx'} \geq 0 \).

### 2.7.2 Optimal Control

We have two types of variables in the previous example: state variable and control variable. We can also discuss the dynamic optimization problem using the analytical framework of optimal control.

The optimal control problem can be generally expressed as follows:

\[
\begin{align*}
\max & \int_{t_0}^{t_1} f[t, x(t), u(t)] dt \\
\text{s.t.} & \quad x'(t) = g(t, x(t), u(t)), \quad (2.7.14) \\
& \quad x(t_0) = x_0. \quad (2.7.15)
\end{align*}
\]
In the above statement, \( x(t) \) is a state variable, and \( u(t) \) is a control variable that affects the change of the state variable, and the objective (2.7.13) is a function of the state variable and the control variable.

The necessary and sufficient conditions for optimal control are given below. Analogously to the optimization problem under static constraints, the dynamic Lagrange equation is established as:

\[
L = \int_{t_0}^{t_1} \left\{ f[t, x(t), u(t)] + \lambda(t) \left[ g(t, x(t), u(t)) - x'(t) \right] \right\} dt, \tag{2.7.16}
\]

where \( \lambda(t) \) is the multiplier of the constraint on the change in state at time \( t \), commonly known as costate variable. Integrating by parts gives:

\[
L = \int_{t_0}^{t_1} \left\{ f[t, x(t), u(t)] + \lambda(t) g(t, x(t), u(t)) + x(t) \lambda'(t) \right\} dt - \lambda(t_1) x(t_1) + \lambda(t_0) x(t_0).
\]

The necessary conditions for the optimal control problem can be derived by using similar process of deducing the calculus of variation. Assuming that \( u^*(t) \) is the optimal control function, we introduce a new control function \( u^*(t) + ah(t) \) which reduces to the original optimal control function when \( a = 0 \). The optimal state function \( x^*(t) \) can be determined by giving the optimal control function \( u^*(t) \) and the initial state \( x(t_0) = x_0 \). Denote the state variable generated by control function \( u^*(t) + ah(t) \) and initial state \( x_0 \) as \( y(t, a) \), which satisfy: \( y(t, a) = x^*(t), y(t_0) = x_0 \), and \( dy(t, a)/dt = g(t, y(t, a), u^*(t) + ah(t)) \). Set the function:

\[
J(a) = \int_{t_0}^{t_1} f[t, y(t, a), u^*(t) + ah(t)]dt
\]

\[
= \int_{t_0}^{t_1} \left\{ f[t, y(t, a), u^*(t) + ah(t)] + \lambda(t) g(t, y(t, a), u^*(t) + ah(t)) + y'(t, a) \lambda'(t) \right\} dt - \lambda(t_1) y(t_1, a) + \lambda(t_0) y(t_0, a). \tag{2.7.17}
\]

The derivative of the function (2.7.17) at \( a = 0 \) is:

\[
J'(a) = \int_{t_0}^{t_1} [(f_x + \lambda g_x + \lambda') y_a + (f_u + \lambda g_u) h] dt - \lambda(t_1) y'(t_1, 0).
\]

Thus, \( \lambda(t) \) is required to be differentiable, and the optimization needs to satisfy the following three conditions:

The first one is the first-order condition with respect to the control variable:

\[
f_u[t, x(t), u(t)] + \lambda g_u(t, x(t), u(t)) = 0. \tag{2.7.18}
\]

The second one is the first-order condition with respect to the costate variable:

\[
\lambda'(t) = -f_x[t, x(t), u(t)] - \lambda(t) g_x[t, x(t), u(t)], \quad \lambda(t_1) = 0. \tag{2.7.19}
\]
The third one is the state function:

\[ x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0. \]  (2.7.20)

The Hamilton equation for optimal control, which is similar to the Lagrange equation for constrained optimizations, is defined as:

\[ H(t, x(t), u(t)) \equiv f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t)). \]  (2.7.21)

We then have: 1) the optimal control condition,

\[ \frac{\partial H}{\partial u} = 0 : \quad \frac{\partial H}{\partial u} = f_u + \lambda g_u = 0, \]  (2.7.22)

i.e., the equation (2.7.18);

2) the costate (multiplier) equation,

\[ -\frac{\partial H}{\partial x} = \lambda' : \quad \lambda'(t) = -\frac{\partial H}{\partial x} = -(f_x + \lambda g_x), \]  (2.7.23)

i.e., the equation (2.7.19);

3) the state equation,

\[ \frac{\partial H}{\partial \lambda} = x' : \quad x'(t) = \frac{\partial H}{\partial \lambda} = g, \]  (2.7.24)

i.e., the equation (2.7.20).

Example 2.7.2  Consider Example 2.7.1 again. The problem is

\[ \min \int_0^T \left[ c_1 u^2(t) + c_2 x(t) \right] dt \]

s.t. \[ x'(t) = u(t), \quad x(0) = 0, \quad x(T) = B, \quad x'(t) \geq 0. \]

It follows from the above three conditions of optimization that:

\[ 2c_1 u(t) = -\lambda(t); \quad \lambda'(t) = -c_2; \quad x'(t) = u(t), \quad x(0) = 0, \quad x(T) = B. \]

and thus we have:

\[ x^{*\prime\prime}(t) = \frac{c_2}{2c_1}, \quad t \in [0, T], \]

\[ x^*(t) = \frac{c_2}{4c_1}t(t - T) + Bt/T, \quad t \in [0, T], \]

\[ u^*(t) = \frac{c_2}{2c_1}t + k, \quad t \in [0, T]; \quad k = \frac{-c_2}{4c_1}T + B/T. \]

The second-order conditions of optimal control can be similarly derived. If the objective function and the state function \( f \) and \( g \) are concave with respect to \( x \) and \( u \), then the first-order necessary conditions are also the sufficient conditions, and one can refer to Kamien & Schwartz (1991) for the proof.
2.7.3 Dynamic Programming

The third method of dealing with the dynamic optimization is dynamic programming proposed by Richard Bellman, and its basic logic can be summarized as Bellman’s principle of optimality. An optimal path satisfies the property that whatever the states and the control variables are before a certain time, the selection of decision function must constitute an optimal policy from now to the end with regard to the current state.

The general form of dynamic programming problems is:

$$\max \int_0^T f(t, x(t), u(t)) dt + \phi(x(T), T) \quad (2.7.25)$$

s.t.  \( x'(t) = g(t, x(t), u(t)), x(0) = a, t \in [0, T]. \) \( (2.7.26) \)

Define the optimal value function \( J(t_0, x_0) \) as the maximal value starting at time \( t_0 \) in state \( x_0 \):

$$J(t_0, x_0) = \max_u \left\{ \int_{t_0}^{t_0+\Delta t} f dt + \int_{t_0+\Delta t}^T f dt + \phi(x(T), T) \right\}.$$ \( (2.7.27) \)

At time \( t_0 + \Delta t \), the state changes to \( x_0 + \Delta x \), and it follows Bellman’s principle of optimality that the equation (2.7.28) is equivalent to:

$$J(t_0, x_0) = \max_u \left\{ \int_{t_0}^{t_0+\Delta t} f dt + \int_{t_0+\Delta t}^T f dt + \phi(x(T), T) \right\}.$$ \( (2.7.29) \)

The equation (2.7.29) depicts Bellman’s principle of optimality. Expanding the right side of (2.7.29) by Taylor’s theorem gets:

$$J(t_0, x_0) = \max_u \left[ f(t_0, x_0, u) \Delta t + J(t_0, x_0) + J_t(t_0, x_0) \Delta t \right.$$ \( + J_x(t_0, x_0) \Delta x + h.o.t. \] \( (2.7.30) \)

Let \( \Delta t \to 0 \), equation (2.7.30) becomes:

$$0 = \max_u \left[ f(t, x, u) + J_t(t, x) + J_x(t, x) x' \right]$$
and then we have
\[ -J_t(t, x) = \max_u [f(t, x, u) + J_x(t, x)g(t, x, u)]. \] (2.7.31)

Compared to the method of optimal control, \( J_x(t, x) \) on the right side of (2.7.31) plays the role of the costate variable \( \lambda \). We just define \( \lambda(t) = J_x(t, x) \), and hence the economic meaning behind the costate variables is the marginal contribution of states to the value function.

The derivative of (2.7.31) with respect to \( x \) gives:
\[ -J_{tx}(t, x^*) = f_x(t, x^*, u^*) + J_x(t, x^*)g_x. \] (2.7.32)

Since
\[ \lambda'(t) = \frac{dJ_x(t, x)}{dt} = J_t + J_{xx}g, \]

we get:
\[ -\lambda'(t) = f_x + \lambda g_x. \] (2.7.33)

The equation (2.7.33) is just the first-order condition for optimal control with respect to the state variable:
\[ -\partial H/\partial x = \lambda'. \]

The derivative of the right side of (2.7.31) with respect to \( u \) gives:
\[ f_u + J_xg_u = 0, \]

and this is the first-order condition for optimal control with respect to control variables:
\[ \frac{\partial H}{\partial u} = f_u + \lambda g_u = 0. \]

Therefore, the optimal control and dynamic programming are essentially consistent.

In the discrete case, the method of dynamic programming may be more convenient. The following results are given only for infinite time horizon.

Suppose that the state set \( S \subseteq \mathbb{R}^n \) is a nonempty and compact set, and \( U : S \times S \to \mathbb{R} \) is a bounded continuous function, which generally represents the utility function in a period. Given the initial state \( s_0 = z \), the general dynamic optimization problem is:
\[ \max_{\{s_t\}} \sum_{t=0}^{\infty} \delta^t U(s_t, s_{t+1}) \] (2.7.34)
\[ \text{s.t. } s_t \in S, \quad \forall t, \]
\[ s_0 = z. \] (2.7.35)
It can be proved by using the contraction mapping theorem that there is a sequence of maximum points in the problem (2.7.34) and hence there exists a maximum value denoted by $V(z)$. Function $V : S \rightarrow \mathbb{R}$ is called the value function of problem (2.7.34). Like function $U(\cdot, \cdot)$, the value function is also continuous. In addition, if $S$ is a convex set and $U(\cdot, \cdot)$ is concave, then $V(\cdot)$ is also concave, and it is equivalent to Bellman’s principle of optimality, that is, it is the solution to the following Bellman equation:

$$V(s) = \max_{\hat{s} \in S} U(s, \hat{s}) + \delta V(\hat{s}).$$

The equivalence results provide the basis for solving the dynamic optimization problem by Bellman method. The following theorem reveals that the value function is the only function satisfying the Bellman equation.

**Theorem 2.7.1**

$$f(s) = \max_{\hat{s} \in S} U(s, \hat{s}) + \delta V(\hat{s}) \quad (2.7.36)$$

that is, $f(\cdot) = V(\cdot)$.

**Proof.** Using (2.7.36) repeatedly gives: for each $T$,

$$f(z) = \max_{\{s_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \delta^t U(s_t, s_{t+1}) + \delta^T f(x_T)$$

s.t. $s_t \in S, \quad \forall t,$

$s_0 = z.$

When $T \rightarrow \infty$, the contribution of $\delta^T f(x_T)$ to the above summation is increasingly negligible, and thus $f(\cdot) = V(\cdot)$. \(\square\)

The above theorem provides a way to calculate the value function. Starting from any continuous function $f_0(\cdot) : S \rightarrow \mathbb{R}$, one can imagine $f_0(\hat{s})$ as a trial “value” function which gives the estimated value from time 0. Then let

$$f_1(s) = \max_{\hat{s} \in S} U(s, \hat{s}) + \delta f_0(\hat{s})$$

holds for any $s \in S$, and thus we obtain a new value function $f_1(\hat{s})$.

Value function $V(\cdot)$ can be also found by the iterative method. If $f_1(t) = f_0(t)$, then $f_0(t)$ satisfies the Bellman equation. It follows from the above theorem that $f_0(t) = V(t)$. If $f_1(t) \neq f_0(t)$, we get a new value function from $f_1(t)$, and get the whole sequence of functions $\{f_r(\cdot)\}_{r=0}^{\infty}$. Then, it can be shown that

$$\lim_{r \rightarrow \infty} f_r(s) = V(s),$$

that is, it will converge to the value function as $r$ increases to infinity.
If the function is differentiable, there is a similar first-order condition called the Euler equation of dynamic optimization:

$$0 = F_y(s_t^*, s_{t+1}^*) + \beta F_x(s_t^*, s_{t+2}^*), \quad t = 0, 1, 2, \ldots$$

(2.7.37)

The first-order condition of optimal decision gives:

$$0 = F_y[x, g(x)] + \beta V'[g(x)],$$

(2.7.38)

where $g(x)$ is the state of next periods determined by $x$ following Bellman’s principle of optimality. It follows from the envelope theorem that

$$V'(x) = F_x[x, g(x)].$$

(2.7.39)

The Euler equation is derived from these two equations above.

### 2.8 Differential Equations

We first provide the general concept of ordinary differential equations defined on Euclidean spaces.

**Definition 2.8.1** An equation,

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

(2.8.40)

which constitutes independent variable $x$, unknown function $y = y(x)$ of the independent variable, and its first derivative $y' = y'(x)$ to the $n$th order derivative $y^{(n)} = y^{(n)}(x)$, is called an **ordinary differential equation**.

If the highest order derivative in the equation is $n$, the equation is also called an $n$th-order ordinary differential equation.

If for all $x \in I$, the function $y = \psi(x)$ satisfies

$$F(x, \psi(x), \psi'(x), \ldots, \psi^{(n)}(x)) = 0,$$

then $y = \psi(x)$ is called a **solution to the ordinary differential equation** (2.8.40).

Sometimes the solutions of the ordinary differential equations are not unique, and there may even exist infinite solutions. For example, $y = C + \frac{1}{5}x^4$ is the solution of the ordinary differential equation $\frac{dy}{dx} + \frac{y}{x} = x^3$, where $C$ is an arbitrary constant. Next we introduce the concept of general solutions and particular solutions of ordinary differential equations.

**Definition 2.8.2** The solution of the $n$th-order ordinary differential equation (2.8.40)

$$y = \psi(x, C_1, \ldots, C_n),$$

(2.8.41)
which contains \( n \) independent arbitrary constants, \( C_1, \ldots, C_n \), is called the **general solution** to ordinary differential equation (2.8.40). Here, independence means that the Jacobi determinant

\[
D[\psi, \psi^{(1)}, \ldots, \psi^{(n-1)}] \overset{\text{def}}{=} \begin{vmatrix}
\frac{\partial \psi}{\partial C_1} & \frac{\partial \psi}{\partial C_2} & \cdots & \frac{\partial \psi}{\partial C_n} \\
\frac{\partial \psi^{(1)}}{\partial C_1} & \frac{\partial \psi^{(1)}}{\partial C_2} & \cdots & \frac{\partial \psi^{(1)}}{\partial C_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \psi^{(n-1)}}{\partial C_1} & \frac{\partial \psi^{(n-1)}}{\partial C_2} & \cdots & \frac{\partial \psi^{(n-1)}}{\partial C_n}
\end{vmatrix}
\]

is not identically equal to 0.

If a solution of an ordinary differential equation, denoted \( y = \psi(x) \), does not contain any constant, it is called a particular solution. Obviously, a general solution becomes a particular solution when the arbitrary constants are determined. In general, the restrictions of some initial conditions determine the value of any constants. For example, for ordinary differential equation (2.8.40), if there are some given initial conditions:

\[
y(x_0) = y_0, y^{(1)}(x_0) = y_0^{(1)}, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)},
\]

then the ordinary differential equation (2.8.40) and the initial value conditions (2.8.42) are said to be the Cauchy problem or initial value problem for the \( n \)-th-order ordinary differential equations. Then the question is what conditions the function \( F \) should satisfy so that the above ordinary differential equations are uniquely solvable. This problem is the existence and uniqueness of solutions for ordinary differential equations.

### 2.8.1 Existence and Uniqueness Theorem of Solutions for Ordinary Differential Equations

We first consider an ordinary differential equation of first-order \( y' = f(x, y) \) that satisfies initial condition \((x_0, y_0)\), that is, \( y(x_0) = y_0 \). Let \( y(x) \) be a solution to the differential equation.

**Definition 2.8.3** Let a function \( f(x, y) \) be defined on \( D \subseteq \mathbb{R}^2 \). We say \( f \) satisfies the **local Lipschitz condition** with respect to \( y \) at the point \((x_0, y_0) \in D\), if there exists a neighborhood \( U \subseteq D \) of \((x_0, y_0)\), and a positive number \( L \) such that

\[
|f(x, y) - f(x, z)| \leq L|y - z|, \forall (x, y), (x, z) \in U.
\]

If there is a positive number \( L \) such that

\[
|f(x, y) - f(x, z)| \leq L|y - z|, \forall (x, y), (x, z) \in D,
\]
we call \(f(x, y)\) satisfies **global Lipschitz condition** in \(D \subseteq \mathbb{R}^2\).

The following lemma characterizes the properties of the function satisfying Lipschitz condition.

**Lemma 2.8.1** Suppose that \(f(x, y)\) defined on \(D \subseteq \mathbb{R}^2\) is continuously differentiable. If there is an \(\epsilon > 0\) such that \(f_y(x, y)\) is bounded on \(U = \{(x, y) : |x - x_0| < \epsilon, |y - y_0| < \epsilon\}\), then \(f(x, y)\) satisfies the local Lipschitz condition. If \(f_y(x, y)\) is bounded on \(D\), then \(f(x, y)\) satisfies the global Lipschitz condition.

**Theorem 2.8.1** If \(f\) is continuous on an open set \(D\), then for any \((x_0, y_0) \in D\), there always exists a solution \(y(x)\) of the differential equation, and it satisfies \(y' = f(x, y)\) and \(y(x_0) = y_0\).

The following is the theorem on the uniqueness of the solution for differential equations.

**Theorem 2.8.2** Suppose that \(f\) is continuous on an open set \(D\), and satisfies the global Lipschitz condition with respect to \(y\). Then for any \((x_0, y_0) \in D\), there always exists a unique solution \(y(x)\) satisfying \(y' = f(x, y)\) and \(y(x_0) = y_0\).

For \(n\)th order ordinary differential equations, \(y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})\), if the Lipschitz condition is changed to for \(y, y', \cdots, y^{(n-1)}\) instead of for \(y\), we have similar conclusions about the existence and uniqueness of solution. See Ahmad and Ambrosetti (2014) for the specific proof of existence and uniqueness.

**2.8.2 Some Common Ordinary Differential Equations with Explicit Solutions**

Generally, we hope to obtain the concrete form of solutions, namely explicit solutions, for differential equations. However, in many cases, there is no explicit solution. Here we give some common cases in which differential equations can be solved explicitly.

**Case of Separable Equations**

Consider a separable differential equation \(y' = f(x)g(y), \text{ and } y(x_0) = y_0\). It can be rewritten as:

\[
\frac{dy}{g(y)} = f(x)dx.
\]

Integrating both sides, then we get the solution to the differential equation.

For example, for \((x^2 + 1)y' + 2xy^2 = 0, y(0) = 1\), using the above solving procedure, we get the solution as

\[
y(x) = \frac{1}{\ln(x^2 + 1) + 1}.
\]

In addition, the differential equation with the form \(y' = f(y)\) is called autonomous system, since \(y'\) is only determined by \(y\).
Homogeneous Type of Differential Equation

Some differential equations with constant coefficients have explicit solutions.

**Definition 2.8.4** We call the function $f(x, y)$ a homogeneous function of degree $n$ if for any $\lambda$, $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Differential equations have the form of homogeneous functions if $M(x, y)dx + N(x, y)dy = 0$, where $M(x, y)$ and $N(x, y)$ are homogeneous functions with the same order.

By variable transformation $z = \frac{y}{x}$, the above differential equations can be transformed into the separable form. Suppose that $M(x, y)$ and $N(x, y)$ are homogeneous functions of degree $n$, $M(x, y)dx + N(x, y)dy = 0$ is transformed to $z + x \frac{dz}{dx} = -\frac{M(1, z)}{N(1, z)}$, and then the final form is $\frac{dz}{dx} = \frac{z + M(1, z)}{N(1, z)}$, where $z + \frac{M(1, z)}{N(1, z)}$ is a function of $z$.

Exact Differential Equation

Given a simply connected and open subset $D \subseteq \mathbb{R}^2$ and two functions $M$ and $N$ which are continuous and satisfy $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$ on $D$, then implicit first-order ordinary differential equation of the form $M(x, y)dx + N(x, y)dy = 0$

is called the exact differential equation, or the total differential equation. The nomenclature of “exact differential equation” refers to the exact derivative of a function. Indeed, when $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$, the solution is $F(x, y) = C$, where the constant $C$ is determined by the initial value, and $F(x, y)$ satisfies $\frac{\partial F}{\partial x} = M(x, y)$ or $\frac{\partial F}{\partial y} = N(x, y)$.

It is clear that a separable differential equation is a special case of an exact differential equation $y' = f(x)g(y)$ or $\frac{1}{g(y)}dy - f(x)dx = 0$, and then we have $M(x, y) = -f(x)$, $N(x, y) = \frac{1}{g(y)}$ and $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} = 0$.

For example, $2xy^3 dx + 3x^2y^2 dy = 0$ is an exact differential equation, of which the general solution is $x^2 y^3 = C$, and $C$ is a constant.

When solving differential equations with explicit solutions, we usually convert differential equations into the form of exact differential equations.
First-Order Differential Linear Equation

Consider the first order linear differential equation of the following form:

\[
\frac{dy}{dx} + p(x)y = q(x).
\]  \hspace{1cm} (2.8.43)

When \( q(x) = 0 \), the above differential equation (2.8.43) is a separable differential equation, and its solution is assumed to be \( y = \psi(x) \).

Suppose that \( \psi_1(x) \) is a particular solution of the differential equation (2.8.43), then \( y = \psi(x) + \psi_1(x) \) is clearly also the solution of the equations (2.8.43).

It is easy to show that the solution to \[ \frac{dy}{dx} + p(x)y = 0 \] is \[ y = Ce^{-\int p(x)dx} \].

Next we find the general solution to the differential equation (2.8.43).

Suppose that \( y = c(x)e^{-\int p(x)dx} \),

and differentiating this gives

\[ y' = c'(x)e^{-\int p(x)dx} + c(x)p(x)e^{-\int p(x)dx}, \]

ten substituting this back into the original differential equation, we have

\[ c'(x)e^{-\int p(x)dx} + c(x)p(x)e^{-\int p(x)dx} = p(x)c(x)e^{-\int p(x)dx} + q(x), \]

and thus

\[ c'(x) = q(x)e^{\int p(x)dx}. \]

We have

\[ c(x) = \int q(x)e^{\int p(x)dx} dx + C. \]

Thus, the general solution is

\[ y(x) = e^{-\int p(x)dx} \left( \int q(x)e^{\int p(x)dx} dx + C \right). \]

Bernoulli Equation

The following differential equation is called the Bernoulli equation:

\[
\frac{dy}{dx} + p(x)y = q(x)y^n,
\]  \hspace{1cm} (2.8.44)

where \( n \) (with \( n \neq 0, 1 \)) is a natural number.

Multiplying both sides by \( (1-n)y^{(-n)} \) gives:

\[ (1-n)y^{(-n)} \frac{dy}{dx} + (1-n)y^{(1-n)}p(x) = (1-n)q(x). \]
Let \( z = y^{(1-n)} \), and get:
\[
\frac{dz}{dx} + (1 - n)zp(x) = (1 - n)q(x),
\]
which becomes a first-order linear differential equation whose explicit solution can be obtained.

The differential equations with explicit solutions have other forms, such as some special forms of Ricatti equations, and the equations similar to
\[
M(x, y)dx + N(x, y)dy = 0,
\]
but not satisfying
\[
\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}.
\]

2.8.3 Higher Order Linear Equations with Constant Coefficients

Consider a differential equation of degree \( n \) with constant coefficients
\[
y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x). \tag{2.8.45}
\]
If \( f(x) \equiv 0 \), then the differential equation (2.8.45) is called constant coefficient homogeneous differential equation of degree \( n \), otherwise it is called constant coefficients nonhomogeneous differential equation.

There is a method for finding the general solution \( y_g(x) \) of a constant coefficient homogeneous differential equation of degree \( n \). The general solution is the sum of \( n \) bases of solutions \( y_1, \cdots, y_n \), that is, \( y_g(x) = C_1 y_1(x) + \cdots + C_n y_n(x) \), where \( C_1, \cdots, C_n \) are arbitrary constants. These arbitrary constants are uniquely determined by initial-value conditions. Find a function \( y(x) \) satisfying
\[
y(x) = y_0; y'(x) = y_1; \cdots; y^{(n-1)}(x) = y_{n-1}, \text{ when } x = x_0,
\]
where \( x_0; y_0, y_1, \cdots, y_{n-1} \) are given initial values.

The procedures for solving the fundamental solution of homogeneous differential equations are given below:

1. Solve the characteristic equation with respect to \( \lambda \):
\[
\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n = 0.
\]
Suppose that the roots of the characteristic equation are \( \lambda_1, \cdots, \lambda_n \). Some roots may be complex and some are multiple.

2. If \( \lambda_i \) is the non-multiple real characteristic root, then the fundamental solution corresponding to this root is \( y_i(x) = e^{\lambda_i x} \).

3. If \( \lambda_i \) is the real characteristic root of multiplicity \( k \), then there are \( k \) fundamental solutions:
\[
y_{i_1}(x) = e^{\lambda_i x}, y_{i_2}(x) = xe^{\lambda_i x}, \cdots, y_{i_k}(x) = x^{k-1}e^{\lambda_i x}.
\]
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(4) If \( \lambda_j \) is the non-multiple complex characteristic root, \( \lambda_j = \alpha_j + i\beta_j, i = \sqrt{-1} \), its complex conjugate denoted by \( \lambda_{j+1} = \alpha_j - i\beta_j \) is also the characteristic root, thus there are two fundamental solutions generated by these complex conjugate roots \( \lambda_j, \lambda_{j+1} \):

\[
y_{j1} = e^{\alpha_j x} \cos \beta_j x, \quad y_{j2} = e^{\alpha_j x} \sin \beta_j x.
\]

(5) If \( \lambda_j \) is the complex characteristic root of multiplicity \( l \), \( \lambda_j = \alpha_j + i\beta_j \), its complex conjugate is also the complex characteristic root of multiplicity \( l \), thus these \( 2l \) complex roots generate \( 2l \) fundamental solutions:

\[
y_{j1} = e^{\alpha_j x} \cos \beta_j x, \quad y_{j2} = xe^{\alpha_j x} \cos \beta_j x, \ldots, y_{jl} = x^{l-1}e^{\alpha_j x} \cos \beta_j x;
\]

\[
y_{j1+1} = e^{\alpha_j x} \sin \beta_j x, \quad y_{j2+1} = xe^{\alpha_j x} \sin \beta_j x, \ldots, y_{2l} = x^{l-1}e^{\alpha_j x} \sin \beta_j x.
\]

The following is a general method for solving nonhomogeneous differential equations.

The general form of solution to nonhomogeneous differential equations is 
\( y_{nh}(x) = y_g(x) + y_p(x) \), where \( y_g(x) \) is the corresponding general solution of the homogeneous equation, and \( y_p(x) \) is the particular solution of the nonhomogeneous equation.

Next are some procedures for solving for particular solutions of nonhomogeneous equations.

(1) If \( f(x) = P_k(x)e^{bx} \), and \( P_k(x) \) is the polynomial of degree \( k \), then the form of particular solutions is:

\[
y_p(x) = x^sQ_k(x)e^{bx},
\]

where \( Q_k(x) \) is also a polynomial of degree \( k \). If \( b \) is not a characteristic root corresponding to the characteristic equation, then \( s = 0 \); if \( b \) is a characteristic root of multiplicity \( m \), then \( s = m \).

(2) If \( f(x) = P_k(x)e^{px} \cos qx + Q_k(x)e^{px} \sin qx \), and \( P_k(x) \) and \( Q_k(x) \) are all polynomials of degree \( k \), then the form of particular solutions is:

\[
y_p(x) = x^sR_k(x)e^{px} \cos qx + x^sT_k(x)e^{px} \sin qx,
\]

where \( R_k(x), T_k(x) \) are also polynomials of degree \( k \). If \( p + iq \) is not a root of the characteristic equation, then \( s = 0 \); if \( p + iq \) is a characteristic root of multiplicity \( m \), then \( s = m \).

(3) A general method for solving nonhomogeneous differential equations is called the variation of parameters or the method of undetermined coefficients.

Suppose that the general solution of a homogeneous equation is given as follows:

\[
y_g = C_1y_1(x) + \cdots + C_ny_n(x),
\]
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where \( y_i(x) \) is the fundamental solution. Regard constants \( C_1, \ldots, C_n \) as the functions with respect to \( x \), such as \( u_1(x), \ldots, u_n(x) \), so the form of particular solutions to the nonhomogeneous equation can be expressed as

\[
y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x),
\]

where \( u_1(x), \ldots, u_n(x) \) are the solutions of the following equations

\[
\begin{align*}
    u_1'(x)y_1(x) &+ \cdots + u_n'(x)y_n(x) = 0, \\
    \vdots \\
    u_1'(x)y_1^{(n-2)}(x) &+ \cdots + u_n'(x)y_n^{(n-2)}(x) = 0, \\
    u_1'(x)y_1^{(n-1)}(x) &+ \cdots + u_n'(x)y_n^{(n-1)}(x) = f(x).
\end{align*}
\]

(4) If \( f(x) = f_1(x) + f_2(x) + \cdots + f_r(x) \), and \( y_{p1}(x), \ldots, y_{pr}(x) \) are the particular solutions corresponding to \( f_1(x), \ldots, f_r(x) \), then

\[
y_p(x) = y_{p1}(x) + \cdots + y_{pr}(x).
\]

Here is an example to familiarize the application of this method.

**Example 2.8.1** Solve \( y'' - 5y' + 6y = t^2 + e^t - 5 \).

The characteristic roots are \( \lambda_1 = 2 \) and \( \lambda_2 = 3 \). Thus, the general solution of the homogeneous equation is:

\[
y(t) = C_1e^{2t} + C_2e^{3t}.
\]

Next, to find a particular solution of the nonhomogeneous equation, its form is written as:

\[
y_p(t) = at^2 + bt + c + de^t.
\]

We first substitute this particular solution in the initial equation to determine the coefficients \( a, b, c, d \):

\[
2a + de^t - 5(2at + b + de^t) + 6(at^2 + bt + c + de^t) = t^2 - 5 + e^t.
\]

The coefficients of both sides should be consistent, thus we get:

\[
6a = 1, \quad -5 \times 2a + 6b = 0, \quad 2a - 5b + 6c = -5, \quad d - 5d + 6d = 1,
\]

Therefore, \( d = 1/2, \quad a = 1/6, \quad b = 5/18 \) and \( c = -71/108 \).

Finally, the general solution of the nonhomogeneous differential equation is:

\[
y(t) = C_1e^{2t} + C_2e^{3t} + \frac{t^2}{6} + \frac{5t}{18} - \frac{71}{108} + \frac{e^t}{2}.
\]
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2.8.4 System of Ordinary Differential Equations

The general form is:

\[ \dot{x}(t) = A(t)x(t) + b(t), \quad x(0) = x_0, \]

where \( t \) (time) is an independent variable, \( x(t) = (x_1(t), \ldots, x_n(t))' \) is a vector of dependent variables, \( A(t) = (a_{ij}(t))_{n \times n} \) is an \( n \times n \) matrix of real varying coefficients, and \( b(t) = (b_1(t), \ldots, b_n(t))' \) is an \( n \)-dimensional varying vector.

Consider the case that \( A \) is a constant coefficient matrix and \( b \) is a constant vector, also called the system of differential equations with constant coefficients:

\[ \dot{x}(t) = Ax(t) + b, \quad x(0) = x_0, \]  \hspace{1cm} (2.8.46)

where \( A \) is assumed to be nonsingular.

The system of differential equations (2.8.46) can be solved by the following two steps.

Step 1: we consider the system of homogeneous equations (i.e. \( b = 0 \)):

\[ \dot{x}(t) = Ax(t), \quad x(0) = x_0. \]  \hspace{1cm} (2.8.47)

And its solution is denoted by \( x_c(t) \).

Step 2: find a particular solution \( x_p \) to the nonhomogeneous equation (2.8.46). The constant vector \( x_p \) is a particular solution so that \( Ax_p = -b \), namely \( x_p = -A^{-1}b \).

Given the general solution of the homogeneous equation and the particular solution to the nonhomogeneous equation, the general solution of the system of differential equations (2.8.47) is:

\[ x(t) = x_c(t) + x_p. \]

There are two methods for solving the system of homogeneous differential equations (2.8.47).

The first one is that we can eliminate \( n - 1 \) dependent variables so that the system of differential equations becomes the differential equation of order \( n \), such as the following example.

Example 2.8.2 The system of differential equation is:

\[ \begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + 4y. \end{cases} \]

We differentiate the first equation to eliminate \( y \) and \( \dot{y} \). Since \( \dot{y} = 3x + 4\dot{x} - 4 \cdot 2x = 3x + 4\dot{x} - 4 \cdot 2x \), we obtain the corresponding quadratic homogeneous differential equation:

\[ \ddot{x} - 6\dot{x} + 5x = 0, \]
thus the general solution is $x(t) = C_1e^t + C_2e^{5t}$. Since $y(t) = \dot{x} - 2x$, $y(t) = -C_1e^t + 3C_2e^{5t}$.

The second method is to rewrite the homogeneous differential equation (2.8.47) as:

$$x(t) = e^{At}x_0,$$

where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots .$$

Now we solve $e^{At}$ in three different cases.

**Case 1: $A$ has different real eigenvalues**

Matrix $A$ has different real eigenvalues, which means that its eigenvectors are linearly independent. Thus $A$ can be diagonalized, that is,

$$A = P\Lambda P^{-1},$$

where $P = [v_1, v_2, \cdots, v_n]$ consists of the eigenvectors of $A$, and moreover $\Lambda$ is a diagonal matrix whose diagonal elements are the eigenvalues of $A$, thus we have

$$e^{At} = Pe^{\Lambda t}P^{-1}.$$  

Therefore, the solution to the system of differential equation (2.8.47) is:

$$x(t) = Pe^{\Lambda t}P^{-1}x_0 = Pe^{\Lambda t}c = c_1v_1e^{\lambda_1t} + \cdots + c_ne^{\lambda_nt},$$

where $c = (c_1, c_2, \cdots, c_n)$ is a vector of arbitrary constants, and it is determined by the initial value, namely $c = P^{-1}x_0$.

**Case 2: $A$ has multiple real eigenvalues, but no complex eigenvalues**

First, consider a simple case that $A$ has only one eigenvalue of multiplicity $m$. In this case, there are at most $m$ linearly independent eigenvectors, which means that the matrix $P$ cannot be constructed as a matrix consisting of linearly independent eigenvectors, so $A$ cannot be diagonalized.

Thus, the solution has the following form:

$$x(t) = \sum_{i=1}^{m} c_i h_i(t),$$
where $h_i(t)$, $\forall i$, are quasi-polynomials, and $c_i$, $\forall i$, are determined by initial conditions. For example, when $m = 3$, we have:

$$h_1(t) = e^{\lambda t}v_1,$$
$$h_2(t) = e^{\lambda t}(tv_1 + v_2),$$
$$h_3(t) = e^{\lambda t}(t^2v_1 + 2tv_2 + 3v_3),$$

where $v_1, v_2, v_3$ are determined by following conditions:

$$(A - \lambda I)v_i = v_{i-1}, v_0 = 0.$$ 

If $A$ has more than one multiple real eigenvalues, then the solution of the differential equation (2.8.47) can be obtained by summing up the solutions corresponding to each eigenvalue.

**Case 3: $A$ has complex eigenvalues**

Since $A$ is a real matrix, complex eigenvalues will be generated in the form of conjugate pairs.

If an eigenvalue of $A$ is $\alpha + \beta i$, then its conjugate complex $\alpha - \beta i$ is also an eigenvalue.

Now consider a simple case: $A$ has only one pair of complex eigenvalues, $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$.

Let $v_1$ and $v_2$ be the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$; then we have $v_2 = \bar{v}_1$, where $\bar{v}_1$ refers to the conjugation of $v_1$. The solution of the differential equation (2.8.47) can be expressed as:

$$x(t) = e^{At}x_0$$
$$= Pe^{\Lambda t}P^{-1}x_0$$
$$= Pe^{\Lambda t}c$$
$$= c_1v_1e^{(\alpha + \beta i)t} + c_2v_2e^{(\alpha - \beta i)t}$$
$$= c_1v_1e^{\alpha t}(\cos \beta t + i \sin \beta t) + c_2v_2e^{\alpha t}(\cos \beta t - i \sin \beta t)$$
$$= (c_1v_1 + c_2v_2)e^{\alpha t} \cos \beta t + i(c_1v_1 - c_2v_2)e^{\alpha t} \sin \beta t$$
$$= h_1e^{\alpha t} \cos \beta t + h_2e^{\alpha t} \sin \beta t,$$

where $h_1 = c_1v_1 + c_2v_2$ and $h_2 = i(c_1v_1 - c_2v_2)$.

If $A$ has many pairs of conjugate complex eigenvalues, then the solution of the differential equation (2.8.47) is obtained by summing up the solutions corresponding to all eigenvalues.

### 2.8.5 Stability of Simultaneous Differential Equations

Consider the following simultaneous differential equations system:

$$\dot{x} = f(t, x),$$

(2.8.48)
where \( t \) (time) is an independent variable, \( x = (x_1, \cdots , x_n) \) are dependent variables, and \( f(t, x) \) is continuously differentiable with respect to \( x \in \mathbb{R}^n \) and satisfies the initial condition \( x(0) = x_0 \). Such simultaneous differential equations are called planar dynamic systems. If \( f(t, x^*) = 0 \), the point \( x^* \) is called the stationary point of the above dynamical system.

Definition 2.8.5 A simultaneous differential equation system \( x^* \) is locally stable if there is \( \delta > 0 \) and a unique path of \( x = \phi(t, x_0) \) such that \( \lim_{t \to \infty} \phi(t, x_0) = x^* \) whenever \( |x^* - x_0| < \delta \).

Consider the case of simultaneous differential equations system with two variables \( x = x(t) \) and \( y = y(t) \):

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y), \\
\frac{dy}{dt} &= g(x, y).
\end{align*}
\]

Let \( \mathcal{J} \) be the Jacobian

\[
\mathcal{J} = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\]
evaluated at \((x^*, y^*)\), and \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of this Jacobian.

Then the stability of stationary point is characterized as follows:

1. It is a (locally) stable (or unstable) node if \( \lambda_1 \) and \( \lambda_2 \) are different real numbers and are negative (or positive);
2. It is a (locally) saddle point if eigenvalues are real numbers but with opposite signs, namely \( \lambda_1 \lambda_2 < 0 \);
3. It is a (locally) stable (or unstable) focus if \( \lambda_1 \) and \( \lambda_2 \) are complex numbers and \( \text{Re}(\lambda_1) < 0 \) (or \( \text{Re}(\lambda_1) > 0 \));
4. It is a center if \( \lambda_1 \) and \( \lambda_2 \) are complex and \( \text{Re}(\lambda_1) = 0 \);
5. It is a (locally) stable (or unstable) improper node if \( \lambda_1 \) and \( \lambda_2 \) are real, \( \lambda_1 = \lambda_2 < 0 \) (or \( \lambda_1 = \lambda_2 > 0 \)) and the Jacobian is not a diagonal matrix;
6. It is a (locally) stable (or unstable) star node if \( \lambda_1 \) and \( \lambda_2 \) are real, \( \lambda_1 = \lambda_2 < 0 \) (or \( \lambda_1 = \lambda_2 > 0 \)) and the Jacobian is a diagonal matrix.

Figure 2.5 below depicts six types of stationary point.

2.8.6 The Stability of Dynamical System

In a dynamic system, Lyapunov method studies the global stability of stationary points.

Let \( \bar{x}(t, x_0) \) be the unique solution of the dynamic system (2.8.48), and \( B_r(x) = \{ x' \in D : |x' - x| < r \} \) be an open ball of radius \( r \) centered at \( x \).

The following is the definition of stability of stationary point.
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Definition 2.8.6 The stationary point \( x^* \) of the dynamic system (2.8.48)

1. is globally stable if for any \( r > 0 \), there is a neighbourhood \( U \) of \( x^* \) such that
\[
\bar{x}(t, x_0) \in B_r(x^*), \forall x_0 \in U.
\]

2. is globally asymptotically stable if for any \( r > 0 \), there is a neighbourhood \( U' \) of \( x^* \) such that
\[
\lim_{t \to \infty} \bar{x}(t, x_0) = x^*, \forall x_0 \in U.
\]

3. is globally unstable if it is neither globally stable nor asymptotically globally stable.

Definition 2.8.7 Let \( x^* \) be the stationary point of the dynamic system (2.8.48),
\( Q \subseteq \mathbb{R}^n \) be an open set containing \( x^* \), and \( V(x) : Q \to \mathbb{R} \) be a continuously differentiable function. If it satisfies:

1. \( V(x) > V(x^*), \forall x \in Q, x \neq x^*; \)
(2) \( \dot{V}(x) \) is defined as:

\[ \dot{V}(x) \overset{\text{def}}{=} \nabla V(x)f(t,x) \leq 0, \forall x \in Q, \quad (2.8.49) \]

where \( \nabla V(x) \) is the gradient of \( V \) with respect to \( x \),

thus it is called Lyapunov function.

The following is the Lyapunov theorem about the stationary point of dynamic systems.

**Theorem 2.8.3** If there exists a Lyapunov function \( V \) for the dynamic system 
(2.8.48), then the stationary point \( x^* \) is globally stable.

If the Lyapunov function \( (2.8.49) \) of the dynamic system satisfies \( \dot{V}(x) < 0, \forall x \in Q, x \neq x^* \), then the stationary point \( x^* \) is asymptotically globally stable.

### 2.9 Difference Equations

Difference equations can be regarded as discretized differential equations, and many of their properties are similar to those of differential equations.

Let \( y \) be a real-valued function defined on natural numbers. \( y_t \) means the value of \( y \) at \( t \), where \( t = 0, 1, 2, \cdots \), which can be regarded as time points.

**Definition 2.9.1** The first-order difference of \( y \) at \( t \) is:

\[ \Delta y(t) = y(t + 1) - y(t). \]

The second-order difference of \( y \) at \( t \) is:

\[ \Delta^2 y(t) = \Delta(\Delta y(t)) = y(t + 2) - 2y(t + 1) + y(t). \]

Generally, the \( n \)th-order difference of \( y \) at \( t \) is:

\[ \Delta^n y(t) = \Delta(\Delta^{n-1} y(t)), \quad n > 1. \]

**Definition 2.9.2** The difference equation is a function of \( y \) and its differences \( \Delta y, \Delta^2 y, \cdots, \Delta^{n-1} y, \)

\[ F(y, \Delta y, \Delta^2 y, \cdots, \Delta^{n} y, t) = 0, t = 0, 1, 2, \cdots. \] \hspace{1cm} (2.9.50)

If \( n \) is the highest order of nonzero coefficient in the formula (2.9.50), the above equation is called an \( n \)th-order difference equation.
If \( F(\psi(t), \Delta \psi(t), \Delta^2 \psi(t), \ldots, \Delta^n \psi(t), t) = 0 \) holds for \( \forall t \), then we call \( y = \psi(k) \) a solution of the difference equation. Similar to differential equations, the solutions of difference equations also have general solutions and particular solutions. The general solutions usually contain some arbitrary constants that can be determined by initial conditions.

The difference equations can also be expressed in the following form by variable conversion:

\[
F(y(t), y(t+1), \ldots, y(t+n), t) = 0, \quad t = 0, 1, 2, \ldots \tag{2.9.51}
\]

The followings are mainly focused on the difference equations with constant coefficients. A common expression is written as:

\[
f_0y(t+n) + f_1y(t+n-1) + \cdots + f_{n-1}y(t+1) + f_ny(t) = g(t), \quad t = 0, 1, 2, \ldots \tag{2.9.52}
\]

where \( f_0, f_1, \ldots, f_n \) are real numbers, and \( f_0 \neq 0, f_n \neq 0 \).

Dividing both sides of the equation by \( f_0 \), and making \( a_i = \frac{f_i}{f_0} \) for \( i = 0, \ldots, n \), \( r(t) = \frac{g(t)}{f_0} \), the \( n \)th order difference equation can be written as the simpler form:

\[
y(t+n) + a_1y(t+n-1) + \cdots + a_{n-1}y(t+1) + a_ny(t) = r(t), \quad t = 0, 1, 2, \ldots \tag{2.9.53}
\]

Here are three procedures that are usually used to solve \( n \)th order linear difference equations:

Step 1: find the general solution of the homogeneous difference equation

\[
y(t+n) + a_1y(t+n-1) + \cdots + a_{n-1}y(t+1) + a_ny(t) = 0,
\]

and let the general solution be \( Y \).

Step 2: find a particular solution \( y^* \) of the difference equation (2.9.53).

Step 3: the solution of the difference equation (2.9.53) is

\[
y(t) = Y + y^*.
\]

The followings are the solutions of the first-order, second-order and \( n \)th-order difference equations, respectively.

### 2.9.1 First-order Difference Equations

The first-order difference equation is defined as:

\[
y(t+1) + ay(t) = r(t), \quad t = 0, 1, 2, \ldots \tag{2.9.54}
\]
The corresponding homogeneous difference equation is:
\[ y(t + 1) + ay(t) = 0, \]
and the general solution is \( y(t) = c(-a)^t \), where \( c \) is an arbitrary constant.

To get a particular solution for a nonhomogeneous difference equation, consider \( r(t) = r \), that is, the case that does not change over time. Obviously, a particular solution is as follows:
\[ y^* = \frac{r}{1 + a}, \quad a \neq -1, \]
\[ y^* = rt, \quad a = -1. \]

Hence, the solution of the nonhomogeneous difference equation (2.9.54) is:
\[ y(t) = \begin{cases} 
  c(-a)^t + \frac{r}{1 + a}, & \text{if } a \neq -1, \\
  c + rt, & \text{if } a = -1.
\end{cases} \quad (2.9.55) \]

If the initial condition \( y(0) = y_0 \) is known, the solution of the difference equation (2.9.54) is:
\[ y(t) = \begin{cases} 
  (y_0 - \frac{r}{1 + a}) \times (-a)^t + \frac{r}{1 + a}, & \text{if } a \neq -1, \\
  y_0 + rt, & \text{if } a = -1.
\end{cases} \quad (2.9.56) \]

If \( r \) depends on \( t \), a particular solution is:
\[ y^* = \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i), \]
thus the solution of the difference equation (2.9.54) is:
\[ y(t) = (-a)^t y_0 + \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i), \quad t = 1, 2, \ldots. \]

For a general function \( r(t) = f(t) \), the coefficients of \( A_0, \ldots, A_m \) can be determined by using the method of undetermined-coefficients, namely considering \( y^* = f(A_0, A_1, \ldots, A_m; t) \). The following is to solve for a particular solution in a case that \( r(t) \) is a polynomial.

**Example 2.9.1** Solve the following difference equation:
\[ y(t + 1) - 3y(t) = t^2 + t + 2. \]

The homogeneous equation is:
\[ y(t + 1) - 3y(t) = 0, \]
The general solution is:

\[ Y = C3^t. \]

Using the method of undetermined-coefficients to get the particular solution of the nonhomogeneous equation, suppose that the particular solution has the form:

\[ y^* = At^2 + Bt + D. \]

Substitute \( y^* \) into the nonhomogeneous difference equation, and get:

\[ A(t+1)^2 + B(t+1) + D - 3At^2 - 3Bt - 3D = t^2 + t + 2, \]

or

\[ -2At^2 + 2(A - B)t + A + B - 2D = t^2 + t + 2. \]

Since equality holds for each \( t \), we must have:

\[
\begin{aligned}
-2A &= 1 \\
2(A - B) &= 1 \\
A + B - 2D &= 2,
\end{aligned}
\]

which gives \( A = -\frac{1}{2}, B = -1 \) and \( D = -\frac{3}{4} \), thus we have a particular solution: \( y^* = -\frac{1}{2}t^2 - t - \frac{3}{4} \). Therefore, a particular solution of the nonhomogeneous equation is \( y(t) = Y + y^* = C3^t - \frac{1}{2}t^2 - t - \frac{3}{4} \).

We can also solve the case with an exponential function by using the method of undetermined-coefficients.

**Example 2.9.2** Consider the first-order difference equation:

\[ y(t+1) - 3y(t) = 4e^t. \]

Suppose that the form of particular solution is \( y^* = Ae^t \), then substituting it into the nonhomogeneous difference equation gives: \( A = \frac{4}{e - 3} \). Therefore, the general solution of the first-order difference equation is: \( y(t) = Y + y^* = C3^t + \frac{4e^t}{e - 3} \).

Here are some of the common ways for finding particular solutions:

1. when \( r(t) = r \), a usual form of particular solution is: \( y^* = A \);
2. when \( r(t) = r + ct \), a usual form of particular solution is: \( y^* = A_1t + A_2 \);
3. when \( r(t) = t^n \), a usual form of particular solution is: \( y^* = A_0 + A_1t + \cdots + A_n t^n \);
(4) when \( r(t) = c^t \), a usual form of particular solution is: \( y^* = Ae^t \);

(5) when \( r(t) = \alpha \sin(ct) + \beta \cos(ct) \), a usual form of particular solution is: \( y^* = A_1 \sin(ct) + A_2 \cos(ct) \).

### 2.9.2 Second-order Difference Equation

The second-order difference equation is defined as:

\[
y(t + 2) + a_1y(t + 1) + a_2y(t) = r(t).
\]

The corresponding homogeneous differential equation is:

\[
y(t + 2) + a_1y(t + 1) + a_2y(t) = 0.
\]

Then, its general solution depends on the roots of the following linear equation:

\[
m^2 + a_1m + a_2 = 0,
\]

which is called the auxiliary equation or characteristic equation of second order difference equations. Let \( m_1 \) and \( m_2 \) be the roots of this equation. Since \( a_2 \neq 0 \), both \( m_1 \) and \( m_2 \) are not 0.

**Case 1:** \( m_1 \) and \( m_2 \) are different real roots.

The general solution of the homogeneous equation is \( Y = C_1m_1^t + C_2m_2^t \), where \( C_1 \) and \( C_2 \) are arbitrary constants.

**Case 2:** \( m_1 \) and \( m_2 \) are the same real roots.

The general solution of the homogeneous equation is \( Y = (C_1 + C_2t)m_1^t \).

**Case 3:** \( m_1 \) and \( m_2 \) are two complex roots, namely \( r(\cos \theta \pm i \sin \theta) \) with \( r > 0, \theta \in (-\pi, \pi] \). The general solution of the homogeneous equation is \( Y = C_1r^t \cos(t\theta + C_2) \).

For a general function \( r(t) \), it can be solved by the method of undetermined-coefficients.

### 2.9.3 Difference Equations of Order \( n \)

The general \( n \)th-order difference equation is defined as:

\[
y(t + n) + a_1y(t + n - 1) + \cdots + a_{n-1}y(t + 1) + a_ny(t) = r(t), t = 0, 1, 2, \cdots
\]

The corresponding homogeneous equation is:

\[
y(t + n) + a_1y(t + n - 1) + \cdots + a_{n-1}y(t + 1) + a_ny(t) = 0,
\]
2.9. DIFFERENCE EQUATIONS

and its characteristic equation is:

\[ m^n + a_1 m^{n-1} + \cdots + a_{n-1}m + a_n = 0. \]

Let its \( n \) characteristic roots be \( m_1, \ldots, m_n \).

The general solutions of the homogeneous equations are the sum of the bases generated by these eigenvalues, and its concrete forms are as follows:

Case 1: The formula generated by a single real root \( m \) is \( C_1 m^k \).

Case 2: The formula generated by the real root \( m \) of multiplicity \( p \) is:

\[ (C_1 + C_2 t + C_3 t^2 + \cdots + C_p t^{p-1}) m^t. \]

Case 3: The formula generated by a pair of nonrepeated conjugate complex roots \( r (\cos \theta \pm i \sin \theta) \) is:

\[ C_1 r^i \cos(t \theta + C_2). \]

Case 4: The formula generated by a pair of conjugate complex roots \( r (\cos \theta \pm i \sin \theta) \) of multiplicity \( p \) is:

\[ r^i [C_{1,1} \cos(t \theta + C_{1,2}) + C_{2,1} t \cos(t \theta + C_{2,2}) + \cdots + C_{p,1} t^{p-1} \cos(t \theta + C_{p,2})]. \]

The general solution of the homogeneous difference equation is obtained by summing up all formulas generated by eigenvalues.

A particular solution \( y^* \) of a nonhomogeneous difference equation can be generated by the method of undetermined-coefficients.

A particular solution is:

\[ y^* = \sum_{s=1}^{n} \theta_s \sum_{i=0}^{\infty} m_s^i r(t - i), \]

where

\[ \theta_s = \frac{m_s}{\prod_{j \neq s} (m_s - m_j)}. \]

2.9.4 Stability of \( n \)th-Order Difference Equations

Consider an \( n \)th-order difference equation

\[ y(t + n) + a_1 y(t + n - 1) + \cdots + a_{n-1} y(t + 1) + a_n y(t) = r(t), \quad t = 0, 1, 2, \cdots. \]  
\[ (2.9.58) \]

The corresponding homogeneous equation is:

\[ y(t + n) + a_1 y(t + n - 1) + \cdots + a_{n-1} y(t + 1) + a_n y(t) = 0, \quad t = 0, 1, 2, \cdots. \]  
\[ (2.9.59) \]
Definition 2.9.3 The difference equation (2.9.53) is asymptotically stable, if an arbitrary solution \( Y(t) \) of the homogeneous equation (2.9.59) satisfies \( Y(t)|_{t \to \infty} = 0 \).

Let \( m_1, \ldots, m_n \) be the solution of their characteristic equation:

\[
m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \tag{2.9.60}
\]

Theorem 2.9.1 Suppose that the modulus of all eigenvalues of the characteristic equation are less than 1. Then, the difference equation (2.9.58) is asymptotically stable.

When the following inequality conditions are satisfied, the modulus of all eigenvalues of the characteristic equation are less than 1.

\[
\begin{vmatrix}
1 & a_n \\
 a_n & 1
\end{vmatrix} > 0,
\]

\[
\begin{vmatrix}
1 & 0 & a_n & a_{n-1} \\
 a_1 & 1 & 0 & a_n \\
 a_n & 0 & 1 & a_1 \\
 a_{n-1} & a_n & 0 & 1
\end{vmatrix} > 0,
\]

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\
a_1 & 1 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & a_n & \cdots & 1 & 0 & 0 & \cdots & a_n \\
a_n & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_{n-1} \\
a_{n-1} & a_n & \cdots & 0 & 0 & 1 & \cdots & a_{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & 1
\end{vmatrix} > 0.
\]

2.9.5 Difference Equations with Constant Coefficients

The difference equation with constant coefficients is defined as:

\[
x(t) = Ax(t-1) + b, \tag{2.9.61}
\]

where \( x = (x_1, \ldots, x_n)' \), \( b = (b_1, \ldots, b_n)' \). Suppose the matrix \( A \) is diagonalizable, the corresponding eigenvalues are \( \lambda_1, \ldots, \lambda_n \) and the matrix \( P \) formed by linearly independent eigenvectors such that

\[
A = P^{-1}
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
P.
\]
Similar to the discussion of obtaining the solutions of ordinary differential equations, we can find the general solutions of the difference equation with constant coefficients. Also, a necessary and sufficient condition for the differential equation (2.9.61) to be (asymptotically) stable is that the modulus of all eigenvalues $\lambda_i$ are less than 1. When the modulus of all eigenvalues $\lambda_i$ are less than 1, the stationary point $x^* = \lim_{t \to \infty} x(t) = (I - A)^{-1}b$.

### 2.10 Basic Probability

Risk and uncertainty, as well as some of their basic operations, are widely used in economics. This section briefly introduces some concepts involved in the book.

#### 2.10.1 Probability and Conditional Probability

Compared with other fields of mathematics, the development of probability theory is rather late. However, the probability theory has developed rapidly and become a very important field in mathematics since its axiomatization.

When dealing with probability problems, we must clearly define the probability space. Classical probability (i.e., explaining probability as the same possibility) is often associated with permutation and combination. Since statistics requires getting data from sampling, the randomness in probability theory is revealed.

Let the probability of random variable $X_a$ be $\pi_{as}$, $s \in S$, where $S$ can be either discrete or continuous. When $S = \{1, \cdots, n\}$, where $n$ can be finite or infinite, this situation is about a discrete random variable. If $S$ is an interval of the real space, then it is called a continuous random variable.

If there is a correlation between two random variables, then the value of a random variable provides information for the value of the other random variable, which gives the concept of conditional probability.

When the two random variables, $X_a$ and $X_b$, share the same probability distribution $\pi_{as'}$, with the known information that $X_a = X_{as}$, the probability of $X_b = X_{bs'}$ is called a conditional probability:

$$P(X_b = X_{bs'}|X_a = X_{as}) = \frac{\pi_{as'}}{\sum_{s' \in S} \pi_{ss'}}.$$

This formula is also called the Bayes rule.
2.10.2 Mathematical Expectation and Variance

The expectation of random variable $X_a$ is the weighted average of all possible values, and is defined and denoted by

$$E(X_a) \equiv \bar{X}_a = \sum_{s \in S} \pi_s X_{as},$$

which in a continuous case is defined by integral instead of summation and it will be discussed in next subsection.

The operation rule of expected utility is that if $X_a$ and $X_b$ are two random variables, then we have

$$E(aX_a + bX_b) = a\bar{X}_a + b\bar{X}_b.$$ 

The variance of a random variable $X_a$ measuring the degree of variation of its value is defined as

$$\text{Var}(X_a) \equiv \sigma^2_{X_a} = \sum_{s \in S} \pi_s (X_{as} - \bar{X}_a)^2.$$

Thus, the larger the variance, the greater the variation degree is.

There may be some correlations between the two random variables, $X_a$ and $X_b$. Suppose that the value space of $X_a$ is $\{X_{as}\}_{s \in S}$, and that of $X_b$ is $\{X_{bs'}\}_{s' \in S'}$. Then their covariance measures the correlations between their values.

Let $\pi_{ss'}$ be the probability of $X_a = X_{as}$ and $X_b = X_{bs'}$. Covariance, denoted by $\text{Cov}(X_a, X_b)$, is defined as:

$$\text{Cov}(X_a, X_b) = \sum_{s \in S, s' \in S'} \pi_{ss'} (X_{as} - \bar{X}_a)(X_{bs'} - \bar{X}_b)$$

or

$$\text{Cov}(X_a, X_b) = E(X_a - \bar{X}_a)(X_b - \bar{X}_b) = E(X_aX_b) - E(X_a)E(X_b).$$

If two random variables $X_a$ and $X_b$ are independent, then $\pi_{ss'} = \pi_s\pi_{s'}$, and thus we have $\text{Cov}(X_a, X_b) = 0$.

The following operation is for deriving the variance of linear combinations:

$$\text{Var} \left( \sum_{a \in A} \alpha_a X_a \right) = \sum_{a \in A, b \in A} \alpha_a\alpha_b \text{Cov}(X_a, X_b).$$

2.10.3 Continuous Distributions

When a random variable $X$ takes values over $[a, b]$, its probability distribution function $F$ on support $[a, b]$ is defined by

$$F(x) = \text{Prob}[X \leq x],$$
which is the probability that \( X \) takes values not exceeding \( x \). By definition, the function \( F \) is nondecreasing and satisfies \( F(a) = 0 \) and \( F(b) = 1 \). Here \( a \) and \( b \) can be any real number so that it is possible that \( a = -\infty \) and \( b = \infty \).

The derivative of \( F \) is called the probability density function and is denoted by \( f \equiv F' \). We assume that \( f \) is continuous and \( f(x) > 0 \) for all \( x \in (a, b) \).

The expectation of \( X \) is then defined by

\[
E(X) = \int_a^b x f(x) \, dx.
\]

If \( u : [a, b] \to \mathbb{R} \) is an arbitrary function, the expectation of \( u(X) \) is defined by

\[
E[u(X)] = \int_a^b u(x) f(x) \, dx,
\]

which can also be written as

\[
E[u(X)] = \int_a^b u(x) \, dF(x).
\]

The conditional expectation of \( X \) given that \( X < x \) is

\[
E[X|X < x] = \frac{1}{F(x)} \int_a^x t f(t) \, dt,
\]

and thus

\[
F(x)E[X|X < x] = \int_a^x t f(t) \, dt = x F(x) - \int_a^x F(t) \, dt,
\]

in which the second equality is obtained by integrating by parts.

### 2.10.4 Common Probability Distributions

Next we review some common distributions, and their expectations and variances.

**Binomial Distribution**

There are many balls in a box with two colors. The proportion of red balls is \( p \) and that of black balls is \( 1 - p \). The value of the random variable \( X \) is 1 if the red ball is drawn, otherwise it is 0. If it is taken only once, the probability distribution of the random variable is \( p(X = 1) = p \) and \( p(X = 0) = 1 - p \).

The expectation and variance are:

\[
E(X) = p; \quad \text{Var}(X) = p(1 - p).
\]
If we draw \( n \) times (the ball is put back to the box at each time), the random variable is defined as the number of times that the red ball is drawn.

The probability distribution of random variables is

\[
p(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}.
\]

Its expectation and variance are:

\[
E(X) = np; \quad \text{Var}(X) = np(1-p).
\]

**Poisson Distribution**

If the probability of a random variable \( X \) is

\[
P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!},
\]

then \( X \) follows a **Poisson distribution** with parameter \( \lambda \), and its expectation and variance are:

\[
E(X) = \lambda; \quad \text{Var}(X) = \lambda.
\]

**Uniform Distribution**

If the probability density function of a random variable \( X \) is

\[
f(x) = \frac{1}{b-a}, \quad x \in [a,b],
\]

then \( X \) follows a **uniform distribution** over \([a, b]\). Its expectation and variance are:

\[
E(X) = \frac{b+a}{2}; \quad \text{Var}(X) = \frac{(b-a)^2}{12}.
\]

**Normal Distribution**

If the probability density function of a random variable \( X \) is

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, \infty),
\]

then \( X \) follows a **normal distribution** with parameters \((\mu, \sigma^2)\).

The expectation and variance are:

\[
E(X) = \mu; \quad \text{Var}(X) = \sigma^2.
\]
Exponential Distribution

If the probability density function of a random variable $X$ is

$$f(x) = \lambda e^{-\lambda x}, x \in [0, \infty),$$

then $X$ follows an **exponential distribution** with parameter $\lambda$, and its expectation and variance are

$$E(X) = \frac{1}{\lambda}; \quad \text{Var}(X) = \frac{1}{\lambda^2}.$$

### 2.11 Stochastic Dominance and Affiliation

#### 2.11.1 Order Stochastic Dominance

**First-Order Stochastic Dominance**

**Definition 2.11.1 (First-Order Stochastic Dominance)** Given two distribution functions $F$ and $G$ with support $[a, b]$, we say that $F$ first-order stochastically dominates $G$ if for all $x \in [a, b], F(x) \leq G(x)$.

The first-order stochastic dominance means that for any outcome $x$, the probability of obtaining at least $x$ under $F(\cdot)$ is at least as high as that under $G(\cdot)$. For example, considering two assets, the first-order stochastic dominance means that when two assets are greater than a certain constant return, the probability of one asset’s return is higher than that of the other. This is analogous to the monotonicity concept under certainty.

There is another test criterion for $F$ to first-order stochastically dominate $G$. The following theorem shows that these two criterions are equivalent.

**Theorem 2.11.1** $F(\cdot)$ first-order stochastically dominates $G(\cdot)$ if and only if for any nondecreasing function $u : [a, b] \to \mathbb{R}$, we have

$$\int_a^b u(z) dF(z) \geq \int_a^b u(z) dG(z).$$

**Proof.** Define $H(z) = F(z) - G(z)$. We need to prove that $H(z) \leq 0$ if and only if $\int_a^b u(z) dH(z) \geq 0$ for any increasing and differentiable function $u(\cdot)$.

**Sufficiency:** We prove this by way of contradiction. Suppose that there is a $\hat{z}$ such that $H(\hat{z}) > 0$. We choose a weakly increasing and differentiable function $u(z)$ as

$$u(z) = \begin{cases} 
0, & z \leq \hat{z}, \\
1, & z > \hat{z},
\end{cases}$$

such that $\int_a^b u(z) dH(z) > 0$. This contradicts the assumption that $H(z)$ is nonnegative. Therefore, $H(z)$ must be nonnegative everywhere.
then immediately \( \int_a^b u(z) dH(z) = -H(\hat{z}) < 0 \), a contradiction.

**Necessity:** Since a monotonic function is differentiable almost everywhere, in finding integration, we may just assume \( u \) is differentiable, and we have

\[
\int_a^b u(z) dH(z) = [u(z)H(z)]_a^b - \int_a^b u'(z) H(z) dz = 0 - \int_a^b u'(z) H(z) dz \geq 0,
\]

in which the first equality is obtained by integration by parts, the second equality is based on \( F(a) = G(a) = 0, \ F(b) = G(b) = 1 \), while the inequality is based on the assumptions that \( u(\cdot) \) is weakly increasing \((u'(\cdot) \geq 0)\) and \( H(z) \leq 0 \).

For any two probability distributions \( F \) and \( G \), as long as an agent’s utility is (weakly) increasing in outcomes, he prefers the one that first-order stochastically dominates the other one.

**Second-Order Stochastic Dominance**

**Definition 2.11.2 (Second-Order Stochastic Dominance)** Given two distribution functions \( F \) and \( G \) defined on \([a, b]\), which have the same expectation, we say that \( F(\cdot) \) second-order stochastically dominates \( G(\cdot) \) if

\[
\int_a^z F(r) dr \leq \int_a^z G(r) dr
\]

for all \( z \).

It is clear that first-order stochastic dominance implies second-order stochastic dominance. The second-order stochastic dominance implies not only monotonicity but also lower risk. To do so, we introduce the notion of “Mean-Preserving Spreads”.

Suppose that \( X \) is a random variable with distribution function \( F \). Let \( Z \) be a random variable whose distribution conditional on \( X = x, H(\cdot | X = x) \), is such that for all \( x \), \( E[Z | X = x] = 0 \). Suppose that \( Y = X + Z \) is the random variable obtained from first drawing \( X \) from \( F \) and then for each realization \( X = x \), drawing a \( Z \) from the conditional distribution \( H(\cdot | X = x) \) and adding it to \( X \). Let \( G \) be the distribution of \( Y \) so defined. We will then say that \( G \) is a **mean-preserving spread** of \( F \).

While random variables \( X \) and \( Y \) have the same mean, namely \( E[X] = E[Y] \), variable \( Y \) is “more spread-out” than \( X \) since it is obtained by adding a “noise” variable \( Z \) to \( X \). Now, suppose that \( u : [a, b] \to \mathbb{R} \) is a concave function. Using Jensen’s inequality

\[
E(u(X)) \leq E(u(Y)),
\]
2.11. STOCHASTIC DOMINANCE AND AFFILIATION

we obtain

\[ E_Y[u(Y)] = E_X[E_Z[u(X + Z)|X = x]] \leq E_X[u(E_Z[X + Z|X = x])] = E_X[u(X)]. \]

As such, similar to Theorem 2.11.1, we have the following conclusion for the second-order stochastic dominance.

**Theorem 2.11.2** If distributions \( F(\cdot) \) and \( G(\cdot) \) defined on \([a, b]\) have the same mean, then the following statements are equivalent.

1. \( F(\cdot) \) second-order stochastically dominates \( G(\cdot) \);
2. for any nondecreasing concave function \( u : \mathbb{R} \to \mathbb{R} \), we have \( \int_a^b u(z) dF(z) \geq \int_a^b u(z) dG(z) \);
3. \( G(\cdot) \) is a mean-preserving spread of \( F(\cdot) \).

**Proof.** (3) \( \Rightarrow \) (2): It is obtained by using

\[
\int_a^b u(z) dF(z) = \int_a^b u \left( \int_a^b (x + z) dH_z(x) \right) dF(z)
\]  
\[
\geq \int_a^b \left( \int_a^b u(x + z) dH_z(x) \right) dF(z)
\]

\[
= \int_a^b u(z) dG(z),
\]

in which the inequality follows from the concavity of \( u(\cdot) \).

(1) \( \Rightarrow \) (2): For expositional convenience, we set \( b = 1 \). We have

\[
\int_a^b u(z) dF(z) - \int_a^b u(z) dG(z)
\]

\[
= -u'(1) \int_a^b (F(z) - G(z)) dz + \int_a^b \left( \int_a^z (F(x) - G(x)) dx \right) u''(z) dz
\]

\[
= \int_a^b (\int_a^z (F(x) - G(x)) dx) u''(z) dz
\]

\[
\geq 0,
\]

in which the inequality follows from the definition of second-order stochastic dominance, namely

\[
\int_a^z F(r) dr \leq \int_a^z G(r) dr,
\]

and also \( u''(\cdot) \leq 0 \) for any \( z \). We thus have

\[
\int_a^b u(z) dF(z) - \int_a^b u(z) dG(z) \geq 0.
\]
(1)⇒(3): We just show the case with discrete distributions. Define

\[ S(z) = G(z) - F(z), \]
\[ T(x) = \int_a^x S(z)dz. \]

By the definition of second-order stochastic dominance, we have \( T(x) \geq 0 \) and \( T(1) \geq 0 \), which imply that there exists some \( \hat{z} \) such that \( S(z) \geq 0 \) for \( z \leq \hat{z} \) and \( S(z) \leq 0 \) for \( z \geq \hat{z} \).

Since the random variable follows a discrete distribution, \( S(z) \) must be a step function. Let \( I_1 = (a_1, a_2) \) be the first interval over which \( S(z) \) is positive, and \( I_2 = (a_3, a_4) \) be the first interval over which \( S(z) \) is negative. If no such \( I_1 = (a_1, a_2) \) exists, then \( S(z) = 0 \) and hence statement (3) is immediate. If \( I_1 = (a_1, a_2) \) does exist, then \( I_2 = (a_3, a_4) \) must exist as well.

So, \( S(z) \equiv \gamma_1 > 0 \) for \( z \in I_1 \), and \( S(z) \equiv -\gamma_2 < 0 \) for \( z \in I_2 \). By \( T(x) \geq 0 \), we must have \( a_2 < a_3 \). If \( \gamma_1(a_2 - a_1) \geq \gamma_2(a_4 - a_3) \), then there exist \( a_1 < \hat{a}_2 \leq a_2 \) and \( \hat{a}_4 = a_4 \) such that \( \gamma_1(\hat{a}_2 - a_1) = \gamma_2(\hat{a}_4 - a_3) \). If \( \gamma_1(a_2 - a_1) < \gamma_2(a_4 - a_3) \), then there exists \( a_3 < \hat{a}_4 \leq a_4 \) such that \( \gamma_1(\hat{a}_2 - a_1) = \gamma_2(\hat{a}_4 - a_3) \).

Letting

\[ S_1(z) = \begin{cases} \gamma_1, & \text{if } a_1 < z < \hat{a}_2, \\ -\gamma_2, & \text{if } a_3 < z < \hat{a}_4, \\ 0, & \text{otherwise.} \end{cases} \]

If \( F_1 = F + S_1 \), then \( F_1 \) is a mean-preserving spread of \( F \). Letting \( S^1 = G - F_1 \), we can similarly construct \( S_2(z) \) and \( F_2 \). Since \( S(z) \) is a step function, then there exists an \( n \) such that \( F_0 = F, F_n = G \), and \( F_{i+1} \) is a mean-preserving spread of \( F_i \). Also, a finite summation of mean-preserving spreads is still a mean-preserving spread.

Though a continuous function can be arbitrarily approximated by step functions, the formal proof is complicated. Rothschild and Stiglitz (1971) provided a complete proof for the case with continuous distributions.

### 2.11.2 Hazard Rate Dominance

Let \( F \) be a distribution function with support \([a, b]\). The hazard rate of \( F \) is the function \( \lambda : [a, b] \to \mathbb{R}_+ \) defined by

\[ \lambda(x) = \frac{f(x)}{1 - F(x)}. \]

If we interpret \( F \) as the probability that some event will happen before time \( x \), then the hazard rate at \( x \) represents the instantaneous probability that the event will happen at \( x \), given that it has not happened until time \( x \). The event may be the failure of some component—a lightbulb. So it is sometimes also called as the “failure rate”.
Solving for $F$, we have

$$F(x) = 1 - \exp\left(-\int_a^x \lambda(t)dt\right). \quad (2.11.62)$$

This shows that any arbitrary function $\lambda : [a, b) \rightarrow \mathbb{R}_+$ such that for all $x < b$,

$$\int_a^x \lambda(t)dt < \infty, \quad \lim_{x \to b} \int_a^x \lambda(t)dt = \infty,$$

is the hazard rate of some distribution that is given by (2.11.62).

**Definition 2.11.3 (Hazard Rate Dominance)** For any two distributions $F$ and $G$ with hazard rates $\lambda_F$ and $\lambda_G$, respectively, we say that $F$ dominates $G$ in terms of the hazard rate if $\lambda_F(x) \leq \lambda_G(x)$ for all $x$. This order is also referred in short as **hazard rate dominance**.

If $F$ dominates $G$ in terms of the hazard rate, then

$$F(x) = 1 - \exp\left(-\int_a^x \lambda_F(t)dt\right) \leq 1 - \exp\left(-\int_a^x \lambda_G(t)dt\right) = G(x),$$

and hence $F$ stochastically dominates $G$. Thus, hazard rate dominance implies first-order stochastic dominance.

### 2.11.3 Reverse Hazard Rate Dominance

A closely related concept to the hazard rate is the reverse hazard rate $\sigma : (a, b] \rightarrow \mathbb{R}_+$ given by

$$\sigma(x) \equiv \frac{f(x)}{F(x)},$$

sometimes is referred to as the **inverse of the Mills’ ratio**. Similarly, solving for $F$ gives

$$F(x) = \exp\left(-\int_x^b \sigma(t)dt\right). \quad (2.11.63)$$

This shows that any arbitrary function $\sigma : (a, b] \rightarrow \mathbb{R}_+$ such that for all $x > a$,

$$\int_x^b \sigma(t)dt < \infty \text{ and } \lim_{x \to a} \int_x^b \sigma(t)dt = \infty,$$

is the “reverse hazard rate” of some distribution that is given by (2.11.63).

**Definition 2.11.4 (Reverse Hazard Rate Dominance)** For two distributions $F$ and $G$ with reverse hazard rates $\sigma_F$ and $\sigma_G$, we say that $F$ dominates $G$ in terms of the reverse hazard rate if $\sigma_F(x) \geq \sigma_G(x)$ for all $x$. This order is also referred in short as **reverse hazard rate dominance**.
If $F$ dominates $G$ in terms of the reverse hazard rate, then
\[ F(x) = \exp\left(-\int_x^b \sigma_F(t)dt\right) \leq \exp\left(-\int_x^b \sigma_G(t)dt\right) = G(x), \]
and hence, again, $F$ stochastically dominates $G$. Thus, reverse hazard rate dominance also implies first-order stochastic dominance.

### 2.11.4 Likelihood Ratio Dominance

**Definition 2.11.5 (Likelihood Ratio Dominance)** We say that the distribution function $F$ dominates $G$ in terms of the likelihood ratio if for all $x < y$,
\[
\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)},
\]
which means that $\frac{f}{g}$ is a nondecreasing function. As such, we refer to this order as **likelihood ratio dominance**.

Rewriting (2.11.64) gives
\[
\frac{f(y)}{f(x)} \leq \frac{g(y)}{g(x)},
\]
and then for all $x$, we have
\[
\int_x^b \frac{f(y)}{f(x)} dy \leq \int_x^b \frac{g(y)}{g(x)} dy,
\]
which in turn implies that
\[
\frac{1 - F(x)}{f(y)} \leq \frac{1 - G(x)}{g(y)}.
\]
Thus, likelihood ratio dominance implies hazard rate dominance.

Similarly, rewriting (2.11.64) gives
\[
\frac{f(x)}{f(y)} \leq \frac{g(x)}{g(y)},
\]
and then for all $x$, we have
\[
\int_a^y \frac{f(x)}{f(y)} dx \leq \int_a^y \frac{g(x)}{g(y)} dx,
\]
which implies that
\[
\frac{F(y)}{f(y)} \leq \frac{G(y)}{g(y)}.
\]
Thus, likelihood ratio dominance implies reverse hazard rate dominance.

Summarizing the above discussions, one can see that likelihood ratio dominance is the strongest, which implies both hazard rate dominance and reverse hazard rate dominance, both of which in turn imply first-order stochastic dominance that again implies second-order stochastic dominance.
2.11.5 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be $n$ random variables independently and randomly drawn from a distribution $F$ with density $f$. Let $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ be a rearrangement of these so that

$$Y_1^{(n)} \geq Y_2^{(n)} \geq \cdots \geq Y_n^{(n)},$$

where $Y_k^{(n)}, k = 1, 2, \ldots, n$ are referred to as order statistics.

Let $F_k^{(n)}$ denote the distribution of $Y_k^{(n)}$, with corresponding probability density function $f_k^{(n)}$. If there is no confusion, we simply denote them as $Y_k, F_k$ and $f_k$. In auction theory, we will typically be interested in properties of the highest and second highest order statistics, namely $Y_1$ and $Y_2$.

**Highest Order Statistic**

The distribution of the highest order statistic $Y_1$ can be obtained as follows. The event that $Y_1 \leq y$ is equivalent to the event: $X_k \leq y$ for all $k$. Since $X_k$ is an independent draw from the same distribution $F$, we have that

$$F_1(y) = F(y)^n.$$

The density function is

$$f_1(y) = nF(y)^{n-1}f(y).$$

Note that if $F$ stochastically dominates $G$, and $F_1$ and $G_1$ are the distributions of the highest order statistics of $n$ draws from $F$ and $G$, respectively, then $F_1$ stochastically dominates $G_1$.

**Second-Highest Order Statistics**

The distribution of the second-highest order statistic $Y_2$ can also be easily derived. The event that $Y_2 \leq y$ is the union of the following disjoint events: (1) all $X_k$’s are less than or equal to $y$; and (2) $n - 1$ of the $X_k$’s are less than or equal to $y$ and one is greater than $y$. There are $n$ different ways in which (2) can occur. Thus we have

$$F_2(y) = F(y)^n + nF(y)^{n-1}(1 - F(y))$$

$$= nF(y)^{n-1} - (n - 1)F(y)^n.$$

The probability density function then is

$$f_2(y) = n(n - 1)(1 - F(y))F(y)^{n-2}f(y).$$

Again, one can verify that if $F$ stochastically dominates $G$ and also $F_2$ and $G_2$ are the distributions of the second-highest order statistics of $n$ draws from $F$ and $G$, respectively, then $F_2$ stochastically dominates $G_2$. 
CHAPTER 2. PRELIMINARY KNOWLEDGE AND METHODS OF MATHEMATICS

2.11.6 Affiliation

Affiliation is a basic assumption used to study auction with interdependent values in which random variables are non-negatively correlated.

Definition 2.11.6 Suppose that random variables \( X_1, X_2, \ldots, X_n \) are distributed on some product of intervals \( D \subseteq \mathbb{R}^n \) according to the joint density function \( f \). \( X = (X_1, X_2, \ldots, X_n) \) are said to be affiliated if for all \( x', x'' \in X \),

\[
f(x' \vee x'') f(x' \wedge x'') \geq f(x') f(x''),
\]

in which

\[
x' \vee x'' = (\max(x'_1, x''_1), \ldots, \max(x'_n, x''_n))
\]

denotes the component-wise maximum of \( x' \) and \( x'' \), and

\[
x' \wedge x'' = (\min(x'_1, x''_1), \ldots, \min(x'_n, x''_n))
\]

denotes the component-wise minimum of \( x' \) and \( x'' \). If (2.11.65) is satisfied, then we also say that \( f \) is affiliated.

Suppose that the density function \( f : D \to \mathbb{R}_+ \) is strictly positive in the interior of \( D \) and twice continuously differentiable. One can verify that \( f \) is affiliated if and only if, for all \( i \neq j \),

\[
\frac{\partial^2}{\partial x_i \partial x_j} \ln f \geq 0,
\]

which means that the off-diagonal elements of the Hessian of \( \ln f \) are non-negative.

Proposition 2.11.1 Let \( X_1, X_2, \ldots, X_n \) be random variables, and \( Y_1, Y_2, \ldots, Y_{n-1} \) be the largest, second largest, ..., smallest order statistics from among \( X_2, X_3, \ldots, X_n \). If \( X_1, X_2, \ldots, X_n \) are symmetrically distributed and affiliated, then we have

1. variables in any subset of \( X_1, X_2, \ldots, X_n \) are also affiliated;
2. \( X_1, Y_1, Y_2, \ldots, Y_{n-1} \) are affiliated.

Monotone Likelihood Ratio Property

Suppose that the two random variables \( X \) and \( Y \) have a joint density \( f : [a, b]^2 \to \mathbb{R} \). If \( X \) and \( Y \) are affiliated, then for all \( x' \geq x \) and \( y' \geq y \), we have

\[
f(x', y') f(x, y') \leq f(x, y) f(x', y') \Leftrightarrow \frac{f(x, y)}{f(x', y')} \leq \frac{f(x, y')}{f(x', y)} \quad (2.11.66)
\]

and

\[
\frac{f(y'| x)}{f(y | x)} \leq \frac{f(y'| x')}{f(y | x')},
\]
so the likelihood ratio
\[ \frac{f(\cdot|x')}{f(\cdot|x)} \]
is increasing and this is referred to as the monotone likelihood ratio property.

Using the same arguments as for order stochastic dominance in previous subsection, it can be deduced that for all \( x' \geq x \), \( F_Y(\cdot|x') \) dominates \( F_Y(\cdot|x) \) in terms of the likelihood ratio and the other dominance relationships then follow as usual. We have the following conclusions.

**Proposition 2.11.2** If \( X \) and \( Y \) are affiliated, the following properties hold:

1. For all \( x' \geq x \), \( F(\cdot|x') \) dominates \( F(\cdot|x) \) in terms of hazard rate; that is,
   \[ \lambda(y|x') = \frac{f(y|x')}{1 - F(y|x')} \leq \frac{f(y|x)}{1 - F(y|x)} = \lambda(y|x). \]
   Or equivalently, for all \( y \), \( \lambda(y|\cdot) \) is nonincreasing.

2. For all \( x' \geq x \), \( F(\cdot|x') \) dominates \( F(\cdot|x) \) in terms of the reverse hazard rate; that is,
   \[ \sigma(y|x') = \frac{f(y|x')}{F(y|x')} \leq \frac{f(y|x)}{F(y|x)} = \sigma(y|x), \]
   or equivalently, for all \( y \), \( \sigma(y|\cdot) \) is nondecreasing.

3. For all \( x' \geq x \), \( F(\cdot|x') \) first-order stochastically dominates \( F(\cdot|x) \); that is,
   \[ F(y|x') \leq F(y|x), \]
   or equivalently, for all \( y \), \( F(y|\cdot) \) is nonincreasing.

4. For all \( x' \geq x \), \( F(\cdot|x') \) second-order stochastically dominates \( F(\cdot|x) \); that is, for all \( y \)
   \[ \int_{a}^{y} F(r|x')dr \leq \int_{a}^{y} F(r|x)dr \]
   or equivalently, for all \( y \), \( \int_{a}^{y} F(y|\cdot) \) is nonincreasing.

All of these results extend in a straightforward manner to the case where the number of conditional variables is more than one. Suppose that \( Y, X_1, X_2, \ldots, X_n \) are affiliated and let \( F_Y(\cdot|x) \) denote the distribution of \( Y \) conditional on \( X = x \), we can also get the above dominance relations.
CHAPTER 2. PRELIMINARY KNOWLEDGE AND METHODS OF MATHEMATICS

2.12 Biographies

2.12.1 Friedrich August Hayek

Friedrich August Hayek (1899-1992), one of the greatest economic thinkers of the 20th century and a representative of the Austrian school, won the 1974 Nobel Prize in Economics for his contributions to the theory of money and economic cycles as well as to the penetrating analysis of the interdependence of economic, political and institutional phenomena. The Nobel Prize Committee believed that Hayek’s in-depth analysis of the economic cycle made him one of the very few economists who had warned about possible great economic depression before 1929. In fact, in the 20th century, both academically and practically speaking, it was the competition between the market economic system and the planned economic system and the disputes between their advantages and disadvantages. Hayek’s penetrating analysis of different economic systems had led him to point out very early that the planned economy is not feasible from the standpoint of information efficiency, incentive compatibility and resource allocation efficiency. The practice results proved Hayek’s extraordinary judgment and insight, and finally it was ended with the demise of the planned economic system, which made him one of the most influential economists of the 20th century.

Hayek was born in an intellectual family in Vienna and received a doctorate from the University of Vienna (1921-1923). When Hayek was at the University of Vienna, he attended classes by Ludwig von Mises (1881-1973). It was Mises’s thorough critique of socialism published in 1922 that eventually pulled Hayek out of the Fabian socialist ideological trend. The best way to understand Hayek’s great contribution to economics and classical liberalism is to analyze it from the perspective of Mises’s paradigm of social collaboration. Hayek taught mainly at the London School of Economics and Political Science (1931-1950), the University of Chicago (1950-1962), and Freiburg University (1962-1968), etc. At the University of Chicago, Hayek was a professor of social and ethical science in the “Committee on Social Thought” and did not get a teaching post in the Department of Economics. Professor Friedman, a friend in the economics department, was also critical of Hayek’s books on economics. When he first arrived at the University of Chicago, Hayek did political studies and did not engage in economics research, and he held a negative attitude to some research methods at the Department of Economics. Even so, Hayek interacted frequently with some of the Chicago School of Economics and his political views were also compatible with many of the Chicago School. Hayek made a remarkable contribution to the University of Chicago. He strongly supported Aaron Director, a Chicago School economist and the founder of law and economics, to carry out “Law and Society” project at the University of
Chicago Law School. While the latter persuaded the University of Chicago Press to publish Hayek’s *The Road to Slavery*, which later became popular around the world. Hayek also collaborated with Friedman and others on the establishment of the International Forum of Liberal Economists and so on.

Hayek had two profound debates in his life: one was the “socialist controversy” in the 1920s and 1930s, which was the debate between Mises-Hayek and Lange-Lerner on a theoretical feasibility of efficient allocations under socialism. He criticized the drawbacks of the planned economy from the perspective of information and incentives. He held that the planned economy was theoretically impracticable, and emphasized the importance of a spontaneous social order based on freedom, competition, and rules. This advanced internal logic judgment was verified before his death. The second was the theoretical debate with Keynes in the 1930s. He pointedly criticized Keynes’s theoretical claims and academic viewpoints put forward in *Monetary Theory*, and thought that Keynes’s economic proposition of achieving full employment by lowering interest rate and increasing money supply was fundamentally wrong. In 1947, Hayek advocated the establishment of the Pilgrimage Mountain Society, an important academic organization of liberals. He advocated thorough economic freedom and opposed any form of state intervention, calling for the “non-nationalization” of currency issuance.

Hayek’s profound thought of revealing the importance of institution will undoubtedly continue to influence and guide the world, especially the next step of reform in China.

### 2.12.2 Joseph Alois Schumpeter

Joseph Alois Schumpeter (1883-1950), an Austrian American political economist (but not a member of the Austrian School) with profound influence, is hailed as the originator of the Innovation Theory. Known as one of the greatest economists in history, his name is closely related with most of the concepts and knowledge about market economy and innovation. He proposed the four most representative and well-known economic terms, namely, innovation, entrepreneurship, corporate strategy, and creative destruction, and believed that “creative destruction” is a double-edged sword which can engender economic growth, but can also impair some values that people have traditionally cherished. “What poverty brings is a tragic life, while it is difficult for prosperity to maintain the peace of mind.”

In 1883, Schumpeter was born into the family of a weaving factory owner in Triesch, Habsburg Moravia (now in Czech, so some people also think Schumpeter as a Czech-American), Austria Hungary. He enrolled in a noble middle school in Vienna. He studied law and sociology at University of Vienna from 1901 to 1906 and received his doctor’s degree in law in
1906. In 1908, he became an associate professor at University of Czernowitz through his instructor’s recommendation just at the beginning of the road as an economist. Czernowitz is a remote city but a good place for learning with tranquility outside modern industrial civilization. Here, Schumpeter wrote his first masterpiece, *The Theory of Economic Development* published in 1912, which touches on “innovation” and its role in economic development, and made a stir in the circles of economics. According to statistics, the concept of “creative destruction” proposed by Schumpeter was cited frequently, second only to “invisible hand” of Adam Smith. *The Theory of Economic Development* has become one of the classical economic literatures in the twentieth Century. Later Schumpeter emigrated to the United States, and had been teaching at Harvard University ever since.

In his famous book, *History of Economic Analysis*, he argued that the difference between an economic scientist and an average economist lies in the adoption of the following three elements in the process of economic analysis. First is the economic theory with inherent logical analysis. Second is history with analysis from historical visions. Third is statistics with data and empirical analysis. Schumpeter’s five innovation concepts are so frequently quoted and mentioned, even to an extent that his name appears in every discussion about innovation. What is more, as the founder of the Innovation Theory and the research of business history, Schumpeter’s influence is also being “rediscovered”.

Innovation refers to an economic process that recombines and integrates the original production factors into new production methods in order to increase efficiency and reduce costs. In Schumpeter’s economic model, those who can successfully innovate can survive the dilemmas of diminishing returns, while those who fail to recombine production factors will be eliminated by the market first. The creative destruction of capitalism means that when the economy cycles to the bottom, it is time that some entrepreneurs have to consider exiting the market and others must innovate to survive. As long as the excess competitors are excluded or some successful “innovations” are created, the economy will improve and the production efficiency will be increased. But when an industry becomes profitable again, it will attract the investment of new competitors. Then it becomes a process of diminishing returns again and returns to the previous state. Therefore, every depression implies the possibility of another technological innovation, or it can be said that the result of technological innovation is another expected depression. In Schumpeter’s view, the creativity and destructiveness of capitalism are homologous. However, Schumpeter did not believe that the superiority of capitalism is due to its own impetus which can promote its own development continually. He believed that the capitalist economy will eventually collapse on its own scale because it cannot withstand the energy of its rapid expansion. The business cycle, also known as the economic cycle, is Schumpeter’s most quoted economic claim. Schumpeter’s concept
of creative destruction has a great influence on the development of modern economics. The combination of the dynamic market mechanism and R&D economics provides economists with a perspective of endogenous technology. Schumpeter’s technological innovation has become a core element of the endogenous growth theory in macroeconomics.

In *Capitalism, Socialism and Democracy*, Schumpeter gave a modern definition of democracy: “the democratic method is that institutional arrangement for arriving at political decisions in which individuals acquire the power to decide by means of a competitive struggle for the people’s vote”. He held that democracy is an process in which political elites compete for power and the people choose political leaders. The essence of democracy lies in a competitive election process. Political elites master political power and implement ruling, but their legitimacy comes from the choice of the people. Schumpeter also took the view that, in the political market with democracy, politicians provide political programs and policies according to the preferences of voters, and compete freely in the election to woo the voters. Schumpeter’s definition of democracy symbolizes the great transformation of democratic theory from the classical democracy directly ruled by the people to the modern election democracy.

The economic development of any country needs to go through three stages, namely, factor-driven, efficiency-driven and innovation-driven. The ideas and theories of Hayek and Schumpeter play a vital role in theoretically guiding and clarifying the way for the two driven stages.

### 2.13 Exercises

**Exercise 2.1** Consider an economy with two sectors: the industrial sector and the monetary sector, characterized by the following equations:

\[ Y = C + I + G, \]

\[ C = a + b(1 - t)Y, \]

\[ I = d - ei, \]

\[ G = G_0, \]

where \( Y, C, I \) and \( i \) (\( i \) is the interest rate) are endogenous variables, \( G_0 \) is an exogenous variable, and \( a, b, d, e \) and \( t \) are all structure parameters.

In the newly introduced monetary market, we have:

- the equilibrium conditions: \( M_d = M_s \),
- the money demand: \( M_d = kY - li \),
- and the money supply: \( M_s = M_0 \),

where \( M_0 \) is the exogenous variable of money stock, \( k \) and \( l \) are parameters.

Given this economy, solve the following problems: (using Cramer’s rule)
1. Equilibrium income $Y^*$;
2. Money supply multiplier;

**Exercise 2.2** $Q$ represents the set of rational numbers, and as a metric s-space, its distance is defined by $d(p, q) = |p - q|$, where $p \in E = \{p \in Q : 2 < p < 40\} \subseteq Q$. 

1. Prove that $E$ is closed and bounded in $Q$.
2. Prove that $E$ is not compact.
3. Is $E$ open in $Q$? Why?

**Exercise 2.3** Given a metric space $X$, consider a series of open sets $\{E_n\}_{n \in \mathbb{N}}$ in $X$.

1. Prove that $\bigcup_{n \in \mathbb{N}} E_n$ is an open set.
2. Prove that it may not be true that the intersection of a series of open sets is open (Give an example).

**Exercise 2.4** Prove the following theorems:

1. The difference of an open set and a closed set is also open, while the difference of a closed set and an open set is also closed.
2. Each closed set is the intersection of a countable number of open sets; each open set is the union of a countable number of closed sets.

**Exercise 2.5** Let $S \subseteq \mathbb{R}^L$. Prove that the following propositions are equivalent:

1. $S$ is compact.
2. $S$ is bounded and closed.
3. Every sequence in $S$ has a convergent subsequence with the limit point in $S$.
4. Every infinite subset of $S$ has a cluster point in $S$.
5. Each closed subset of the set $S$ with finite intersection property (i.e., the intersection over any finite subcollection is nonempty) is nonempty.

**Exercise 2.6** Prove the following propositions:
2.13. EXERCISES

1. Every closed subset of a compact set is compact.

2. If \( f : X \to Y \) is continuous and \( K \) is compact in \( X \), then \( f(K) \) is compact in \( Y \).

3. \( S_i \) is compact, \( i \in I \), if and only if \( \prod_{i \in I} S_i \) is compact.

4. \( S_i \) is compact, \( i = 1, 2, \ldots, m \), if and only if \( \sum_{i=1}^{m} S_i \) is compact.

Exercise 2.7 (Shapley-Folkman Theorem) Prove the theorem: Let \( S_i, (i = 1, \ldots, n) \) be \( n \) non-empty subsets of \( \mathbb{R}^m \) and \( S = \sum_{i=1}^{n} S_i \). Then each \( x \in Co(S) \) has a representation \( x = \sum_{i=1}^{n} x_i \) such that \( x_i \in Co(S_i) \) for all \( i \), and \( x_i \in S_i \) for at least \( (n - m) \) indices \( i \).

Exercise 2.8 Prove the following theorems:

1. If \( f \) is a differentiable function defined on \( \mathbb{R}^1 \), then \( f \) is concave if and only if the first order condition \( f'(x) \) is non-increasing.

2. If \( f \) is a twice differentiable function defined on \( \mathbb{R}^1 \), then \( f \) is concave if and only if the second order condition \( f''(x) \) is non-positive.

3. If \( f \) is a differentiable function defined on \( \mathbb{R}^1 \), then \( f \) is concave if and only if \( f(y) \leq f(x) + f'(x)(y-x) \) for any \( x, y \in \mathbb{R}^1 \).

Exercise 2.9 Suppose that \( f(x) = \frac{1}{2} x^T A x + b^T x + c \), where \( x \in \mathbb{R}^n \), \( x^T \) is the transpose of vector \( x \), \( A \) is an \( n \times n \) symmetric matrix, \( b \) is an \( n \)-dimensional vector, and \( c \) is a constant.

1. Prove that if \( A \) is a positive semi-definite matrix, then \( f(x) \) is a convex function.

2. Prove that if \( A \) is a positive definite matrix, then \( f(x) \) is a strictly convex function.

Exercise 2.10 Determine whether Kuhn-Tucker conditions is applicable for the following optimization problems and solve it.

\[
\begin{align*}
\max & \quad x_1 \\
\text{s.t.} & \quad x_1^3 - x_2 \leq 0, \\
& \quad x_2 \leq 0.
\end{align*}
\]

Exercise 2.11 Solve the following optimization problems using Kuhn-Tucker conditions.

\[
\begin{align*}
\max & \quad xyz \\
\text{s.t.} & \quad x^2 + y^2 + z^2 \leq 6, \\
& \quad x \geq 0, y \geq 0, z \geq 0.
\end{align*}
\]
Exercise 2.12 The maximization problem is as follows:

$$\max f(x)$$

subject to

$$g^1(x) = 0, \ldots, g^m(x) = 0,$$

where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) are increasing functions with respect to \( x \), and \( m < n \). Prove that: If \( f \) is quasi-concave and all \( g^j \) are quasi-convex functions, then any local optimum is the global optimal solution.

Exercise 2.13 Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a function, \( p, x, y \in \mathbb{R}^n \). Consider the following optimization problem:

$$\max x u(x)$$

subject to

$$px = y.$$

Suppose that there is an optimum solution \( x^*(p, y) > 0 \) such that \( v(x, y) = u(x^*(p, y)) \).

1. Prove that \( v(p, y) \) is homogeneous of degree zero.
2. Prove that \( v(p, y) \) is a quasi-concave function.

Exercise 2.14 Suppose that a Cobb-Douglas utility function \( u : \mathbb{R}^2 \to \mathbb{R} \) is defined as:

$$u(x_1, x_2) = x_1^\alpha x_2^\beta, \alpha, \beta > 0.$$

Prove:

1. If \( \alpha + \beta \leq 1 \), then \( u \) is a concave function.
2. If \( \alpha + \beta > 1 \), then \( u \) is a quasi-concave function, but not concave.
3. For any \( \alpha > 0 \) and \( \beta > 0 \), \( h(x_1, x_2) = \ln(u(x_1, x_2)) \) is a concave function.

Exercise 2.15 Suppose that \( X \) is a nonempty, closed and convex set in \( \mathbb{R}^n \), \( x_0 \notin \overline{X} \). Prove that the following propositions are true.

1. There is a point \( a \in X \) such that \( d(x_0, a) < d(x_0, x) \) for all \( x \in \overline{X} \), and \( d(x_0, a) > 0 \).
2. There is a point \( p \in \mathbb{R}^n, p \neq 0 \), \( ||p|| = (\sum_{i=1}^n p_i^2)^{1/2} < \infty \) and \( \alpha \in \mathbb{R} \) such that

$$p \cdot x \geq \alpha, \text{ for all } x \in \overline{X} \text{ and } p \cdot x_0 < \alpha.$$

Namely, \( \overline{X} \) and \( x_0 \) is separated by a hyperplane \( H = \{ x : p \cdot x = \alpha, x \in \mathbb{R}^n \} \).

Exercise 2.16 Consider following functions:
2.13. EXERCISES

(1) $3x^5y + 2x^2y^4 - 3x^3y^2$.
(2) $3x^5y + 2x^2y^4 - 3x^3y^4$.
(3) $x^{3/4}y^{1/4} + 6x + 4$.
(4) $\frac{x^2 - y^2}{x^2 + y^2} + 3$.
(5) $x^{1/2}y^{-1/2} + 3xy^{-1} + 7$.
(6) $x^{3/4}y^{1/4} + 6x$.

1. Find homogeneous functions among them and determine their orders of degree.

2. Test whether the above functions satisfy the Euler theorem.

**Exercise 2.17** There is a simple application of upper (lower) hemi-continuity of correspondences. Suppose that $f : X \times Y \to \mathbb{R}$,

$$G(x) = \{ y \in \Gamma(x) : f(x, y) = \max_{y' \in \Gamma(x)} f(x, y') \}.$$ 

1. Suppose that $X = \mathbb{R}$, $\Gamma(x) = Y = [-1, 1]$. For all $x \in X$, $f(x, y) = xy^2$. Draw the graph of $G(x)$ and prove that $G(x)$ is upper hemi-continuous at $x = 0$, but not lower hemi-continuous.

2. Suppose that $x = \mathbb{R}$ and $\Gamma(x) = Y = [0, 4]$ for all $x \in X$. Define that

$$f(x, y) = \max\{ 2 - (y - 1)^2, x + 1 - (y - 2)^2 \}.$$ 

Draw the graph of $G(x)$ and prove that: $G(x)$ is upper hemi-continuous but not lower hemi-continuous, and specify at which points it is not lower hemi-continuous.

3. Suppose that $X = \mathbb{R}_+$, $\Gamma(x) = Y = \{ y \in \mathbb{R} : -x \leq y \leq x \}$. For all $x \in X$, define that $f(x, y) = \cos(y)$, then draw the graph of $G(x)$ and prove that: $G(x)$ is upper hemi-continuous but not lower hemi-continuous, and specify at which points it is not lower hemi-continuous.

**Exercise 2.18** Let $S = \{ x \in \mathbb{R}^2 : \| x \| = 4 \}$ be the boundary of a circle with a radius of 2. The mapping $\psi : \mathbb{R}^2 \to S$ is defined as:

$$\psi(x) = \arg \min_{x' \in S} d(x, x'),$$ 

namely, $\psi(x)$ contains the closest point in $S$ to $x$. Discuss the upper and lower hemi-continuity of $\psi(x)$.
Exercise 2.19 Consider a correspondence $\Gamma : D \subseteq \mathbb{R}^l \rightarrow \mathbb{R}^k$ of which the graph is defined as

$$G(\Gamma) = \{(x, y) \in D \times \mathbb{R}^k : y \in \Gamma(x)\}.$$ 

If $G(\Gamma)$ is a closed set, then we call $\Gamma$ has a closed graph; If $G(\Gamma)$ is a bounded and closed set, we call $\Gamma$ is compact-valued. Suppose that $\Gamma$ is compact-valued. Prove:

1. If $\Gamma$ is upper hemi-continuous, then it has a closed graph.
2. If $\Gamma$ is locally bounded and the graph of it is closed, then $\Gamma$ is upper hemi-continuous. (Hint: The definition of locally bounded correspondence $\Gamma$: $G(\Gamma) = \{(x, y) \in D \times \mathbb{R}^k : y \in \Gamma(x)\}$, if for each $x \in D$, there is an $\epsilon > 0$ and a bounded set $Y(x) \subseteq \mathbb{R}^k$ such that for all $x' \in N_{\epsilon}(x) \cap D$, $\Gamma(x') \subseteq Y(x)$.)

Exercise 2.20 Suppose that $X \subseteq \mathbb{R}_+$ is a nonempty compact set. Prove that:

1. If $f : X \rightarrow X$ is a continuous increasing function, then $f$ has a fixed point.
2. Specially, suppose that $X = [0, 1]$. If $f : X \rightarrow X$ is an increasing function (not necessarily continuous), does $f$ has another fixed point?

Exercise 2.21 Suppose that $X$ is a complete metric space, $T$ is the mapping from $X$ to $X$. Denote

$$a_n = \sup_{x \neq x'} \frac{d(T^n x, T^n x')}{d(x, x')}, \quad n = 1, 2, \ldots.$$ 

Prove that: If $\sum_{n=1}^{\infty} a_n < \infty$, then the mapping $T$ has a unique fixed point.

Exercise 2.22 Consider $n \in \mathbb{N}$ and an $n$th order square matrix $A = (a_{ij})_{n \times n}$. For any $x \in \mathbb{R}^n$, we have

$$Ax = (\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j)^T.$$ 

Suppose that $f$ is a differentiable mapping from $\mathbb{R}$ to $\mathbb{R}$ such that

$$s = \sup\{|f'(t)| : t \in \mathbb{R}\} < \infty.$$ 

Define a mapping $F$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, namely,

$$F(x) = (f(x_1), \ldots, f(x_n))^T.$$ 

For a given $n$-dimensional vector $w$, we can solve the following system of nonlinear equations:

$$z = AF(z) + w. \quad (*)$$
1. Prove that: if \( \max \{ \sum_{j=1}^{n} |a_{ij}| : i = 1, \cdots, n \} < \frac{1}{\pi} \), then there is a unique \( z \in \mathbb{R}^n \) satisfying the above system of equations (*)

2. Prove that: if \( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| < \frac{1}{\pi} \), then there is a unique \( z \in \mathbb{R}^n \) satisfying the above system of equations. (*)

**Exercise 2.23** Suppose that \( h \) is a mapping from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \), and \( H : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) is a bounded function such that there is a \( K \in (0, 1) \)

\[ |H(x, y) - H(x, z)| < K|y - z|, \text{ for any } x \geq 0, \ y, \ z \in \mathbb{R}. \]

Prove that: there is a unique bounded function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) such that

\[ f(x) = H(x, f(h(x))), \text{ for any } x \geq 0. \]

**Exercise 2.24** Find the extremum curve of the following functional:

1. \( V(y) = \int_{0}^{1} (t^2 + y'^2)dt, \ y(0) = 0, \ y(1) = 2; \)
2. \( V(y) = \int_{0}^{1} (y + yy' + y' + 0.5y'^2)dt, \ y(0) = 2, \ y(1) = 5; \)
3. \( V(y) = \int_{0}^{T} (1 + y^2)^{0.5} dt, \ y(0) = A, \ y(T) = Z. \)

**Exercise 2.25** Solve the following optimal control problem:

\[
\max \int_{0}^{3} (x - 2)^2(x'(t) - 1)^2 dt \\
\text{s.t.} \quad x(0) = 0, \ x(3) = 2.
\]

**Exercise 2.26** Consider the following optimal control problem, write the Hamilton equation, and solve the optimal function.

\[
\max \int_{0}^{1} (x + u) dt \\
\text{s.t.} \quad x'(t) = 1 - u^2, \ x(0) = 1.
\]

**Exercise 2.27** Consider the following optimization problem:

\[ v(q) = \max_{x \in \mathbb{R}^+} \ln(2x + q) - 6x + 2q, \]

where \( q \in (0, 2) \).

1. Solve \( v(q) \) and its derivative \( v'(q) \).
2. Verify the envelope theorem holds.

**Exercise 2.28** Find the general solution to the extremum curve of the following functional:

\[ V(y, z) = \int_{a}^{b} (y'^2 + z'^2 + 3y'z') dt. \]
Exercise 2.29 In the problem of functional \( \int_0^T F(t, y, z, y', z') dt \), suppose that \( y(0) = A, z(0) = B, y_T = C, z_T = D, T \) are free, and \( A, B, C \) and \( D \) are constants.

1. How many transversal conditions are required for the problem? Why?
2. Write these transversal conditions.

Exercise 2.30 The integrand function of the target functional is \( F(t, y, y') = 4y^2 + 4yy' + y'^2 \).

1. Write the Euler equation.
2. Is the above Euler equation sufficient for the maximization or minimization problems? Why?

Exercise 2.31 Solve the paths of \( y(t) \) and \( z(t) \) of extremum curves of \( V(y, z) = \int_0^T (y'^2 + z'^2) dt \) subject to \( y - z' = 0 \).

Exercise 2.32 Solve the optimal paths of control variables, state variables and costate variables as follows:

1. \( \max \int_0^T -(t^2 + 2u^2) dt \) subject to \( y' = u, y(0) = 2, \) and \( y(T) = 3, T \) is free.
2. \( \max \int_0^T -(u^2 + y^2 + 3uy) dt \) subject to \( y' = u, y(0) = y_0, \) and \( y(t) \) is free.
3. \( \max \int_0^4 2ydt \) subject to \( y' = y + u, y(0) = 3, y(4) \geq 200. \)

Exercise 2.33 Find the optimal consumption path of the following exhaustible resource problem:

\[
\max \int_0^T \ln qe^{-\delta t} dt \\
\text{s.t. } s' = -q, s(0) = s_0, s(t) \geq 0.
\]

Exercise 2.34 Using the revised transversal conditions expressed by the present value Hamilton function, solve the problems

1. with the end curve \( y_T = \phi(t) \).
2. with truncated vertical end line.
3. with truncated horizontal end line.

Exercise 2.35 In a maximization problem, there are two known state variables, \( (y_1, y_2) \), two control variables \( (u_1, u_2) \), an inequality constraint and an inequality integral constraint. The initial state is fixed, but the final state is free at fixed \( T \).
2.13. EXERCISES

1. State the maximization problem.
2. Define the Hamilton’s equation and the Lagrange function.
3. Suppose that there is an internal solution, then write the conditions of the maximum principle.

**Exercise 2.36** Consider the problem of “eating cake” as follows. The agent has a $A_0 > 0$ units commodity for consumption in the period 0 and can save the commodity to the next period without costs, and its utility function is $\sum_{t=0}^{\infty} \beta^t \ln c_t$.

1. Write the Bellman equations of the problem.
2. Define the state variable and control variable.
3. Find the value function.

**Exercise 2.37** Consider the following problem of “tree cutting”: the growth a tree can be represented by the function $h$, that is, $k_{t+1} = h(k_t)$, where $k_t$ is the scale of the tree at time $t$.

There is no cost for cutting trees, and the timber price is $p = 1$. Interest rate $r$ remains unchanged, $\beta = 1/(1 + r)$.

1. Assuming that trees cannot be replanted, we write the maximization problem of present value as $v(k) = \max\{k, \beta v[h(k)]\}$. Under what conditions about $h$ is there a simple rule that can be used to describe when to cut trees?
2. Suppose that another tree can be planted where the original tree is cut down, and the replanting cost $c \geq 0$ remains unchanged for a long term, under what conditions about $h$ and $c$ is there a simple rule that can be used to describe when to cut trees?

**Exercise 2.38** Solving following dynamic programming problems by three methods: value function iteration, guessing value function and guessing policy function respectively:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

s.t. $c_t + k_{t+1} = A k^\alpha_t$,

where $k_0$ is given.

**Exercise 2.39** Solve the following differential equations:

1. $y' = t^2 y$. 
2. \( y'' - 4y' + 5y = 0. \)
3. \( y'' - 2y' - 3y = 9t^2. \)

**Exercise 2.40** Consider the following two dimensional autonomous differential equations:

\[
\frac{dx}{dt} = x(4 - x - y), \\
\frac{dy}{dt} = y(6 - y - 3x).
\]

1. Solve the equilibrium of the power system.
2. Verify the stability of each equilibrium.

**Exercise 2.41** Solve the following difference equations:

1. \( y(t+1) - 2y(t) = 4^t. \)
2. \( y(t+2) + 3y(t+1) + 2y(t) = 0. \)
3. \( y(t+2) - y(t+1) - 6y(t) = t + 2. \)

**Exercise 2.42** Suppose that \( X_1, X_2, \ldots, X_n \) are \( n \) independent and identically distributed random variables. The distribution function is \( F \), and probability density function is \( f \). Let \( Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)} \) be the corresponding order statistics satisfying \( Y_1^{(n)} \geq Y_2^{(n)} \geq \cdots \geq Y_n^{(n)}. \)

1. Find the distribution function and probability density function of \( Y_n^{(n)}. \)
2. Find \( E(Y_n^{(n)}) \) and \( Var(Y_n^{(n)}). \)
3. Find \( \text{Cov}(Y_1^{(n)}, Y_n^{(n)}). \)

**Exercise 2.43** Let \( X \) and \( Y \) be two random variables in the range \([a, b]\. Prove:

1. If \( X \) first-order stochastically dominates \( Y \), then \( X \) is necessarily second-order stochastically dominates \( Y \).
2. If \( X \) second-order stochastically dominates \( Y \), then \( EX \geq EY \).
3. If \( X \) second-order stochastically dominates \( Y \) and \( EX = EY \), then \( Eu(X) \geq Eu(Y) \), for all concave and twice differentiable functions (whether increasing or not).
4. If \( X \) second-order stochastically dominates \( Y \) and \( EX = EY \), then \( Var(X) \leq Var(Y) \).
Exercise 2.44 Suppose that $X$ is a non-negative random variable, the distribution function and density function are $F$ and $f$ respectively. The risk rate of random variable $X$ is defined as

$$\lambda_X : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \lambda_X(t) = \frac{f(t)}{1 - F(t)}.$$ 

If $\lambda_X(\cdot) \leq \lambda_Y(\cdot)$, we say that the random variable $X$ stochastically dominates random variable $Y$ in terms of risk rate. Suppose that $G$ and $g$ are respectively the distribution function and density function of random variable $Y$. If $f(\cdot)/g(\cdot)$ is a non-decreasing function, then we say that $X$ stochastically dominates $Y$ in terms of likelihood ratio. Prove the following statements:

1. $\lambda_X(\cdot) \leq \lambda_Y(\cdot)$ if and only if $1 - G(t)/(1 - F(t))$ is a non-increasing function.

2. If $X$ dominates $Y$ in terms of likelihood ratio, then $X$ must stochastically stochastically dominate $Y$ in terms of risk rate.

Exercise 2.45 Prove that: if $X_1, X_2, \cdots, X_N$ are correlated, and $\gamma(\cdot)$ is an increasing function, then for $x'_1 > x_1$, we have

$$E[\gamma(Y_1)|X_1 = x'_1] \geq E[\gamma(Y_1)|X_1 = x_1].$$

2.14 References

Books and Monographs:


**Papers:**


Part V

Externalities and Public Goods
The rest of the book will examine the allocation of resources closer to the real-world environment. The main theme is how to solve the issue of “market failure” to remedy market defects. These will be the topics discussed in the remaining chapters of this book.

The market we have discussed so far is basically a frictionless ideal market economy. In addition, we directly or implicitly assume that the markets are perfectly competitive except for monopolistic competition, oligopoly and monopoly discussed in Chapter 9.

Chapters 3-10 and Chapter 13 are mainly descriptive positive analysis of the market economy, discussing how rational consumers and firms make optimal decisions and how the market operates in various structures (perfect competition, monopolistic competition, oligopoly). Chapters 11-12 mainly make normative analysis on the value orientation of the perfectly competitive market, and discuss the market’s optimality, rationality, uniqueness and universality from different aspects.

In Chapters 11 and 12, we have introduced the internal logic relations between competitive equilibrium (Walrasian equilibrium) and efficiency (Pareto optimality). The concept of competitive equilibrium provides us with an appropriate notion of market equilibrium for competitive market economies. The concept of Pareto optimality offers a minimal and uncontroversial test that any socially optimal economic outcome should pass since it is a formulation of the idea that there is no further improvement in society, and it conveniently separates the issue of economic efficiency from more controversial (and political) questions regarding the ideal distribution of well-being across individuals.

The important results and insights obtained in Chapters 10-12 are the First and Second Fundamental Theorems of Welfare Economics, Economic Core Theorem, Core Equivalence Theorem, and the Fairness Theorem. These results demonstrate the optimality, rationality and uniqueness of a perfectly competitive free market economy from different aspects. The First Fundamental Theorems of Welfare Economics provides a set of conditions, i.e., perfect competition, personal interests-pursuing (local non-satiation preferences), no externalities, no public goods, no increasing returns to scale, complete information, etc., under which the market economy achieves Pareto optimal allocations. When these conditions are not satisfied, the market often fails, resulting in inefficient allocations of resources. It is, in some sense, the formal expression of Adam Smith’s claim about the “invisible hand” of the market.

The Second Fundamental Theorems of Welfare Economics goes even further. It states that under the same set of conditions plus convexity and continuity conditions, all Pareto optimal outcomes can in principle be implemented through the market mechanism by appropriate redistribution of initial endowments and then “letting the market work”. The First and Second Fundamental Theorems of Welfare Economics prove that Pareto op-
timal allocations and competitive market equilibrium allocations are equivalent in a certain sense, while the Economic Core Theorem reveals that the free and competitive market institution is also socially stable.

The Fairness Theorem discussed in Chapter 12 gives specific suggestions on how to achieve efficiency and equity of resource allocations simultaneously. If all economic agents have equal starting point for competition, even if this starting point is not Pareto efficient, but through the operation of the competitive market, it can achieve Pareto efficient and equitable allocations. The Core Equivalence Theorem is even deep and profound which reveals a market economic institution is a unique institution that results in efficient allocations when giving people sufficient economic freedom and competition. That is, it demonstrates that under the objective and realistic constraints of individual pursuing their self-interests, as long as individuals choose to voluntarily cooperate and exchange freely under perfect competition, even if no economic system arrangement is arranged beforehand, the result is the same as that of the competitive market.

All these conclusions show the effective role of market economy in resolving problems associated to fairness and justice. Through the joint role of government, market and society, we can achieve resource allocations with both efficiency and equity. These results provide theoretical support for inheritance tax, compulsory education, environmental protection, antimonopoly, and financial regulation, thus providing inspirations for solving the problem of excessive gap between the rich and the poor as well as market failures.

As mentioned above, all these conclusions are based on their preconditions, and the perfectly competitive market is just an ideal state without friction, which basically does not exist in reality and belong to benchmark economic theories. But this research method, like natural science, has its necessity. Like the ideal state without friction in physics, it does not exist at all in reality, but it provides a benchmark point and reference system for the study of practical problems with various frictions. Similarly, the perfect competition situation analyzed by the general equilibrium theory has important theoretical and practical significance for us to think, study and test the market economy in reality. It provides direction and strategy for the choice and reform of economic systems, establishes a benchmark point and reference system for us to study more realistic markets, and it is the starting point for us to think about and test the results of market economy. In particular, if the market economy fails to achieve the efficiency of resource allocation, the market that is intervened to achieve Pareto improvement will inevitably violate at least some of the conditions of the First Fundamental Theorem of Welfare Economics.

Of course, from the micro- and incomplete information perspectives, market still faces many problems that often lead to the phenomenon of "market failure". It is important to analyze the circumstances under which
a market will fail and how a government should perform. As long as the
governance boundary of the market mechanism is clarified, we do not un-
conditionally decline the role of the market, or go from one extreme to
the other, but know the circumstances under which the market can play
a decisive role in allocating resources, and how a government could play
a good, appropriated but not unlimited role in cases of market failures.
Of course, the basic premise is still that individuals are rational, and gov-
ernment should not directly intervene in economic activities, but establish
rules or systems to solve the problem of market failure. This is due to
that with incomplete information, direct intervention in economic activi-
ties (such as a large number of state-owned enterprises and arbitrary re-
strictions on market access and interference with commodity prices) often
does not work well. In this regard, the last two parts of the book, which
are devoted to the study of how to design rules and institutional incentive
compatible mechanisms, can play a significant role in making the market
more efficient and solving market failure problems.

Therefore, the contents of the rest of the book can be seen as a further
expansion of these topics. From Chapter 14 to the end of the book, our
discussion will turn from how the superiority of the frictionless and freely
competitive market system to the issues that may arise, and discuss how
to correct for these failures. To this end, we will examine the various situa-
tions in which the actual market deviates from the ideal perfect competition
situation and the so-called “market failure” issues resulting in Pareto inef-
ficient market equilibrium, and we will provide solutions to market failure
problems.

In the current part, we will study externalities and public goods in
Chapter 13 and Chapter 14, respectively. In both cases, the actions of one a-
gent directly affect the utility or production of other agents in the economy.
We will see these nonmarketed “goods” or “bads” lead to Pareto inefficient
outcomes in general. It turns out that private markets often do not work
well in the presence of externalities and public goods. We will consider
situations of incomplete information which also result in Pareto inefficient
outcomes in general in Part VI.
Chapter 14

Externalities

14.1 Introduction

In this chapter we deal with economic environments in the presence of externalities. The so-called externality refers to situations that economic activities (production or consumption activities) of some individuals affect the utility or production levels of other individuals, which further affect their own economic activities. The basic conclusion is that the existence of externality generally leads to Pareto inefficient allocations, and thus there is a market failure. The fundamental reason is that due to externality in the economy, some factors affecting economic activities are not properly considered, and thus even if perfect competition and free choice are present, allocation is often inefficient, and thus other institutional arrangements or mechanism designs are necessary to improve the allocation of resources.

Externality is an objective and ubiquitous phenomenon. Specifically, externality consists of two categories: consumer externality and production externality, and they are the same in nature.

14.1.1 Consumption Externality

In the previous analysis, utility, satisfaction or welfare of individuals are only related to their own consumption levels, but not to the consumption of others. But in fact, in many cases, the utility level of an agent will also be affected the consumptions of others, while you can not control the consumption of others, thereby an agent’s utility level is passively affected. Externality can be either negative, which may hurt an agent, or positive, which may benefit the agent. Such examples abound.

Example 14.1.1 The following examples about consumption externalities are commonly seen in reality:

(i) One person’s quiet environment is disturbed by another person’s noise.
(ii) Mr. A hates Mr. D smoking next to him.

(iii) Mr. B’s satisfaction decreases as Mr. C’s consumption level increases, because Mr. B envies Mr. C’s affluent life, which leads to the mentality of hating the rich.

(iv) It does not matter if you are feeling well or not. You do want to others feel well. If someone else is feeling well, you will not feel well. Seeing others feel well makes you unhappy.

(v) You watch the TV bought by your roommates.

(vi) Take your colleague’s free ride to work.

(vii) Your dress is influenced by the style of other people.

(viii) The utility from using phone, WeChat and email depends on whether other people have the corresponding device.

Environmental impacts or pollution in (i) and (ii) are typical externalities, and individual behaviors that pollute the environment can have a negative impact on the health of others. Also, there are always some people who are jealous of others around us. Benefiting oneself at the cost of others in (iv) may be understandable, and harming others at the cost of oneself (all perish together) may also be understandable, but (iii) harming others without benefiting oneself is commonplace. Although it is somewhat hard to understand, it may be interpreted by negative consumption externalities. (v) and (vi) are examples of positive externalities, while (vii) and (viii) are examples of consumption network externalities.

These examples illustrate that the existence of externalities is a very common phenomenon. Some may deny the existence of externality with the example of looking at beauty without cost. In fact, there are two misunderstandings in this assertion. Firstly, whether the price is zero is not used in defining externality. Secondly, to see a beauty more often, such as model shows or stars in the movie, you need to pay.

Formally, we express the existence or absence of consumption externalities as:

\[ u_i(x_i) : \text{without preference externality}; \]
\[ u_i(x_1, \ldots, x_n) : \text{with preference externality}, \]

in which other individuals’ consumptions affect an individual’s utility.

### 14.1.2 Production Externality

Production externality means that the level of one firm’s output is affected by the production activities of other economic agents. Production externality effects can be either negative or positive.
The following lists some examples for production externality:

**Example 14.1.2** Here are some typical examples about production externalities in reality.

(i) Sewage discharge from chemical plants can affect the production of surrounding fishermen. In particular, downstream fishing can be adversely affected by pollutants emitted from an upstream chemical plant.

(ii) The machine noise of the factory near to you disturbs your quietness.

(iii) Breathing in Beijing city’s air is somehow equivalent to smoking a pack of cigarettes a day.

(iv) Problem with a bank has caused panic and a significant decline in currency liquidity. In the face of bank failures caused by the financial crisis, everyone is afraid to deposit, thus affecting the real economy.

(v) Beekeeping and farms have mutually positive production externalities. The flowers in the farm benefit the beekeepers, which in turn facilitates pollination of crops.

(vi) University brands can benefit all students, including those with low GPA, making it relatively easier to find jobs.

(vii) R&D of an enterprise may increase the productivity and in turn output levels of other firms. In the IT industry, the fixed cost of production is large, and the marginal cost is small, but the positive externality of the product is very large. As such, knowledge has a typical externality. In order to facilitate innovation, intellectual property protection is required; otherwise, the enterprise may not have incentives to conduct R&D. Monopoly is not altogether a bad thing. One of its benefits is that it can stimulate firms to carry out R&D and innovations (and thus obtain monopoly profits).

(viii) The output of a firm is influenced by the stock of knowledge in the entire economy (production network externality).

In addition, the decision of government and its officials might have a huge positive or negative externality to individuals, both in terms of production and consumption. This is the fundamental reason why there is a need for supervision or checks and balances on government and its officials.

This leads to an examination of various suggestions for alternative ways to allocate resources that may lead to more efficient outcomes. Achieving
an efficient allocation in the presence of externalities essentially involves making sure that agents face the correct pricing for their activities. Ways of solving externality problems include taxation, regulation, property rights, merges, etc.

14.2 Consumption Externality

When there are no consumption externalities, agent $i$’s utility function is a function of only his own consumption:

$$u_i(x_i).$$

(14.2.1)

In this case, for any $l = 1, 2, \ldots, L$ and $h = 1, 2, \ldots, L$, the first-order conditions for the competitive equilibrium are given by

$$MRS_{x_1, x_1} = \frac{p^l}{p^h} = MRS_{x_2, x_2} = \cdots = MRS_{x_L, x_L},$$

(14.2.2)

and from Chapter 11, we know that the first-order conditions for Pareto efficiency are also given by:

$$MRS_{x_1, x_1} = \cdots = MRS_{x_L, x_L},$$

(14.2.3)

As a result, because of price-taking behavior, every competitive equilibrium implies Pareto efficiency if utility functions are quasi-concave.

The main purpose of this section is to show that a competitive equilibrium allocation is not in general Pareto efficient when there exists an externality in consumption. We show this by examining that the first-order conditions for a competitive equilibrium are not in general the same as the first-order conditions for Pareto efficient allocations in the presence of consumption externalities. The following discussion is mainly drawn from Tian and Yang (2009).

Consider the following simple two-person and two-good exchange economy.

$$u_A(x_A, x_B, y_A),$$

(14.2.4)

$$u_B(x_A, x_B, y_B),$$

(14.2.5)

which are assumed to be strictly increasing in his own goods consumption, quasi-concave. To have interior solutions, we assume that $u_i$ satisfies the Inada condition $\frac{\partial u}{\partial x_i}(0) = +\infty$, and also $\lim_{x_i \to 0} \frac{\partial u}{\partial x_i} x_i = 0$. Note that here good $x$ results in consumption externalities.

The first order conditions for the competitive equilibrium are the same as before:

$$MRS_{x_A} = \frac{p_x}{p_y} = MRS_{x_B}.$$  

(14.2.6)
14.2. CONSUMPTION EXTERNALITY

We now find the first order conditions for Pareto efficient allocations in exchange economies with consumption externalities. Pareto efficient allocations \( x^* \) can be completely characterized by the FOCs of the following problem.

\[
\max_{x \in \mathbb{R}^4_+} u_B(x_A, x_B, y_B)
\]

s.t.

\[
\begin{align*}
&x_A + x_B \leq w_x, \\
y_A + y_B \leq w_y, \\
u_A(x_A, x_B, y_A) \geq u_A(x_A^*, x_B^*, y_A^*).
\end{align*}
\]

The first order conditions are

\[
\begin{align*}
x_A : &\quad \frac{\partial u_B}{\partial x_A} - \lambda_x + \mu \frac{\partial u_A}{\partial x_A} = 0, \\
y_A : &\quad -\lambda_y + \mu \frac{\partial u_A}{\partial y_A} = 0, \\
x_B : &\quad \frac{\partial u_B}{\partial x_B} - \lambda_x + \mu \frac{\partial u_A}{\partial x_B} = 0, \\
y_B : &\quad \frac{\partial u_B}{\partial y_B} - \lambda_y = 0,
\end{align*}
\]

(14.2.8)–(14.2.11)

\[
\begin{align*}
\lambda_x : &\quad w_x - x_A - x_B \geq 0, \lambda_x \geq 0, \lambda_x (w_x - x_A - x_B) = 0, \\
\lambda_y : &\quad w_y - y_A - y_B \geq 0, \lambda_y \geq 0, \lambda_y (w_y - y_A - y_B) = 0, \\
\mu : &\quad u_A - u_A^* \geq 0, \mu \geq 0, \mu (u_A - u_A^*) = 0.
\end{align*}
\]

(14.2.12)–(14.2.14)

By (14.2.11), \( \lambda_y = \frac{\partial u_B}{\partial y_B} > 0 \), and thus by (14.2.13),

\[
y_A + y_B = w_y,
\]

(14.2.15)

which means there is never disposal (or destruction) of the good which does not exhibit a negative externality. Also, by (14.2.9) and (14.2.11), we have

\[
\mu = \frac{\partial u_B}{\partial y_B}.
\]

(14.2.16)

Then, by (14.2.8) and (14.2.9), we obtain

\[
\frac{\lambda_x}{\lambda_y} = \frac{\left[ \frac{\partial u_A}{\partial x_A} + \frac{\partial u_B}{\partial x_B} \right]}{\left[ \frac{\partial u_B}{\partial y_B} + \frac{\partial u_A}{\partial y_A} \right]},
\]

(14.2.17)

and by (14.2.10) and (14.2.11), we have

\[
\frac{\lambda_x}{\lambda_y} = \frac{\left[ \frac{\partial u_B}{\partial x_B} + \frac{\partial u_A}{\partial x_A} \right]}{\left[ \frac{\partial u_B}{\partial y_B} + \frac{\partial u_A}{\partial y_A} \right]}.
\]

(14.2.18)
Thus, by (14.2.17) and (14.2.18), we get
\[
\frac{\partial u_A}{\partial x_A} + \frac{\partial u_B}{\partial x_B} = \frac{\partial u_B}{\partial y_B} + \frac{\partial u_A}{\partial y_A},
\]
which means the social marginal rates of substitution of good $x$ for good $y$ for the two consumers equal at Pareto efficient allocations. We call it social marginal rate of substitution because the social welfare function can be written as the sum of individual utilities. From the above marginal equality condition, we know that, in order to evaluate the relevant marginal rates of substitution for the optimality conditions, we must take into account both the direct and indirect effects of consumption activities in the presence of externalities. That is, to achieve Pareto optimality, when one consumer increases the consumption of good $x$, not only does the consumer’s consumption of good $y$ need to change, the other consumer’s consumption of good $y$ must also be changed. Therefore the social marginal rate of substitution of good $x$ for good $y$ by consumer $i$ equals \( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} \). Thus, since the first order conditions for competitive equilibrium and Pareto optimality are not the same, we immediately have the following conclusion:

**Proposition 14.2.1** If there is consumption externality, competitive equilibrium allocations are in general not Pareto efficient.

From the above marginal equality conditions, we can see that in order to correctly calculate the relevant marginal rate of substitution for the marginal equality conditions, we must consider the direct and indirect effects of consumption activities in the presence of consumption externalities.

Solving (14.2.8) and (14.2.10) for $\mu$ and $\lambda_x$, we have
\[
\mu = \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_A} > 0,
\]
and
\[
\lambda_x = \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B}.
\]

When the consumption externality is positive, from (14.2.17) or (14.2.18), we can easily see that $\lambda_x$ is always positive since $\lambda_y = \frac{\partial u_B}{\partial y_B} > 0$. Also, when no externality or a one-sided externality\(^1\) exists, by either (14.2.17) or (14.2.18), $\lambda_x$ is positive. Thus, the marginal equality condition (14.2.19)

\(^1\)Only one consumer imposes an externality on another consumer.
and the balanced conditions completely determine all Pareto efficient allocations for these cases. However, when there are negative externalities for both consumers, the Kuhn-Tucker multiplier $\lambda_x$ directly given by (14.2.21) or indirectly given by (14.2.17) or (14.2.18) is the sum of negative and positive terms, and thus the sign of $\lambda_x$ may be indeterminate. However, unlike the claim in some textbooks such as Varian (1992, pp.438), the marginal equality condition, (14.2.19) and the balanced conditions along may not guarantee finding Pareto efficient allocations correctly.

To guarantee that an allocation be Pareto efficient in the presence of negative externalities, we must require $\lambda_x = 0$ at efficient points, which in turn requires that social marginal rates of substitution of good $x$ for good $y$ be nonnegative, that is,

$$\frac{\partial u_A}{\partial x_A} + \frac{\partial u_B}{\partial x_B} = \frac{\partial u_B}{\partial x_B} + \frac{\partial u_A}{\partial y_A} \geq 0,$$

or equivalently requires both (14.2.19) and

$$\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} \geq \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$$

for all Pareto efficient points.

We can interpret the term in the left-hand side of (14.2.23), $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B}$, as the joint marginal benefit of consuming good $x$, and the term in the right-hand side, $\frac{\partial u_B}{\partial x_B} \frac{\partial u_A}{\partial x_A}$, as the joint marginal cost of consuming good $x$ because the negative externality hurts the consumers. To consume the goods efficiently, a necessary condition is that the joint marginal benefit of consuming good $x$ should not be less than the joint cost of consuming good $x$.

Thus, the following conditions

\[
\begin{cases}
\frac{\partial u_A}{\partial x_A} + \frac{\partial u_B}{\partial x_B} = \frac{\partial u_B}{\partial x_B} + \frac{\partial u_A}{\partial y_A} \geq 0, \\
y_A + y_B = w_y, \\
x_A + x_B \leq w_x, \\
\left(\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}\right)(w_x - x_A - x_B) = 0,
\end{cases}
\]

constitute a system (PO) from which all Pareto efficient allocations should be satisfied. We can further analyses the solution for Pareto efficiency by considering three cases.

Case 1. When $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} > \frac{\partial u_B}{\partial x_B} \frac{\partial u_A}{\partial x_A}$, or equivalently $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} = \frac{\partial u_B}{\partial x_B} \frac{\partial u_A}{\partial x_A} + \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B}$,
system (PO) reduce to \( x_A + x_B = w_x \). In this case, there is no destruction for good \( x \). Substituting \( x_A + x_B = w_x \) and \( y_A + y_B = w_y \) into the marginal equality condition (14.2.19), it provides a relationship between \( x_A \) and \( y_A \), from which we can find Pareto efficient allocations.

Case 2. When the joint marginal benefit equals the joint marginal cost:

\[
\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} = \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A},
\]

(14.2.24)

then

\[
\frac{\partial u_A}{\partial x_A} + \frac{\partial u_A}{\partial y_A} + \frac{\partial u_B}{\partial x_B} + \frac{\partial u_B}{\partial y_B} = 0
\]

(14.2.25)

and thus \( \lambda_x = 0 \). In this case, when \( x_A + x_B \leq w_x \), the necessity of destruction is indeterminant (Note that no destruction means that \( x_A + x_B = w_x \)). However, even when destruction is necessary, we can still determine the set of Pareto efficient allocations by using \( y_A + y_B = w_y \) and the zero social marginal equality condition (14.2.25). Indeed, after substituting \( y_A + y_B = w_y \) into (14.2.25), we can solve for \( x_A \) in terms of \( y_A \).

Case 3. When \( \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} < \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} \), for any allocation that satisfies \( x_A + x_B = w_x \), \( y_A + y_B = w_y \), and the marginal equality condition (14.2.19), the social marginal rates of substitution must be negative. Hence, the allocation will not be Pareto efficient. Thus, there must be a destruction (free disposal) for good \( x \) for Pareto efficiency, and there exists Pareto efficient allocations that satisfy (14.2.25).

Summarizing the above three cases, we conclude that one can use the following set of conditions

\[
\begin{align*}
\frac{\partial u_A}{\partial x_A} + \frac{\partial u_A}{\partial y_A} + \frac{\partial u_B}{\partial x_B} + \frac{\partial u_B}{\partial y_B} & = 0 \\
x_A + x_B & = w_x \\
y_A + y_B & = w_y
\end{align*}
\]

(14.2.26)

together with whether \( \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} \geq \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} \), to judge there is destruction (free disposal) or not.

Indeed, if \( \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} \geq \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} \) is also satisfied, then no free disposal in achieving Pareto efficient allocations. If \( \frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} < \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} \), although utility functions are strictly increasing in \( x \), there must be the case of destruction of some amount of \( x \) in achieving Pareto efficient allocations.

We then have the following proposition that characterizes whether or not there is destruction (free disposal) of endowments \( w_x \) in achieving Pareto efficient allocations although utility functions are strictly increasing in \( x \).

**Proposition 14.2.2** For \( 2 \times 2 \) pure exchange economies, suppose that utility functions \( u_i(x_A, x_B, y_i) \) are continuously differentiable, strictly quasi-concave, and \( \frac{\partial u_i(x_A, x_B, y_i)}{\partial x_i} > 0 \) for \( i = A, B \).
14.2. CONSUMPTION EXTERNALITY

(1) If the social marginal rates of substitution of good $x$ for good $y$ are positive at a Pareto efficient allocation $x^*$, then there is no free disposal for $w_x$ in achieving Pareto efficient allocation $x^*$.

(2) If the social marginal rates of substitution of good $x$ for good $y$ are negative for any allocation $(x_A, x_B)$ satisfying $x_A + x_B = w_x$, $y_A + y_B = w_y$, and the marginal equality condition (14.2.19), then there must be free disposal for $w_x$ in achieving any Pareto efficient allocation $x^*$. That is, $x_A^* + x_B^* < w_x$ and $x^*$ is determined by $y_A + y_B = w_y$ and (14.2.25).

Example 14.2.1 Consider the following utility function:

$$u_i(x_A, x_B, y_i) = \sqrt{x_i y_i} - x_j, \quad i \in \{A, B\}, \ j \in \{A, B\}, j \neq i.$$  

By the marginal equality condition (14.2.19), we obtain

$$\left(\frac{\sqrt{y_A}}{x_A} + 1\right)^2 = \left(\frac{\sqrt{y_B}}{x_B} + 1\right)^2$$  

(14.2.26)
and thus

$$\frac{y_A}{x_A} = \frac{y_B}{x_B}. \quad (14.2.27)$$

Let $x_A + x_B \equiv \bar{x}$. Substituting $x_A + x_B = \bar{x}$ and $y_A + y_B = w_y$ into (14.2.27), we have

$$\frac{y_A}{x_A} = \frac{w_y}{\bar{x}}. \quad (14.2.28)$$

Then, by (14.2.27) and (14.2.28), we get

$$\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} = \frac{1}{4} \sqrt{\frac{y_A}{x_A} \frac{y_B}{x_B}} = \frac{y_A}{4x_A} = \frac{w_y}{4\bar{x}}$$  

(14.2.29)
and

$$\frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} = 1. \quad (14.2.30)$$

Thus, $\bar{x} = w_y/4$ is the critical point that makes $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} = 0$, or equivalently $\frac{\partial u_A}{\partial x_A} + \frac{\partial u_A}{\partial y_A} = \frac{\partial u_B}{\partial x_B} + \frac{\partial u_B}{\partial y_B} = 0$. Hence, if $w_x > \frac{w_y}{4}$, then $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} < 0$, and thus there must be destruction in any Pareto efficient allocation. If $w_x < \frac{w_y}{4}$, then $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} > 0$, and then no Pareto optimal allocation requires no free disposal. Finally, when $w_x = \frac{w_y}{4}$, any allocation that satisfies the marginal equality condition (14.2.19) and the balanced conditions $x_A + x_B = w_x$ and $y_A + y_B = w_y$ also satisfies (14.2.23) since $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} = 0$, and thus it is a Pareto efficient allocation with no free disposal.

---

2 As we discussed above, this is true if the consumption externality is positive, or there is no externality or only one side externality.
Note that, since \( \frac{\partial u_A}{\partial x_A} \) and \( \frac{\partial u_B}{\partial x_B} \) represent marginal benefit, they are usually diminishing with increase in consumption in good \( x \). Since \( \frac{\partial u_A}{\partial x_B} \) and \( \frac{\partial u_B}{\partial x_A} \) are in the form of marginal cost, their absolute values are typically increasing in good \( x \). Hence, when total endowment \( w_x \) is small, the social marginal benefit would exceed the social marginal cost so that there is no destruction of good \( x \). As the total endowment of \( w_x \) increases with the total endowment of \( w_y \) fixed (i.e., \( y \) becomes relatively scarce when \( x \) becomes abundant), the social marginal cost will ultimately outweigh the marginal social benefit, which results in the destruction of the endowment of \( w_x \).

Alternatively, we can get the same result by using social marginal rates of substitution. When utility functions are strictly quasi-concave, marginal rates of substitution are diminishing. Therefore, in the presence of negative consumption externalities, social marginal rates of substitution of good \( x \) for good \( y \) may become negative when the consumption of good \( x \) becomes sufficiently large. When this occurs, it is better to destroy some resources for good \( x \). As the destruction of good \( x \) increases, which will, in turn, decrease the consumption of good \( x \), social marginal rates of substitution will increase. Eventually they will become nonnegative.

When there is a negative externality, it is seemly strange that some commodities need to be fee disposed or destroyed in order to achieve Pareto efficient allocations of resources, but this phenomenon is not only important in theory, but also related to reality. Indeed, Tian and Yang (2012) used the above theoretical results to explain a well-known puzzle of the happiness-income relationship in the economics and psychology literatures: people’s happiness rises with income up to a point, but not beyond it. For example, mean life satisfaction in the United States has been declining in roughly past fifty years, while that in the United Kingdom has run approximately flat across the same period in Britain. If we interpret income as a good, when the good becomes an inferior good or people envy each other’s income level (e.g. low-income people envy high-income people), then according to the above results, when income exceeds a certain threshold level, if all income is spent, people’s happiness decreases with the increase of consumption, which leads to Pareto inefficient allocations. Therefore, when economic growth reaches a certain level, if other aspects (such as spiritual civilization and political civilization) can not follow up, increase in income does not increase people’s satisfaction, which is the so-called puzzle of the happiness-income.

To illustrate this point, go back to Example 14.2.1, as analyzed by Tian and Yang (2012), and interpret \( x \) as the composite of material goods or GDP index, and \( y \) as the composite non-material goods or non-GDP index. If we do not increase the level of non-material goods \( w_y \) in a comprehensive and balanced way, and only focus on the economic growth of GDP, eventually
we will have $w_x > \frac{w_y}{T}$. As a result, we can see in reality that as people’s income level constantly rises, but happiness continues to decline.

The above results have a strong policy implication, that is, the government’s pursuit of GDP growth does not always improve people’s happiness, but may reduce people’s satisfaction, resulting in Pareto inefficient allocations. This is the fundamental reason that in the past few decades, people’s happiness in many countries has risen and then began to decline as income continues to rise. A similar phenomenon is also recently seen in China. But according to the above discussions, an individual’s happiness comes from two levels: material and non-material (spiritual civilization and political civilization). In fact, the level of an individual’s happiness is determined by multiple factors: (1) material factors, such as income levels and differences; (2) mental factors, such as career achievement, work stress, unemployment, leisure time, friendships and family harmony; (3) social and political factors, such as social equity, political stability, democratic rights, etc.; (4) ecological factors, such as control of environmental pollution and ecological damage, which are related to individual health and even survival. It can be seen that the factors listed in (1) are GDP products and the factors listed in (2)-(4) are non-material goods or non-GDP products.

Therefore, happiness comes from material civilization, spiritual civilization, political civilization, and ecological civilization. When people’s living standards are limited, people care more about pursuing material civilization. When their living standard reaches a certain level, people will tend to pursue spiritual civilization, political civilization and ecological civilization. That is, people first need to solve the problem of food, clothing, shelter, and transportation, then the superstructure of art, poetry, philosophy, life comfort and quality, physical health, democratic politics, and protection of one’s own rights. Due to the negative externalities of income, the construction of spiritual civilization, political civilization and ecological civilization is also very important. Therefore, both material and non-material goods need to be balanced and fully developed, otherwise, it will not lead to social harmony and efficiency.

Happiness economics is the subject of psychology, ethics, and economics. Many economists believe that mainstream economics cannot solve the problem of human happiness. However, the above results show that happiness economics can also be incorporated into the framework of mainstream economics. It can still be assumed that individuals are self-interested in pursuing their personal interests, and Pareto optimality or social welfare maximization is still a necessary and basic criterion of whether the resource allocation is efficient. It just adds to the assumption that people’s income generally has negative externalities (this is a reasonable assumption, which is common in reality, for example, some people think that their utility will be decreased if others’ income is higher than their own income). For a de-
tailed discussion of this issue, see Tian and Yang in (2009, 2012).

14.3 Production Externality

We now discuss that when production externalities exist, competitive markets may also lead to inefficient allocations of resources. To illustrate this, consider a simple economy with two firms. Firm 1 produces an output \( x \) that will be sold in a competitive market. However, production of \( x \) imposes an externality cost, denoted by \( e(x) \), to firm 2, which is assumed to be convex and strictly increasing.

Let \( y \) be the output produced by firm 2, which is sold in competitive market. Let \( c_x(x) \) and \( c_y(y) \) be the cost functions of firms 1 and 2 which are both convex and strictly increasing.

The profits of the two firms are:

\[
\pi_1 = p_x x - c_x(x),
\]

(14.3.31)

\[
\pi_2 = p_y y - c_y(y) - e(x),
\]

(14.3.32)

where \( p_x \) and \( p_y \) are the prices of \( x \) and \( y \), respectively. Then, by the first order conditions, we have for positive amounts of outputs:

\[
p_x = c'_x(x), \quad \text{ (14.3.33)}
\]

\[
p_y = c'_y(y). \quad \text{ (14.3.34)}
\]

However, the profit maximizing output \( x_c \) from the first order condition is over produced from a social point of view. The first firm only takes account of the private cost – the cost that is imposed on itself– but it ignores the social cost – the private cost plus the cost it imposes on the other firm.

What’s the socially efficient output?

The social profit, \( \pi_1 + \pi_2 \), is not maximized at \( x_c \) and \( y_c \) which satisfy (14.3.33) and (14.3.34). If the two firms merged so as to internalize the externality

\[
\max_{x,y} p_x x + p_y y - c_x(x) - c(x) - e(x) - c_y(y)
\]

(14.3.35)

which gives the first order conditions:

\[
p_x = c'_x(x^*) + e'(x^*),
\]

\[
p_y = c'_y(y^*),
\]

where \( x^* \) is an efficient amount of output; it is characterized by price being equal to the marginal social cost. Thus, production of \( x^* \) is less than the competitive output in the externality case by the convexity of \( e(x) \) and \( c_x(x) \).
14.4 Solutions to Externalities

From the above discussion, we know that a competitive market in general may not result in Pareto efficient outcomes in the presence of externalities, and one needs to seek some other alternative mechanisms to solve the market failure problem. There are many methods proposed to remedies the market failure of externality such as:

1. Pigovian taxes;
2. Voluntary negotiation (Coase Approach);
3. Compensatory tax/subsidy;
4. Creating a missing market with property rights;
5. Direct intervention;
6. Merges of firms;
7. Creating a market for the exchange of emission rights;
8. Incentive mechanism design.

Any of the above solution may result in Pareto efficient outcomes, but may lead to different income distributions. Also, it is important to know what kinds of information are required to implement a solution listed above. Most of the above proposed solutions need to make the following assumptions:

1. The source and degree of the externality are identifiable.
2. The recipients of the externality are identifiable.
3. The causal relationship of the externality can be established objectively.
4. The cost of preventing (by different methods) an externality is perfectly known to everyone.
5. The cost of implementing taxes and subsides is negligible.
6. The cost of voluntary negotiation is negligible.

We will discuss the advantages and disadvantages of the above schemes below, and find out which are feasible and which are not. In addition, it is important to know what kind of information is needed to perform each of these solutions. Most of the above schemes require information symmetry, such as Pigovian tax, Coase theorem, and so on. There will be major problems in the implementation of these methods in case of incomplete information. Therefore, incentive mechanism design is necessary to implement these solutions or to provide new solutions.

### 14.4.1 Pigovian Tax

The Pigovian tax was proposed by Arthur Cecil Pigou (1877–1959, see his biography in Section 14.6.1). For those externality-producing firms, the government imposes a tax on the marginal cost of externality as the tax rate \( t = e'(x^*) \). In the case of complete information, the externality and the tax rate \( t \) can be determined, and the first-order optimal conditions of the enterprise are the same as the socially optimal first-order conditions, so as to achieve efficient allocation of resources.

To see this, set a tax rate, \( t \), such that \( t = e'(x^*) \). This tax rate to firm 1 would internalize the externality. Indeed, the net profit of firm 1 would be:

\[
\pi_1 = p_x \cdot x - c_x(x) - t \cdot x, \tag{14.4.36}
\]

which leads to the first order condition:

\[
p_x = c'_x(x) + t = c'_x(x) + e'(x^*), \tag{14.4.37}
\]

which is the same as the one for social optimality. That is, when firm 1 faces the wrong pricing of its action, and a tax \( t = e'(x^*) \) should be imposed for each unit of firm 1’s production. This will lead to a socially optimal outcome that is less than that of competitive equilibrium outcome. Such correction taxes are called Pigovian taxes.

The problem with this solution is that it requires that the taxing authority knows the externality cost \( e(x) \). But, how does the authority know the externality and how do they estimate the value of externality in real world? If the authority knows this information, it would work such as imposing a Pigovian tax on gasoline since emission of automobile is relatively easier to determine. But, in most cases, it does not work well. Therefore, it is only applicable to scenarios where \( e(x) \) is relatively easier to identify. For example, the emissions of a car can be easily quantified, so that the emission tax per gallon of gasoline can be easily determined.
In addition, as pointed out by Ng (2004), if \( e(x) \) is an assessment function of environmental disruption, this often involves many people (even globally) and the future, thus it is difficult to estimate. But if \( e(x) \) is a cost function on abatement spending, it is often easier to estimate. Ng argues that in the case of serious pollution (and therefore there is abatement investment), it is not necessary to estimate the damage of pollution, but to tax according to the marginal cost of abatement spending. However, when \( e(x) \) is private information and is difficult to identify, it is hard for the tax collector to accurately obtain information about the cost \( e(x) \), so this method cannot be directly adopted. In order to obtain information, some effective methods are necessary, but they all involve a cost. If the cost is too high, it is difficult to adopt in practice.

### 14.4.2 Coase’s Approach

A different approach to the externality problem relies on the parties to negotiate a solution to the problem themselves.

Nobel laureate Ronald Harry Coase (1910-2013, see his biography in Section 14.6.2) raised two problems of Pigou’s tax: first, government intervention interferes with economic freedom; second, taxpayers are unlikely to know about the cost of \( e(x) \) in most situations.

The greatest novelty of Nobel laureate Ronald Coase’s contribution was the systematic treatment of trade in property rights. To solve the externality problem, Coase in his famous article, “The Problem of Social Cost”, in 1960 emphasized that whether externality can be effectively solved depends on whether property rights are clearly defined. For this reason, Coase put forward clear definition of property rights and methods of voluntary exchange and negotiation. The so-called Coase Theorem asserts that as long as property rights are clearly defined, the outcome of negotiations between the two parties will result in the efficient level of production in the presence of production externality.

The term “Coase Theorem” originated with George Stigler, who explained Coase’s ideas in his textbook, “The Theory of Price”, pp. 110-114. Stigler asserted that the Coase theorem actually contains two claims in the absence of transaction costs:

1. **Claim 1 (Coase Efficiency Theorem):** Voluntary negotiations over externalities will lead to a Pareto-optimal outcome.

2. **Claim 2 (Coase Neutrality Theorem or Independence Theorem):** The level of externality is the same regardless of who the property rights are given and how they are allocated.

Stigler’s Claim 2 would follow from Claim 1 if it were true that every Pareto optimal allocation has the same level of externality-producing activities, regardless of the way that private goods are distributed. Thus, the
so-called Coase’s Theorem assesses that as long as property rights are clearly assigned and transition cost is zero, the two parties will negotiate in such a way that the optimal level of the externality-producing activity is implemented. As a policy implication, a government should simply rearrange property rights with appropriately designed property rights. Market then could take care of externalities without direct government intervention.

Coase illustrates his assertion through various examples of the two-person economy with externalities. The following simple examples depict Coase’s core ideas and ideas.

Example 14.4.1 Two firms: One is chemical factory that discharges chemicals into a river and the other is the fisherman. Suppose that the river can produce a value of $100,000. If the chemicals pollute the river, the fish cannot be eaten. How does one solve the externality? Coase’s method states that as long as the property rights of the river are clearly assigned, it results in efficient outcomes. That is, the government should give the ownership of the lake either to the chemical firm or to the fisherman, then it will yield an efficient output. To see this, assume that:

The cost of the filter is denoted by $c_f$.

Case 1: The lake is given to the factory.

i) $c_f < 100,000$. The fisherman is willing to buy a filter for the factory. The fisherman will pay for the filter so that the chemical cannot pollute the lake.

ii) $c_f > 100,000$. The chemical is discharged into the lake. The fisherman does not want to install any filter.

Case 2: The lake is given to the fisherman, and the firm’s net product revenue is greater than $100,000$.

i) $c_f < 100,000$. The factory buys the filter so that the chemical cannot pollute the lake.

ii) $c_f > 100,000$. The firm pays $100,000 to the fisherman before the chemical is discharged into the lake.

In this way, no matter who owns the property, two cases lead to the same efficient result: as long as $C_f < 500,000$, pollution will not happen, otherwise it will happen. Only differences is the income distribution effect is different.

Like the above example, Cases’s own examples supporting his claims are about negotiations between firms rather than those between individuals. Because firm are profit maximization rather than utility maximization, their economic behavior seem to be fiduciaries behavior. This difference is important because profit maximization has no income effect, while utility
maximization generally has income effect. So, we have to make some re-
stricted assumptions on consumers’ utility functions to make Coase’s The-
orem be true.

Now consider an economy with two consumers with \( L \) goods. Furthermore, consumer \( i \) has initial wealth \( w_i \), and his utility function is given by

\[
u_i(x_1^i, \ldots, x_L^i, h).
\]

That is, the utility of each consumer is related to the quantity of goods consumed as well as the activity \( h \) carried out by consumer 1.

Activity \( h \) is something that has no direct monetary cost for person 1. For example, \( h \) is the quantity of loud music played by person 1. In order to play it, the consumer must purchase electricity, but electricity can be captured as one of the components of \( x_i \). From the point of view of consumer 2, \( h \) represents an external effect of consumer 1’s action. In the model, we assume that

\[
\frac{\partial u_2}{\partial h} \neq 0.
\]

Thus the externality in this model lies in the fact that \( h \) affects consumer 2’s utility, but it is not priced by the market. Let \( v_i(p, w_i, h) \) be consumer \( i \)’s indirect utility function:

\[
v_i(w_i, h) = \max_{x_i} u_i(x_i, h)
\]

s.t. \( px_i \leq w_i \).

To rule out income effect resulted from the assignment of property right-
s, we assume that preferences are quasi-linear with respect to some nu-
meraire commodity. Thus, the consumer’s indirect utility function takes
the form:

\[
v_i(w_i, h) = \phi_i(h) + w_i.
\]

We further assume that utility is strictly concave in \( h : \phi_i''(h) < 0 \). Again, the competitive equilibrium outcome in general is not Pareto optimal. In order to maximize utility, consumer 1 chooses \( h \) in order to maximize \( v_1 \) so that the interior solution satisfies \( \phi_1'(h^*) = 0 \). Even though consumer 2’s utility depends on \( h \), it cannot affect the choice of \( h \).

On the other hand, the socially optimal level of \( h \) will maximize the sum
of the consumers’ utilities:

\[
\max_h \phi_1(h) + \phi_2(h).
\]

The first-order condition for an interior maximum is:

\[
\phi_1'(h^{**}) + \phi_2'(h^{**}) = 0,
\]
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where \( h^{**} \) is the Pareto optimal amount of \( h \). Thus, the social optimum is where the sum of the marginal benefit of the two consumers equals zero. In the case where the externality is bad for consumer 2 (loud music), the level of \( h^* > h^{**} \). That is, too much \( h \) is produced. In the case where the externality is good for consumer 2, too little will be provided, \( h^* < h^{**} \).

Now we show that, as long as property rights are clearly determined, the two parties will negotiate in such a way that the optimal level of the externality-producing activity is implemented. We first consider the case where consumer 2 has the right to prohibit consumer 1 from undertaking activity \( h \). But, this right is contractible. Consumer 2 can sell consumer 1 the right to undertake \( h_2 \) units of activity \( h \) in exchange for some transfer, \( T_2 \). The two consumers will bargain both over the size of the transfer \( T_2 \) and over the number of units of the externality-producing good, \( h_2 \).

In order to determine the outcome of the bargaining, we first specify the bargaining mechanism as follows:

1. Consumer 2 offers consumer 1 a take-it-or-leave-it contract specifying a payment \( T_2 \) and an activity level \( h_2 \).
2. If consumer 1 accepts the offer, that outcome is implemented. If consumer 1 does not accept the offer, consumer 1 cannot produce any of the externality-producing good, i.e., \( h_2 = 0 \).

To analyze this, begin by considering which offers, \((h_2, T_2)\), will be accepted by consumer 1. Since in the absence of agreement, consumer 1 must produce \( h_2 = 0 \) by noting that the rights are given to consumer 2, consumer 1 will accept \((h_2, T_2)\) if and only if it satisfies the participation constraint, that is, consumer 1 accepts if and only if:

\[
\phi_1(h_2) - T_2 \geq \phi_1(0).
\]

Given this constraint on the set of acceptable offers, consumer 2 will choose \((h_2, T_2)\) in order to solve the following problem:

\[
\begin{align*}
\max_{h_2, T_2} & \quad \phi_2(h_2) + T_2 \\
\text{s.t.} & \quad \phi_1(h_2) - T_2 \geq \phi_1(0).
\end{align*}
\]

Since consumer 2 prefers higher \( T_2 \), the constraint is binding at the optimum. Thus the problem becomes:

\[
\max_{h_2} \phi_1(h_2) + \phi_2(h_2) - \phi_1(0).
\]

The first-order condition for this problem is given by:

\[
\phi'_1(h_2) + \phi'_2(h_2) = 0.
\]
This is the same condition that results in the socially optimal level of \( h_2 \). Thus consumer 2 chooses \( h_2 = h^{**} \), and, using the constraint, \( T_2 = \phi_1(h^{**}) - \phi_1(0) \). And, the offer \((h_2, T_2)\) is accepted by consumer 1. Thus this bargaining process implements the social optimum.

Now we consider the case where consumer 1 has the right to produce as much of the externality as she wants. We maintain the same bargaining mechanism. Consumer 2 makes consumer 1 a take-it-or-leave-it offer \((h_1, T_1)\), where the subscript indicates that consumer 1 has the property right in this situation. However, now, in the event that consumer 1 rejects the offer, she can choose to produce as much of the externality as she wants, which means that she will choose to produce \( h^* \). Thus the only change between this situation and the first case is what happens in the event that no agreement is reached. In this case, consumer 2’s problem is:

\[
\max_{h_1, T_1} \phi_2(h_1) - T_1 \\
\text{s.t. } \phi_1(h_1) + T_1 \geq \phi_1(h^*).
\]

Again, we know that the constraint is binding, and so consumer 2 chooses \( h_1 \) and \( T_1 \) in order to maximize

\[
\max \phi_1(h_1) + \phi_2(h_1) - \phi_1(h^*),
\]

which is also maximized at \( h_1 = h^{**} \), since the first-order condition is the same. The only difference is in the transfer. Here \( T_1 = \phi_1(h^*) - \phi_1(h^{**}) \).

While the outcomes of both property-rights implement \( h^{**} \), they have different distributional consequences. The transfer payment is positive in the case where consumer 2 has the property rights, while it is negative when consumer 1 has the property rights. The reason for this is that consumer 2 is in a better bargaining position when the non-bargaining outcome is that consumer 1 is forced to produce 0 unit of the externality-producing good.

However, note that in the quasi-linear framework, redistribution of the numeraire commodity has no effect on social welfare. The fact that regardless of how the property rights are assigned, that bargaining leads to Pareto optimal allocations is an example of the Coase Theorem: If trade of the externality can occur, then bargaining will lead to an efficient outcome no matter how property rights are assigned (as long as they are clearly assigned). Note that well-defined, enforceable property rights are essential for bargaining to work. If there is a dispute over who has the right to pollute (or not pollute), then bargaining may not lead to efficiency. An additional requirement for efficiency is that the bargaining process itself is costless. Note that the government does not need to know about individual consumers here - it only needs to define property rights. However, it is critical that they are clearly defined. Thus the Coase Theorem provides an argument in favor of having clear laws and well-developed courts.
However, we know that quasi-linear function is a very restrictive assumption, which means there is no income effect. If Coase’s theorem is only valid for quasilinear utility function, then it has great limitations on the conclusion that consumption externality is valid. So, a natural question is, does Coase Theorem hold for other types of utility functions? Hurwicz’s conclusions are surprising and regrettable. Hurwicz (Japan and the World Economy, 7, 1995, pp. 49-74) argued that, even when the transaction cost is zero, the absence of income effects in the demand for the good with externality is not only sufficient (which is well known) but also necessary for Coase Neutrality Theorem to be true, i.e., when the transaction cost is negligible, the level of pollution will be independent of the assignments of property rights if and only if preferences of the consumers are quasi-linear with respect to the private good, leading to absence of income effects in the demand for the good with externality.

Unfortunately, as shown by Chipman and Tian (2012), the proof of Hurwicz’s claim on the necessity of parallel preferences for “Coase’s conjecture” is not right. To see this, consider the following class of utility functions:

\[ U_i(x_i, h) = x_i e^{-h} + \phi_i(h), \quad i = 1, 2 \]  

(14.4.38)

where

\[ \phi_i(h) = \int e^{-h} b_i(h) dh. \]  

(14.4.39)

\( U_i(x_i, h) \) is clearly not quasi-linear in \( x_i \). It is further assumed that for all \( h \in (0, \eta], b_1(h) > \xi, b_2(h) < 0, b_i'(h) < 0 \) \((i = 1, 2)\), \( b_1(0) + b_2(0) \geq \xi \), and \( b_1(\eta) + b_2(\eta) \leq \xi \).

We then have

\[
\begin{align*}
\frac{\partial U_i}{\partial x_i} & = e^{-h} > 0, \quad i = 1, 2, \\
\frac{\partial U_1}{\partial h} & = -x_1 e^{-h} + b_1(h) e^{-h} > e^{-h} [\xi - x_1] \geq 0, \\
\frac{\partial U_2}{\partial h} & = -x_2 e^{-h} + b_2(h) e^{-h} < 0
\end{align*}
\]

for \((x_i, h) \in (0, \xi) \times (0, \eta), i = 1, 2\). Thus, by the mutual tangency equality condition for Pareto efficiency, we have

\[
0 = \frac{\partial U_1}{\partial h} \left/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial h} \right/ \frac{\partial U_2}{\partial x_2} = -x_1 - x_2 + b_1(h) + b_2(h) = b_1(h) + b_2(h) - \xi,
\]

(14.4.40)

which is independent of \( x_i \). Hence, if \((x_1, x_2, h)\) is Pareto optimal, so is \((x'_1, x'_2, h)\) provided \( x_1 + x_2 = x'_1 + x'_2 = \xi \). Also, note that \( b'_i(h) < 0 \) \((i = 1, 2)\), \( b_1(0) + b_2(0) \geq \xi \), and \( b_1(\eta) + b_2(\eta) \leq \xi \). Then \( b_1(h) + b_2(h) \) is strictly monotone and thus there is a unique \( h \in [0, \eta] \), satisfying (14.4.40). Thus, the contract curve is horizontal even though individuals’ preferences need not be parallel.
Example 14.4.2 Suppose that \( b_1(h) = (1+h)^\alpha \eta + \xi \) with \( \alpha < 0 \), and \( b_2(h) = -h^\eta \). Then, for all \( h \in (0, \eta] \), \( b_1(h) > \xi \), \( b_2(h) < 0 \), \( b'_i(h) < 0 \) (i = 1, 2), \( b_1(0) + b_2(0) > \xi \), and \( b_1(\eta) + b_2(\eta) < \xi \). Thus, \( \phi_i(h) = \int e^{-h} b_i(h) dh \) is concave, and \( U_i(x_i, h) = x_i e^{-h} + \int e^{-h} b_i(h) dh \) is quasi-concave, \( \partial U_i / \partial x_i > 0 \) and \( \partial U_1 / \partial h > 0 \), and \( \partial U_2 / \partial h < 0 \) for \( (x_i, h) \in (0, \xi) \times (0, \eta) \), i = 1, 2, but it is not quasi-linear in \( x_i \).

Chipman and Tian (2012) then investigate the necessity for the “Coase conjecture” that the level of pollution is independent of the assignments of property rights. This reduces to developing the necessary and sufficient conditions that guarantee that the contract curve is horizontal so that the set of Pareto optima for the utility functions is \( h \)-constant. This in turn reduces to finding the class of utility functions such that the mutual tangency (first-order) condition does not contain \( x_i \) and consequently it is a function, denoted by \( g(h) \), of \( h \) only:

\[
\frac{\partial U_1}{\partial h} \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial h} \frac{\partial U_2}{\partial x_2} = g(h) = 0. \tag{14.4.41}
\]

Let \( F_i(x_i, h) = \frac{\partial U_i}{\partial h} / \frac{\partial U_i}{\partial x_i} \) (i = 1, 2), which can be generally expressed as

\[
F_i(x_i, h) = x_i \psi_i(h) + f_i(x_i, h) + b_i(h),
\]

where \( f_i(x_i, h) \) are nonseparable and nonlinear in \( x_i \). \( \psi_i(h), b_i(h) \), and \( f_i(x_i, h) \) will be further specified below.

Let \( F(x, h) = F_1(x, h) + F_2(\xi - x, h) \). Then the mutual tangency equality condition can be rewritten as

\[
F(x, h) = 0. \tag{14.4.42}
\]

Thus, the contract curve, i.e., the locus of Pareto-optimal allocations, can be expressed by a function \( h = f(x) \) that is implicitly defined by (14.4.42).

Then, the Coase Neutrality Theorem, which is characterized by the condition that the set of Pareto optimal allocations (the contract curve) in the \((x, h)\) space for \( x_i > 0 \) is a horizontal line \( h = \text{constant} \), implies that

\[
h = f(x) = \bar{h}
\]

with \( \bar{h} \) constant, and thus we have

\[
\frac{dh}{dx} = -\frac{F_x}{F_h} = 0
\]

for all \( x \in [0, \xi] \) and \( F_h \neq 0 \), which means that the function \( F(x, h) \) is independent of \( x \). Then, for all \( x \in [0, \xi] \),

\[
F(x, h) = x \psi_1(h) + (\xi - x) \psi_2(h) + f_1(x, h) + f_2(\xi - x, h) + b_1(h) + b_2(h) \equiv g(h). \tag{14.4.43}
\]
Since the utility functions $U_1$ and $U_2$ are functionally independent, and $x$ disappears in (14.4.43), we must have $\psi_1(h) = \psi_2(h) \equiv \psi(h)$ and $f_1(x, h) = -f_2(\xi - x, h) = 0$ for all $x \in [0, \xi]$. Therefore,

$$F(x, h) = \xi \psi(h) + b_1(h) + b_2(h) \equiv g(h), \quad (14.4.44)$$

and

$$\frac{\partial U_i}{\partial h} \int \frac{\partial U_i}{\partial x_i} = F_i(x_i, h) = x_i \psi(h) + b_i(h), \quad (14.4.45)$$

which is a first-order linear partial differential equation. It can be verified that the principal integral $U_i(x_i, h)$ of (14.4.45) is given by

$$U_i(x_i, h) = x_i e \int \psi(h)dh + \phi_i(h), \quad i = 1, 2 \quad (14.4.46)$$

with

$$\phi_i(h) = \int e \int \psi(h)dh b_i(h)dh. \quad (14.4.47)$$

The general solution of (14.4.45) is then given by $\hat{U}_i(x, y) = \psi(U_i)$, where $\psi$ is an arbitrary function. Since a monotonic transformation preserves orderings of preferences, we can regard the principal solution $U_i(x_i, h)$ as a general functional form of utility functions that is fully characterized by (14.4.45).

Note that (14.4.46) is a general utility function that contains quasi-linear utility in $x_i$ and the utility function given in (14.4.38) as special cases. Indeed, it reduces to quasi-linear utility function when $\psi(h) \equiv 0$ and to the utility function given by (14.4.38) when $\psi(h) = -1$.

To make the mutual tangency (first-order) condition (14.4.41) be also sufficient for the contract curve to be horizontal in a pollution economy, we assume that for all $h \in (0, \eta], x_1 \psi(h) + b_1(h) > 0, x_2 \psi(h) + b_2(h) < 0, \psi'(h) \leq 0, b'_1(h) < 0$ $(i = 1, 2), \xi \psi(0) + b_1(0) + b_2(0) \geq 0$, and $\xi \psi(\eta) + b_1(\eta) + b_2(\eta) \leq 0$.

We then have for $(x_i, h) \in (0, \xi) \times (0, \eta), i = 1, 2$,

$$\frac{\partial U_i}{\partial x_i} = e \int \psi(h) > 0, \quad i = 1, 2,$$

$$\frac{\partial U_1}{\partial h} = e \int \psi(h)[x_1 \psi(h) + b_1(h)] > 0,$$

$$\frac{\partial U_2}{\partial h} = e \int \psi(h)[x_2 \psi(h) + b_2(h)] < 0,$$

and thus

$$0 = \frac{\partial U_1}{\partial h} \left/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial h} \right/ \frac{\partial U_2}{\partial x_2}$$

$$= (x_1 + x_2) \psi(h) + b_1(h) + b_2(h)$$

$$= \xi \psi(h) + b_1(h) + b_2(h), \quad (14.4.48)$$

which does not contain $x_i$. Hence, if $(x_1, x_2, h)$ is Pareto optimal, so is $(x'_1, x'_2, h)$ provided $x_1 + x_2 = x'_1 + x'_2 = \xi$. Also, note that $\psi'(h) \leq 0$, so $b_i(h)$ is non-increasing.
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\[ b_i'(h) < 0 \quad (i = 1, 2), \quad \xi \psi(0) + b_1(0) + b_2(0) \geq 0, \text{ and } \xi \psi(\eta) + b_1(\eta) + b_2(\eta) \leq 0. \]

Then \( \xi \psi(h) + b_1(h) + b_2(h) \) is strictly monotone and thus there is a unique \( h \in [0, \eta] \) that satisfies (14.4.48). Thus, the contract curve is horizontal even though individuals’ preferences need not be parallel.

The formal statement of Coase Neutrality Theorem obtained by Chipman and Tian (2012) thus can be set forth as follows:

**Proposition 14.4.1 (Coase Neutrality Theorem)** In a pollution economy considered in the chapter, suppose that the transaction cost equals zero, and that the utility functions \( U_i(x_i, h) \) are differentiable and such that \( \partial U_i / \partial x_i > 0 \), and \( \partial U_1 / \partial h > 0 \) but \( \partial U_2 / \partial h < 0 \) for \( (x_i, h) \in (0, \xi) \times (0, \eta), i = 1, 2 \). Then, the level of pollution is independent of the assignments of property rights if and only if the utility functions \( U_i(x, y) \), up to a monotonic transformation, have a functional form given by

\[
U_i(x_i, h) = x_i e^{\int \psi(h)} + e^{\int \psi(h) dh} b_i(h) dh, \tag{14.4.49}
\]

where \( h \) and \( b_i \) are arbitrary functions such that the \( U_i(x_i, h) \) are differentiable, \( \partial U_1 / \partial x_i > 0 \), and \( \partial U_1 / \partial h > 0 \) but \( \partial U_2 / \partial h < 0 \) for \( (x_i, h) \in (0, \xi) \times (0, \eta), i = 1, 2 \).

Although the above Coase neutrality theorem includes quasilinear utility function as a special case, due to the special form of such function, Coase Neutrality Theorem has no generality, which is not true for general utility function. Therefore, Hurwicz’s insight on the limitations of Coase’s theorem is still valid. Coase’s theorem is more applicable to production externalities rather than consumption externalities.

It is important to understand the limitations of the Coase Theorem. Otherwise, it is easy to oversimplify or over apply the Coase Theorem. Many may think that with clear property rights, free exchange and voluntary cooperation, the market can operate efficiently without considering the preconditions of Coase’s Theorem, especially the two basic prerequisites: (1) zero transaction cost; (2) no income effect. In reality, costs of negotiation and organization, in general, are not negligible, and the income effect may not be zero. Thus, a privatization is optimal only in case of zero transaction cost, no income effect, and perfectly competitive economic environments. But in reality, these conditions are often not satisfied. For example, the privatization of state-owned enterprises tends to be costly, and there is a lot of debate about how to privatize and who should get a share or benefit. If there is no corresponding system as the basis and the transaction cost is too large, then radical privatization may not be desirable.

Does the clearly defined private property rights necessarily lead to the optimal allocation of resources, and is it impossible for other ownership? Tian (2000, 2001) showed that private ownership (resp. state ownership...
and collective ownership) may be (resp. relatively or constrainedly) efficient, depending on the development level of underlying institutional environment. If the institutional environment is much underdeveloped, state and collective ownership could be relatively more efficient compared to private ownership. Only when the market system is mature, and has a good governance, the private property right system can be efficient, or the private property rights system is optimal. So, all three property rights systems may be (relatively) optimal, depending on the standardization of the institutional environment. Therefore, instead of discussing the privatization of state-owned enterprises, it is better to continuously improve the institutional environment and allow private enterprises to flourish. China’s economic reform and opening-up over the past 40 years has fully demonstrated this point.

The problem of the Coase Efficiency Theorem is more serious. First, as Arrow (1979, p. 24) pointed out, the basic postulate underlying Coase’s theory appears to be that the process of negotiation over property rights can be modelled as a cooperative game, and this requires the assumption that each player know the preferences or production functions of each of the other players. When information is not complete or asymmetric, in general, it results in Pareto inefficient outcomes. For instance, when there is one polluter and there are many pollutees, a “free-rider” problem arises and there is an incentive for pollutees to misrepresent their preferences. Whether the polluter is liable or not, the pollutees may be expected to overstate the amount they require to compensate for the externality. Thus, we may need to design an incentive compatible mechanism to solve the free-rider problem.

Secondly, even if the information is complete, there are several circumstances that have led a number of authors to question the conclusion in the Coase Efficiency Theorem:

(1) The economic core may be empty, and hence no Pareto optimum exists. An example of this for a three-agent economy was presented by Aivazian and Callen (1981).

(2) There may be a fundamental non-convexity that prevents a Pareto optimum from being supported by a competitive equilibrium. Starrett (1972) showed that externalities are characterized by “fundamental non-convexities” that may preclude existence of competitive equilibrium.

(3) When an agent possesses the right to pollute, there is a built-in incentive for extortion. As Andel (1966) pointed out, anyone with the right to pollute has an incentive to extract payment from potential pullutees, e.g., threat to blow a bugle in the middle of the night.
(4) In his 1979 article, Arrow argued that Coase’s theorem relies on a bargaining process and finally forms a cooperative game, which depends on the assumption of complete information. Obviously in reality the information is not complete and may lead to free-rider problems.

Thus, the hypothesis that negotiations over externalities will mimic trades in a competitive equilibrium is, as Coase himself has conceded, not one that can be logically derived from his assumptions, but must be regarded as an empirical conjecture that may or may not be confirmed by the data. A lot of theoretical work therefore still remains in order to provide Coasian economics with the rigorous underpinning.

### 14.4.3 Missing Market

We can regard externality as a lack of a market for an “externality”. For the above example in Pigovian taxes, a missing market is a market for pollution. Adding a market for firm 2 to express its demand for pollution - or for a reduction of pollution - will provide a mechanism for efficient allocations. By adding this market, firm 1 can decide how much pollution it wants to sell, and firm 2 can decide how much pollution it wants to buy.

Let \( r \) be the price of pollution.

\( x_1 = \) the units of pollution that firm 1 wants to sell;

\( x_2 = \) the units of pollution for firm 2 wants to buy.

 Normalize the output of firm 1 to \( x_1 \).

The profit maximization problems become:

\[
\pi_1 = p_x x_1 + r x_1 - c_1(x_1),
\]

\[
\pi_2 = p_y y - r x_2 - e_2(x_2) - c_y(y).
\]

The first order conditions are:

\[
p_x + r = c_1'(x_1) \quad \text{for Firm 1},
\]

\[
p_y = c_y'(y) \quad \text{for Firm 2},
\]

\[
-r = e'(x_2) \quad \text{for Firm 2}.
\]

At the market equilibrium, \( x_1^* = x_2^* = x^* \), we have

\[
p_x = c_1'(x^*) + e'(x^*) \quad (14.4.50)
\]

which results in a socially optimal outcome.

In this model, we assume that the price of pollution is given to all firms. When the number of firms is small, this assumption is not necessarily
true, and thus may still lead to inefficiency of pollution level. Therefore, in the real world, the method of auctioning pollution rights is used to greatly increase competition.

The basic idea of this model has been used to deal with a large number of externalities in reality, establishing a warrants market for the consumption and production of many externality-producing goods. In addition to establishing a market for pollution rights transactions, it is also widely applied in establishing licenses trading markets for many externality-producing goods such as radio frequency spectrum. As an application, we will discuss this issue by considering emissions trading next section.

14.4.4 The Compensation Mechanism

The Pigovian taxes were not adequate in general to solve externalities due to the information problem: the tax authority cannot know the cost imposed by the externality. How can one solve this incomplete information problem?

Varian (1994) proposed an incentive mechanism which encourages the firms to correctly reveal the costs they impose on the other. Here, we discuss this mechanism. In brief, a mechanism consists of a message space and an outcome function (rules of game). We will introduce in detail the mechanism design theory in Part VI. Varian’s incentive mechanism allows firms to form a Pareto efficient tax rate through the game. The regulatory department does not know the individual’s information, so it is necessary to induce individuals’ information about their economic characteristics to implement the efficient tax rates $t_1$ and $t_2$ through an incentive compatible mechanism, thus achieving efficient resource allocations.

Varian’s mechanism is designed in a way that firms proposes a tax rate for each other. If the tax rates set by the two parties are different, they then will be punished. The mechanism proposed by Varian is divided into two stages. In the first stage, firms independently propose tax rates for each other, which introduces the idea of competitive markets, and no firm can control the tax rate imposed on itself. If one can determine its tax rate, then rent-seeking appears. Under this kind of consideration, individuals need to weigh each other’s decisions and adjust their own decisions. In the second stage, the mechanism designer uses the idea of “inducing with profit” to distribute interests according to the information of both parties. Finally, the individual makes the decision of production and output according to the rules determined by the mechanism, and the equilibrium outcome is Pareto efficient.

Strategy Space (Message Space): $M = M_1 \times M_2$ with $M_i = \{(t_i, x_i)\}$, $i = 1, 2$, where $t_1$ is interpreted as a Pigovian tax proposed by firm 1 and $x_1$ is the proposed level of output by firm 1, and $t_2$ is interpreted as a Pigovian tax proposed by firm 2 and $y_2$ is the proposed level of output by firm 2.
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The mechanism has two stages:

**Stage 1** (Announcement stage): Firms 1 and 2 name Pigovian tax rates, \( t_i, i = 1, 2 \), which may or may not be the efficient level of such a tax rate.

**Stage 2** (Choice stage): If firm 1 produces \( x \) units of pollution, firm 1 must pay \( t_2x \) to firm 2. Thus, each firm takes the tax rate as given. Firm 2 receives \( t_1x \) units as compensation. Each firm pays a penalty, \( (t_1 - t_2)^2 \), if they announce different tax rates.

Thus, the payoffs of two firms are:

\[
\pi_1^* = \max_x p_x x - c_x(x) - t_2x - (t_1 - t_2)^2, \\
\pi_2^* = \max_y p_y y - c_y(y) + t_1x - e(x) - (t_1 - t_2)^2.
\]

Because this is a two-stage game, we may use the subgame perfect equilibrium, i.e., an equilibrium in which each firm takes into account the repercussions of its first-stage choice on the outcomes in the second stage. As usual, we solve this game by looking at stage 2 first.

At stage 2, firm 1 will choose \( x(t_2) \) to satisfy the first order condition:

\[
p_x - c'_x(x) - t_2 = 0. \tag{14.4.51}
\]

Note that, by the convexity of \( c_x \), i.e., \( c''_x(x) > 0 \), we have

\[
x'(t_2) = -\frac{1}{c''_x(x)} < 0. \tag{14.4.52}
\]

Firm 2 will choose \( y \) to satisfy \( p_y = c'_y(y) \).

Stage 1: Each firm will choose the tax rates \( t_1 \) and \( t_2 \) to maximize their payoffs.

For Firm 1,

\[
\max_{t_1} p_x x - c_x(x) - t_2x(t_2) - (t_1 - t_2)^2, \tag{14.4.53}
\]

which leads to the following first order condition:

\[
2(t_1 - t_2) = 0,
\]

and thus

\[
t_1^* = t_2. \tag{14.4.54}
\]

For Firm 2,

\[
\max_{t_2} p_y y - c_y(y) + t_1x(t_2) - e(x(t_2)) - (t_1 - t_2)^2 \tag{14.4.55}
\]

so that the first order condition is

\[
t_1x'(t_2) - e'(x(t_2))x'(t_2) + 2(t_1 - t_2) = 0,
\]
and then we have
\[ [t_1 - e'(x(t_2))]x'(t_2) + 2(t_1 - t_2) = 0. \tag{14.4.56} \]

By (14.4.52), (14.4.54) and (14.4.56), we get
\[ t^* = e'(x(t^*)) \quad \text{with} \quad t^* = t^*_1 = t^*_2. \tag{14.4.57} \]

Substituting the equilibrium tax rate, \( t^* = e'(x(t^*)) \), into (14.4.51), we obtain
\[ p_x = c'_x(x^*) + e'(x^*), \tag{14.4.58} \]
which is the condition for social efficiency of production.

Remark 14.4.1 This mechanism works by setting opposing incentives for two agents. Firm 1 always has an incentive to match the announcement of firm 2. But consider firm 2’s incentive. If firm 2 thinks that firm 1 will propose a large compensation rate \( t_1 \) for him, he wants firm 1 to be taxed as little as possible so that firm 1 will produce as much as possible. On the other hand, if firm 2 thinks firm 1 will propose a small \( t_1 \), it wants firm 1 to be taxed as much as possible. Thus, the only point where firm 2 is indifferent about the level of production of firm 1 is where firm 2 is exactly compensated for the cost of the externality.

In general, individuals’ personal goals are different from a social goal, which are in general incentive incompatible. However, we may be able to construct an appropriate incentive-compatible mechanism so that individuals’ personal interests-maximizing goals are consistent with a social goal such as efficient allocations. Tian (2003) also gave the solution to the consumption externalities by giving the incentive mechanism that results in Pareto efficient allocations. Tian (2004) studied the informational efficiency problem of the mechanisms that results in Pareto efficient allocations for consumption externalities.

14.5 Emissions Trading and Efficient Allocation of Pollution Rights

This section deals with emissions trading. Emissions trading is also called cap and trade (CAT) that is a market-based approach to controlling pollution by providing economic incentives for achieving reductions in the emissions of pollutants. In contrast to command-and-control environmental regulations such as best available technology (BAT) standards and government subsidies, emissions trading programs are a type of flexible environmental regulation that allows organizations to decide how best to meet policy targets.
We will discuss the governance of pollution and focuses on how to achieve efficient allocation of pollution rights through markets, so that pollution can be efficiently controlled. In the 1990s, some European and American countries established emission permit markets for pollutants and gained somewhat successful experience. Their application has also been gradually extended to other countries in the world. This section focuses on the intrinsic mechanism, efficiency, and possible limitations of the emissions trading market. The discussion in this section refers to the analysis of emissions trading markets by Leach (2004) and Newell and Stavins (2003).

An important issue in the way of pollution charges and emission cap is information. Since the government does not have the information on technology of firms, discharge of pollutants usually do not reach the level of Pigovian tax, nor can it achieve the most efficient level of pollution abatement. In addition, in the environmental pollution, the negotiations between firms and residents are faced with excessive transaction costs, such as free-rider, and thus cannot achieve efficient pollution control. Cap-and-trade can usually reduce information requirements and transaction costs through market mechanism.

Here is a simple example to discuss the efficiency of the emission rights market.

Consider an economy with two firms. There is no externality between two firms and they may discharge pollutants. Without control, firm $i$ will produce pollution of $\bar{e}_i$. Let $a_i$ denote the volume of emission abatement. Assume that the cost of emission abatement is $C_i = c_i \frac{a_i^2}{2}$. Two firms have different costs of emission reduction. Suppose that $c_1 < c_2$, the total social emission regulated by the government is $2\hat{e}$, and the emission cap for every firm is $\hat{e} < \bar{e}$.

### 14.5.1 Emission Reduction without Trading Market

If the emission rights of two firms cannot be exchanged, then emission abatement cost for firm $i$ is $C_i = c_i \frac{(\bar{e}_i - \hat{e})^2}{2}$, and the total cost of emission abatement is $(c_1 + c_2) \frac{(\bar{e} - \hat{e})^2}{2}$. The marginal costs of emission abatements for firm 1 and firm 2 are $c_1(\bar{e} - \hat{e})$ and $c_2(\bar{e} - \hat{e})$, respectively. Since two firms’ marginal costs of emission abatement are different, the total cost of social emission abatement is not minimized. If firm 2 transfers 1 unit of emission rights to firm 1, social emission abatement cost is reduced by $(c_2 - c_1)(\bar{e} - \hat{e})$.

The establishment of an emission rights market will reduce the total cost of emission abatement without affecting the total emissions.
14.5.2 Emissions Trading

Below we discuss market equilibrium under emissions trading. Assume that the emissions trading market is competitive, and the number of firm $i$ is continuum. The total number of firms in each category is standardized to 1.

For firm $i$, the optimization problem is as below:

$$\begin{align*}
\min_{a_1} & \quad \frac{a_1^2}{2} + p(\bar{e} - \hat{e} - a_1) \\
\text{s.t.} & \quad a_1 \leq \bar{e},
\end{align*}$$

(14.5.59)

(14.5.60)

where $a_1$ is the actual emission abatement by firm 1 and $p$ is the price of emission rights. Emission abatement needed for firm 1 is $\bar{e} - \hat{e}$. If the actual emission abatement is $a_1 < \bar{e} - \hat{e}$, firm 1 needs to purchase $\bar{e} - \hat{e} - a_1$ of emission rights. If its actual emission abatement is $a_1 > \bar{e} - \hat{e}$, firm 1 can supply $-(\bar{e} - \hat{e} - a_1)$ of emission rights. Therefore, the objective function for firm 1 is to minimize the pollution cost under emission cap $\bar{e}$.

Condition (14.5.60) represents that the highest possible abatement of emissions by firm 1 will not exceed the pollution it produces.

So, the optimal decision of firm 1 is

$$a_1 = \begin{cases} \frac{p}{c_1}, & \text{if } p < c_1 \bar{e}, \\ \bar{e}, & \text{otherwise}. \end{cases}$$

(14.5.61)

The demand of emission rights for firm 1 is denoted as $d_1(p) = \bar{e} - \hat{e} - a_1$.

Here we see the supply as a negative demand. By (14.5.61), we obtain

$$d_1(p) = \begin{cases} \bar{e} - \hat{e} - \frac{p}{c_1}, & \text{if } p < c_1 \bar{e}, \\ -\hat{e}, & \text{otherwise}. \end{cases}$$

(14.5.62)

Similarly, we can get the demand of emission rights for firm 2, $d_2(p)$. When $d_1(p) + d_2(p) = 0$, emissions rights market reaches equilibrium and there are two cases of equilibrium.

(1) Equilibrium 1: firm 1 reserves part of emission rights. That is, $a_1 < \bar{e}$. Market clears and satisfies

$$2(\bar{e} - \hat{e}) - \left[ \frac{p}{c_1} + \frac{p}{c_2} \right] = 0,$$

the equilibrium price is

$$p = 2(\bar{e} - \hat{e}) \left[ \frac{1}{c_1} + \frac{1}{c_2} \right]^{-1},$$

(14.5.63)

and satisfies

$$p < c_1 \bar{e}.$$
14.5. EMISSIONS TRADING AND EFFICIENT ALLOCATION OF POLLUTION RIGHTS

Substituting the equation (14.5.63) into the above equation gives

\[ \hat{e} > \hat{e} \frac{c_2 - c_1}{2c_2}. \]

In other words, if firm 1’s cap is large enough, then the firm will retain part of the emission rights. In this equilibrium, emissions trading is

\[-d_1 = d_2 = (\hat{e} - \hat{e}) \frac{c_2 - c_1}{c_1 + c_2}. \tag{14.5.64}\]

From the formula (14.5.64), we can see that the greater the difference in the emissions technologies of the two firms, the greater the scale of the emissions trading. Through emissions trading, the social emission abatement costs can be minimized. In equilibrium, the emission abatement cost of the society is

\[ C_1 + C_2 = 2(\bar{e} - \hat{e}) \left[ \frac{1}{c_1} + \frac{1}{c_2} \right]^{-1} < (\bar{e} - \hat{e})^2 \frac{c_1 + c_2}{2}. \]

(2) Equilibrium 2: firm 1 sells all emission rights.
Market clears and satisfies

\[ \bar{e} - 2\hat{e} - \frac{p}{c_2} = 0, \]

the equilibrium price is

\[ p = c_2(\bar{e} - 2\hat{e}), \]

and satisfies

\[ p > c_1 \bar{e}, \]

getting

\[ \hat{e} < \bar{e} \frac{c_2 - c_1}{2c_2}. \]

The volume of emissions trading is

\[-d_1 = d_2 = \hat{e}. \]

The total social emission abatement cost is

\[ \frac{c_1}{2} (\bar{e})^2 + \frac{c_2}{2} (\bar{e} - 2\hat{e})^2 < (\bar{e} - \hat{e})^2 \frac{c_1 + c_2}{2}. \]

In the emissions trading market, when firms have different technologies in emission abatement, the transaction will allow them to remain the same in the marginal cost of emission abatement, resulting in the efficient allocation of pollution rights. In addition, in the case of emission rights market, firms will also have greater incentives in the innovation of emission abatement technologies. This is because, market transactions can make
the benefits of emission abatement innovations greater, rather than merely lower their own emission abatement costs. Many studies have found that through market transactions, emission abatement costs of the society can be greatly saved. For example, the study on sulfur dioxide emissions trading among electric utilities by Carlson et al. (2000) shows that emissions trading can lower abatement cost curves for the American power industry by over 50% since 1985, as opposed to the command-and-control approach (requiring a uniform emission rate standard), and the trading can reduce annual abatement costs by 700 million ~ 800 million dollars.

However, the operation of the emission rights market will also have its costs. For example, how to allocate emission rights among firms and how to allocate for new participants. Of course, a way to deal with the distribution on emission rights is by auction. However, setting up an auction market among different pollutants is also a big challenge, because in some industries, there may be intrinsic links among different pollutants. Joskow et al. (1998) studied the transaction costs in the operation of emission rights market, and discussed the impact of auctions on the price of emission rights in the sulfur dioxide emission rights market in the context of the United States.

In addition, the allocation of emission rights may lead to rent-seeking and social risks. Different firms may have different borrowing capabilities and may suffer from efficiency losses in auctions and market trading. Besides, for many developing countries, the supervision of emission has always been a ticklish problem. If the pollution discharge cannot be efficiently monitored, then the market for emission rights will inevitably lack clear property rights. Stavins (1995) discussed the impact of transaction costs of emission rights on pollution control efficiency. In addition, Tietenberg (1995) discussed the spatial allocation of emission rights. For more information on the emissions trading market, readers can refer to Gayer and Horowitz (2005). They discussed in detail some of the important theoretical and practical issues on the emissions trading market.

### 14.6 Biographies

#### 14.6.1 Arthur Pigou

Arthur Cecil Pigou (1877 - 1959) was an English economist, the “father of welfare economics”, and one of the leading representatives of the Cambridge School. Pigou was born in a military family in England and was admitted to the University of Cambridge where he first studied history. Later, he came to economics under the influence of Marshall. In 1908 Pigou became a Professor of Political Economy at the University of Cambridge in succession to Alfred Marshall, and held the post until 1943. Pigou inherited
Marshall’s academic tradition and analytical framework to a large extent. In addition, he also served as a Fellow of the Royal Society, Honorary President of the International Economic Association, member of the Cunliffe Committee on the Currency and Foreign Exchange and the Royal Commission on Income Tax. He proposed the concept of “economic welfare” in his representative works The Economics of Welfare, Industrial Fluctuations, and A Study in Public Finance. He advocated the equalization of national income and established the cardinal utility theory. Pigou was the first to systematically study externality from the perspective of welfare economics. Based on the concept of “external economy” put forward by Marshall, he expanded the concept and content of “external diseconomy”. The study turned from the effect of external factors on the business to the impact of the business or residents on other businesses or residents.

The The Economics of Welfare published in 1920 was Pigou’s most famous representative work. This book systemized welfare economics and marked the establishment of Pigou’s complete theoretical system. Its interpretation of welfare economics has always been regarded as “classic”, and Pigou was therefore also known as the “father of welfare economics”. Pigou believed that the purpose of this book was to study the important factors that affect economic welfare in real life. The whole book was centered on how to increase social welfare. Pigou proposed the Pigovian Tax, which advocates subsidies for activities that have positive externalities. Pigou was known for his “Pigovian tax”.

Pigou was an important model of neoclassical thought. In fact, Keynes criticized Pigou as a representative of a full employment analysis perspective in the neoclassical school. Pigou also replied by saying that Keynes’s The General Theory of Employment, Interest, and Money was a mixture of wrong ideas. To respond to Keynes, he tried to restore the position of neoclassical employment theory through a logically complete demonstration under the classical assumption about wages and price elasticity.

### 14.6.2 Ronald Coase

Ronald Harry Coase (1910 - 2013) was the originator of the new institutional economics, the founder of the property rights theory, and one of the representatives of the Chicago School of Economics. He was awarded the 1991 Nobel Memorial Prize in Economic Sciences for his discovery and analysis of the role of transaction costs and property rights in the institutional structure and operation.

Coase was born on 29 December 1910 in a small town named Willesden outside London. In his childhood, Coase had to wear leg-irons to help support his legs. Due to physical limitations, young Coase had to attend the school for physical defective. Through his own unremitting efforts, Coase successfully entered the London School of Economics and obtained a Bach-
elor of Commerce degree at the age of 22. After 6 years of teaching at this school, Coase received his doctoral degree from the University of London in 1951. He then came to the United States and taught at the University at Buffalo and the University of Virginia. After that, he became a professor at the University of Chicago.

Coase had written only a few papers in his life. The most famous of these were “The Nature of the Firm” published in 1937 and “The Problem of Social Cost” published in 1960. They were probably the most widely cited among all economics literature. Although he seldom used mathematics, his articles were logically clear. He introduced and adopted the concept of transaction costs and clarified property rights to study the boundaries and externalities of the firm. He introduced institution and firm into the mainstream economics that previously focused on interpreting how the market price system works, and demonstrated the relationship of firm, property right, contract, and market, as well as the important role of these factors in economic development. His economic ideas were extremely profound, and had far-reaching influence on the development of modern economics. Many economic disciplines, such as the economics of property rights, information economics, mechanism design theory, contract theory, and transition economics, have all been influenced by Coase’s ideas which promoted the development of these fields.

In his paper titled “The Nature of the Firm”, he explained how the firm was formed from a distinctive perspective. This paper was later widely considered as having epoch-making significance on economics. From the perspective of “transaction costs”, Coase gave his reasons for how firms emerged. Coase believed that there were costs in the market transaction behavior. These costs include bargaining, costs of contracts formation and implementation, and time costs. Coase also believed that when market transaction costs are higher than coordination costs within the firm, then firm emerges. The existence of the firm is to save market transaction costs by replacing higher-cost market transactions with lower-cost intra-firm transactions. This distinctive research perspective has still been marveled by the economics community.

Coase’s research seldom involves mathematics. In the 1960 famous paper “The Problem of Social Cost”, he used a written discourse to deal with the economic problem of externalities, and to demonstrate the definition of property rights and the importance of property rights arrangement in economic transactions. George Joseph Stigler (1911 - 1991), winner of the 1982 Nobel Prize in Economics, further classified Coase’s theory as “under perfect competition, private and social costs will be equal”, and eventually formed the “Coase Theorem”. The importance of Coase Theorem lies in that apart from the price, the theorem reveals the influence of property rights arrangement and transaction costs on institutional arrangements. Coase Theorem is divided into two parts. As long as the transaction cost
is zero and the property rights are clearly defined, (1) the level of the externality will be the same regardless of the assignment of property rights, known as the Coase Neutrality Theorem; (2) with voluntary exchanges and voluntary negotiations, clearly defined property rights will lead to efficient allocation of resources. That is, with market mechanisms, through voluntary trading and negotiations, the contractual arrangements that implement the best interests of all individuals can be found. This conclusion is called the Coase Efficiency Theorem. Coase further argued that even if there are transaction costs, the parties involved in the interaction will find a less costly institutional arrangement through the contract when the property rights are clearly defined.

Coase’s economic theory and his insights have spread widely in China, which has always been in the process of economic reforms, making him one of the most cited contemporary economists among Chinese economists. Coase also enjoyed a long life and died at the age of 102.

14.7 Exercises

Exercise 14.1 There is an orchard next to an apiary. The orchard produces fruit, and the apiary supplies honey. The flowers of the fruit tree provide honeybees with nectar, and the bees also promote pollen transmission. Suppose that the price of fruit is $2 per unit, and the price of honey is $8 per unit. Let $H$ be the output of honey and $A$ be the yield of fruit. The orchard’s cost function is $C_A(A, H) = A^2 / 2 - 6H$ and the apiary’s cost function is $C_H(A, H) = H^2 / 2 - 3A$.

1. If the orchard and the apiary make independently decisions, what is the output of fruit and honey, respectively?

2. If the orchard and the apiary merge, what is the output of fruit and honey, respectively?

3. Do the markets for fruit and honey result in Pareto efficient allocations? Why?

Exercise 14.2 Consider an economy with two goods and two consumers. The utility functions of the two consumers are

$u_1(x) = 0.5 \ln(x_1^1 + x_1^2) + 0.5 \ln x_1^2,$

$u_2(x) = 0.5 \ln x_2^1 + 0.5 \ln x_2^2.$

Their consumption spaces $Z_i = \mathbb{R}^2_+$, $i = 1, 2$, and the initial endowments are $w_1 = (1, 2)$ and $w_2 = (2, 1)$.

1. Solve for Pareto efficient allocations and competitive equilibria.
2. Is a competitive equilibrium allocation Pareto efficient? Why?

**Exercise 14.3** Consider a pure exchange economy with two goods and two consumers. The first good is “music” and the second good is “bread”. Consumption space is \( X_i = \mathbb{R}^2_+, i = 1, 2 \). The aggregate initial endowment is \((w_m, w_b)\). The utility functions of the two consumers are

\[
\begin{align*}
    u_1(m_1, b_1) &= m_1^{3/5} b_1^{2/5} - k_1, \\
    u_2(m_2, b_2) &= m_2^{3/5} b_2^{2/5} - k_2,
\end{align*}
\]

in which \(m_1\) and \( m_2\) are music consumptions, and \( b_1\) and \( b_2\) are bread consumptions of consumer 1 and 2, respectively, and \( k_1\) and \( k_2\) are constant parameters.

1. What is the set of Pareto optimal allocations?

2. Suppose that the initial endowments of consumers 1 and 2 are \( w_1 = (3/2, 1/2) \) and \( w_2 = (1/2, 3/2) \). Solve for the competitive equilibrium.

3. Verify whether the equilibrium allocation in the question 2 is Pareto optimal.

4. Now suppose that the consumers’ utility functions change to

\[
\begin{align*}
    \hat{u}_1(m_1, m_2, b_1) &= m_1^{3/5} b_1^{2/5} - m_2, \\
    \hat{u}_2(m_1, m_2, b_2) &= m_2^{3/5} b_2^{2/5} - m_1.
\end{align*}
\]

One explanation to the above utility functions is that while one person’s consumption of music increases his own utility, it also interferes with the quiet environment of the other, thereby reducing the utility of the other.

(a) What is the critical value \( \bar{w}_m \) of the aggregate initial endowment such that exhaustion of resources beyond the critical value will result in Pareto inefficient allocation?

(b) What is the set of interior-point Pareto efficient allocations? (Discuss two situations: \( w_m^* \leq \bar{w}_m \) and \( w_m^* > \bar{w}_m \)). Compare the result with that in question 1 above.

(c) Suppose that \( w_1 = (3/2, 1/2) \) and \( w_2 = (1/2, 3/2) \). Solve for the competitive equilibrium. Is the competitive allocation Pareto optimal?
**Exercise 14.4** Consider the pure exchange economy of two commodities and two consumers. Consumption space is $X_i = \mathbb{R}^2_+$, $i = 1, 2$, and the aggregate endowment is given by $(w_x, w_y)$, where $x$ and $y$ represent two commodities. The utility functions of the two consumers are

$$u_1(x_1, y_1) = x_1^{0.3} y_1^{0.7} - x_2,$$

$$u_2(x_2, y_2) = x_2^{0.3} y_2^{0.7} - x_1.$$

1. Solve for $w_x$ such that if $w_x > \bar{w}_x$, any attainable (balanced) allocation is not Pareto efficient.

2. Solve for interior-point Pareto efficient allocations.

3. Suppose that $w_1 = (2, 1), w_2 = (1, 2)$, solve for competitive equilibrium. Is the competitive allocation Pareto efficient? Why?

**Exercise 14.5** Consider the economy of two commodities and two consumers. Consumption space is $X_i = \mathbb{R}^2_+$, $i = 1, 2$. Commodity $m$ represents all commodities that can be purchased with money. Commodity $n$ characterizes all commodities that cannot be purchased with money, such as freedom, and family life. In other words, commodity $m$ represents the composite of all goods that can be included in GDP, while commodity $n$ can not. The utility function of consumer $i$ is:

$$u_i(m_i, n_i, m_j) = m_i^\alpha n_i^{1-\alpha} - \beta m_j, \quad 0 < \alpha < 1, \beta > 0.$$

1. Prove that if the total endowment of $m$ exceeds a certain amount, there must be some free disposal in $m$ (i.e., $m$ can not be exhausted completely) in reaching Pareto optimal allocations.

2. Solve for Pareto efficient allocations.

3. Do you think the conclusion of this question can explain the following paradox: The individual’s happiness index may not increase with the country’s wealth?

**Exercise 14.6** Consider the economy of $n$ consumers. Each consumer $i$ chooses an action $h_i \in \mathbb{R}_+$ and his utility function is $\phi_i(h_i; \sum_i h_i) + w_i$. Suppose that $\phi_i(\cdot)$ is strictly concave. $w_i$ is his initial endowment.

1. Characterize Pareto optimal actions $h_1, \cdots, h_n$.

2. Characterize Nash equilibrium on actions.

3. Compare the Pareto efficient outcome with the Nash equilibrium outcome. What kind of tax rate can lead to Pareto optimal outcomes?
Exercise 14.7 There is a common grazing land in a mountain village where the villager can herd sheep. The cost of raising each sheep is 4. The total revenue of raising sheep on this grazing land is \( f(x) = 20x - 3x^2/2 \), where \( x \) denotes the number of sheep.

1. Prove that free grazing does not maximize the total welfare of the grazing land.

2. The government now decides that a license is necessary for raising sheep. How should the government decide on the price of the license so as to maximize its revenue?

Exercise 14.8 Two persons have to decide separately how fast they should drive the car. The individual \( i \) chooses driving speed \( x_i \) to obtain the utility of \( u_i(x_i) \) and \( u'_i(x_i) > 0 \). However, the faster the car, the more likely there will be a car accident. Let \( P(x_1, x_2) \) be the probability of an accident and it is an increasing function of \( x_1 \) and \( x_2 \). Let \( c_i > 0 \) be the cost to individual \( i \) in the event of the accident. Each person’s utility is linear with regard to currency.

1. Prove that the individual’s choice of driving speed is faster than the requirement for social welfare maximization.

2. If the penalty for the individual \( i \) was \( t_i \) in the event of accident, solve for the \( t_i \) that can internalize the externality.

3. Now suppose that the utility of individual \( i \) changes to 0 in the event of an accident. Find the penalty for the internalization of the externality.

Exercise 14.9 A manufacturer’s cost function \( c(q, h) \) is differentiable and strictly convex, \( q \geq 0 \) is its output level, and \( h \) is the negative externality level of production. This externality affects the consumer, whose indirect utility function is \( \phi(h) + w \), where \( \phi(h) \) is differentiable and satisfies \( \phi'(h) < 0 \).

1. Derive the first-order conditions that the manufacturer chooses \( q \) and \( h \).

2. Derive the Pareto-optimal first-order conditions of \( q \) and \( h \).

3. Suppose that the government imposes a tax on the producer’s output. Prove that it cannot achieve Pareto optimality.

4. Suppose that the government directly levies taxes on the externality and prove that this approach can achieve Pareto optimality.
5. Suppose that \( h = \gamma q \) is constant for \( \gamma > 0 \). Prove that the taxation on production can achieve Pareto optimality.

**Exercise 14.10 (Tragedy of the Commons)** Fishermen can fish freely in the lake. The cost of a fishing boat is \( c > 0 \). When there are \( b \) fishing boats in the lake, a total of \( f(b) \) fish are captured. The fishing amount for each fishing boat is \( f(b)/b \). For \( b \geq 0 \), we have \( f'(b) > 0 \) and \( f''(b) < 0 \). The price of fish is \( p > 0 \) per unit.

1. Solve for the equilibrium quantity of boats.
2. Solve for the Pareto optimal quantity of fishing boats and prove that it is less than the equilibrium quantity.
3. What kind of fishing tax should be imposed on fishing boats to achieve the Pareto optimal quantity?
4. Suppose that the lake belongs to someone. How does he choose the number of fishing boats?

**Exercise 14.11** Consider an economy with two consumers, \( A \) and \( B \), and two commodities, 1 and 2. \( y_j^i \) represents individual \( i \)'s consumption of the commodity \( j \), and \( I_i \) represents the individual’s income level. The prices of the two commodities are \( p_1 \) and \( p_2 \), respectively. The utility function for consumer \( A \) is \( u_A(y_1^A, y_2^A, y_2^B) \), and the utility function for consumer \( B \) is \( u_B(y_1^B, y_2^B) \), which means the consumption of good 2 by consumer \( B \) has an externality to consumer \( A \).

1. Write down the utility maximization problems of consumers and get the conditions that should be satisfied under an equilibrium allocation.
2. If \( \partial u_A / \partial y_B < 0 \), is the equilibrium allocation Pareto efficient?
3. Now suppose that specific duty, denoted \( t_B^2 \), is levied on the consumption of commodity 2 by consumer \( B \), what will happen to the utility maximization problem of consumer \( B \)?
4. Can the method of taxing restore the Pareto efficient allocation? Why?

**Exercise 14.12** A local government proposes to implement a sewage tax system with a minimum discharge standard. Each firm is allowed to emit a certain amount \( \bar{h} \) of pollutants without being taxed, while those beyond \( \bar{h} \) will be taxed.

1. Write down the objective function of the firm.
2. Explain why the system is not efficient in general and cannot encourage minimal-cost abatements, and under what circumstances the system is efficient.

**Exercise 14.13** Consider the constant marginal abatement cost function of two firms:

\[-C'_1(h_1) = a, \; h_1 < \hat{h}_1;\]
\[-C'_2(h_2) = b, \; h_2 < \hat{h}_2.\]

1. When the damage function of pollution is convex and linear, find socially optimal emissions of the two firms.

2. Under these circumstances, is it possible to use economic incentive policy tools to achieve the socially optimal allocation?

**Exercise 14.14 (Macho-Stadler and Pérez-Castrillo, 2006)** Under the linear sewage tax rate and incomplete supervision, a firm decides to report its emissions of \(z\) while its actual emissions are \(h\). The linear tax rate is \(\tau\), and the firm pays a sewage fee of \(z\tau\). The firm’s return is a function of the actual amount of pollutants discharged, which is denoted by \(g(h)\). When firms are not subject to any regulation, their emissions are \(\overline{h}\), and \(g'(\overline{h}) = 0\); when \(h \in [0, \overline{h})\), \(g(h)\) is an increasing and concave function: \(g'(h) > 0\), \(g''(h) < 0\).

The probability that a firm’s real emissions are detected by environmental protection agencies is \(\rho \in [0, 1]\). A penalty to a firm that is found to make a false report is \(\theta(h - z)\), and the penalty is a monotonically increasing and convex function of the difference between actual emissions and reported emissions: for \(x > 0\), \(\theta'(x) > 0\) and \(\theta''(x) > 0\). Since the penalty of unit false report should be higher than the tax rate, we assume \(\theta'(0) > \tau\).

1. How does the firm’s optimal reported emissions and actual emissions change with the regulatory intensity \(\rho\)?

2. Does the increase of regulatory intensity for the firm that reports greater than zero emissions reduce its actual emissions?

**Exercise 14.15** There is a chemical plant in the upper reaches of a river, and its production will cause pollution to two downstream fishermen. The chemical plant can spend \(\$5,000\) to purchase equipment to avoid pollution. The pollution will result in losses of \(\$2,500\) and \(\$4,000\) for the two fishermen, respectively. The fishermen can purchase the decontamination unit alone or jointly for \(\$6,000\) to eliminate the pollution.

1. Suppose that the property rights are not clearly defined and pollution has already occurred, and the two fishermen can negotiate. What is the result?
2. Suppose that the property right is owned by the chemical plant. Chemical plant and the two fishermen can negotiate. What is the result?

3. Now suppose that property right belong to the two fishermen. Chemical plant and the two fishermen can also negotiate. What is the result?

4. Which result of the above three questions is Pareto efficient?

5. If the “tax-subsidy” mechanism is introduced, how should the regulator achieve Pareto efficient outcome?

**Exercise 14.16 (Coase Neutrality Theorem)** Coase Neutrality Theorem concludes that as long as the property rights are clearly defined, the equilibrium level of the externality will be the same regardless of the assignment of property rights. Consider the pure exchange economy of two types of commodities and two consumers. One commodity is “money” and both of them are fond of it. The other commodity is “music”. Music consumption will increases one’s own utility while reduce the utility of the other.

1. What special assumptions about consumer preferences will lead to the Coase neutrality theorem? Demonstrate your claim on two situations concerning the definition of property rights: (a) the musician has the right to play music without his neighbors’ approval; (b) the musician must be authorized by his neighbor to have music. (In order to reach an agreement, one person can compensate another person.) Use a diagram to illustrate your answer.

2. Suppose that both have the Cobb-Douglas utility function. Does the argument of Coase Neutrality Theorem still hold? Illustrate your answer with a diagram.

**Exercise 14.17** The Coase efficiency theorem states as follows: If the property rights are clearly defined and the transaction costs are zero, the negotiation of externalities will lead to Pareto optimal outcomes.

1. Prove that the quasilinear utility function is a sufficient condition for the Coase Theorem to hold.

2. Is the quasilinear utility function a necessary condition for the Coase Theorem to hold? If yes, give a proof; if not, give a counterexample.

**Exercise 14.18 (Kolstad, 2000)** Suppose that there are two polluting firms with hidden characteristics $\theta$. For both firms, $\theta$ does not have to be equal. Suppose that $\theta$ can take a value of 1 or 2. The revenue of firm $i$ is $S_i(h_i, \theta_i) = 1 - \frac{(1-\theta_i h_i)^2}{2\theta_i}$. The damage caused by pollution is $D(h_1 + h_2) = (h_1 + h_2)^2/2$. 
1. Suppose that the regulator knows each firm’s \( \theta \): \( \theta_1 \) and \( \theta_2 \). For all possible combinations of \( \theta_1 \) and \( \theta_2 \), what is the socially optimal pollution for each firm: \( h_1^*(\theta_1, \theta_2) \), \( h_2^*(\theta_1, \theta_2) \)?

2. Now suppose the regulator does not know \( \theta \), but asks each firm report \( \theta \). After receiving reports from each firm, each firm \( i \) will be charged a fee of \( T_i(h_i, \theta_i) \). This fee is based on the reported \( \theta_i \) by firm \( i \), the reported \( \theta_j \) by firm \( j \), and real emissions \( h_i \):

\[
T_i(h_i, \theta_i) = D[h_i + h_j^*(\theta_1, \theta_2)] - S_j[h_j^*(\theta_1, \theta_2), \theta_j].
\]

Before firms report their \( \theta \) values, all of above specifications are common knowledge. Prove that it is in the best interest of each firm to report the true \( \theta \) and take the socially optimal pollution level \( h^* \).

### 14.8 References

**Books and Monographs:**


**Papers:**

14.8. REFERENCES


Chapter 15

Public Goods

15.1 Introduction

In the previous chapter, we discussed the economic environments with externalities. It is very important to consider this issue when analysing the efficiency of a market (or even doing anything). The possibility of externality determines whether or not market failure may occur, and whether or not intervening and supervisory measures are needed. In the presence of externalities, the market may fail to achieve efficient allocation even under perfect competition and freedom of choice. Therefore, some remedies need to be adopted. These measures include: Pigouvian tax, Coasian’s approach, building a market for emissions trading, and designing a proper incentive mechanism, etc.

The presence of public goods is another main case that results in market failure. Once public goods are present in an economy, externalities and thus market failure may occur. It is well-known that financing a public project via voluntary donation is difficult. This is because public good is essentially different from externality. Two main differences are the non-exclusivity and non-rivalry. A good is *excludable* if some individuals can be excluded from consuming it. A good is *non-rival* if one person’s consumption does not reduce the amount available to other consumers.

A **pure public good** is a good in which consuming one unit of the good by an individual in no way prevents others from consuming the same unit of the good. Thus, the good is nonexcludable and non-rival. Examples of public goods include street lights, policemen, fire protection, highway system, national defence, flood-control project, public television and radio broadcast, public parks, and a public project, etc. The purest (most representable) public good is national defense. It protects all citizens from aggression.

*Local Public Goods*: when there is a location restriction for the service of a public good.

The non-exclusivity of public good may result in free-rider problem.
For example, individuals want to get benefits from but do not want to contribute to a public project. The inefficiency of state-owned enterprises also originated from the free-rider problem. These enterprises lack proper incentive mechanisms, therefore, everyone wants to enjoy the efforts provided by others. Even if the competitive market is an efficient social institution for allocating private goods in an efficient manner, it turns out that a private market is not a good mechanism for allocating public goods.

There are three theoretical solutions to this problem: (1) forming social norms and cultures of donation habits, but it is difficult to achieve in the short term, and the effect is limited; (2) remoulding one’s ideology by taking altruism and work as happiness and joy, but the reality is cruel and it is ineffective unless the genes that pursue personal interests are altered; (3) in situations where the social norms and cultures such as donation habits are difficult to form at one time and individuals’ ideological consciousness cannot be greatly improved, incentive mechanism design should be adopted by taking the fact that individuals’ ideological realm is limited as a constraint condition on a case-by-case basis. A comprehensive governance approach that combines incentive mechanism with social norms and regulations can better solve the free-rider problem.

15.2 Notations and Basic Settings

A general setting of public goods economy includes consumers, producers, private goods, public goods, and economic characteristics of consumers and produces.

Let

- \( n \): the number of consumers.
- \( L \): the number of private goods.
- \( K \): the number of public goods.
- \( Z_i \subseteq \mathbb{R}_+^{L} \times \mathbb{R}_+^{K} \): the consumption space of consumer \( i \).
- \( Z \subseteq \mathbb{R}_+^{nL} \times \mathbb{R}_+^{K} \): consumption space.
- \( x_i \in \mathbb{R}_+^L \): a consumption of private goods by consumer \( i \).
- \( y \in \mathbb{R}_+^K \): a consumption/production of public goods.
- \( w_i \in \mathbb{R}_+^L \): the initial endowment of private goods for consumer \( i \). For simplicity, it is assumed that there is no public goods endowment, but they can be produced from private goods.
- \( Y \subseteq \mathbb{R}_+^{L+K} \): the set of production possibilities of the firm. For simplicity, we assume there is only one firm to produce the public goods.
15.2. NOTATIONS AND BASIC SETTINGS

- \((y, -v) \in Y\): a production plan, where \(v \in \mathbb{R}^L_+\) is the vector of private goods input.
- \(f: \mathbb{R}^L_+ \to \mathbb{R}^K_+\): production function with \(y = f(v)\).
- \(\theta_i\): the profit share of consumer \(i\) from the production.
- \((x_i, y) \in Z_i\): a consumption of private goods and public goods by consumer \(i\).
- \((x, y) = (x_1, ..., x_n, y) \in Z\): an allocation.
- \(\succ_i\) (or \(u_i\) if exists): a preference ordering (it is complete and transitive).
- \(e_i = (Z_i, \succ_i, w_i, \theta_i)\): the characteristic of consumer \(i\).
- \(e = (e_1, ..., e_n, f)\): a public goods economy.

The above is a relatively simple depicted class of public goods economic environments. Analogous to the discussion in general equilibrium problem, the economic environments can be more general to allow for production possibility sets of general form, an arbitrary number of firms, either public or private goods as input or output, see Foley (1970) and Milleron (1972) as detailed discussion.

**Definition 15.2.1** Allocation \(z \equiv (x, y) = (x_1, ..., x_n, y) \in Z\) is feasible, if 
\[\left(\sum_{i=1}^{n} x_i - \hat{w}, y\right) \in Y,\]
where \(\hat{w} = \sum_{i=1}^{n} w_i\). If technology can be represented by function \(y = f(v)\), feasibility condition can be written as:
\[\sum_{i=1}^{n} x_i + v \leq \sum_{i=1}^{n} w_i,\]  
(15.2.1)
and
\[y = f(v).\]  
(15.2.2)

**Definition 15.2.2** An allocation \((x, y)\) is Pareto efficient for a public goods economy \(e\) if it is feasible and there is no other feasible allocation \((x', y')\) such that \((x_i', y_i') \succ_i (x_i, y)\) for all consumers \(i\) and \((x'_k, y') \succ_k (x_k, y)\) for some \(k\).

**Definition 15.2.3** An allocation \((x, y)\) is weakly Pareto efficient for the public goods economy \(e\) if it is feasible and there is no other feasible allocation \((x', y')\) such that \((x_i', y_i') \succeq_i (x_i, y)\) for all consumers \(i\).

**Remark 15.2.1** Unlike private goods economies, even though under the assumptions of continuity and strong monotonicity, a weakly Pareto efficient allocation may not be Pareto efficient for the public goods economies. The following proposition is due to Tian (1988).
Proposition 15.2.1 For the public goods economies, a weakly Pareto efficient allocation may not be Pareto efficient even if preferences satisfy strong monotonicity and continuity.

**Proof.** The proof is by way of a counter-example. Consider an economy with $(n, L, K) = (3, 1, 1)$, constant returns in producing $y$ from $x$ (the input-output coefficient normalized to one), and the following endowments and utility functions: $u_1 = u_2 = u_3 = 1$, $u_1(x_1, y) = x_1 + y$, and $u_i(x_i, y) = x_i + 2y$ for $i = 2, 3$. Then $z = (x, y)$ with $x = (0.5, 0, 0)$ and $y = 2.5$ is weakly Pareto efficient but not Pareto efficient because $z' = (x', y') = (0, 0, 0, 3)$ Pareto-dominates $z$ by consumers 2 and 3. □

However, under an additional condition of strict convexity, they are equivalent. The proof is left to readers.

15.3 Discrete Public Goods

15.3.1 Efficient Provision of Public Goods

For simplicity, consider a public good economy with $n$ consumers and two goods: one private good and one public good.

Discrete public good, also called public project, is indivisible. It is assumed that the units of public goods provided is normalized to 1. This can also be interpreted as a logical variable of 0 or 1, 1 for providing public project, and 0 for not providing public project.

Let $g_i$ be the contribution made by consumer $i$, so that

\[
x_i + g_i = w_i;
\]
\[
\sum_{i=1}^{n} g_i = v.
\]

Let $c$ be the cost of providing the public project so that the production technology is given by

\[
y = \begin{cases} 
1 & \text{if } \sum_{i=1}^{n} g_i \geq c \\
0 & \text{otherwise}
\end{cases}.
\]

Assume that $u_i(x_i, y)$ is strictly monotonically increasing and continuous. We first want to know under what conditions providing the public good will Pareto dominate the case of not providing it, i.e., there exists $(g_1, \ldots, g_n)$ such that $\sum_{i=1}^{n} g_i \geq c$ and

\[
u_i(w_i - g_i, 1) > u_i(w_i, 0), \quad \forall i.
\]

Let $r_i$ be the **maximum willingness-to-pay** (reservation price) of consumer $i$, i.e., $r_i$ must satisfy

\[
u_i(w_i - r_i, 1) = u_i(w_i, 0).
\]
15.3. DISCRETE PUBLIC GOODS

Inequality (15.3.3) implies that providing the public project will bring higher utilities for all consumers than not providing the public project, where \( u_i \) is the utility function of consumer \( i \), and it is continuous and strictly monotonic increasing. Under these two conditions, public goods should be provided from the perspective of social optimality. Therefore, as long as we know the utility function of each individual, we know their willingness-to-pay, thus knowing whether providing the public good is Pareto dominated.

If providing the public project Pareto dominates not providing the public project, we have

\[
    u_i(w_i - g_i, 1) > u_i(w_i, 0) = u_i(w_i - r_i, 1), \quad \forall i.
\]

(15.3.5)

By strong monotonicity of \( u_i \), we have

\[
    w_i - g_i > w_i - r_i \quad \forall i.
\]

(15.3.6)

Then, we have

\[
    r_i > g_i,
\]

(15.3.7)

and thus

\[
    \sum_{i=1}^{n} r_i > \sum_{i=1}^{n} g_i \geq c.
\]

(15.3.8)

That is, the sum of the willingness-to-pay for the public good must exceed the cost of providing it. This condition is necessary. In fact, this condition is also sufficient. In summary, we have the following proposition.

**Proposition 15.3.1** Providing a public good Pareto dominates not providing the public good if and only if \( \sum_{i=1}^{n} r_i > \sum_{i=1}^{n} g_i \geq c \).

The problem is that information related to individual preferences and utility functions is unknown to a social planner and thus it may result in inefficient outcomes as we discussed below. As such, it becomes an important issue how to design an incentive mechanism to induce individuals to truthfully report their private information.

15.3.2 Free-Rider Problem

First of all, we want to know whether a free competitive market results in the efficient provision of public goods. The following example shows that, due to the problem of free-rider, we generally cannot expect independent individual decision-making to lead to the efficient provision of public projects.

To see this, consider a simple economy with only two participants, each of whom pays \( r_i = 100, i = 1, 2 \). Suppose that the cost for providing the public project is \( c = 150 \). The participants decide on their own how much
they will pay for the public project. If both are willing to pay 75, the public project will be provided, and each participant will receive 25 units of benefit. If only one person pays 150, the public project is provided, but the benefit is 50, while the benefit of another person is 100. Formally, we have

\[
\begin{align*}
    r_i &= 100 \quad i = 1, 2; \\
    c &= 150 \text{ (total cost)}; \\
    g_i &= \begin{cases} 
        150/2 = 75 & \text{if both agents make contributions;} \\
        150 & \text{if only agent } i \text{ makes contribution.}
    \end{cases}
\end{align*}
\]

Each person decides independently whether or not to contribution for providing the public good. As a result, each one has an incentive to be a free-rider on the other as shown by the payoff matrix in Table 15.1.

<table>
<thead>
<tr>
<th>Person 2</th>
<th>Contribute</th>
<th>Not Contribute</th>
</tr>
</thead>
<tbody>
<tr>
<td>Person 1</td>
<td>(25,25)</td>
<td>(-50,100)</td>
</tr>
<tr>
<td>Not Contribute</td>
<td>(100,-50)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

Table 15.1: Private Provision of a Discrete Public Good

Note that net payoffs are defined by \( r_i - g_i \). Thus, it is given by 100 - 150/2 = 25 when both consumers are willing to produce the public project, and 100-150 = -50 when only one person wants to contribute, but the other person does not.

The dominant strategy equilibrium in this game is (no contribution, no contribution). Thus, although the public project benefits everyone, no body wants to share the cost of producing the public project, but wants to free-ride on the other consumer. As a result, the public good is not provided at all even though it would be more efficient to do so. Thus, voluntary contribution in general does not result in the efficient level of the public good.

The above-mentioned problem is typically a prisoner’s dilemma. This phenomenon is common in reality, causing both worse off. It illustrates the basic conclusion that inefficient allocation often results from conscious and self-dedication only.

15.3.3 Voting for a Discrete Public Good

The amount of a public good is also often determined by a voting. Will this generally result in an efficient provision? The answer in general is negative.

Voting does not result in efficient provision. Consider the following example.
Example 15.3.1

\[ c = 120 \]
\[ r_1 = 80, \quad r_2 = 35, \quad r_3 = 35. \]

Clearly, \( r_1 + r_2 + r_3 > c \). \( g_i = 120/3 = 40 \). So the efficient provision of the public project should be yes. However, under the majority rule, only consumer 1 votes "yes" since she receives a positive net benefit if the good is provided. The 2nd and 3rd persons vote "no" to provide public project, and therefore, the public project will not be provided so that we have inefficient provision of the public project. The problem with the majority rule is that it only measures the net benefit for the public good, whereas the efficient condition requires a comparison of willingness-to-pay.

This example also shows that democracy and efficiency are often incompatible in specific decision-making, because the voters are usually driven by their own interests, that is so-called buttock often determines the head. Through democratic voting, it often leads to inefficient provision of public goods. To overcome the possible inconsistency between democracy and efficiency, a criterion whether democratic decision-making should be adopted is that the higher the level, the more respect for public opinion in the election of leaders, and the more democratic decision-making should be adopted in the selection of leaders. Because leaders’ deciding direction and major decisions has tremendous externalities, we need to elect/select a person who respects public opinion, cares about the total welfare of a society, and be responsible to the voters. Otherwise, they will not be elected/selected in the next term for public office.

However, once elected/selected, since he is accountable to the voters, the implementation of his goals and specific decisions should be efficient. Otherwise, his decisions are always rejected by his staff or team members. How can he be responsible for the voters? So constantly applying the simple majority rule in every specific issue may often lead to inefficient outcomes. Therefore, even in a democratic system, the major leader of an organization (say, presidents of a country or of a university) usually has the power to nominate his deputy and the whole team of leadership.

Of course, a leader himself will be responsible to all the voters and pursue a chance of being reelected via good performance. An example is the professors’ committee in universities. Its duty is to evaluate the academic performance and promotion of faculties rather than getting involved in the detailed or all daily executive works. If every professor has voting right to support his own field of speciality, then the inefficient outcome as described in the above example may arise.

The above analysis shows that neither market nor democratic voting procedure could lead to efficient provision of public goods. The solution to
CHAPTER 15. PUBLIC GOODS

this problem is much hard, which depends on the design of proper mechanisms. We will discuss the VCG (Vickrey-Clarke-Groves) mechanism in Chapter 18, which may elicit efficient provision of public goods and truth-telling of voters.

15.4 Continuous Public Goods

15.4.1 Efficient Provision of Public Goods

Similar results to continuous public goods can also be obtained. Again, for simplicity, we assume there is only one public good and one private good that may be regarded as money, and \( y = f(v) \), where \( y \) is the production of public goods, and \( v \) is the input of private good used in producing public goods.

The welfare maximization approach shows that Pareto efficient allocations can be characterized by

\[
\max_{(x,y)} \sum_{i=1}^{n} a_i u_i(x_i, y) \\
\text{s.t.} \sum_{i=1}^{n} x_i + v \leq \sum_{i=1}^{n} w_i, \\
y \leq f(v).
\]

Define the Lagrangian function:

\[
L = \sum_{i=1}^{n} a_i u_i(x_i, y) + \lambda \left( \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} x_i - v \right) + \mu (f(v) - y). \tag{15.4.9}
\]

When \( u_i \) is strictly quasi-concave and differentiable and \( f(v) \) is concave and differentiable, the set of Pareto optimal allocations is characterized by the first-order conditions:

\[
\begin{align*}
    a_i \frac{\partial u_i}{\partial x_i} - \lambda &\leq 0, \quad \text{with equality if } x_i > 0; \tag{15.4.10} \\
    \mu f'(v) - \lambda &\leq 0, \quad \text{with equality if } v > 0; \tag{15.4.11} \\
    \sum_{i=1}^{n} a_i \frac{\partial u_i}{\partial y} - \mu &\leq 0, \quad \text{with equality if } y > 0. \tag{15.4.12}
\end{align*}
\]

So at an interior solution, by (15.4.10) and (15.4.11)

\[
\frac{a_i}{\mu} = \frac{f'(v)}{\frac{\partial u_i}{\partial x_i}}. \tag{15.4.13}
\]
Substituting (15.4.13) into (15.4.12), we have
\[
\sum_{i=1}^{n} \frac{\partial u_i}{\partial y} \frac{\partial u_i}{\partial x_i} = \frac{1}{f'(v)}.
\tag{15.4.14}
\]
Thus, we obtain the well-known Lindahl-Samuelson condition. This condition is different from the Pareto optimality for economies with private goods only. This condition indicates that the sum of the marginal rate of substitution of public goods for private goods of all economic agents is equal to the marginal rate of technical substitution, while for the private goods economy, the marginal rate of substitution of any two goods for every agent \( i \) is equal to the marginal rate of technical substitution at the Pareto optimality.

Thus, the conditions for Pareto efficiency are given by
\[
\begin{align*}
\sum_{i=1}^{n} MRS_{x_i y_i} &= MRTS_{y v}, \\
\sum x_i + v &\leq \sum_{i=1}^{n} w_i, \\
y &= f(v).
\end{align*}
\tag{15.4.15}
\]

The result shows that the provision level of public goods and the consumption of private goods are jointly determined.

**Example 15.4.1** Consider an economy with one public good, one private good, and \( n \) consumers. The utility function of consumer \( i \) is:
\[
\begin{align*}
u_i &= a_i \ln y + \ln x_i, \\
y &= v.
\end{align*}
\]
The Lindahl-Samuelson condition is
\[
\sum_{i=1}^{n} \frac{\partial u_i}{\partial y} = 1, \quad (15.4.16)
\]
and thus
\[
\sum_{i=1}^{n} \frac{a_i}{x_i} = \sum_{i=1}^{n} \frac{a_i x_i}{y} = 1 \Rightarrow \sum a_i x_i = y, \quad (15.4.17)
\]
which implies the level of the public good depends on the private good consumptions of all agents and is not uniquely determined.

Thus, in general, the marginal willingness-to-pay for a public good depends on the amount of private goods consumption, and therefore, the efficient level of \( y \) depends on \( x_i \). However, in the case of quasi-linear utility functions,
\[
u_i(x_i, y) = x_i + u_i(y), \quad (15.4.18)\]
the Lindahl-Samuelson condition becomes
\[
\sum_{i=1}^{n} u'_i(y) = \frac{1}{f'(v)} \equiv c'(y),
\] (15.4.19)
and thus \( y \) is uniquely determined.

**Example 15.4.2** Suppose that
\[
u_i = a_i \ln y + x_i, \]
y = v.

The Lindahl-Samuelson condition is
\[
\sum_{i=1}^{n} \frac{\partial u_i}{\partial y} \frac{\partial u_i}{\partial x_i} = 1,
\] (15.4.20)
and thus
\[
\sum_{i=1}^{n} \frac{a_i}{y} = 1 \Rightarrow \sum a_i = y,
\] (15.4.21)
which implies the level of the public good is uniquely determined.

### 15.4.2 Lindahl Mechanism and Equilibrium

We have given the conditions for Pareto efficiency in the presence of public goods. The next problem is how to achieve a Pareto efficient allocation under decentralized decision-making of individuals. In an economy with only private goods, as long as the local non-satiation assumption is satisfied, any competitive equilibrium allocation is Pareto efficient.

However, with public goods, a competitive mechanism does not help. Indeed, if the public goods are allocated through the competitive market, we can immediately know that the equilibrium outcome is the same as the one for private goods economies, that is, the marginal rate of substitution of two goods for all individuals is equal to the price ratio of the corresponding goods and then equal to the marginal rate of technical substitution, which does not satisfy the Lindahl-Samuelson condition. As such, competitive mechanism leads to inefficient allocations in the presence of public goods.

For instance, if we solve for competitive equilibrium in an economy with one private good, one public good and two consumers, then utility maximizing behavior would equalize the MRS of \( y \) for \( x \) and its relative price, e.g.,
\[
MRS_{yx}^A = MRS_{yx}^B = \frac{p_y}{p_x},
\]
which immediately violates the Lindahl-Samuelson condition. Therefore, in public goods economy, market failure occurs.
Then, what kind of economic mechanism should be adopted to achieve Pareto efficient allocations in public goods economies? We know that in the private goods economy, the Walras mechanism can lead to the efficient resource allocation. In the case of public goods, one possible institutional arrangement is the Lindahl mechanism.

At the earlier of the 20th century, Lindahl proposed an institutional arrangement based on the Lindahl-Samuelson condition. Lindahl suggested to use a tax method to provide public goods by which the tax rates may be different for different individuals. Each individual should pay a specific “personalized price” for public goods, which means, for the same amount of the consumption of public goods, different prices are assigned to different individuals. Thus, the Lindahl solution is a way to mimic the competitive solution in the presence of public goods with a difference that the consumption level of a public good is the same to all consumers, but the price of the public goods is personalized due to different preferences of consumers.

To see this, consider an economy with \( x_i \in \mathbb{R}^L \) (private goods) and \( y \in \mathbb{R}^K \) (public goods). For simplicity, we assume the production possibility set of public goods \( Y \) is a closed convex cone. Thus, production technology \( y = f(v) \) exhibits constant returns to scale (CRS). A feasible allocation satisfies

\[
\sum_{i=1}^{n} x_i + v \leq \sum_{i=1}^{n} w_i. \tag{15.4.22}
\]

Let \( q_i \in \mathbb{R}^K_+ \) be the personalized price vector of consumer \( i \) for consuming the public goods.

Let \( \hat{q} = \sum_{i=1}^{n} q_i \) be the market price vector of \( y \).

Let \( p \in \mathbb{R}^L_+ \) be the price vector of private goods.

The profit is defined as \( \pi = qy - pv \).

**Definition 15.4.1 (Lindahl Equilibrium)** We say that an allocation \((x^*, y^*) \in Z\), a price vector of private goods \( p^* \in \mathbb{R}^L_+ \), and a personalized price vector of public goods \( q^*_i \in \mathbb{R}^K_+ \), one for each individual \( i = 1, \ldots, n \), constitute a Lindahl equilibrium if the following conditions are satisfied:

(i) \( p^* x_i^* + q^*_i y^* \leq p^* w_i \) for all \( i \);

(ii) If \((x_i, y) \succ_i (x_i^*, y^*)\), then \( p^* x_i + q^*_i y > p^* w_i \) for all \( i \);

(iii) for all \((y, -v) \in Y\), there is \( \hat{q}^* y^* - p^* v^* \geq \hat{q}^* y - p^* v \);

(iv) \((y^*, -v^*) \in Y\),

where \( v^* = \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} x_i^* \), \( \sum_{i=1}^{n} q_i^* = \hat{q}^* \).
The first condition above is budget constraint, the second is utility maximization condition, the third is profit maximization condition, and the fourth is feasibility condition.

Remark 15.4.1 Because the production function exhibits constant returns to scale, the maximum profit is zero at the Lindahl equilibrium. That is, \( q^* y^* - p^* v^* = 0 \), therefore

\[
\sum_{i=1}^{n} p^* x_i^* = \sum_{i=1}^{n} p^* w_i + q^* y^* .
\]

Thus the budget constraint (i) holds with equality at Lindahl equilibrium for every consumer.

We may regard a Walrasian equilibrium as a special case of a Lindahl equilibrium when there are no public goods. In fact, the concept of Lindahl equilibrium in economies with public goods is, in many ways, a natural generalization of the Walrasian equilibrium notion in private goods economies, with attention to the well-known duality that reverses the role of prices and quantities between private and public goods, and between Walrasian and Lindahl allocations. In the Walrasian mechanism, prices for all commodities are the same for all consumers, but the quantities of their private goods consumption are personalized; in the Lindahl mechanism, the quantities of public goods consumed are the same for all consumers, while prices charged for public goods are personalized. In addition, the concepts of Walrasian and Lindahl equilibria are both relevant to private-ownership economies. Furthermore, they are both characterized by purely price-taking behavior on the part of consumers. The Lindahl solution for efficient provision of public goods is essentially an informationally decentralized decision-making process.

Similar to Walrasian equilibrium, Lindahl equilibrium has various good properties. In fact, by redefining the consumption space, an economy with public goods can be regarded as an economy with private goods only. So, the Lindahl equilibrium can then be regarded as Walrasian equilibrium under this redefinition of the consumption space. This method is adopted in the following to prove the existence of Lindahl equilibrium, where the first and second fundamental theorems of welfare economics still hold. The proof is similar to that of Walrasian equilibrium.

**Theorem 15.4.1 (Existence Theorem on Lindahl Equilibrium)** For public goods economy \( e = (\{X_i, w_i, \succ_i\}, \{Y_j\}, \{\theta_i\}) \), there exists a Lindahl equilibrium if the following conditions are satisfied:

(i) \( Z_i = \mathcal{R}_+^{L+K} \);

(ii) \( w_i > 0 \);
(iii) \( \succeq_i \) is continuous, strictly convex, and monotonic;

(iv) \( Y \) is a closed and convex cone, \( 0 \in Y \), \((-\mathbb{R}_+^L, 0) \subseteq Y \) (free disposal property).

**Proof.** We prove this theorem by constructing an economy with only private goods to which the existence theorem of Walrasian equilibrium (CE) can be applied. Specifically, treating the consumptions of different consumers of public goods as different commodities and changing the original commodity space of consumer \( i \) to \( Z_i = (Z_i, \{0\}) \subseteq \mathbb{R}^{L+K} \times \mathbb{R}^{(n-1)K} \), where 0 is null element of \((n - 1)K\) dimensional space. The consumption bundle of \( i \) is \( \bar{z}_i = (x_i, y_i, 0, \cdots, 0) \). The consumption space constructed above and the conditions of this theorem satisfy all requirements of Theorem 10.4.7 (Existence Theorem III for competitive equilibrium), and thus the existence of CE is guaranteed. Therefore, a Lindahl equilibrium exists for the original public goods economy.

Similarly, we can enhance the monotonicity assumption to strong monotonicity and relax the assumption of interior-point initial endowments of private goods.

For a public goods economy with one private good and one public good \( y = \frac{1}{\hat{q}}v \), the definition of Lindahl equilibrium becomes much simpler.

**Definition 15.4.2** An allocation \((x^*, y^*)\) is a Lindahl equilibrium allocation if there exist \( q_i^* \), \( i = 1, \cdots, n \) such that

(i) \( x_i^* + q_i^* y^* \leq w_i \);

(ii) if \((x_i, y) \succ_i (x_i^*, y^*)\), then \( x_i + q_i^* y > w_i \);

(iii) \( \sum_{i=1}^n q_i^* = \hat{q} \).

In fact, the feasibility condition is automatically satisfied when the budget constraints (i) is satisfied.

If \((x^*, y^*)\) is an interior point of the Lindahl equilibrium allocation, then we can have the first order condition of utility maximization:

\[
\frac{\partial u_i}{\partial y} \frac{\partial u_i}{\partial x_i} = \frac{q_i}{1},
\]

which means the Lindahl-Samuelson condition holds:

\[
\sum_{i=1}^n MRS_{yx_i} = \hat{q},
\]

which is the necessary condition for Pareto efficiency.
Example 15.4.3 Solve for the Lindahl equilibrium of the following public goods economy:

\[ u_i(x_i, y) = x_i^{\alpha_i} y^{(1-\alpha_i)}, \quad 0 < \alpha_i < 1, \]
\[ y = \frac{1}{q}. \]

The budget constraint is:

\[ x_i + q_i y = w_i. \]

The demand functions for private goods \( x_i \) and public goods \( y_i \) of each consumer \( i \) are given by

\[ x_i = \alpha_i w_i, \quad (15.4.24) \]
\[ y_i = \frac{(1 - \alpha_i) w_i}{q_i}. \quad (15.4.25) \]

Since there is \( y_1 = y_2 = \cdots = y_n = y^* \) at the equilibrium point, we have by (15.4.25)

\[ q_i y^* = (1 - \alpha_i) w_i. \quad (15.4.26) \]

Making summation leads to

\[ \hat{q} y^* = \sum_{i=1}^{n} (1 - \alpha_i) w_i. \]

Then, we have

\[ y^* = \frac{\sum_{i=1}^{n} (1 - \alpha_i) w_i}{\hat{q}}. \]

And thus, by (15.4.26), we have

\[ q_i = \frac{(1 - \alpha_i) w_i}{y^*} = \frac{\hat{q}(1 - \alpha_i) w_i}{\sum_{i=1}^{n} (1 - \alpha_i) w_i}. \quad (15.4.27) \]

If we want to find a Lindahl equilibrium, we must know the preferences or MRS of each consumer. But, the information about individuals’ preferences are in general unknown, because of the free-rider problem, each consumer will have the incentive to not truthfully reveal their preferences in order to contribute less. What is more, as each consumer has a personalized price system, when the preferences of consumers are not public information, it is difficult to regard the personalized price system of each consumer as given because his report will affect his price.

15.5 Welfare Properties of Lindahl Equilibrium

15.5.1 The First Fundamental Theorem of Welfare Economics

Similarly, we have the following First Fundamental Theorem of Welfare Economics for public goods economies.
Theorem 15.5.1 (The First Welfare Theorem for Public Goods Economy)

For a public goods economy \( e = (e_1, \ldots, e_n, Y) \), every Lindahl allocation \((x^*, y^*)\) with the price system \((q^*_1, \cdots, q^*_n, p^*)\) is weakly Pareto efficient. Furthermore, if local non-satiation is satisfied, it is Pareto efficient.

**Proof.** The proof of the first conclusion of the theorem is included in the proof of the second one. We only need to prove that under local non-satiation, every Lindahl allocation \((x^*, y^*)\) is Pareto efficient. Suppose not. Then there exists another feasible allocation \((x_i, y_i)\) such that \((x_i, y_i) < (x^*_i, y^*_i)\) for all \(i\) and \((x_k, y) >_k (x^*_k, y^*_k)\) for some \(k\).

We first show

\[
p^* x_i + q^*_i y \geq p^* w_i, \text{ for all } i = 1, 2, \cdots, n,
\]

If not, then there is some \(i\) such that

\[
p^* x_i + q^*_i y < p^* w_i.
\]

Then, by local non-satiation, there is \((x'_i, y')\) such that \((x'_i, y') > (x_i, y) \geq (x^*_i, y^*_i)\) and \(p^* x'_i + q^*_i y' < p^* w_i\), contradicting the fact that \((x^*_i, y^*_i)\) is consumer \(i\)'s utility maximizing consumption bundle.

For consumer \(k\), since \((x^*_k, y^*_k)\) is consumer \(k\)'s optimal choice, by \((x_k, y) >_k (x^*_k, y^*_k)\), we have

\[
p^* x_k + q^*_k y > p^* w_k \text{ for some } k.
\]

Thus, by summation, we have

\[
\sum_{i=1}^{n} p^* x_i + \sum_{i=1}^{n} q^*_i y > \sum_{i=1}^{n} p^* w_i, \quad (15.5.28)
\]

that is,

\[
\hat{q}^* y + p^* \sum_{i=1}^{n} (x_i - w_i) = \hat{q}^* y + p^* v > 0.
\]

However, according to profit maximization condition, for all \((y, v) \in Y\), we have

\[
\hat{q}^* y + p^* v \leq 0.
\]

This contradiction proves the theorem. \(\square\)

Similarly, we can define Lindahl equilibrium with transfers.

Definition 15.5.1 (Lindahl Equilibrium with Transfers) For a public goods economy \( e = (e_1, \ldots, e_n, Y) \), an allocation \((x^*, y^*) \in Z\), a price vector of private goods \(p^* \in \mathbb{R}^L_+\) and a personalized price vector of public good \(q^*_i \in \mathbb{R}^K_+, \forall i\), constitute a Lindahl equilibrium with transfers if there exists an assignment of wealth levels \((I_1, \ldots, I_n)\) with \(\sum_i I_i = p \cdot \sum_i w_i\), such that
(i) \( p^* x^*_i + q^*_i y^* \leq I_i \);
(ii) if \((x_i, y) \succ_i (x^*_i, y^*)\), then \( p^* x_i + q^*_i y > I_i \);
(iii) for all \((y, -v) \in Y\), \( \hat{q}^* y^* - p^* v^* = \hat{q}^* y - p^* v \);
(iv) \((y^*, -v^*) \in Y\),

where \( v^* = \sum_{i=1}^n w_i - \sum_{i=1}^n x^*_i, \sum_{i=1}^n q^*_i = \hat{q}^* \).

**Theorem 15.5.2 (The First Welfare Theorem of LE with Transfers)** For a public goods economy \( e = (e_1, \ldots, e_n, Y) \), every Lindahl equilibrium allocation \((x^*, y^*)\) with transfers and price system \((q^*_1, \ldots, q^*_n, p^*)\) is weakly Pareto efficient. Moreover, if consumers’ preferences are locally non-satiated, then it is Pareto efficient.

**Proof.** The proof is analogous to the proof of Theorem 15.5.1 and is thus omitted.

### 15.5.2 Economic Core and Lindahl Equilibrium

Similar to the private goods economy, we can define the economic core as follows:

**Definition 15.5.2 (Blocking Coalition)** A group of economic agents \( S \subseteq N \) is said to block (improve upon) a given allocation \((x, y)\), if the coalition can be Pareto improved through their own endowments, i.e., there exists another allocation \((x', y')\) such that

1. \( (x', y') \) is feasible with respect to \( S \), that is, \( (y, \sum_{i \in S} (x'_i - w_i)) \in Y \),
2. \( (x'_i, y') \succ_i (x_i, y) \) for all \( i \in S \), and \( (x'_k, y') \succ_k (x_k, y) \) for some \( k \in S \).

**Definition 15.5.3 (Economic Core)** A feasible allocation \((x, y)\) is said to have the core property if it cannot be blocked by any coalition. The set of all allocations in core is called economic core or simply core.

**Remark 15.5.1** Every allocation in core is Pareto efficient and individually rational, that is, \( (x_i, y) \succ_i (w_i, y), \forall i = 1, 2, \ldots, n \).

We can similarly show that every Lindahl equilibrium allocation has the core property under local non-satiation of preferences.

**Theorem 15.5.3** When local non-satiation condition holds, if \((x, y, p)\) is a Lindahl equilibrium, then \((x, y)\) is in the core.

Though every Lindahl equilibrium allocation is in the core under local non-satiation of preferences, the core convergence theorem does not hold necessarily. See Milleron (1972) for a counterexample.
15.5.3 The Second Fundamental Theorem of Welfare Economics

Similarly, we have the Second Fundamental Theorem of Welfare Economics for public goods economy.

**Theorem 15.5.4 (The Second Welfare Theorem for Public Goods Economy)**

For public goods economy \( e = (e_1, \ldots, e_n, \{Y_j\}) \), suppose that \( \succ_i \) is continuous, convex, and strongly monotonic, \( Y \) is a closed convex set, and \( 0 \in Y \). Then for any Pareto efficient allocation \( (x^*, y^*) \) with interior-point private consumption (that is \( x^* \in R_{++}^n \)), there exists a nonzero price vector \( (q_1, \ldots, q_n, p) \in R_{++}^{L+nK} \) such that \( (x, y), (q_1, \ldots, q_n), p \) is a Lindahl equilibrium with transfers, that is, there exists an assignment of wealth levels after transfers \( (I_1, \ldots, I_n) \) satisfying \( \sum_i I_i = p \cdot \sum_i w_i \) such that

1. if \( (x_i, y_i) \nsucceq_i (x_i^*, y_i^*) \), then \( px_i + q_i y_i > I_i \equiv px_i^*, i = 1, \ldots, n, \)
2. for all \( (y, -v) \in Y \), \( \hat{q}y^* - pv^* \geq \hat{q}y - pv \),

where \( v^* = \sum_{i=1}^n w_i - \sum_{i=1}^n x_i^* \) and \( \sum_{i=1}^n q_i = \hat{q} \).

**Proof.** Define two subsets \( W \) and \( P(x^*, y^*) \) of space \( R^{nK+L} \) as follows:

\[ W = \{(y, \cdots, y; -v) : (y, -v) \in Y\}. \]

Because \( Y \) is a closed convex set and \( 0 \in Y \), \( W \) is non-empty, closed and convex.

\[ P(x^*, y^*) = \{(y', \cdots, y'; -v') : v' = \sum_{i=1}^n (x'_i - w_i) \& (x'_i, y'_i) \nsucceq_i (x_i^*, y_i^*)\}. \]

Since \( \nsucceq_i \) is convex, \( P(x^*, y^*) \) is convex.

Since \( (x^*, y^*) \) is a Pareto optimal allocation, we must have \( W \cap P(x^*) = \emptyset \); otherwise, there may exist a Pareto improvement. Therefore, applying the hyperplane separating theorem introduced in Chapter 2, we obtain that there exists \( (q_1, \ldots, q_n, p) \neq 0 \) such that for all \( (y, \cdots, y; -v) \in W \) and for all \( (y', \cdots, y'; -v') \in P(x^*, y^*) \), we have

\[ \sum_{i=1}^n q_i y_i' - pv' \geq \sum_{i=1}^n q_i y_i - pv. \quad (15.5.29) \]

We proceed with the proof in five steps.

1. Profit-maximization, that is, for all \( (y, -v) \in Y \), we have \( \hat{q}y^* - pv^* \geq \hat{q}y - pv \).

Let \( \hat{z} = (y^*, \cdots, y^*; -v') \), where \( v' = \sum_{i=1}^n (w_i - x'_i) \) and \( x'_i \geq x_i^* \) (i.e., \( x'_i \geq x_i^* \) and \( x'_i \neq x_i^* \)). Then, by strong monotonicity of preferences, we have \( (x'_i, y_i^*) \nsucceq_i (x_i^*, y_i^*) \), and therefore \( (y^*, \cdots, y^*; -v') \in \)
\[ P(x^*, y^*) \]. Hence, it follows from (15.5.29) that, for any \((y, -v) \in Y\), we have
\[ \hat{q} y^* - pv' \geq \hat{q} y - pv. \]
Let \(v' \rightarrow v^*\), then we have \(\hat{q} y^* - pv^* \geq \hat{q} \cdot y - pv, \ \forall (y, -v) \in Y\).

2. \((q_1, \ldots, q_n, p) \geq 0, \text{ and } p \neq 0.\)
Firstly, we prove \(q_i \geq 0, \ i = 1, \ldots, n.\) Let
\[ \tilde{z} = (y, \ldots, y; -v) + e_{yi}, \]
where \((y, \ldots, y; -v)\) is an element in \(W\), \(e_{yi} = (0, \ldots, 1, 0, \ldots, 0)\) is a vector in \(\mathbb{R}^{nK + L}\) such that the element of public goods \(k\) associated with \(q^i_k\) is 1 and the other elements are all zero. Then, by the strong monotonicity of preferences and the fact that \(e^i_y\) is equally distributed among all economic agents, we have \(\tilde{z} \in P(x^*, y^*)\). Therefore, from (15.5.29), we have
\[ \hat{q} y - pv + q^i_k \geq \hat{q} y - pv. \quad (15.5.30) \]
Consequently,
\[ q^i_k \geq 0, \ k = 1, 2, \ldots, K; i = 1, 2, \ldots, n. \quad (15.5.31) \]

Now we prove that \(p \geq 0.\) Let \(e^l_x = (0, \ldots, 1, 0, \ldots, 0)\) be a vector in \(\mathbb{R}^{nK + L}\) such that the element associated with good \(l\) is 1 and other elements are zero. Repeating the above procedure, we have
\[ p^l_i \geq 0, \ l = 1, 2, \ldots, L. \quad (15.5.32) \]
Lastly, we prove \(p \neq 0\) by contradiction. If \(p = 0\), since \((q_1, \ldots, q_n, p) \neq 0\), then for some public good \(k\), we must have its price \(q^k = \sum_{i=1}^n q^i_k > 0.\) Since the production of public goods exhibits constant returns to scale, when \(p = 0\), the costs for all private goods as inputs are zero. Then the profit may be infinitely large, which contradicts with the fact that \(y\) is a profit-maximizing plan.

3. For all \(i, (x_i, y) \succ (x_i^*, y^*),\) then \(\sum_i px_i + \hat{q} y \geq \sum_i px_i^* + \hat{q} y^*.\)
For every \(i\) and \((x_i, y) \succ (x_i^*, y^*),\) by strong monotonicity of preferences, there exists \((x'_i, y')\) that is arbitrarily close to \((x_i, y),\) such that \((x'_i, y') \succ_i (x_i^*, y^*),\) and thus \((y', \ldots, y'; \sum_i (x'_i - w_i)) \in P(x^*, y^*).\) Also, note that \((y^*, \ldots, y^*; \sum_i (x_i^* - w_i)) \in W.\) Thus, by (15.5.29), we have
\[ \sum_i px'_i + \hat{q} y' \geq \sum_i px_i^* + \hat{q} y^*. \]
Let \(x'_i \rightarrow x_i.\) We have \(\sum_i px_i + \hat{q} y \geq \sum_i px_i^* + \hat{q} y^*.\)
15.6. **FREE-RIDER PROBLEM**

When the MRS is known, Pareto efficient allocation \((x, y)\) can be determined from the Lindahl-Samuelson condition or the Lindahl solution. After that, the contribution of each consumer is given by \(g_i = w_i - x_i\). However, since individuals’ preferences are private information, normally a social planner cannot know the information about MRS. Of course, it will be naive to think that each individual will truthfully reveal his preferences and determine the willingness-to-pay based on the revealed information. Since all economic agents are self-interested, generally they will not tell the true MRS so they may be able to make less contribution.

Indeed, if consumers realize that their shares of the contribution for producing public goods (or the personalized prices) depend on their report of MRS, they have “incentives to cheat.” That is, when the consumers are asked to report their preferences or MRSs, they will have incentives to report their economic characteristics so that they can pay less to consume the
public goods, resulting in insufficient provision of public goods and leading to Pareto inefficient allocations. This is the so-called problem of free riders. That is why it is difficult to raise enough fund for the provision of public goods through voluntary contribution.

To see this, notice that the social goal is to reach Pareto efficient allocations for the public goods economy, but from the perspective of personal interest, the utility maximization problem of each person is the following:

$$\max u_i(x_i, y)$$  \hspace{1cm} (15.6.34)

subject to

$$g_i \in [0, w_i];$$
$$x_i + g_i = w_i;$$
$$y = f\left(g_i + \sum_{j \neq i} g_j\right).$$

That is, each consumer $i$ maximizes his payoffs when others’ strategies $g_{-i}$ are given. From this problem, we can form a non-cooperative game:

$$\Gamma = (G_i, \phi_i)_{i=1}^n,$$

where $G_i = [0, w_i]$ is the strategy space of consumer $i$ and $\phi_i : G_1 \times G_2 \times \cdots \times G_n \to R$ is the payoff function of consumer $i$ which is defined by

$$\phi_i(g_i, g_{-i}) = u_i \left[(w_i - g_i), f(g_i + \sum_{j \neq i} g_j)\right].$$  \hspace{1cm} (15.6.35)

Remark 15.6.1 Note that the difference between Nash equilibrium (NE) and dominant strategy is that at NE, given optimal strategy of others, each consumer chooses his optimal strategy so that one’s own optimal strategy depends on others’ strategies; while dominant strategy means that the strategy chosen by each consumer is optimal regardless of others’ strategies so that each one’s optimal strategy is independent of any strategy of others. Thus, a dominant strategy equilibrium is clearly a Nash equilibrium, but the converse may not be true. Only for a very special payoff functions, there is a dominant strategy while a Nash equilibrium exists for a continuous and quasi-concave payoff functions that are defined on a compact set.
15.6. FREE-RIDER PROBLEM

For Nash equilibrium, if \( u_i \) and \( f \) are differentiable, then the first order condition is:

\[
\frac{\partial \phi_i(g^*)}{\partial g_i} \leq 0, \text{ with equality if } g_i > 0, \; \forall \; i = 1, \ldots, n. \tag{15.6.36}
\]

Thus, we have

\[
\frac{\partial \phi_i}{\partial g_i} = \frac{\partial u_i}{\partial x_i} (-1) + \frac{\partial u_i}{\partial y} f' \left( g_i^* + \sum_{j \neq i} g_j \right) \leq 0, \text{ with equality if } g_i > 0.
\]

So, at an interior-point solution \( g^* \), we have

\[
\frac{\partial u_i}{\partial y} \frac{\partial u_i}{\partial x_i} = \frac{1}{f'(g_i^* + \sum_{j \neq i} g_j)}.
\]

and thus

\[
MRS_{yx_i} = MRTS_{yv},
\]

which does not satisfy the Lindahl-Samuelson condition. Hence, the Nash equilibrium in general does not result in Pareto efficient allocations.

The above equation implies that a lower level of public good is provided rather than the Pareto efficient level of the public good when utility functions are quasi-concave because of diminishing MRS (see Figure 15.1). Therefore, Nash equilibrium allocations are in general not consistent with Pareto efficient allocations. How can one solve this free-rider problem? We will answer this question in Part VI on mechanism design theory. Vickrey-Clarke-Groves mechanism of demand revelation can solve the problem of efficient provision of public goods.
CHAPTER 15. PUBLIC GOODS

Figure 15.1: Free-rider results in a lower provision of public goods than the level of Pareto efficient provision of public goods.

15.7 Biographies

15.7.1 Ludwig Mises

Ludwig von Mises (1881-1973), the third generation head of the Austrian School and a member of the Mont Pelerin Society, enrolled at the University of Vienna in 1900, where he was greatly influenced by Carl Menger (1840 - 1921) and got his doctoral degree from the school of law in 1906. From 1909 to 1934, he was an economist for the Vienna Chamber of Commerce. After the World War I, he also served as a legal advisor to a government agency, responsible for drafting the terms of the final-war treaty - to resolve pre-war private debt problem between belligerents. On New Year’s Day in 1927, Austrian Institute for Business Cycle Research he founded was formally established and Hayek became the first director.

In 1934 - 1940, he moved to Geneva as a professor at the Graduate Institute of International Studies. In 1940 he moved to New York in USA. At pre, Keynesianism in the American academic world was prevalent, Mises’ liberalism was clearly out of the mainstream, and he was not employed by any academic organization. In 1945, through the recommendation of the Lawrence Fertig & William Volker Foundation, Mises entered New York University, but he could only serve as a visiting professor. In 1949, Mises published “Human Behavior.” Even so, he was only able to find a visiting professor position until retirement in 1969.

For a long time, even though Mises’ ideas had not been accepted by mainstream economists, his ideological influence and knowledge contribution to the 20th century human society cannot be ignored. To a certain extent, it could be said that the history of economic thought of human society in the 20th century can not tell a complete story if Mises is missing.
In 2000, America’s “Freedom” magazine referred to Mises as “the century figure of libertarianism.”

The reason why Mises occupied such an important position in the history of contemporary human society is mainly because Mises made numerous remarkable theoretical contributions in understanding the basic principles of human economic and social operations. In addition to his theoretical contributions in inflation, economic cycles, economic epistemology and methodology, and his own unique catalactics (i.e., a theory of the way the free market system reaches exchange ratios and prices) and praxeology, his main theoretical contributions lie in the early 1920s. He presented such a major theoretical insight with a discerning eye: In the absence of a market price mechanism, the impossibility of economic computations will result in the infeasibility of the centrally planned economy.

15.7.2 Douglass North

Douglass C. North (1920 - 2015), as the founder and pioneer of New Economic History (Cliometrics), New Institutional Economics, and New Political Economy, was one of the most important and influential economists in the late 20th century. In 1942 and 1952, respectively, he received a bachelor degree and a Ph.D. degree from the University of California, Berkeley. He began teaching at the University of Washington in 1951, taught at Rice University in 1979, at Cambridge University in 1981-1982, and returned to the University of Washington in 1982. North was awarded the Nobel Memorial Prize in Economic Sciences for renewed research in economic history by applying economic theory and quantitative methods in order to explain economic and institutional change in 1993.

The main contribution of North was innovation in research methodologies, that is, to study new objects with the methods of neoclassical economics. In other words, he used neoclassical economics and econometrics to study the issue of economic history. His early research on ocean shipping and the balance of payments of the U.S., in step with the school of new economic history represented by Robert W. Fogel (1926 - 2013), combined the neoclassical production theory with the data found in the research of economic history. This new method has revolutionized the study of economic history. North was not satisfied with this. He also used property rights theory to explain the impact of institutional change on economic performance in the American history. North’s early work, such as The Economic Growth of the United States: 1790-1860 and Growth and Welfare in the American Past: A New Economic History, have fully reflected this.

From the 1980s, North began to use the property rights theory of the new institutional economics to analyze the more general theory of industrialization in the western world in the last two centuries with the purpose to explore the causes of economic growth in the western world, the inter-
nal relation between economic growth and institutional changes, the trend of interaction between property rights system and economic development, and the inherent requirements of economic development for the institution. North’s works in this area include *The Rise of the Western World: A New Economic History* and *Institutional Change and the American Economic Growth*.

From the 1990s, North began to summarize his experience of 30 years’ research on economic history and extracted some theories that became important contribution to economics, especially the new institutional economics. His works in this area mainly include *Institutions, Institutional Change, and Economic Performance*. To sum up, North’s contribution to economics mainly includes three aspects: first, he used the method of institutional economics to explain economic growth in history; second, as one of the founders of the new institutional economics, North re-examined the role of institutions including the property rights system; third, as an economist, North applied institutions, a content that was not involved in neoclassical economics, as an endogenous variable in economic research. In particular, property rights system, ideology, state, and ethics are taken as variables in economic evolution and economic development. The theory of institutional change has thus been greatly developed.

Property rights theory, state theory and ideology theory are the three cornerstones of North’s institutional change theory. Through the examination and analysis of the evolution of market economy, North produced the thought of institutional change theory and built his analytical framework with the three cornerstone theories. As he stated, “My research focuses on institutional theory. The cornerstones of this theory are: the theory of property rights that describes incentives for individuals and groups in institution; the theory of the state that defines and enforces property rights; the theory of ideology that influences people’s different reaction on the change of objective existence, which explains why people have different understandings of reality.” It is worth mentioning that, in the process of clarifying the above analytical framework, North always used the cost-benefit analysis to demonstrate the rationality of the choice of property rights structure, the necessity of the existence of state, and the importance of ideology. Such analysis makes North’s institutional change theory enormously persuasive. In 1993, North was awarded the Nobel Memorial Prize in Economic Sciences for his establishment of the “Institutional Change Theory” including property rights theory, state theory, and ideology theory.

### 15.8 Exercises

**Exercise 15.1** Prove that for the public goods economy, weak Pareto efficient allocation is Pareto efficient when preferences satisfy strong monotonicity, continuity, and strict convexity.
Exercise 15.2 (Pareto efficiency of Lindahl equilibrium) Consider a public goods economy with \( n \) individuals who consume one private good \( x \) and one public good \( y \) in consumption space \( Z_i = \mathbb{R}^2_+ \). Each individual \( i \) is endowed with \( w_i \) units of private good. There are no initial endowments for the public good, but the public good can be produced from the private good, according to a production technology 

\[
y = \frac{1}{q}.
\]

The utility function of individual \( i \) is denoted by \( u_i(x_i, y) \) which is continuously differentiable and satisfies 

\[
\frac{\partial u_i}{\partial x_i} > 0 \quad \text{for all} \quad i = 1, 2, \ldots, n.
\]

1. Define the Lindahl equilibrium and Pareto efficiency for this economy, without assuming representation of preferences by utility functions.

2. For interior Pareto efficiency, is it necessary that all individuals have the same marginal rate of substitution, i.e., that

\[
\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} = \cdots = \frac{\partial u_n}{\partial y}, \quad \frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \cdots = \frac{\partial u_n}{\partial x_n}
\]

If yes, give proof. If not, give a counter example.

3. Now suppose the utility functions are of the quasi-linear form \( u_i = x_i + v_i(y) \) with \( v_i \) strictly increasing, strictly concave and continuously differentiable.

   (a) At interior Pareto efficient solutions \((x^*_1, x^*_2, \ldots, x^*_n, y^*)\), does \( y^* \) vary with \((x^*_1, x^*_2, \ldots, x^*_n)\)? Why or why not?

   (b) Is the answer to part (a) different when quasi-linearity is not assumed? Why or why not?

   (c) Suppose that \( n = 2 \), \( w_1 = 5 \), \( w_2 = 4 \), production function is given by \( y = 1/2v \), utility function is given by

\[
\begin{align*}
u_1 &= x_1 + \alpha \ln y, \\
u_2 &= x_2 + \beta \ln y
\end{align*}
\]

Find the Lindahl equilibrium for this economy.
Exercise 15.3 Consider an economy with two goods, a private (rivalrous) good \( x \), say leisure, and a public (non-rivalrous) good \( y \), say radio broadcast music. Both goods are measured in hours per day. There are two consumers, 1 and 2, and one firm. The firm produces \( y \), using labor \( v \) as input. (Thus, if consumer \( i \) supplies \( v_i \) units of labor to the firm, then the amount of leisure left to \( i \) is \( x_i = w_i - v_i \), where \( w_i \) is \( i \)'s initial endowment of leisure.) Let the production function of the firm be linear (constant returns to scale), with \( k \) units of \( v \) needed to produce one unit of \( y \) at any scale of output \( (k > 0) \). There is no free disposal. The initial endowments \( w_i \) of \( x \) are positive, but the initial endowment of \( y \) is zero.

Assume that each consumption set is \( Z_i = \mathcal{R}_+^{L+K} \), and the utility function of consumer \( i \) is given by:

\[
\begin{align*}
  u_i &= x_i + \phi_i(y),
\end{align*}
\]  

(15.8.37)

where the valuation function \( \phi_i(y) \) is twice differentiable, and has a positive derivative \( \phi_i'(y) > 0 \), and a negative second derivative \( \phi_i''(y) < 0 \) for all \( y \geq 0 \), for \( i = 1, 2 \). (Remember it is assumed \( w_i > 0 \) for \( i = 1, 2 \).)

Suppose that each consumer \( i \) voluntarily chooses to contribute an amount of labor \( v_i \geq 0 \) toward the production of the public good \( y \), with \( v_i < w_i \). By definition, at a Nash equilibrium allocation, written \((\bar{x}_1, \bar{x}_2, \bar{v}_1, \bar{v}_2, \bar{y})\), each consumer \( i \) maximizes \( u_i \), treating \( v_j \) as given (for \( j \neq i \)), and taking into account the equality

\[
\begin{align*}
  k\bar{y} &= \bar{v}_1 + \bar{v}_2.
\end{align*}
\]  

(15.8.38)

Answer the following questions (a)-(e) and explain your answers to these questions as fully as possible.

1. Find the conditions that characterize Pareto efficient allocations. (These will be equations in \( x_1, x_2, y \) and the original endowments.)

2. Suppose that, at a Nash equilibrium, consumer 2 contributes a positive amount of labor, but is still left with positive amounts of leisure, i.e., \( w_2 > \bar{v}_2 > 0 \), while consumer 1 contributes nothing, i.e., \( \bar{v}_1 = 0 \). Could such an equilibrium be Pareto optimal?

3. Suppose that, at a Nash equilibrium, both consumers contribute positive amounts of labor, but are still left with positive amounts of leisure. Could such an equilibrium be Pareto optimal?

4. Suppose that, a Nash equilibrium, neither consumer contributes any labor. Could such an equilibrium be Pareto optimal?

5. Define the Lindahl equilibrium and prove that every Lindal equilibrium is Pareto efficient under local non-satiation.
Exercise 15.4 Consider a public economy with $n$ consumers, one private goods $x$ and one public goods $y$. Each consumer has a consumption space $Z_i = \mathbb{R}_+^2$, an endowment of private goods $w_i = 10$, and his preferences can be represented by $u_i(x_i, y) = x_i + \theta_i \ln y$. The production technology of public goods is $f(q) = q$, where $q$ represents the input of private good in production.

1. Find the Pareto optimal allocation. How does this answer change with $n$?

2. If each person contributes some of his endowment to produce public good, what is the Nash equilibrium of voluntary contributions? How does this answer change with $n$?

3. If a government chooses to impose an individual tax of $\epsilon$ on each person to produce $n\epsilon$ units of public good, and each individual decides whether to make an extra contribution to public good, what is the total amount of public good provided in this way (Assume that $\epsilon$ is very small)?

4. If the government can only collect an income tax at the rate of $\tau$, and all taxes are used to produce public goods, what is the tax rate that can guarantee the efficient provision of public goods? If consumers vote to determine the tax rate, what is the tax rate determined by the majority voting? What is the difference compared to the situation where all consumers have the same preference parameter $\theta$?

Exercise 15.5 (Public goods and group size) Consider a public good economy with $n$ identical economic agents, one private good and one public good. Suppose that consumer $i$’s consumption space $Z_i = \mathbb{R}_+^2$ and the utility function is $u_i(x_i, y) = x_i + h(y)$, where $x_i$ represents private good consumed by consumer $i$ and $y$ represents public good. Suppose that $h$ is concave, differentiable, monotonically increasing, and satisfies $\lim_{y \to 0} h'(y) > 1$ and $\lim_{y \to \infty} h'(y) = 0$. Each agent has an endowment $\omega$ of private good, and the amount of endowment is sufficiently large such that the non-negativity constraint of private consumption is always not binding. The public goods production exhibits constant returns to scale. One unit of private goods can produce one unit of public goods, and only symmetric allocations are discussed.

1. Prove that the optimal provision level of public goods is an increasing function of $n$.

2. The level of public good provided under the voluntary contribution equilibrium is independent of the number of people $n$. And comment on this.
Exercise 15.6 Consider a public good economy with \( n \) identical economic agents, one private good and one public good. One way to resolve the free-rider problem proposed in the economics literature is to prize the contribution for public goods in the form of lottery: if the individual \( i \) contributes \( z_i \), then there is a probability of \( \frac{z_i}{\sum_{j=1}^{n} z_j} \) to win lottery worthy \( R \) units of private good. Personal contributions are used both to raise funds for public good and to provide lottery bonuses, so only \( \sum_{j=1}^{n} z_j - R \) of the contribution is put into public good production. Suppose that the lottery bonus \( R \) is independent of \( n \). Prove that under the symmetric Nash equilibrium in which each individual contributes \( z \) (\( z \) is a function of \( n \) and \( R \)):

1. If \( n > 1 \), the provision level of public good \( y = nz - R \) is always greater than the voluntary provision level in the previous question.
2. The provision level of public good \( y = nz - R \) increases with \( n \).
3. When \( n \to \infty \), the provision level of public good approaches to a finite value.
4. Now suppose that when \( n \to \infty \), the level of optimal public good also approaches to infinity. The conclusion in the above question 3 is disappointing. So assume that the total bonus increases with \( n \): \( R = nr \). Prove:
   (a) The provision level of public good \( y = n(z - r) \) increases with \( n \).
   (b) When \( n \to \infty \), the public good level also approaches to infinity, and comment on this result.

Exercise 15.7 (Tragedy of the Commons) Suppose that there are \( n \) peasant households in a village and each household has the right to raise dairy cows in a public pasture. The number of dairy cows raised by the farmer \( i \) is \( x_i \). The amount of milk a cow can produce depends on the number of cows \( \hat{x} \) grazing on the pasture. Assume that the income of the farmer \( i \) raising \( x_i \) cows is \( x_i v(\hat{x}) \). \( v(\hat{x}) > 0 \) when \( \hat{x} < \hat{x}_0 \), \( v(\hat{x}) = 0 \) when \( \hat{x} > \hat{x}_0 \), \( v(0) > 0 \), \( v' < 0 \) and \( v'' < 0 \). The cost per cow is \( c \), and assume that the cow can be perfectly segmented and that \( v(0) > c \). Each household also decides how many cows to buy at the same time, and all the cows bought will graze on the public pasture.

1. Determine the game for peasant households.
2. Find the Nash equilibrium and compare it with the socially optimal result.

Exercise 15.8 Consider a public good economy with \( n \) individuals, one private good \( x \) and one public good \( y \). The total endowment of private good is
and public good can be produced from the private good with cost function \( c(y) \). The utility function for consumer \( i \) is \( u_i(x_i, y) = x_i + v_i(y) \), where \( v_i \) is an arbitrary function defined on \( \mathbb{R}_+ \). Each consumer’s consumption set is \( \mathbb{R} \times \mathbb{R}_+ \), and the consumption of public good is non-negative. The private good cannot be used for public consumption.

1. Suppose that destruction is cost-free, write down all inequalities that describe the feasible allocation \((x_1, \cdots, x_n, y)\).

2. The public good consumption \( y \) is said to be “surplus maximizing” if the following condition is satisfied:

\[
y \in \arg \max_{y' \geq 0} \sum_i v_i(y') - c(y').
\]

Consider whether the proposition is correct:

“If \( y \) is the surplus maximizing public good consumption, then any feasible allocation that produces \( y \) units of public good must be Pareto optimal.”

Prove your answer.

3. Consider whether the proposition is correct:

“If the allocation \((x_1, \cdots, x_n, y)\) is Pareto optimal, then \( y \) is surplus-maximizing.”

Prove your answer.

**Exercise 15.9** Suppose that there are \( n \) fishermen in a fishing village. Some fishermen are fishing in the sea. Since the sea is big enough, no matter how many fishermen go fishing, every fisher can catch \( k \) fish. There are some other fishermen who go fishing in a lake (The fish in the sea and the fish in the lake are perfect substitutes). If \( x \) fishermen go fishing in the lake, then each fisherman can catch \( x^{-1/2} \) fish (That is, these fishermen can catch \( x^{1/2} \) fish in total, and every fisherman catches the same amount of fish).

1. For fishermen, it is cost-free to choose between fishing in the lake or in the sea, and no fishermen go where they believe they will catch less fish. How many fishermen will go fishing in the sea? How many fishermen will go fishing in the lake? How many fish will they catch on average?

2. If the government restricts fishing in the lake, how many fishermen should be allowed to go fishing in the lake so as to maximize the fishing capacity in this community?
3. If the demand function of the fish is assumed to be

\[ Q = A - BP, \]

compare the price of fish in the market without restriction to that under efficient allocation.

4. Now suppose that the fish in the lake and the fish in the sea are not perfect substitutes. The price of marine fish is $20 each, and the demand for fish in the lake is

\[ Q_L = A' - B'P_L. \]

If there are no restrictions on fishermen, how many fishermen will go fishing in the lake at equilibrium? If the government collects a fixed license fee on fishermen who go fishing in the lake, will the price of fish in the lake rise or fall? Write the derivation process.

**Exercise 15.10** Consider a public good economy with one private good \( x \), one public good \( y \), and \( n \) consumers whose consumption choice sets are non-negative in each dimension. Each consumer \( i \) owns \( w_i \) units of private good, and they do not own public good which can be produced with production function is \( y = v \). The utility function for each consumer \( i \) is represented by \( u_i(x_i, y) \), which may not be differentiable (note that it is then not possible to answer the following questions 1 and 2 with differential methods).

1. Define Lindahl equilibrium and Pareto efficient allocation of the economy.

2. Prove that every Lindahl equilibrium allocation is Pareto efficient. (Hint: If you need additional assumptions, make it clear in the proof.)

3. Now suppose that \( u_i \) is differentiable. Then give the Lindahl-Samuelson first-order conditions for Pareto efficient allocation.

4. When the consumer’s utility function is \( u_i(x_i, y) = (x_i + 1)^{\alpha_i}y^{1-\alpha_i} \)
   and \( 0 < \alpha_i < 1 \), find Lindahl equilibrium. Is it Pareto efficient?

**Exercise 15.11** Suppose that \( n \) economic agents have the same Cobb-Douglas utility function \( u_i(x_i, y) = x_i^{\alpha}y^{1-\alpha} \) and the consumption set \( Z_i = \mathbb{R}_+^2 \). The total amount of wealth is \( w \), and they are divided among \( k \leq n \) individuals. How many public goods are provided? How does the quantity of public goods change when \( k \) increases?
**Exercise 15.12** An ancient village uses some goods (say sheep) for two purposes: either as food or as a public religious sacrifice. Suppose that villager $i$'s initial endowment of sheep is $w_i > 0$. Let $x_i \geq 0$ be the consumption of sheep, and $g_i \geq 0$ be the amount for public sacrifice. The total amount of sheep used for sacrifice is $y = \sum_{i=1}^{n} g_i$. The utility function for villager $i$ is given by:

$$u_i(x_i, y) = x_i + a_i \ln y,$$

where $a_i > 1$.

1. When deciding on their sacrifice, each villager $i$ regards that the sacrifice of other villagers remain fixed, and on this basis he decides the sacrifice he would offer. Let $y_{-i} = \sum_{j \neq i} g_j$ be the sacrifice except villager $i$. Write out the utility-maximizing problem that determines the sacrifice of villager $i$.

2. Recall that for all individuals $i$, $y = g_i + y_{-i}$. What is the equilibrium amount of public good? (Hint: Not everyone will contribute positive public good.)

3. Who will be a free-rider in this problem?

4. In this economy, what is the Pareto efficient quantity of public good to be provided?

**Exercise 15.13** A town has a population of 1,000, and each citizen’s utility function is $u_i(x_i, y) = (x_i + k_i)y^{\alpha}$, where $y$ is the size of the town’s ice-skating rink measured in square meters, and $x_i$ is the number of bread consumed each year. Suppose that the price of bread is $1$ each, and the price of maintaining a square meter of skating rink is also $1$. Different citizens have different income $w_i$. Find the Lindahl equilibrium for this town. Under the Lindahl equilibrium, how much should the government raise from resident $i$?

**Exercise 15.14 (The First Welfare Theorem of Lindahl Allocation with Transfers)** Prove Theorem 15.4.3: given the public goods economy $e = (e_1, \ldots, e_n, \{Y_j\})$, every Lindahl equilibrium allocation $(x^*, y^*)$ with transfers and the price system $(q^*_1, \ldots, q^*_n, p^*)$, is weakly Pareto efficient. If the consumers’ preferences also satisfy local non-satiation, it is Pareto efficient.

**Exercise 15.15 (The First Welfare Theorem of Welfare Economics with strictly convex preference)** Suppose that $\succ_i$ are strictly convex. Let allocation $(x, y) \in X \times Y$ and non-zero price vector $(q_1, \ldots, q_n, p) \in \mathbb{R}^{L+nK}_{+}$ constitute a Lindahl equilibrium. Prove the Lindahl equilibrium allocation is Pareto efficient.
Exercise 15.16 (Lindahl equilibrium, constrained Lindahl equilibrium, Lindahl quasi-equilibrium and Pareto optimality) For the public goods economy $e = (e_1, \cdots, e_n, \{Y_j\})$, suppose that for all $i$, $0 \neq w_i \in X_i = \mathbb{R}_+^L$ and $\succ_i$ is preference ordering. Allocation $(x, y) \in X \times Y$ and non-zero price vector $(q_1, \cdots, q_n, p) \in \mathbb{R}^{L+nK}_+$ constitute a constrained Lindahl equilibrium, if the other conditions in the definition remain the same, except that (ii) is replaced by

$$(i'i') (x_i, y) \succ_i (x_i^*, y^*) \text{ and } x_i + y \leq \sum_{t=1}^n w_i \text{ implies } p^x x_i + q_i^y y > p^* w_i, \forall i = 1, \cdots, n.$$

Allocation $(x, y) \in X \times Y$ and non-zero price vector $(q_1, \cdots, q_n, p) \in \mathbb{R}^{L+nK}_+$ constitute a Lindahl quasi-equilibrium, if the other conditions in the definition remain the same, except that (ii) is replaced by

$$(i'i'') (x, y) \succ_i (x^*, y^*) \text{ implies } p^x x_i + q_i^y y \geq p^* w_i, \forall i = 1, \cdots, n.$$

1. We know that if preferences $\succ_i$ satisfy local non-satiation, every Lindahl equilibrium allocation is Pareto optimal. What if the local non-satiation is not satisfied?

2. Prove: If $\succ_i$ satisfies convexity, then the interior-point constrained Lindahl equilibrium is a Lindahl equilibrium. Can the convexity be relaxed to local non-satiation?

3. Prove: If $\succ_i$ satisfies strong monotonicity, then Lindahl equilibrium is Lindahl quasi-equilibrium. Can strong monotonicity be relaxed to monotonicity?

4. Prove: If a Lindahl equilibrium allocation is a Lindahl quasi-equilibrium allocation, it is Pareto efficient.

5. From the previous question, if $\succ_i$ are strictly convex, the Lindahl equilibrium allocation is Pareto optimal. Then, if $\succ_i$ is strictly convex, is Lindahl equilibrium necessarily a Lindahl quasi-equilibrium?

6. Suppose that $\succ_i$ satisfies continuity for every individual $i$. Prove: If $p \in \mathbb{R}_+^L$, the Lindahl quasi-equilibrium is a Lindahl equilibrium.

7. Suppose that for any individual $i$, $\succ_i$ satisfies continuity and strong monotonicity. Prove: If $(p, x)$ is a Lindahl quasi-equilibrium and $x_i \in \text{int} \mathbb{R}_+^L$ for some $i$, then $p \in \mathbb{R}_+^L$.

Exercise 15.17 (Economic core theorem in public economy) Prove Theorem 15.4.4: Under the local non-satiation of preferences, if $(x, y, p)$ is a Lindahl equilibrium, then $(x, y)$ is in the core.
Exercise 15.18 (The Second Theorem of Welfare Economics in public goods economy with non-satiated preferences) Prove the theorem: for given public goods economy \( e = (e_1, \cdots, e_n, \{Y_j\}) \), suppose that preferences \( \succ_i \) are continuous, convex, and non-satiated. \( Y \) is a closed convex set and \( 0 \in Y \). Then, for any Pareto optimal allocation \((x^*, y^*)\), there exists non-zero price vector \((q_1, \cdots, q_n, p) \in R^{L+nK}\) such that \((x, y, (q_1, \cdots, q_n), p)\) is a Lindahl quasi-equilibrium with transfers. That is, there is an assignment of wealth levels \((I_1, \cdots, I_n)\) with \( \sum_i I_i = p \sum_i w_i \) such that

1. if \((x_i, y) \succ_i (x_i^*, y^*)\), then \( px_i + q_i y \geq I_i \equiv px_i^* + q_i y^*\),
2. for all \((y, -v) \in Y\), we have \( qy^* - pv^* \geq qy - pv\),

where \( v^* = \sum_{i=1}^n w_i - \sum_{i=1}^n x_i^*; \sum_{i=1}^n q_i = \hat{q} \).

Furthermore, if for all \( i, 0 \in X_i \) and \( px_i^* + q_i y^* > 0 \), then \((x^*, y^*, p)\) is a Lindahl equilibrium with transfers.

15.9 References

Books and Monographs:


Papers:

CHAPTER 15. PUBLIC GOODS


Part VI

Information, Incentives, and Mechanism Design
Most of the above chapters focus on the market system, discuss how it works, what are its advantages and limitations. In reality, people are always exploring whether there is a better economic system. This is especially true when markets fail. Although some previous chapters have discussed the design of mechanisms for specific problems, we need a more general analytical framework to study the design of economic mechanisms, and can compare the advantages and disadvantages of all economic mechanisms under certain criterions, which is the theory of economic mechanism design to be discussed in this part.

Information economics, principal-agent theory (also called optimal contract theory), the theory of mechanism design, and market design theory (mainly consisting of auction theory and matching theory) have become very important and active research fields in economics in the last five decades. They have a wide range of applications in various disciplines such as finance, management, corporate law, and political science. Because of this, more than twenty economists of founding contributors of mechanism design and the associated fields of game theory so far have won the Nobel prize in economics, including Friedrich A. Hayek, Kenneth Arrow, George J. Stigler, Gerard Debreu, Ronald Coase, Herbert Simon, John Nash, Reinhard Selten, William Vickrey, James Mirrlees, George Akerlof, Joseph Stiglitz, Michael Spence, Robert Aumann, Leo Hurwicz, Eric Maskin, Roger Myerson, Peter Diamond, Oliver Williamson, Alvin E. Roth, Lloyd S. Shapley, Jean Tirole, Oliver Hart, and Bengt Holmstrom.

As we mentioned earlier, pursuing individual interest (utility or profit) is not only the most basic assumption of economics, but also is the most relevant reality besides informational incompleteness and asymmetry. As such, if a self-interested agent has said something, others cannot tell if this is the truth. Even if a pair of eyes stare at us, we do not know if he is listening, so that real-world economic problems are extremely complicated and difficult to cope with, and economics which focuses on studying economic phenomena and economic human behavior has become a very difficult discipline. So, how can we solve the problem of incentive distortions caused by individual self-interestedness and informational asymmetry? That is, when information is asymmetric, how to induce self-interested individuals to report truthfully their economic characteristics and actions through proper mechanism design so that the goal of the society or the designer is achieved? If this goal can be achieved, are individual interest and social interest incentive compatible? This is the main content covered by the economic mechanism design theory.

Incentive or incentive compatibility has become one of the most basic and core concepts in modern economics. To many economists, economics is to a large extent a matter of incentives: incentives to work hard, to produce good quality products, to study, to invest, to save, etc.

Until 1970s, when discussing incentive-compatibility issues, tradition-
al economic analysis took economic mechanisms as given and was mostly concerned with understanding the theory of value in large economies. A central question asked in general equilibrium theory is whether a certain mechanism (especially the competitive mechanism) generates Pareto-efficient allocations, and if so – for what categories of economic environments. In a perfectly competitive market, the pressure of competitive markets solves the problem of incentives for consumers and producers. The major project of understanding how prices are formed in competitive markets can proceed without worrying about incentives.

The question was then reversed in the economics literature: instead of regarding mechanisms as given and seeking the class of environments for which they work, one seeks mechanisms which will implement some desirable outcomes (especially those that result in Pareto-efficient and individually rational allocations) for a given class of environments without destroying participants’ incentives, and have a low cost of operation and other desirable properties. In a sense, the theorists went back to basics.

The reverse question was stimulated by two major lines in the history of economics. Within the capitalist/private-ownership economics literature, a stimulus arose from studies focusing upon the failures of the competitive market to function as a mechanism for implementing efficient allocations in various kinds of new nonclassical economic environments such as the presence of externalities, public goods, indivisible goods, incomplete information, imperfect competition, increasing return to scale, etc. At the beginning of seventies of the last century, works by Akerlof (1970), Hurwicz (1972), Spence (1974), and Rothschild and Stiglitz (1976) showed in various ways that asymmetric information was posing a much greater challenge and could not be satisfactorily imbedded in a proper generalization of the Arrow-Debreu general equilibrium theory.

A second stimulus arose from the socialist/state-ownership economics literature, as evidenced in the “socialist controversy” — the debate between Mises-Hayek and Lange-Lerner in twenties and thirties of the last century. The controversy was provoked by von Mises’s skepticism as to even a theoretical feasibility of rational allocation under socialism. This debate clarified many problems and reached many consensuses. However, because both sides of the debate lack precise definitions of key economic terms, the definition is unclear or inconsistent, coupled with the lack of scientific quantitative analysis tools, they are unable to give formal and clear conclusions with rigorous mathematical logic. It is Hurwicz who was the first to abstract the most essential issues in this debate, one of which is information in economic systems, and the other is incentives. He studies these two issues by establishing rigorous analytical frameworks.

Thus the structures of incentives and information are two basic features of any economic system. The study of these two features is attributed to these two major lines, culminating in the theory of mechanism design. The
theoretical framework of economic mechanism design that was originated by Hurwicz is very general. All economic mechanisms and systems (including those known and unknown, private-ownership, state-ownership, and mixed-ownership systems) can be studied with the framework.

At the micro level, the development of the theory of incentives has also been a major advance in economics in the last fifty years. Before, by treating the firm as a black box the theory had remained silent on how the owners of firms succeed in aligning the objectives of its various members, such as workers, supervisors, and managers, with profit maximization.

When economists began to look more carefully at the firm, either in agricultural or managerial economics, incentives became the central focus of their analysis. Indeed, delegation of a task from the principal to an agent who has different objectives is problematic when information about the agent is asymmetric or incomplete. This problem is the essence of incentive questions. Thus, conflicting objectives and decentralized information are the two basic ingredients of incentive theory.

We will discover that, in general, these informational problems prevent society from achieving the first-best allocation of resources that could be possible in a world where all information would be common knowledge.\(^1\) The additional costs that must be incurred because of the strategic behavior of privately informed economic agents can be viewed as one category of the transaction costs. Although they do not exhaust all possible transaction costs, economists have been rather successful in modeling and analyzing these types of costs and providing a good understanding of the limits set by these on the allocation of resources. This line of research also provides a whole set of insights on how to begin to take into account agents’ responses to the incentives provided by institutions.

The three words — contracts, mechanisms and institutions are to a large extent synonymous.\(^2\) They all mean “rules of the game,” which describe what actions the parties can undertake, and what outcomes these actions would be obtained. In most cases the rules of the game are given by designer: in chess, basketball, etc. The rules are designed to achieve better outcomes. But there is one difference. While the theory of mechanism design may be able answer “big” questions at nation’s level, such as “market economy vs planned economy” and “socialism vs capitalism,” contract the-

\(^1\)The term of the first-best is relatively to the second-best. The theory of the second-best concerns what happens when one or more optimality conditions (typically the case of incomplete information) cannot be satisfied. Canadian economist Richard Lipsey and Australian economist Kelvin Lancaster showed in a 1956 paper that if one optimality condition in an economic model cannot be satisfied, it is possible that the next-best solution involves changing other variables away from the ones that are usually assumed to be optimal, see: Lipsey and Lancaster (1956), “The General Theory of Second Best”. Review of Economic Studies 24(1): 11-32.

\(^2\)However, a significant difference is that a contract must be voluntary while a mechanism may not require this constraint.
ory is developed and useful for more manageable, or micro level questions, concerning specific voluntary contracting practices and mechanisms.

Thus, mechanism design is normative economics, in contrast to game theory which is positive economics. Game theory is important because it predicts how a given game will be played by agents. The theory of mechanism design goes one step further: given the physical environment and the constraints faced by the designer, what goal can be realized or implemented? What mechanisms are optimal among those that are feasible? In designing mechanisms one must take into account incentive constraints (e.g., consumers may not report truthfully how many pairs of shoes they need or how productive they are).

This part considers the design of economic mechanism in which one or more parties have private characteristics or hidden actions. The party who designs the mechanism is called the designer or principal, while the other parties will be called agents, individuals, or participants. For the most part we will focus on the situation where the designer has no private information and the agents do. This framework is called screening, because the principal will in general try to screen different types of agents by inducing them to choose different outcomes. The opposite situation, in which the designer has private information and agents do not, is called signaling, since the designer could signal his type with the design of his contract or mechanism.

We will briefly present the mechanism design theory in four chapters. Chapters 16 and 17 discuss the principal-agent theory/optimal contract theory where the principal delegates an action to a single agent with exogenous or endogenous private information and subject to two kinds of constraints: participation constraint and incentive compatibility constraint. This private information of the agent can be of two classes: either the agent has some private information about his cost or valuation that is ignored by the principal, the case of adverse selection or hidden information; or the agent can take an action unobserved by the principal, the case of moral hazard or hidden action. The design of the principal’s optimal contract can be simply regarded as a constrained optimization problem. This simple focus will turn out to be enough to highlight the various trade-offs between allocative efficiency and distribution of information rents for the case of adverse information or between efficiency and insurance (i.e., trade-off between incentives and risk) for moral hazard arising under incomplete information. The mere existence of informational constraints may generally prevent the principal from achieving allocative efficiency. We will characterize the allocative distortions that the principal finds desirable to implement in order to mitigate the impact of informational constraints. The main references for Chapters 16 and 17 are Laffont and Martimort (2002) and Bolton and Dewatripont (2005).

Chapter 18 will consider situations with one principal and multiple a-
agents. Moreover, maintaining the hypothesis that agents adopt an individualistic behavior, those organizational contexts require a solution concept of equilibrium, which describes the strategic interaction between agents under complete information. The solution concepts of equilibrium we consider are mainly dominant equilibrium and Nash equilibrium.

Chapter 19 will discuss the case of incomplete information in which asymmetric information may not only affect the relationship between the principal and each of his agents, but it may also plague the relationships between agents. As such, agents do not know each other's characteristics, and we need to consider Bayesian incentive compatible mechanism.

Chapter ?? will briefly study dynamic contract theory. We will discuss long-term incentive contract in a dynamic principal-agent setting mainly with one agent and adverse selection. We will first consider the case where the principal (designer) can commit to a contract forever, and then consider what happens when she cannot commit against modifying the contract as new information arrives.
Chapter 16

Principal-Agent Theory: Hidden Information

16.1 Introduction

The optimal contract design is also called the principal-agent theory. When a principal assigns a task to an agent who has private information, the agent’s utility, technology, cost, and ability are all likely to be his private information. By misreporting the private information, the agent may obtain extra benefits, and then the incentive issue appears. We call such a principal-agent situation “hidden information” or “adverse selection” problem.

Some Examples
The principal-agent problem exists almost everywhere. Here are some specific examples.

(1) The agency of environmental protection wants to reduce haze pollution, but does not know the cost of pollution abatement of a firm.

(2) The township government in China entrusts the collective land to farmers who privately observe the farming skills and local weather conditions.

(3) Shareholders delegate the daily decisions of a company to a professional manager who knows more about markets and production technologies.

(4) Professors teach students, but do not know the students’ learning ability.

(5) A client delegates his defense to an attorney who has professional knowledge of various legal provisions and the difficulty of the case.
(6) Venture capital companies lend funds to a founder of a high-tech company who will be the only one master new technologies.

(7) The central government delegates administrative power to a local government who knows more about local situations.

(8) National Development and Reform Commission (NDRC) of China entrusts petroleum resources to China National Petroleum Corporation (CNPC), but NDRC does not participate in the specific camp.

(9) An insurance company provides car insurance to an agent whose driving skill is his private information.

(10) A regulatory agency contracts with a public utility company without having complete information about its technology.

All of the above issues show that the difference in information between a principal and an agent has a significant influence on contract design. In order to make optimal use of resources and to allow the agent to have incentives to reveal his private information, the agent needs to be given some information rent. In order to induce the agent to truthfully report the private information, the principal needs to make a tradeoff between allocative efficiency and rent extraction when designing the optimal contract. The implicit assumption is that the contractual relationship is conducted within certain legal framework. As the contract can be enforced by the court, the agent is bound by the terms of the contract.

The main objective of this chapter is to characterize the optimal rent extraction-efficiency trade-off faced by the principal when designing her contract offer to the agent under the set of incentive feasible constraints: incentive compatibility and participation constraints. If an incentive constraint binds at the optimal solution, then adverse selection limits the efficiency of the transaction. The main lesson of this chapter is that the optimal second-best contract calls for a distortion in the volume of trade away from the first-best, and the principal must transfer some information rent to the more efficient type of an agent.

### 16.2 Basic Model

In this section, we introduce the simplest model to capture the essence of adverse selection. This problem was first studied by Mirrlees in 1971 (for Mirrlees’ biography, see Section 17.11.2). We assume that there are two types of participants, one is called a principal who has some sort of monopoly power in the transaction, while the other is called the agent who
16.2. BASIC MODEL

has private information in the transaction. Different from the market equilibrium discussed before, here we focus on how information distribution affects the efficiency of market transactions.

16.2.1 Economic Environment

For the ease of understanding, we discuss the issue of adverse selection in a monopolistic market. This problem was first proposed by Mussa and Rosen (1978), and a more deeper discussion was conducted by Maskin and Riley (1984). Consider an economy in which a monopolist offers a product to a consumer. For the case of a continuum of consumers, the proportion of certain type of consumers can be considered as the probability that there is one such consumer. According to the law of large numbers, these two approaches are equivalent.

The consumer’s evaluation of the product is private information that cannot be observed by the monopolist. The purpose of the monopolist is to get maximum surplus from the consumer, and this depends on how he distinguishes the consumer’s type from the consumer’s behavior. If we put this example into the principal-agent framework, the monopolist corresponds to principal, and consumer corresponds to agent. Judging the role of participants in the principal-agent framework is usually based on the fact that the principal provides a menu of contracts, and the agent chooses one from these contracts.

Assume the consumer’s utility function is

\[ u(q, T, \theta) = \theta v(q) - T, \]

where \( q \) is the quantity of good purchased by the consumer; \( T \) is consumer’s transfer payment; and \( \theta \) is the type of consumer’s evaluation on the good with \( \theta \in [\theta_H, \theta_L] \) and \( \theta_H > \theta_L \). Let \( \Delta \theta = \theta_H - \theta_L \). Assume that the probability that \( \theta_L \) happens is \( \beta \), which means the probability of utility function \( u(q, T, \theta_L) = \theta_L v(q) - T \) is \( \beta \), and the probability of utility function \( u(q, T, \theta_H) = \theta_H v(q) - T \) is \( 1 - \beta \). As such, the private information is exogenously given. The utility function above is quasi-linear, which usually satisfies the following assumptions:

\[ v(0) = 0, \quad v'(q) > 0, \quad v''(q) < 0. \]

The monopolist’s objective function is

\[ \pi = T - cq, \]

where \( c \) is the production cost per unit.

16.2.2 Outcome Space and Contracting Variables

The contracting variables are the quantity \( q \) produced by the principal and the transfer \( T \) received by the agent. Let \( A \) be the set of feasible outcomes
that is given by
\[ A = \{ (q, T) : q \in \mathbb{R}_+, T \in \mathbb{R} \} \].
(16.2.1)

These variables are both observable and verifiable by a third party such as a benevolent court of law.

16.2.3 Information Structure and Timing

The information structure can be divided into three categories according to the timing when the principal and the agent know the type. It is called the ex ante stage when both participants do not know the type of the agent. It is called the interim stage when the agent knows his type while the principal does not know. When both participants know the type of the agent, it is called the ex post stage.

In the principal-agent theory, contracts are often offered at the interim stage. As such, unless explicitly stated, two participants will proceed in the following order:

- \( t = 0 \): the consumer is informed about his own type \( \theta \);
- \( t = 1 \): the monopolist provides a menu of contracts;
- \( t = 2 \): the consumer accepts or rejects the contract;
- \( t = 3 \): the contract is implemented.

16.2.4 Optimal Contract under Complete Information

To analyze the influence of adverse selection on the decisions, we usually use the complete information case as a benchmark in which the first-best outcome can be achieved.

If the monopolist knows the consumer’s type, he will provide a separate contract for each type, that is, the contract \((T_i, q_i)\) corresponds to \(\theta_i, \ i \in \{H, L\}\). The goal of the monopolist is to obtain the consumer’s surplus as much as she can on the basis that the consumer accepts the contract. If the consumer does not accept the contract, his utility level is \(\bar{u} = 0\). Thus, the monopolist’s goal is to solve the following maximization problem:

\[
\max_{T_i, q_i} T_i - cq_i,
\]

subject to participation constraint:

\[
\theta_i v(q_i) - T_i \geq 0.
\]

Since the principal prefers higher \(T_i\) as possible, the constraint is binding at the optimum:

\[
T_i = \theta_i v(q_i).
\]

By substituting the constraint into the objective function, the problem becomes:
16.2. BASIC MODEL

\[ \max_{T_i, q_i} \theta_i v(q_i) - c q_i. \]

The first-order condition for this problem is given by:

\[ \theta_i v'(q_i^*) = c. \]

In this case, the outcome is Pareto optimal, that is, the marginal social value of a unit of consumption is equal to the marginal cost of production. At the same time, the monopolist obtains the maximum surplus

\[ \pi = \beta (T^*_L - c q^*_L) + (1 - \beta) (T^*_H - c q^*_H), \]

and the consumer’s surplus is 0.

Implementation of the First-Best Contract

The monopolist can design the following contract in order to obtain the above profit level. First of all, it requires that the consumer should obtain a level of utility no lower than his reservation utility \( \bar{u} \). That is, \( \theta_i v(q_i) - T_i \geq \bar{u} \), which is the participation constraint. Secondly, since principal can distinguish different types under complete information, the monopolist can make the take-it-or-leave-it offer \((T^*_i, q^*_i)\) to the agent. It will be accepted by consumer \( \theta_i \) because the utility he received is no less than his reservation utility.

Graphical Representation of Optimal Contract

![Figure 16.1: Indifference curves of two different types of the consumer](image)

The optimal contract under complete information is the first-best, namely the points \( A^* \) and \( B^* \) shown in Figure 16.2.
Figure 16.1 depicts the indifference curves of two types of the consumer. Compared to $\theta_L$, the indifference curve of $\theta_H$ has a larger slope; moreover, the two types of indifference curves intersect only once. This feature is called the single crossing property or Spence-Mirrlees property, which is a very important feature in mechanism design.

In the optimal contract, each type of the consumer only obtains his reservation utility, and at the same time, the slopes of the optimal points equal to the marginal cost $c$ as shown in Figure 16.2.

### 16.2.5 Incentive Feasible Contracts

With the result of the complete information as a benchmark, we can analyze the optimal contract in the presence of incomplete information. We will see that under incomplete information, the optimal contract under the complete information will lead to incentive incompatibility issues.

When the monopolist cannot observe the consumer’s type, the above contract $(T^*_i, q^*_i), i \in \{H, L\}$ is not implementable. From Figure 16.2, one can see that, under incomplete information, the low type will choose point $B^*$, but the high type (the $\theta_H$ type) will also choose point $B^*$ for a higher profit, instead of choosing point $A^*$ that is optimal for the high type under the complete information, which means that the high type has incentive to misreport himself as the low type. This is because, for the high type, the utility of choosing $(T^*_H, q^*_H)$ is 0, but if he chooses $(T^*_L, q^*_L)$, the utility level is:

$$u(T^*_L, q^*_L, \theta_H) = \theta_H v(q^*_L) - T^*_L > \theta_L v(q_L) - T_L = 0.$$ 

Thus, under asymmetric information, $(T^*_i, q^*_i), i \in \{H, L\}$ cannot be implemented by the “take-it-or-leave-it” contract. The high type $\theta_H$ will mimic
the low type consumer $\theta_L$.

Then an appropriate contract must be designed so that different type has the incentive to choose the contract proposed for his type, that is, the incentive compatibility constraint must be met.

**Definition 16.2.1** A menu of contracts $A_i = (q_i, T_i)$, $i \in \{H, L\}$ is incentive compatible if it satisfies:

$$u(q_i, T_i, \theta_i) \geq u(q_j, T_j, \theta_i), \forall i, j \in \{H, L\}.$$  

Incentive-compatible contracts mean that each type chooses his own contract. Under asymmetric information, when a monopolist designs a contract, the contract must satisfy the following two sets of constraints:

\begin{align*}
U_H &= \theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L = U_L + \Delta \theta v(q_L), \quad (16.2.2) \\
U_L &= \theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H = U_H - \Delta \theta v(q_H), \quad (16.2.3) \\
U_H &= \theta_H v(q_H) - T_H \geq 0, \quad (16.2.4) \\
U_L &= \theta_L v(q_L) - T_L \geq 0. \quad (16.2.5)
\end{align*}

We call constraints (16.2.2) and (16.2.3) the incentive compatibility constraints, and constraints (16.2.4) and (16.2.5) the participation constraints, of types $\theta_H$ and $\theta_L$, respectively.

**Definition 16.2.2** A menu of contracts $A_i = (q_i, T_i)$, $i \in \{H, L\}$ is incentive feasible if it satisfies both incentive compatibility and participation constraints (16.2.2) through (16.2.5).

Solving the asymmetric information optimal contract is actually solving the following maximization problem:

$$\max_{(q_i, T_i)} \beta(T_L - c q_L) + (1 - \beta)(T_H - c q_H),$$

subject to the incentive compatibility constraints (16.2.2) and (16.2.3), and participation constraints (16.2.4) and (16.2.5).

Before formally solving the optimization problem, we first discuss the possible choices of incentive feasible contracts of the monopolist so that we know the set of incentive feasible contracts is not empty. We consider the following two special contracts.

(1) Bunching contracts or pooling contracts: the first special case of incentive feasible contract is obtained when the contracts targeted for each type coincide, that is, $T_L = T_H = T$, $q_L = q_H = q$, and both types accept the contract. In this case, the incentive compatibility constraint is automatically established. The only participation constraint matters. If the monopolist chooses the same $T$ for two types, $\theta_L v(q) - T \geq 0$ is the only constraint.
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(2) Shutdown of the low type: the nonzero contract \((T_H, q_H)\) is provided for type \(\theta_H\), the null contract \((T_L, q_L) = (0, 0)\) is for type \(\theta_L\), and it also satisfies \(\theta_L v(q_H) - T_H \leq 0\). With such a contract, the low type is not be served.

As with the pooling contract, the benefit of the \((0, 0)\) option is that it somewhat reduces the number of constraints since the incentive and participation constraints take the same form. The cost of such a contract may be an excessive screening of types, leading to non-optimal contracting. Here, the screening of types takes the rather extreme form of the least-efficient type. Hence, these two contracts are not optimal although they are incentive feasible contracts.

16.2.6 Monotonicity Constraints

Incentive compatibility constraints reduce the set of feasible outcomes so that output levels must satisfy a monotonicity condition that is not required under complete information. This condition implies that the Spence-Mirrlees single-crossing condition

\[
\frac{\partial^2 u}{\partial \theta \partial q} > 0
\]

is satisfied, which means that two different types of indifference curves intersect at most once.

Indeed, adding (16.2.2) and (16.2.3), we immediately have:

\[
\Delta \theta(v(q_H) - v(q_L)) \geq 0.
\]

Since \(v'(q) > 0\), when \(q_H \neq q_L\), we must have \(q_H > q_L\), which means that the consumption of low type will not be higher than the consumption of high type. This condition is not only necessary but also sufficient for implementability.

In order to derive sufficiency, assume \(q_H \geq q_L\). We need to show that there exists transfers \(T_H\) and \(T_L\) such that the incentive constraints hold. It is enough to take those transfers such that:

\[
\theta_L(v(q_H) - v(q_L)) \leq T_H - T_L \leq \theta_H(v(q_H) - v(q_L)).
\]

(16.2.6)

16.2.7 Information Rent

To understand the characteristics of the optimal contract under asymmetric information it is useful to introduce the concept of information rent that is the difference between the utility under asymmetric information and the utility under complete information.

Under complete information, since the monopolist has full negotiating power, she can fully extract consumer surplus so that the consumer only
gets reservation utility levels $U^*_H$ and $U^*_L$:

$$U^*_H = \theta_H v(q^*_H) - T^*_H = 0,$$

$$U^*_L = \theta_L v(q^*_L) - T^*_L = 0.$$ 

So the information rent of high type $\theta_H$ and low type $\theta_L$ can respectively be denoted by

$$U_H = \theta_H v(q_H) - T_H,$$

and

$$U_L = \theta_L v(q_L) - T_L.$$

Under asymmetric information, how much benefit (utility) would a $\theta_H$-agent get by mimicking a $\theta_L$-agent? The high type would get:

$$\theta_H v(q^*_L) - T^*_L = \Delta \theta v(q^*_L) + U_L.$$ 

Even if the expected utility obtained by low type is equal to the reservation utility, that is, $U_L = \theta_L v(q^*_L) - T^*_L = 0$, in order to have an incentive-compatible contract $(T_H, q_H; T_L, q_L)$, the high type should obtain the information rent that is greater than or equal to the benefit from mimicking the low type, i.e., it should satisfy

$$U_H \geq \triangle \theta v(q^*_L).$$

Thus, as long as the principal insists on a positive output for the low type, $q^*_L > 0$, the principal must give up a positive rent to the high type to have an incentive-compatible contract $(T_H, q_H; T_L, q_L)$. This information rent is generated by the informational advantage of the high-type agent over the principal. The problem for the monopolist is thus to choose an optimal way to transfer some information rent to certain types so as to get a maximum profit.

### 16.2.8 Optimal Contract under Asymmetric Information

According to the timing of the contractual setting, the monopolist must offer a menu of contracts before knowing the type she is facing. Thus, the monopolist wants to maximize the expected payoffs under incentive compatible contract $(T_H, q_H; T_L, q_L)$, and then the monopolist’s optimal problem can be written as:

$$\max_{(T_H, q_H; T_L, q_L)} (1 - \beta)(T_H - cq_H) + \beta(T_L - cq_L)$$

s. t. 

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L,$$

$$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H,$$

$$\theta_H v(q_H) - T_H \geq 0,$$

$$\theta_L v(q_L) - T_L \geq 0.$$
Using the notation of information rents $U_H = \theta_H v(q_H) - T_H$ and $U_L = \theta_L v(q_L) - T_L$, we can replace the transfers $T_H$ and $T_L$ with information rent and output, for which the monopolist’s optimization problem is to solve for $(U_H, q_H; U_L, q_L)$. The focus on information rents enables us to assess the distributive impact of asymmetric information, and the focus on outputs allows us to analyze its impact on allocative efficiency and the overall gains from trade. Thus an allocation corresponds to a level of output and a distribution of the gains from trade between the monopolist (principal) and the consumer (agent).

With this change of variables, the principal’s objective function can be rewritten as:

$$
\text{max}(1 - \beta)(\theta_H v(q_H) - cq_H) + \beta(\theta_L v(q_L) - cq_L) - ((1 - \beta)U_H + \beta U_L) \quad (16.2.7)
$$

s.t.

$$
U_H \geq U_L + \Delta \theta v(q_L), \quad (16.2.8)
$$

$$
U_L \geq U_H - \Delta \theta v(q_H), \quad (16.2.9)
$$

$$
U_H \geq 0, \quad (16.2.10)
$$

$$
U_L \geq 0. \quad (16.2.11)
$$

Expression (16.2.7) can be divided into two parts:

$$
(1 - \beta)(\theta_H v(q_H) - cq_H) + \beta(\theta_L v(q_L) - cq_L) - ((1 - \beta)U_H + \beta U_L).
$$

The first term denotes expected allocative efficiency, and the second term denotes expected information rent. This expression implies that the principal is ready to accept some distortions away from efficiency in order to decrease the agent’s information rent. We denote the solution to the problem (16.2.7) with a superscript SB, meaning second-best.

### 16.2.9 The Rent Extraction-Efficiency Trade-Off

For the above optimization problem (16.2.7), a technical difficulty is how to deal with these constraints. One way is to use the Lagrangian method to incorporate constraint equations into Lagrangian equations through Lagrangian multipliers, and then to deal with the optimization problem of inequality constraints according to the Kuhn-Tucker theorem. The other way that is more convenient to solve the optimal contract is to analyze these inequality constraints by identifying which are the equality (binding) constraints, and which are the strict inequality constraints so that they do not need to be considered. Let us first consider the contracts without shutdown, i.e., $q_L > 0$. This is true when the so-called Inada condition $v'(0) = +\infty$ is satisfied plus the condition $\lim_{q \to 0} v'(q)q = 0$. 


The high type’s incentive compatibility constraint (16.2.8) and participation constraint (16.2.10) must be at least one binding, otherwise we can reduce the utility $U_H$ of high type, thereby increasing monopoly profits; similarly, the low type’s incentive compatibility constraint (16.2.9) and participation constraint (16.2.11) must be at least one of the equality constraints.

From constraint (16.2.8), we know that $U_H = U_L + \Delta \theta v(q_L) > 0$, resulting in a strict inequality for high type’s participation condition (16.2.8). Thus, the incentive compatibility constraint for high type (16.2.8) must be an equality constraint.

By the monotonicity condition $q_H > q_L$ and the binding constraint equation (16.2.8), we obtain that the low type’s incentive compatibility constraint (16.2.9) must be a strict inequality constraint, therefore, the low type’s participation constraint (16.2.11) must be an equality constraint.

For this reason, in the above maximization problem (16.2.7), the binding constraints facing the monopolist are: the participation constraint for low type (16.2.11), and the incentive compatibility constraint for the high type $\theta_H$. Thus we have:

$$U_H = \Delta \theta v(q_L),$$
$$U_L = 0.$$  \hspace{1cm} (16.2.13) \hspace{1cm} (16.2.14)

Substituting these two equality constraints into (16.2.7), we have

$$\max_{q_H, q_L} (1-\beta)(\theta_H v(q_H) - cq_H) + \beta(\theta_L v(q_L) - cq_L) - (1-\beta)(\Delta \theta v(q_L)),$$  \hspace{1cm} (16.2.15)

which leads to the following first-order conditions:

$$\theta_H v'(q^{SB}_H) = c,$$  \hspace{1cm} (16.2.16)

$$\theta_L v'(q^{SB}_L) = \frac{c}{1 - (\frac{1-\beta}{\beta} \frac{\Delta \theta v}{\Delta \theta v})} > c.$$  \hspace{1cm} (16.2.17)

Obviously, in the above two first-order conditions, the interior solution satisfies: $q^{SB}_H = q^*_H$ and $q^{SB}_L < q^*_L$.

Summarizing the above discussion, when a monopolist provides a positive amount of products to both types, there is no allocation distortion for the efficient type $\theta_H$ in the separating equilibrium, but the cost is to pay for the information rent which comes from the gains screening the low type. For the low type $\theta_L$, his consumption is lower than his first-best consumption so that the principal can pay less information rent to the high type, thus there is an allocation distortion with no information rent to the low type. Formally, we have the following proposition:

**Proposition 16.2.1** Under asymmetric information, the optimal contracts entail:
(1) No output distortion for the high type with respect to the first-best, \( q_{SB}^H = q_H^* \); A downward output distortion for the low type, \( q_{SB}^L < q_L^* \) with:

\[
\left[ \theta_L - \left( \frac{1-\beta}{\beta} \right) \Delta \theta \right] v'(q_{SB}^L) = c. \tag{16.2.18}
\]

(2) Only the high type gets a positive information rent which is equal to the information rent gained from mimicking a low type:

\[
U_{SB}^H = \Delta \theta v(q_{SB}^L). \tag{16.2.19}
\]

(3) The second-best transfers are respectively given by:

\[
T_{SB}^H = \theta_H v(q_H^*) - \Delta \theta v(q_{SB}^L), \tag{16.2.20}
\]

\[
T_{SB}^L = \theta_L v(q_{SB}^L). \tag{16.2.21}
\]

The above conclusions are correlated with each other. In order to allow the high type to choose the consumption designed for him, the principal should give him some information rent that is determined by the consumption \( q_{SB}^L \) of the low type and the valuation gap of the two types, \( \theta_H - \theta_L \). The reason for the principal to reduce the consumption of the low type is to minimize the information rent. In addition, the distortion of the low type depends on the valuation gap between the two types.

When \( \theta_H - \theta_L \to 0 \), the high type’s information rent goes to zero, and the low type’s consumption will go to the efficient level. When \( \theta_H - \theta_L \to \infty \), the high type’s information rent goes to infinity. In this case, the monopolist may adopt an exclusive contract to shut down the low type to avoid paying high information rent. Thus, under asymmetric information, the monopolist faces a basic trade-off when choosing contracts: the extraction of high type’s information rent and the efficiency concern of low type’s consumption distortion.

The points \( A_{SB} \) and \( B_{SB} \) in Figure 16.3 specify the second-best contract for \( \theta_H \) and \( \theta_L \), respectively. In the second-best contract, there is no allocation distortion in \( A_{SB} \) at which the consumer \( \theta_H \) obtains an information rent of \( T_H^* - T_{SB}^H \), and in \( B_{SB} \) of second-best contract, the consumption of \( \theta_L \) is lower than social optimality with zero information rent, which is located on the indifference curve of \( \theta_L \)’s reservation utility.

The above proposition gives one of the most important conclusions of the adverse selection problem: the optimal contract under incomplete information in general is only a second-best but not first best contract, and no distortion at the top. That is, under asymmetric information, the consumption of the high type does not appear to be distorted with respect to the first-best contract under complete information, while the consumption of the low type may be distorted downward. This rule exists in almost all asymmetric information environments. For example, a capital owner (bank) faces a large
number of borrowers, but does not know their operating efficiency, so he
can only set loan thresholds or differentiate interest rates based on their
“declaration.”

In order to prevent a high type from mimicking a low type, the loan
threshold or interest rate level of low type must be distorted. This is why
small corporations often do not have sufficient credit support as large cor-
porations. They often face stricter approval and strength conditions or need
to pay higher interest rates. This phenomenon is called credit rationing, see
the classic paper by Stiglitz and Weiss (1981) for more details.

**Remark 16.2.1** As early as 2,600 years ago, Sun Tzu already had insight in-
to the basic idea of principal-agent theory and the above basic conclusions
are given in *The Art of War*, which is one of ancient Chinese war books. It
is the earliest, most outstanding and complete theory about war. Dubbed
“the Bible of Military Science”, which is also the earliest work on military s-
trategies in the world. He considered the critical importance of information
and its symmetry, and gave the basic conclusion of information economics:
in the complete information situation, the best is first-best; when information
is asymmetric, the best is second-best. It says that: if “know the enemy
and yourself, you can fight a hundred battles with no danger of defeat; if
you only know yourself, but not your opponent, you may win or may lose,
and if you know neither yourself nor your enemy, you will always endan-
ger yourself.”
16.2.10 Shutdown Policy

When there is no positive solution in (16.2.17) due to \( \beta < \frac{\Delta \theta}{\theta^L} \), \( q^S_{L} = 0 \), the monopolist will choose to shutdown the low type contract. When the low type’s distribution probability is below certain the critical level, i.e., \( \beta < \frac{\Delta \theta}{\theta^L} \), offering products to low type, or increasing the allocation efficiency of low type, becomes unworthy for the monopolist. In addition, if the difference of two types is very big and there is a low type consumption, it is necessary to provide high information rent for high type. In this case, the monopolist may also choose to shutdown the low type market, and the constraint condition becomes \( T = \theta_H v(q) \). The monopolist’s optimal choice is:

\[
\max_q \theta_H v(q) - cq.
\]

Then, we have the exclusive contract \( (q^c, T^c) \) that satisfies:

\[
\theta_H v'(q^c) = c, \quad T^c = \theta_H v(q^c).
\]

Then, the consumption of \( \theta_H \) is efficient with zero information rent, while the low type does not have any consumption.

Summarizing the discussion of this section, we conclude that under asymmetric information, the optimal outcome that one can obtain is in general only the second-best. Of course, there are some ways to improve the allocation efficiency, such as signaling, or the agent himself does not have information advantages.

16.3 Applications

This section proposes several classical settings where the basic model of this chapter is useful. Introducing adverse selection in each of these contexts has proved to be a significant improvement of standard microeconomic analysis.

16.3.1 Regulation

In a market economy, some technical conditions make the market itself unable to operate efficiently, such as in the natural monopoly industries, market competition outcomes in the occurrence of redundant construction, or the emergence of industry monopolies. As such, the government regulation may be helpful to improve the market efficiency, but we need to pay attention to the problem of information asymmetry. Unlike traditional regulatory theory, since the 1980s, the new regulatory theory has discussed the government’s optimal regulatory policy under the asymmetric information environment. We here use the principal-agent theory with adverse
selection to discuss optimal regulation. Baron and Myerson (1982) were among the earliest contributors to incentive regulation theory.

In the Baron and Myerson (Econometrica, 1982) regulation model, the principal is a regulator (such as government) who maximizes a weighted average of the consumer’s surplus $S(q) - t$ and of a regulated monopoly’s profit $U = t - \theta q$, where $\theta$ is marginal cost, and therefore, the smaller $\theta$ is, the more efficient firms are. There are two types of firm, $\theta_H, \theta_L$, where the probability of the $\theta_L$ type is $\nu$, and also $\Delta \theta = \theta_H - \theta_L$. With a weight $\alpha < 1$ for the firm profit and a weight 1 for consumer’s net revenue, the principal’s objective function is written now as

$$V = (S(q) - t) + \alpha U = S(q) - \theta q - (1 - \alpha)U.$$

Because $\alpha < 1$, it is socially costly to give up a rent to the firm. The monopoly firm owns private information about the cost. To encourage the monopolist to truthfully report the cost, the principal needs to offer a second-best incentive contract. Let $(t_H, q_H; t_L, q_L)$ be the incentive contract and $U_i = t_i - \theta_i q_i$ be the utility of the $\theta_i$ type.

The social planner’s problem is

$$\max\{\nu[(S(q_L) - \theta_L q_L - (1 - \alpha)U_L] + (1 - \nu)[S(q_H) - \theta_H q_H - (1 - \alpha)U_H]\}$$

subject to

$$U \geq \bar{U} + \Delta \theta \bar{q};$$
$$\bar{U} \geq U - \Delta \theta \bar{q};$$
$$U \geq 0;$$
$$\bar{U} \geq 0.$$

The analysis is the mirror image of that of the standard model discussed in the previous section. Now the efficient type is the one with the lower marginal cost $\theta_L$ for producing the good. Hence, the incentive compatibility constraint for efficient type and the participation constraint for inefficient type are the only binding constraints. Using the previous method, we can get

$$U_H = 0 \text{ and } U_L = \Delta \theta \bar{q}_H.$$

Maximizing expected social welfare under these two incentive compatibility and participation constraints leads to no output distortion with respect to the first-best outcome $q^{SB}_L = q^*_L$ for the efficient type and a downward distortion of the inefficient type’s output with respect to the first-best outcome, i.e., $q^{SB}_H < q^*_H$, which is given by:

$$S'(q^{SB}_H) = \theta_H + \frac{\nu}{1 - \nu}(1 - \alpha)\Delta \theta.$$

(16.3.22)

Note that a higher value of $\alpha$ reduces the output distortion, because the regulator is less concerned about the distribution of rents within society as
\( \alpha \) increases. If \( \alpha = 1 \), the firm’s rent is no longer costly and the regulator behaves as a pure efficiency maximizer implementing the first-best output in all states of nature.

Laffont and Tirole (1986, 1993) discussed a related but different incentive regulation issue. Unlike Baron and Myerson (1982), Laffont and Tirole (1986, 1993) discussed that the regulation institution can observe the cost of the firm, but does not know the type of the firm and the effort invested in reducing the cost. At this point, in order to encourage the firm to reduce cost, regulation institution needs to design an optimal incentive contract.

In a public project, the value of the society is \( S \), and the cost of the project completed by the regulated firm is \( C = \theta - e \), where \( e \) is the effort invested by the firm in reducing costs, assuming that the cost function for effort is \( \psi(e) \) which satisfies \( \psi'(e) > 0, \psi''(e) > 0 \). \( \theta \) is the type of firm. Suppose that there are two types \( \theta_H \) and \( \theta_L \), where the probability of the \( \theta_L \) type is \( \nu \).

Let \( T \) be the payment to the firm, the the utility of the firm is \( U = T - \psi(e) \). \( \lambda \) is the shadow cost of public funds, which means that if one unit is paid to the firm, the social cost is \( 1 + \lambda \). The regulator’s goal is to maximize social welfare, i.e., to maximize \( S - (1 + \lambda)(T + \theta - e) + T - \psi(e) = S - (1 + \lambda)\theta - e + \psi(e) - \lambda U \).

The regulation institution designs an incentive compatible contract that stipulates the cost of the firm and the compensation to the firm, \( (T_H, C_H; T_L, C_L) \), which satisfies \( U_i = T_i - \psi(\theta_i - C_i) \geq T_j - \psi(\theta_j - C_j) \). Using the previous principle’s tradeoff between rent extraction and efficiency, we get:

\[
\begin{align*}
\psi'(\theta_L - C_L) &= 1, \text{ or } e_L = e^*, \\
\psi'(\theta_H - C_H) &= 1 - \frac{\lambda \nu}{1 + \lambda (1 - \nu)}, \Phi'(\theta_H - C_H) < 1,
\end{align*}
\]

where \( \Phi \equiv \psi(e) - \psi(e - \Delta \theta) \). In other words, for the efficient type of firms, the effort to reduce cost under the optimal contract is socially optimal, while for the inefficient type of firms, there exists a distortion in effort.

With regard to the application of the principal-agent model in regulation theory, interested readers may refer to the classic book of Laffont and Tirole (1993) for more details.

### 16.3.2 Financial Contracts

Asymmetric information significantly affects the financial markets. For instance, the difficulty of loaning small and medium-sized firms is a worldwide problem. In the case of asymmetric information, it is even worse. Freixas and Laffont (1990) discussed the issue of optimal credit decision when information is asymmetric. The principal is a bank, and the agent is an investor that has private information about project profitability.
16.4. THE REVELATION PRINCIPLE

The bank also has borrowing cost in the credit process, such as paying interest to depositors. Assume that the interest cost per unit is $R$. Assume the bank provides a loan of size $k$ to an investor, and there are two types of productivity: $\Theta = \{\theta_h, \theta_L\}$, and the probabilities of taking $\theta_L$ and $\theta_H$ are $1 - \nu$ and $\nu$, respectively. The cost of the bank is $Rk$. Let $T(k)$ be the credit return to the bank within the agreed time. Therefore, the lender’s utility function is $V = T(k) - Rk$, where we do not consider the actual factors of some financial markets, such as credit risk and default, but only focus on the issue of credit incentives. The bank gives the investor an incentive-compatible credit contract $(T_L, k_L; T_H, k_H)$.

Let $U_i = \theta_i f(k_i) - T_i$ denote the utility of the investor $\theta_i$ under the incentive contract. If $(T_L, k_L; T_H, k_H)$ is an incentive feasible contract, then it needs to satisfy: $U_i = \theta_i f(k_i) - T_i \geq \theta_i f(k_j) - T_j = U_j + (\theta_i - \theta_j) f(k_j)$ and $U_i \geq 0$. Using the trade-off between rent extraction and efficiency in adverse selection, we easily get: for the high type, the optimal capital borrowing scale is $k_{SB}^H = k_H^*$ such that $\theta_H f'(k_H^*) = R$, and for the low type, there is a downward distortion in the size of the optimal capital lending:

$$k_{SB}^L < k_L^*$$

with

$$\theta_L f'(k_{SB}^L) = \frac{R}{1 - \frac{\Delta \theta}{\theta_L}} > R = \theta_L f'(k_L^*).$$

In this way, the loan given to a low productivity firm is even lower when information is asymmetric. Because of economic scale, small and medium-sized firms have lower productivity compared with large-scale firms. Coupled with large repayment risks, loans are more difficult. The principal-agent model can then explain the difficulty of small and medium-sized firms in loaning.

16.4 The Revelation Principle

In the above analysis, the consumer (agent) is required to report his own “type.” A natural question is whether a higher utility of principal could be achieved with a more complex contract allowing the agent possibly to choose among more options. The answer is negative. Allowing the consumer (agent) to send more general information does not result in a better outcome but only increases the complexity of the contract.

The revelation principle introduced below ensures that there is no loss of generality in restricting the principal to offer a simple menu of contracts that has at most as many options as the cardinality of the type space. This simple menu of contracts is actually an example of direct revelation mechanisms.
A **direct revelation mechanism** is a mapping \( g(\cdot) \) from the type space \( \Theta \) to the outcome space \( A \).

In the principal-agent model, a direct revelation mechanism can be written as \( g(\theta) = (q(\theta), T(\theta)) \), \( \forall \theta \in \Theta \). If the consumer (agent) reports that his type is \( \tilde{\theta} \in \Theta \), then the quantity of goods purchased by the consumer is \( q(\tilde{\theta}) \) and the payment is \( T(\tilde{\theta}) \). In a direct revelation mechanism, an important concept is to “telling the truth.”

A direct revelation mechanism \( g(\cdot) \) is **truthful** if the agent reports his type truthfully, i.e., the following incentive compatibility constraints are satisfied:

\[
\begin{align*}
\theta_H v(q(\theta_H)) - T(\theta_H) &\geq \theta_H v(q(\theta_L)) - T(\theta_L), \\
\theta_L v(q(\theta_L)) - T(\theta_L) &\geq \theta_L v(q(\theta_H)) - T(\theta_H).
\end{align*}
\] (16.4.23)

(16.4.24)

If letting \( q_H = q(\theta_H), q_L = q(\theta_L); T_H = T(\theta_H), T_L = T(\theta_L) \), we get back to the original form.

When the communication between the monopolist (principal) and the consumer (agent) is more complicated (the principal is more than just asking the agent to report his type), we can get a more general mechanism. Let \( M \) be the message space of the agent under the more general mechanism.

A **general mechanism** \( \langle M, \tilde{g}(\cdot) \rangle \) is composed of the mapping \( \tilde{g}(\cdot) \) from the message space \( M \) to \( A \):

\[
\tilde{g}(m) = (\tilde{q}(m), \tilde{T}(m)), \forall m \in M.
\] (16.4.25)

Under such a mechanism, the agent of type \( \theta \) will report the optimal information \( m^*(\theta) \) that is determined by:

\[
\theta v(\tilde{q}(m^*(\theta))) - \tilde{T}(m^*(\theta)) \geq \theta v(\tilde{q}(\tilde{m})) - \tilde{T}(\tilde{m}), \quad \forall \tilde{m} \in M.
\] (16.4.26)

Then, the mechanism \( \langle M, \tilde{g}(\cdot) \rangle \) determines the allocation rule \( a(\theta) = (\tilde{q}(m^*(\theta)), \tilde{T}(m^*(\theta))) \) that maps \( \Theta \) to \( A \).

The question is whether we can design a more complex mechanism to make the principal more profitable. The answer is negative. Consider a general mechanism \( \langle M, \tilde{g}(\cdot) \rangle \). By choosing an optimal decision, one can see that the outcome can be compounded into a direct revelation mechanism through a composite function. This is illustrated by the “principle of direct revelation” of the single agent scenario below. This result greatly reduces the complexity of finding the optimal mechanism. It shows that we only need to seek for direct revelation mechanism.
Proposition 16.4.1 (Revelation Principle) Any allocation rule $a(\theta)$ obtained by a general mechanism $⟨M, \tilde{g}⟩$ can also be implemented with a truthful direct revelation mechanism.

**Proof.** The indirect mechanism $⟨M, \tilde{g}⟩$ induces an allocation rule $a(\theta) = (\tilde{q}(m^*(\theta)), \tilde{T}(m^*(\theta)))$ from $\Theta$ into $A$. By composition of $\tilde{g}(\cdot)$ and $m^*(\cdot)$, we can construct a direct revelation mechanism $g(\cdot)$ mapping $\Theta$ into $A$, namely $g = \tilde{g} \circ m^*$, or more precisely $g(\theta) = (q(\theta), T(\theta)) \equiv (\tilde{q}(m^*(\theta)), \tilde{T}(m^*(\theta))), \forall \theta \in \Theta$.

We check now that the direct revelation mechanism $g(\cdot)$ is truthful. Indeed, since (16.4.26) is true for all $\tilde{m}$, it holds in particular for $\tilde{m} = m^*(\theta'), \forall \theta' \in \Theta$. Thus we have

\[ \theta v(\tilde{q}(m^*(\theta))) - \tilde{T}(m^*(\theta)) \geq \theta v(\tilde{q}(m^*(\theta'))) - \tilde{T}(m^*(\theta')), \forall (\theta, \theta') \in \Theta^2, \]  

(16.4.27)

Finally, by the definition of $g(\cdot)$, we get

\[ \theta v(q(\theta)) - T(\theta) \geq \theta v(q(\theta')) - T(\theta'), \forall (\theta, \theta') \in \Theta^2. \]  

(16.4.28)

Hence, the direct revelation mechanism $g(\cdot)$ is truthful. \hfill \square

The revelation principle can be illustrated by Figure 16.4.

![Figure 16.4: Revelation principle](image)

Importantly, the revelation principle provides a considerable simplification of contract design. It enables us to restrict the analysis to a well-defined family of contracts among truthful direct revelation mechanisms.

16.5 Extensions of the Basic Model

In this section, we extend the basic model to the one with a more general type space. We first discuss the case of finite number of discrete types, and then discuss the case of continuum types. Through the discussion of multiple types of adverse selection, we can further deepen the understanding of adverse selection.
16.5.1 Finite Types

We again discuss the monopolist’s choice. The consumer’s objective function is the same as before:

\[ u(q, T, \theta_i) = \theta_i v(q) - T. \]

However, here we assume that there are at least three types:

\[ \theta_n > \theta_{n-1} > \cdots > \theta_1, \quad n \geq 3. \]

Assume that the probability of taking \( \theta_i \) is \( \beta_i \). Consider that the monopolist offers different contracts \((q_i, T_i)\) for different types, and the monopolist’s optimal choice is given by the following maximization problem:

\[
\max_{\{(q_i, T_i)\}} \sum_{i=1}^{n} (T_i - c q_i) \beta_i
\]

s.t.

\[ \theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j, \quad \forall i, j, \quad (16.5.30) \]

\[ \theta_i v(q_i) - T_i \geq 0, \quad \forall i, \quad (16.5.31) \]

where (16.5.30) are incentive compatibility constraints for type \( \theta_i \), and (16.5.31) are participation constraints for type \( \theta_i \). Below we focus on the separating equilibrium.

Under the Spence-Mirrlees single-crossing condition, as consumer’s utility is increasing in \( \theta_i \), the consumption of the agent is also increasing.

We can add the incentive compatibility conditions of the \( \theta_i, \theta_j \) to get:

\[ (\theta_i - \theta_j)(v(q_i) - v(q_j)) \geq 0. \]

In this way, once \( \theta_i > \theta_j \), there will be \( q_i \geq q_j \).

Now we want to verify that the participation constraint for the least efficient type and the local downward incentive compatibility constraints for other types are binding in the monopoly’s profit-maximizing contract.

**Definition 16.5.1** A menu of contracts satisfies the **local downward incentive constraints** if

\[ \theta_i v(q_i) - T_i \geq \theta_i v(q_{i-1}) - T_{i-1}, \quad \forall \theta_i > \theta_1. \quad (16.5.32) \]

It satisfies the **downward incentive constraints** if

\[ \theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j, \quad \forall \theta_i > \theta_j. \]

Let \( q_0 = T_0 = 0 \) denote a contract that does not provide goods. Then \( \theta_1 \)'s participation constraint can be written in the form of a local downward incentive constraint:

\[ \theta_1 v(q_1) - T_1 \geq \theta_1 v(q_0) - T_0 \equiv 0. \]
Provided the incentive compatibility constraints hold, by the participation constraint of $\theta_1$ and

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_1) - T_1 \geq \theta_1 v(q_1) - T_1 \geq 0,$$

the participation constraint of other types $\theta_i > \theta_1$ is automatically satisfied. If $\theta_1$’s participation constraint is not binding, a small amount added for all $T_i$ will still make all the participation constraints and incentive compatibility constraints be satisfied, but this will increase the monopolist’s profit, so at optimal contract, $\theta_1$’s participation constraint must be binding.

We now verity that as long as local downward incentive constraints hold, global incentive constraints also hold. We first verify that if local downward incentive constraints and the monotonicity condition hold, then downward incentive constraints are satisfied. That is, $\theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j, \forall \theta_i > \theta_j$.

If the local downward incentive constraints hold for $\theta_i, \cdots, \theta_j > \theta_1$, then for all $\forall \theta_k \in \{\theta_i, \cdots, \theta_{j+1}\}$, we have

$$\theta_k v(q_k) - T_k \geq \theta_k v(q_{k-1}) - T_{k-1}.$$

Since $\theta_k \leq \theta_i$, from monotonicity condition $q_k \leq q_i$ and the above inequality, we have

$$\theta_i v(q_k) - T_k \geq \theta_i v(q_{k-1}) - T_{k-1}$$

for all $k \in \{i, i-1, \cdots, j+1\}$.

Letting $k = i, i-1, \cdots, j+1$, and making summation, we have:

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j, \forall \theta_i > \theta_j.$$

Thus downward incentive constraints are satisfied.

Second, we verify that at the maximum profit, for all $\theta_i > \theta_1$, the local downward incentive constraints must be binding:

$$\theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1},$$

otherwise, for all $j \geq i$, we can increase $T_j$ slightly, all incentive compatibility constraints and participation constraints still hold, but will increase the monopolist’s profits.

Finally, we verify that if all local downward incentive constraints are binding and the monotonicity condition is satisfied, then all upward incentive constraints also hold. This is because

$$\theta_k v(q_k) - T_k = \theta_k v(q_{k-1}) - T_{k-1},$$

$q_k \leq q_{k-1}$, $\theta_k \geq \theta_i$ and $v(q_k)$ is monotonic, then for all $k \in \{i+1, \cdots, j\}$, we have

$$\theta_i v(q_k) - T_k \leq \theta_i v(q_{k-1}) - T_{k-1}.$$
Letting $k = i + 1, \ldots, J$ in the above inequality and making the summation, we have

$$\theta_i v(q_j) - T_j \leq \theta_i v(q_i) - T_i, \forall \theta_j > \theta_i.$$  

Therefore, in a finite number of types, the monopolist’s optimal choice of contract is actually the solution of the following equality-constraint optimization problem:

$$\max_{\{q_i, T_i\}} \sum_{i=1}^{n} (T_i - cq_i) \beta_i$$

s.t.  

$$\theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1}, q_0 = T_0 = 0, \forall i, (16.5.34)$$

$$q_i \geq q_j, i > j.$$  

The Lagrangian function is

$$L = \sum_{i=1}^{n} [T_i - cq_i] \beta_i + \sum_{i=1}^{n} \lambda_i [\theta_i v(q_i) - \theta_i v(q_{i-1}) - T_i + T_{i-1}],$$

and we then have the following first-order conditions:

$$(\lambda_i \theta_i - \lambda_{i+1} \theta_{i+1}) v'(q_i) = c \beta_i, \quad i < n;$$

$$\lambda_n \theta_n v'(q_n) = c \beta_n;$$

$$\beta_i = \lambda_i - \lambda_{i+1}, \quad i < n;$$

$$\beta_n = \lambda_n.$$

Therefore, we have the following conclusions:

**Proposition 16.5.1** Under asymmetric information, for the case of finite number of types, the optimal contracts entail:

1. For the highest-type consumer, there is $\theta_n v'(q_n) = c$, so there is no consumption distortion.

2. For other types $\theta_i < \theta_n$, there is consumption distortion

$$\theta_i v'(q_i) = c - \frac{\lambda_i - \lambda_{i+1}}{\lambda_i - \lambda_{i+1} \theta_{i+1}^{-1}} > c.$$  

3. For the lowest-type consumer, there is no information rent, while other types’ information rents are positive, and as the type changes from the lowest to the highest, its information rent also increases. For $\theta_i > \theta_1$, the information rent is $\sum_{j=2}^{i} (\theta_j - \theta_{j-1}) v(q_{j-1}).$

From two types to more than two types, we find that the basic trade-off between information rent extraction and allocative efficiency still holds.
16.5. EXTENSIONS OF THE BASIC MODEL

16.5.2 Continuum of Types

We now discuss the case of a continuum of types. Most work in the principal-agent literature is done within this setting.

Consider our standard model with \( \theta \) in the interval \( \Theta = [\underline{\theta}, \overline{\theta}] \). Since the revelation principle is still valid with a continuum of types, and we can restrict our analysis to direct revelation mechanisms \( \{(q(\theta), T(\theta))\} \).

Assume that the type of consumer (agent) follows the density function \( f(\theta) \) (distribution function is \( F(\theta) \)) and the range of \( \theta \) is \( [\underline{\theta}, \overline{\theta}] \). The monopolist (principal) chooses the contract that solves the following optimization problem:

\[
\max_{\{(q(\theta), T(\theta))\}} \int_{\underline{\theta}}^{\overline{\theta}} (T(\theta) - cq(\theta)) f(\theta) d\theta \quad (16.5.36)
\]

subject to:

\[
\theta v(q(\theta)) - T(\theta) \geq \theta v(q(\hat{\theta})) - T(\hat{\theta}), \quad \forall (\theta, \hat{\theta}) \in \Theta^2, \quad (16.5.37)
\]

\[
\theta v(q(\theta)) - T(\theta) = 0, \quad \forall \theta \in \Theta. \quad (16.5.38)
\]

For the participation constraint (16.5.38), the same as before, it is enough that only the lowest type is required to satisfy the participation constraint, i.e.,

\[
\theta v(q(\theta)) - T(\theta) \geq 0.
\]

For incentive compatibility constraint, from the inequality (16.5.37), we can get:

\[
(\theta - \hat{\theta})(v(q(\theta)) - v(q(\hat{\theta}))) \geq 0.
\]

Thus, incentive compatibility alone requires that \( q(\cdot) \) must be nondecreasing, which implies that \( q(\cdot) \) is differentiable almost everywhere. So we will restrict the analysis to differentiable outcome functions. In this way, monotonicity condition implies \( \frac{dq(\theta)}{d\theta} \geq 0 \).

Incentive compatibility requirement is equivalent to the following optimization problem:

\[
\theta = \arg\max_{\hat{\theta}} \{\theta v(q(\hat{\theta})) - T(\hat{\theta})\}.
\]

The first-order necessary condition is:

\[
\theta v'(q(\theta)) \frac{dq(\theta)}{d\theta} - T'(\theta) = 0. \quad (16.5.39)
\]

(16.5.39) is also called the local incentive compatibility condition.

The second-order necessary condition is:

\[
\theta v''(q(\theta)) \left( \frac{dq(\theta)}{d\theta} \right)^2 + \theta v'(q(\theta)) \frac{d^2 q(\theta)}{d\theta^2} - T''(\theta) \leq 0. \quad (16.5.40)
\]
Differentiating (16.5.39) with respect to $\theta$, we have
\[ \theta v''(q(\theta)) \left( \frac{dq(\theta)}{d\theta} \right)^2 + v'(q(\theta)) \frac{dq(\theta)}{d\theta} + \theta v'(q(\theta)) \frac{d^2 q(\theta)}{d\theta^2} - T''(\theta) = 0. \] (16.5.41)

By comparing (16.5.40) with (16.5.41) and using $v'(q(\theta)) > 0$, the second-order condition turns out to be equivalent to the monotonicity condition $\frac{dq(\theta)}{d\theta} \geq 0$.

We now show that the local incentive compatibility implies that the global incentive compatibility constraints must be satisfied:
\[ \theta v(q(\theta)) - T(\theta) \geq \theta v(q(\hat{\theta})) - T(\hat{\theta}), \quad \forall (\theta, \hat{\theta}) \in \Theta^2. \] (16.5.42)

By the first-order condition (16.5.39), we have:
\[ T(\theta) - T(\hat{\theta}) = \int_{\theta}^{\hat{\theta}} \tau v'(q(\tau)) \frac{dq(\tau)}{d\tau} d\tau \]
\[ = \theta v(q(\theta)) - \hat{\theta} v(q(\hat{\theta})) - \int_{\theta}^{\hat{\theta}} v(q(\tau)) d\tau, \] (16.5.43)

or
\[ \theta v(q(\theta)) - T(\theta) = \hat{\theta} v(q(\hat{\theta})) - T(\hat{\theta}) + \int_{\theta}^{\hat{\theta}} v(q(\tau)) d\tau - (\theta - \hat{\theta}) v(q(\hat{\theta})). \] (16.5.44)

Since $q(\cdot)$ is non-decreasing, $\int_{\theta}^{\hat{\theta}} v(q(\tau)) d\tau - (\theta - \hat{\theta}) v(q(\hat{\theta})) \geq 0$. Thus, if the local incentive compatibility constraint (16.5.39) holds, the global incentive compatibility constraint (16.5.42) also holds.

With the above settings, continuum incentive compatibility constraints (16.5.42) simplify to a differential equation and a monotonicity constraint. As such, the first-order condition (16.5.39) and the monotonicity condition $\frac{dq(\theta)}{d\theta} \geq 0$ fully characterize the truthful revelation mechanism.

Then the monopolist’s optimization problem can be rewritten as follows:
\[ \max_{\{(q(\theta), T(\theta))\}} \int_{\theta}^{\hat{\theta}} (T(\theta) - cq(\theta)) f(\theta) d\theta \] (16.5.45)

s.t. \[ \theta v'(q(\theta)) \frac{dq(\theta)}{d\theta} - T'(\theta) = 0, \quad \forall (\theta, \hat{\theta}) \in \Theta^2, \] (16.5.46)
\[ \frac{dq(\theta)}{d\theta} \geq 0, \quad \forall \theta \in \Theta, \] (16.5.47)
\[ \theta v(q(\theta)) - T(\theta) \geq 0, \quad \forall \theta \in \Theta. \] (16.5.48)

To solve the above optimization problem, the usual procedure is to ignore the monotonicity constraints first. Define information rent function for $\theta$:
\[ U(\theta) = \theta v(q(\theta)) - T(\theta) = \max_{\theta} \theta v(q(\hat{\theta})) - T(\hat{\theta}). \]
Using the envelope theorem, we get:

\[
\frac{dU(\theta)}{d\theta} = v(q(\theta)).
\]

By monotonicity, the participation constraint of the lowest type is binding, which implies that his information rent is 0, that is, \(U(\theta) = 0\). Thus, we have

\[
U(\theta) = \int_\theta^\theta v(q(\tau))d\tau.
\]

Since \(T(\theta) = \theta v(q(\theta)) - U(\theta)\), we can write the monopolist’s objective function as:

\[
\Pi = \int_\theta^\theta \left[ \theta v(q(\theta)) - \int_\theta^\theta v(q(\tau))d\tau - cq(\theta) \right] f(\theta)d\theta.
\]

By integrating by parts, we have

\[
\Pi = \int_\theta^\theta \left[ \theta v(q(\theta)) - \theta v(q(\theta))f(\theta) - v(q(\theta)[1 - F(\theta)]) \right] d\theta.
\]

The first-order condition for \(q(\theta)\) is then given by

\[
\left[ \theta - \frac{1 - F(\theta)}{f(\theta)} \right] v'(q(\theta)) = c. \tag{16.5.49}
\]

Therefore, we get a basic conclusion: for the highest type \(\theta = \bar{\theta}\), there is no distortion in consumption, while for other types of \(\theta < \bar{\theta}\), there is a downward distortion in consumption.

We further examine the characterization of the optimal contract. \(h(\theta) \equiv \frac{f(\theta)}{1 - F(\theta)}\) is called the hazard rate. A sufficient condition for the monotonicity constraint is \(\frac{dh(\theta)}{d\theta} \geq 0\), so that the hazard rate is monotonically non-decreasing. Let \(g(\theta) = \theta - (h(\theta))^{-1}\), which is so-called virtual valuation function.

The first-order condition (16.5.49) then can be rewritten as:

\[g(\theta)v'(q(\theta)) = c,\]

which means that for the principal to maximize her profit, she will adopt the virtual value \(g(\theta)\) rather than the true type of the agent in case of asymmetric information. The virtual value of the agent is less than his true value because the latter includes the information rent of the agent and the principal cannot extract all the information rent.

Differentiating the above equation with respect to \(\theta\), we get

\[
\frac{dq(\theta)}{d\theta} = -\frac{g'(\theta)v'(q(\theta))}{v''(q(\theta))g(\theta)}.
\]
Thus, if \( \frac{dh(\theta)}{d\theta} \geq 0 \), then \( q'(\theta) \geq 0 \) so monotonicity condition is satisfied. For many continuous distributions, such as uniform distribution, normal distribution, exponential distribution, and other frequently used distributions, the monotonicity of the hazard rate is satisfied.

### 16.5.3 Bunching and Ironing

We now briefly discuss how to handle the optimal choice problem if the monotonicity of \( q(\theta) \) is not satisfied. The usual method is to iron out intervals that do not satisfy monotonicity.

Without considering the monotonicity, the previous optimal contract \( q^*(\theta) \) should satisfy:

\[
\left[ \theta - \frac{1 - F(\theta)}{f(\theta)} \right] v'(q^*(\theta)) = c.
\]

Assume that in the interval \([\hat{\theta}_1, \hat{\theta}_2]\), \( q^*(\theta) \) does not satisfy monotonicity, as shown in Figure 16.5.

![Figure 16.5: Contracts that violate monotonicity](image)

The appearance of this non-monotonic contract may be due to the fact that the hazard rate \( h(\theta) \) is not monotonic. The monopolist’s optimal contract (denoted by \( \bar{q}(\theta) \)) is the solution to the following maximization prob-
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\[ \max_{(q(\theta), T(\theta))} \int_0^\theta \left[ \theta v(q(\theta)) - cq(\theta) - \frac{v(q(\theta))}{h(\theta)} \right] f(\theta) d\theta \]

s.t.\[ \frac{dq(\theta)}{d\theta} = \mu(\theta), \]
\[ \mu(\theta) \geq 0. \]

The Hamiltonian is:

\[ H(\theta, q(\theta), \mu, \lambda) = \left[ \theta v(q(\theta)) - cq(\theta) - \frac{v(q(\theta))}{h(\theta)} \right] f(\theta) + \lambda(\theta) \mu(\theta). \]

Using the Pontryagin’s maximum principle, the optimal solution \((\bar{q}(\theta), \bar{\mu}(\theta))\) satisfies the following conditions:

\[ H(\theta, \bar{q}(\theta), \bar{\mu}(\theta), \lambda(\theta)) \geq H(\theta, q(\theta), \mu(\theta), \lambda(\theta)); \]

\[ \frac{d\lambda(\theta)}{d\theta} = - \left[ \left( \theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta), \]

\[ \lambda(\theta) = \lambda(\bar{\theta}) = 0 \quad \text{(transversality conditions)} \]

for all continuous points.

From Kamien and Schwartz (1991, pp.133), when \( H(\theta, q(\theta), \mu, \lambda) \) is a concave function with respect to \( q(\theta) \), the above necessary conditions are also sufficient conditions.

Applying the formula of integration by parts to (16.5.51) shows:

\[ \lambda(\theta) = - \int_0^\theta \left[ \left( \theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta. \]

The transversality conditions are:

\[ 0 = \lambda(\theta) = \lambda(\bar{\theta}) = - \int_0^\theta \left[ \left( \theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta. \]

The first-order condition for optimization requires \( \bar{\mu}(\theta) \geq 0 \) when maximizing \( H(\theta, q(\theta), \mu, \lambda) \), which implies \( \lambda(\theta) \leq 0 \) or

\[ \int_0^\theta \left[ \left( \theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta \geq 0. \]

As long as \( \lambda(\theta) < 0 \), we must have

\[ \bar{\mu}(\theta) = \frac{dq(\theta)}{d\theta} = 0. \]

We then have complementary slackness condition

\[ \bar{\mu}(\theta) \lambda(\theta) = 0, \]
which implies that

\[
\frac{d\bar{q}(\theta)}{d\theta} \int_{\theta}^{\theta} \left[ \left( \theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta = 0.
\]

From the above equation, we know that, if \( \bar{q}(\theta) \) is increasing in some interval, \( \bar{q}(\theta) \) must be the same as \( q^*(\theta) \). Indeed, if \( \bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d\theta} > 0 \), then \( \lambda(\theta) = 0 \), and so \( \frac{d\lambda(\theta)}{d\theta} = 0 \). Then we have \( [(\theta - \frac{1}{h(\theta)})v'(\bar{q}(\theta)) - c] = 0 \), which is the condition for \( q^*(\theta) \). Thus we must have \( \bar{q}(\theta) = q^*(\theta) \).

Thus what we need to do is to determine under which interval \( \bar{q}(\theta) \) is constant. As shown in Figure 16.6, for \( \theta_1 \) on the left side of \( \hat{\theta}_1 \) and \( \theta_2 \) on the right side of \( \hat{\theta}_2 \), we have \( \lambda(\theta) = 0 \) and \( \bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d\theta} = \frac{d\lambda(\theta)}{d\theta} > 0 \) for \( \theta < \theta_1 \) or \( \theta > \theta_2 \). But for \( \theta \in [\theta_1, \theta_2] \), we have \( \lambda(\theta) < 0 \) and \( \bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d\theta} = 0 \), which means that \( \bar{q}(\theta) \) is constant on \( [\theta_1, \theta_2] \).

By the continuity of \( \lambda(\theta) \), we know \( \lambda(\theta_1) = \lambda(\theta_2) = 0 \), and thus

\[
\int_{\theta_1}^{\theta_2} \left[ \left( \theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta = 0. \tag{16.5.53}
\]

On the other hand, by the continuity of \( \bar{q}(\theta) \), we have

\[
q^*(\theta_1) = q^*(\theta_2). \tag{16.5.54}
\]

By two equations (16.5.53) and (16.5.54), we can determine two unknowns \( \theta_1 \) and \( \theta_2 \). The new optimal contract \( \bar{q}(\theta) \) is shown in Figure 16.6.

![Figure 16.6: Optimal contract with bunching property](image-url)
Thus, the optimal contract actually “irons” the original \( q^*(\theta) \) in the interval \([\theta_1, \theta_2]\). This interval is called the bunching interval since \( \bar{q}(\theta) \) is constant on it. For further discussion, see Bolton and Dewatripont (2005, pp. 91-93).

In addition, another extension is a multi-dimension adverse selection. One of the classic problems is the monopoly pricing of multi-products. For example, in reality, many merchants often bundle different products. The multi-dimensional type of adverse selection provides an analytical framework for understanding similar bundled sales mechanisms. A detailed discussion of this can be found in Bolton and Dewatripont (2005, Chapter 6), as well as a survey by Rochet and Stole (2003).

16.6 Ex Ante Participation Constraints

The contracts we consider so far are offered at the interim stage, i.e., the agent already knows his type. However, sometimes the principal and the agent can contract at the ex ante stage, i.e., before the agent is informed of his type. For instance, the contracts of the firm may be designed before the agent receives the exact information on his productivity, say a newly graduated student does not know his own productivity and whether he can be competent for the job.

Also, in the above example of monopoly sales, if the monopoly upstream firm provide a new production tool for the downstream firm, the downstream firm does not know the true effectiveness of this new production tool before signing a sales contract, and their interactions will have different characteristics as before. In this section, we characterize the optimal contract for this alternative timing structure under various assumptions about the risk attitude of the two players.

Unlike the previous model, before signing the contract, the buyer did not know the type of value he had for the product, the timing of interaction is:

- \( t = 0 \): the seller provides contract;
- \( t = 1 \): the buyer accepts or rejects contract;
- \( t = 2 \): the buyer is informed of his own type \( \theta \);
- \( t = 3 \): contract is enforced.

To simplify the discussion, we still assume that there are only two types, \( \theta \in \{\theta_H, \theta_L\} \). The agent’s risk attitude will affect the contract design. For this reason, we discuss the optimal incentive compatible contract when the agent is risk-neutral and risk-averse, respectively.
**16.6.1 Risk Neutrality**

Suppose that the principal and the agent meet and contract ex ante. If the agent is risk-neutral, his ex ante participation constraint is now written as

\[(1 - \beta)U_H + \beta U_L \geq 0.\]  

(16.6.55)

This ex ante participation constraint replaces the two interim participation constraints.

Since the principal’s objective function is decreasing in the agent’s expected information rent, the principal wants to impose a zero expected rent to the agent and have (16.6.55) to be binding. Moreover, the principal must structure the rents \(U_H\) and \(U_L\) to ensure that the two incentive constraints remain satisfied.

\[U_H \geq U_L + \Delta \theta v(q_L),\]  

(16.6.56)

\[U_L \geq U_H - \Delta \theta v(q_H).\]  

(16.6.57)

A typical optimal contract of such a rent distribution that is both incentive compatible and satisfies the ex ante participation constraint with an equality is:

\[U^*_H = \beta \Delta \theta v(q^*_H) > 0;\]  

\[U^*_L = -(1 - \beta)\Delta \theta v(q^*_L) < 0.\]  

(16.6.58)

With such a rent distribution, the optimal contract implements the first-best outputs without cost from the principal’s point of view as long as the first-best is monotonic as requested by the implementability condition. In the contract defined by (16.6.58), the agent is rewarded when he is efficient and punished when he turns out to be inefficient.

In fact, the principal has many more options in structuring the rents \(U_H\) and \(U_L\) in such a way that the incentive compatibility constraints hold and the ex ante participation constraint (16.6.55) is binding. As shown in Figure 16.7, the information rent distribution is not unique, and in fact, there are infinitely many possibilities.

In summary, we then have the following proposition.

**Proposition 16.6.1** When the agent is risk-neutral and contracting takes place ex ante, the optimal incentive contract implements the first-best outcomes.

Consider the following contracts \(\{(T^*_H, q^*_H); (T^*_L, q^*_L)\}\), where \(q^*_H\) and \(q^*_L\) are optimal production, \(T^*_H = c q^*_H + T^*\) and \(T^*_L = c q^*_L + T^*\), with \(T^*\) being a lump-sum payment to be defined below. By definition of \(q^*_H\), we have

\[\theta_H v(q^*_H) - T^*_H = \theta_H v(q^*_H) - c q^*_H - T^* > \theta_H v(q^*_L) - c q^*_L - T^* = \theta_H v(q^*_L) - T^*_L.\]  

(16.6.59)
By definition of $q_L^*$, we have:

$$\theta_L v(q_L^*) - T_L^* = \theta_L v(q_L^*) - cq_L^* - T^*$$

$$> \theta_L v(q_H^*) - cq_H^* - T^* = \theta_L v(q_H^*) - T_H^*.$$ (16.6.60)

Therefore, the contract is incentive-compatible.

Note that the incentive compatibility constraints are now strict inequalities. Moreover, the fixed-fee $T^*$ can be used to make the agent’s ex ante participation constraint binding by choosing

$$T^* = \beta(\theta_L v(q_L^*) - cq_L^*) + (1 - \beta)(\theta_H v(q_H^*) - cq_H^*).$$

This implementation of the first-best outcome amounts to having the principal selling the benefit of the relationship to the risk-neutral agent for a lump-sum payment $T^*$. The agent benefits from the full value of the good and trades off the value of any production against its cost just as if he was an efficiency maximizer. We will say that the agent is residual claimant for the profit. From later discussion, it is known that even a principal who is risk-averse will get the same result.

In reality, many contracts are of such features.

(1) After reform and opening-up in China, the contract was implemented for contracting land and households, and the production team contracted the farmland to the farmers. The farmers needed to pay a certain amount of products to the government each year, and the rest was kept to the farmers. The subsequent production responsibility system is the same.
A building owner rents out the building to do business, the renter pays a certain rent, and the remaining profits or losses are borne by the renter himself.

A bank lends funds to enterprises at a fixed loan interest rate, and companies bear business risks. Profits or losses are all their own.

### 16.6.2 Risk Aversion

We just showed that the implementation of the first-best is feasible with risk-neutrality. What happens if the agent is risk-averse?

Consider now a risk-averse agent with a Von Neumann-Morgenstern utility function $u(\cdot)$ defined on his monetary gains $v(q) - T$, such that $u' > 0$, $u'' < 0$, and $u(0) = 0$. Again, the contract between the principal and the agent is signed before the agent discovers his type. The incentive compatibility constraints are the same as before, but the agent’s ex ante participation constraint is now written as:

$$\beta u(U_L) + (1 - \beta) u(U_H) \geq 0. \quad (16.6.61)$$

As usual, one can check that incentive compatibility constraint for the low type is slack (not binding) at the optimum, high type’s incentive compatibility constraint (16.6.56) is binding, and ex ante participation constraint (16.6.61) is also binding so that

$$\beta u(U_L) + (1 - \beta) u(U_L + \Delta v(q_L)) = 0. \quad (16.6.62)$$

and thus the principal’s program reduces now to

$$\max_{\{U_H,q_H;U_L,q_L\}} (1 - \beta)(\theta_H v(q_H) - cq_H - U_H) + \beta(\theta_L v(q_L) - cq_L - U_L)$$

subject to (16.6.56), (16.6.57) and (16.6.61).

We then have the following proposition.

**Proposition 16.6.2** When the agent is risk-averse and contracting takes place ex ante, the optimal menu of contracts entails:

1. No output distortion for the high type, $q_{SB}^H = q^*_H$. A downward output distortion for the low type, $q_{SB}^L < q^*_L$, with

   $$\theta_L v'(q_{SB}^L) = \frac{c}{1 - \frac{(1 - \beta)[u'(U_L) - u'(U_H)] \Delta \theta}{\beta u'(U_L) + (1 - \beta)u'(U_H) \theta_L}}. \quad (16.6.63)$$

2. High type’s incentive compatibility constraint (16.6.56) and ex ante participation constraint (16.6.61) are the only binding constraints. The high (resp. low) type receives a strictly positive (resp. negative) ex post information rent, $U_{SB}^H > 0 > U_{SB}^L$. 


The rent distribution is shown in Figure 16.8. Note that when the agent is risk-neutral, the second term in the right of (16.6.63) is zero, and thus we get the same conclusion as in Proposition 16.6.1: the optimal incentive contract implements the first-best outcome.

Figure 16.8: Information rent of risk-averse agent

Thus, with risk-aversion, the principal can no longer costlessly structure the agent’s information rents to ensure that the high type’s incentive compatibility constraint is satisfied. Creating a difference between $U_H$ and $U_L$ to satisfy (16.6.56) makes the risk-averse agent bear some risk. To guarantee the participation of the risk-averse agent, the principal must now pay a risk premium. Reducing this premium calls for a downward reduction in the low type’s output so that the risk borne by the agent is lower. As expected, the agent’s risk-aversion leads the principal to weaken the incentives.

When the agent becomes infinitely risk-averse, the agent’s ex ante individual rationality constraint is the same as his an ex post individual rationality constraint for the worst state of the world such that $U_{SB}^H = 0$ but not $U_{SB}^L < 0$. As such, in the limit, the agent’s outputs $q_{SB}^H$ and $q_{SB}^L$ and the utility levels $U_{SB}^H$ and $U_{SB}^L$ all converge toward the same solution as for the interim case discussed in the previous sections. So, the previous model at the interim stage can also be interpreted as a model with an ex ante infinitely risk-averse agent at the zero utility level.

### 16.6.3 Risk-Averse Principal

Consider now a risk-averse principal with a von Neumann-Morgenstern utility function $\nu(\cdot)$ defined on her monetary gains from trade, $T - cq$, such that $\nu' > 0$, $\nu'' < 0$ and $\nu(0) = 0$. Again, the contract between the principal and the risk-neutral agent is signed before the agent knows his type.
In this context, the first-best contract obviously calls for the first-best outputs \( q^*_H \) and \( q^*_L \). It also calls for the principal to be fully insured between both states of nature and for the agent’s ex ante participation constraint to be binding. This leads to the following two conditions that must be satisfied by the agent’s rents \( U^*_H \) and \( U^*_L \):

\[
\theta H v(q_H) - U^*_H - cq_H = \theta L v(q_L) - U^*_L - cq_L
\]  

(16.6.64)

and

\[
\beta U^*_L + (1 - \beta) U^*_H = 0.
\]  

(16.6.65)

Solving this system of two equations with two unknowns \((U^*_H, U^*_L)\) yields

\[
U^*_H = \beta \{[\theta H v(q^*_H) - cq^*_H] - [\theta L v(q^*_L) - cq^*_L]\};
\]  

(16.6.66)

\[
U^*_L = -(1 - \beta) \{[\theta H v(q^*_H) - cq^*_H] - [\theta L v(q^*_L) - cq^*_L]\}.
\]  

(16.6.67)

By the definition of \( q^*_H \),

\[
U^*_H - U^*_L = [\theta H v(q^*_H) - cq^*_H] - [\theta L v(q^*_L) - cq^*_L] > \Delta \theta v(q^*_H).
\]

By the definition of \( q^*_L \),

\[
U^*_L - U^*_H = [\theta L v(q^*_L) - cq^*_L] - [\theta H v(q^*_H) - cq^*_H] > -\Delta \theta v(q^*_H).
\]

Hence, the profile of rents \( U^*_H \) and \( U^*_L \) is incentive compatible and the first-best allocation is easily implemented in this framework. We can thus generalize the proposition for the case of risk-neutral as follows:

**Proposition 16.6.3** Suppose that the principal is risk-averse over the monetary gains \( T - cq \), the agent is risk-neutral, and contracting takes place ex ante. Then the optimal incentive contract implements the first-best outcome.

**Remark 16.6.1** It is interesting to note that \( U^*_H \) and \( U^*_L \) obtained in (16.6.66) and (16.6.67) are also the levels of rent obtained in (16.6.59) and (16.6.60). Indeed, the lump-sum payment \( T^* = \beta (\theta L v(q^*_L) - cq^*_L) + (1 - \beta) (\theta H v(q^*_H) - cq^*_H) \), which allows the principal to make the risk-neutral agent residual claimant for the hierarchy’s profit, also provides full insurance to the principal. By making the risk-neutral agent the residual claimant for the value of trade, ex ante contracting allows the risk-averse principal to get full insurance and implement the first-best outcome despite the informational friction.

Of course this result does not hold anymore if the agent’s interim participation constraints must be satisfied. In this case, the incentive compatibility constraint (16.6.57) is slack at the optimum, and then the principal’s program reduces to:

\[
\max (1 - \beta) \nu (\theta H v(q_H) - cq_H - U_H) + \beta \nu (\theta L v(q_L) - cq_L - U_L),
\]  

(16.6.68)
subject to high type’s incentive compatibility constraints (16.6.56), as well as low type’s participation constraints:

\[ U_L \geq 0. \]  

By substituting \( U_H \) and \( U_L \) obtained from (16.6.56) and (16.6.69) into the principal’s objective function and maximizing the objective function, we have \( q_{SB}^H = q_H^* \), which is the same with the result in the risk-neutral buyer situation. High type buyer has no consumption distortion. However, the buyer of low utility type has a downward output distortion, \( q_{SB}^L < q_L^* \), which satisfies

\[ \theta_L v'(q_{SB}^L) = \frac{c}{1 - \beta} \frac{(V_H - \Delta \theta)}{\beta v'(V_L) \theta_L}, \]  

where \( V_{SB}^H = \theta_H v(q_{SB}^H) - cq_{SB}^H - \Delta \theta v(q_{SB}^H) \) and \( V_{SB}^L = \theta_L v(q_{SB}^L) - cq_{SB}^L \) are the returns of the principal in two nature states. By the definition of \( q_{SB}^H \), we have

\[ \theta_H v(q_{SB}^L) - cq_{SB}^L < \theta_H v(q_{SB}^H) - cq_{SB}^H = \theta_H v(q_{SB}^*) - cq_{SB}^*. \]

Thus, it can be verified that \( V_{SB}^L < V_{SB}^H \). In particular, it is easy to see that the distortion on the right side of equation (16.6.70) is always less than \( \frac{c}{1 - \beta \Delta \theta} \), in the case of a risk-neutral principal. The economic implications are obvious. By increasing \( q_{SB}^L \), the gap between \( V_{SB}^H \) and \( V_{SB}^L \) can be reduced, which will provide the principal with a certain amount of insurance and increase his ex-ante earnings.

### 16.7 Extensions of the Classical Model

As shown in the preceding sections, the canonical principal-agent model makes several simplifying assumptions. First, there is no externality among agents, that is, the agent’s behavior is assumed to have no impact on the welfare of others; second, the agent is assumed to obtain a minimal type-independent level of utility if he rejects the offer made by the principal. Under these assumptions, the optimal contract exhibits no-distortion for the “best” type agent and downward distortions for all other types.

However, as Meng and Tian (2009) showed, challenges to this “no distortion at the top” convention may arise if we relax either of these two assumptions. To introduce these results, in this section we will discuss these two “challenges” in an integrated nonlinear pricing model.
16.7.1 Network Externalities

An externality is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of others in the economy.

A special case of externality is the so-called the "network externalities" that might arise for any of the following reasons: because the usefulness of the commodity depends directly on the size of the network (e.g., telephones, emails, WeChat etc.); or because of bandwagon effect, which means the desire to be in style: to have a good because almost everyone else has it; or indirectly through the availability of complementary goods and services (often known as the "hardware-software paradigm") or of the after-sale services (e.g., for automobiles). Although "network externalities" are often regarded as positive impact on each others’ consumption, it may display negative properties in some cases. For example, an individual has the desire to own exclusive or unique goods, which is called “Snob effect.” The quantity demanded of a “snob” good is higher the fewer the people who own it. (e.g., garment specially designed by a fashion designer, special design of watches, cars, or buildings.)

Consider a principal-agent model in which the principal is a monopolist of a network good with marginal production cost \( c \) and total output \( q \) who faces a continuum of consumers. The principal’s payoff function is given by \( V = t - cq \), where \( t \) is the payment received from a consumer. Consumer’s preference of the good is characterized by \( \theta \in \Theta = \{ \theta, \overline{\theta} \} \), \( \Pr(\theta = \theta) = v \), \( \Pr(\theta = \overline{\theta}) = 1 - v \), and then \( v \) can be regarded as the frequency of type \( \theta \) by the Law of Large Numbers.

A consumer of \( \theta \) type is assumed to have a utility function of \( U(\theta) = \theta V(q(\theta)) + \Psi(Q) - t(\theta) \), where \( q(\theta) \) is the amount of the good he consumes, \( Q = vq + (1 - v)\overline{q} = E[q(\theta)] \) is the total amount of consumption (network size) and \( t(\theta) \) is the tariff charged by the principal. \( \theta V(q(\theta)) \) is the intrinsic value of consuming, while \( \Psi(Q) \) is the network value. Note that we assume the network effect is homogeneous among all the consumers, namely, the network value is independent of individual preference \( \theta \) and individual consumption \( q(\theta) \). It is assumed that \( V'(q) > 0 \) and \( V''(q) < 0 \).

**Definition 16.7.1** The network is decreasing, constant, or increasing if and only if \( \Psi''(Q) < 0 \), \( \Psi''(Q) = 0 \), or \( \Psi''(Q) > 0 \), respectively.

**Remark 16.7.1** When the network capacity is large and the maintaining technology is advanced enough, an increase in one consumer’s consumption will increase the marginal utilities of others, so \( \Psi''(Q) > 0 \). When network capacity and maintaining technology are limited, consumers are rivals to one another in the sense that an increase in one consumer’s consumption will decrease the marginal utilities of others, thus \( \Psi''(Q) < 0 \).
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When network expansion benefits all the consumers with constant margin, the network value term is a linear function: $\Psi''(Q) = 0$.

The objective of the monopolist is to design a menu of incentive-compatible and self-selecting quantity-price pairs $\{q(\hat{\theta}), t(\hat{\theta})\}$ to maximize his own revenue, where $\hat{\theta} \in \Theta$ is the consumer’s announcement. The network magnitude is $Q = v\bar{q} + (1 - v)\bar{q}$. Under complete information, the monopolist’s problem is:

\[
\begin{align*}
\text{(P2)} & \quad \max_{\{U, q\}} \quad v[U(q) - cq] + (1 - v)[\bar{V}(\bar{q}) - c\bar{q}] + \Psi(Q) - \left[vU + (1 - v)\bar{U}\right] \\
& \text{s.t.} \quad IR(\hat{\theta}) : U \geq 0, \\
& \quad IR(\bar{\theta}) : \bar{U} \geq 0.
\end{align*}
\]

The first-best consumption is thus:

\[
\begin{align*}
\hat{\theta}V'(\bar{q}) + \Psi' \left(v\bar{q} + (1 - v)\bar{q}\right) &= c, \\
\bar{\theta}V'(\bar{q}) + \Psi' \left(v\bar{q} + (1 - v)\bar{q}\right) &= c.
\end{align*}
\] (16.7.71)

Under asymmetric information, two incentive compatibility constraints should be added to the above program, then we get:

\[
\begin{align*}
\text{(P3)} & \quad \max_{\{U, q\}} \quad v[U(q) - cq] + (1 - v)[\bar{V}(\bar{q}) - c\bar{q}] + \Psi(Q) - \left[vU + (1 - v)\bar{U}\right] \\
& \text{s.t.} \quad IR(\hat{\theta}) : U \geq 0, \\
& \quad IR(\bar{\theta}) : \bar{U} \geq 0, \\
& \quad IC(\hat{\theta}) : U \geq U - \Delta \theta V(\bar{q}), \\
& \quad IC(\bar{\theta}) : \bar{U} \geq \bar{U} + \Delta \theta V(\bar{q}).
\end{align*}
\]

The incentive compatibility constraint $IC(\bar{\theta})$ of high demand type and the participation constraint of the low-demand type $IR(\hat{\theta})$ are binding, then the consumptions in the optimal contract (that is second-best) are characterized by the following first-order conditions:

\[
\begin{align*}
\left(\hat{\theta} - \frac{1 - v}{v} \Delta \theta\right) V'(\bar{q}^S) + \Psi' \left(v\bar{q}^S + (1 - v)\bar{q}^S\right) &= c, \\
\bar{\theta}V'(\bar{q}^S) + \Psi' \left(v\bar{q}^S + (1 - v)\bar{q}^S\right) &= c.
\end{align*}
\] (16.7.72)

We synthesize the first-best and second-best solution by considering them as solution to the following parameterized form:

\[
\max_{\{\bar{q}, q, \alpha\}} \Pi(q, \bar{q}, \alpha)
\] (16.7.73)
where
\[ \Pi(q, \eta, \alpha) = v[\alpha V(q) - cq] + (1 - v)[\theta V(\eta) - c\eta] + \Psi(Q). \]

Note that we have the first-best contract under complete information given in (16.7.71) when \( \alpha = \theta \), and the second-best contract under asymmetric information given in (16.7.72) when \( \alpha = \theta - \frac{1-v}{v} \Delta \theta \).

We then have the following proposition.

**Proposition 16.7.1** In the presence of network externalities and asymmetric information, the direction of distortion in consumptions depends on the sign of \( \Psi''(Q) \).

1. If the network is mildly increasing, i.e., \( \Psi''(Q) > 0 \) but is not too large such that \( \Pi_{qq} \) is negative definite for all \( \alpha \in [\theta - \frac{1-v}{v} \Delta \theta, \theta] \), then the consumptions of all types exhibit one-way distortion: \( q^{SB} < q^{FB} \) and \( q^{SB} < q^{FB} \).
2. If the network is decreasing, i.e., \( \Psi''(Q) < 0 \), then the consumptions exhibit two-way distortion: \( q^{SB} < q^{FB} \) and \( q^{SB} > q^{FB} \).
3. If the network is constant, i.e., \( \Psi''(Q) = 0 \), then the rules “no distortion on the top” and “one-way distortion” in canonical settings are still available: \( q^{SB} < q^{FB} \), \( q^{SB} = q^{FB} \).

For all these cases, the network magnitude is downsized: \( Q^{SB} < Q^{FB} \).

**Proof.** The first order condition to (16.7.73) is:
\[ \Pi_q(q, \alpha) = 0, \quad (16.7.74) \]

this implies,
\[ \begin{cases} 
\alpha V'(q) + \Psi'\left(vq + (1-v)\eta\right) = c, \\
\theta V'(\eta) + \Psi'\left(vq + (1-v)\eta\right) = c.
\end{cases} \quad (16.7.75) \]

Differentiating (16.7.74) with respect to parameter \( \alpha \), we get:
\[ \Pi_q \frac{dq}{d\alpha} + \Pi_{q\alpha} = 0, \quad (16.7.76) \]

that is,
\[ \begin{pmatrix} \alpha V''(q) + v^2 \Psi''(Q) \\
v(1-v)\Psi''(Q) \end{pmatrix} \begin{pmatrix} \frac{dq}{d\alpha} \\
\frac{d\eta}{d\alpha} \end{pmatrix} + \begin{pmatrix} v(1-v)\Psi''(Q) \\
(1-v)\theta V''(\eta) + (1-v)^2 \Psi''(Q) \end{pmatrix} \begin{pmatrix} \frac{dq}{d\alpha} \\
\frac{d\eta}{d\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\
0 \end{pmatrix}. \]
Solving the above equations, we have
\[
\begin{align*}
\frac{dq}{d\alpha} &= -V'(q) \left[ \Psi'(q) + (1 - v)\Psi''(Q) \right] \\
\frac{d\eta}{d\alpha} &= \frac{\alpha V''(q) \left[ \Psi'(q) + (1 - v)\Psi''(Q) \right] + v\Psi''(\eta)\Psi''(Q)}{v'V'(q)\Psi''(Q)} \\
\frac{dQ}{d\alpha} &= v \frac{dq}{d\alpha} + (1 - v) \frac{d\eta}{d\alpha} = \frac{- v'\Psi''(q)\Psi''(\eta)}{\alpha V''(q) \left[ \Psi'(q) + (1 - v)\Psi''(Q) \right] + v\Psi''(\eta)\Psi''(Q)}.
\end{align*}
\]
(16.7.77)

From the assumption that the Hessian matrix \(\Pi_{qq}\) is negative definite, it can be verified that the 2\textit{th} diagonal element of \(\Pi_{qq}\) is negative, and thus
\[
\Psi''(\eta) + (1 - v)\Psi''(Q) < 0,
\]
and the determinant of \(\Pi_{qq}\) is positive,
\[
\det(\Pi_{qq}) = v(1 - v) \left\{ \alpha V''(q) \left[ \Psi'(q) + (1 - v)\Psi''(Q) \right] + v\Psi''(\eta)\Psi''(Q) \right\} > 0.
\]
(16.7.79)

The signs of derivatives in (16.7.77) can be determined, which are \(\frac{dq}{d\alpha} > 0\) and \(\frac{dQ}{d\alpha} > 0\), which means \(Q^{\text{SB}} < q^{\text{FB}}\) and \(Q^{\text{SB}} < Q^{\text{FB}}\). The sign of \(\frac{d\eta}{d\alpha}\) and thus the distortion direction of \(\eta\) depends on the sign of \(\Psi''(Q)\): if \(\Psi''(Q) > 0\), then \(\eta^{\text{SB}} < \eta^{\text{FB}}\); if \(\Psi''(Q) < 0\), then \(\eta^{\text{SB}} > \eta^{\text{FB}}\); if \(\Psi''(Q) = 0\), then \(\eta^{\text{SB}} = \eta^{\text{FB}}\).

**Remark 16.7.2** \(\Psi''(\cdot) > 0\) implies that the marginal value from an increase in individual consumption is higher at a higher level of others’ consumption: \(\frac{\partial V}{\partial q_i} > 0\) for \(i \neq j\). Its interpretation is that the externalities in bigger network is larger than in small network so that an agent is more eager to consume more when other agent consume more. It is consistent with the critical “strategic complementarity” assumption in Segal (1999, 2003) and Csorba (2008). This condition allows them to characterize the optimal contracts by applying monotone comparative static tools, pioneered by Topkis (1978) and Milgrom and Shannon (1994).1

### 16.7.2 Countervailing Incentives

We now discuss another cause of the failure of “no distortion on the top” and “one-way distortion” rules: the countervailing incentive problem. We

---

1By Topkis (1978) and Milgrom and Shannon (1994), a twice continuously differentiable function \(\Pi(q_1, q_2, \ldots, q_n; \epsilon)\) defined on a lattice \(Q\) is supermodular if and only if for all \(i \neq j\), \(\frac{\partial^2 \Pi}{\partial q_i \partial q_j} > 0\); furthermore, if \(\frac{\partial^2 \Pi}{\partial q_i \partial \epsilon} > 0\), then function \(\Pi\) has strictly increasing differences in \((q, \epsilon)\). Let \(q(\epsilon) = \max_{q \in Q} \Pi(q, \epsilon)\). Then for a supermodular function with increasing differences in \((q, \epsilon)\), \(q(\epsilon)\) is a strictly increasing function of \(\epsilon\) for all \(i\).
assume that the consumers can bypass the network offered by the incumbent firm and enter a competitive market including many homogenous firms. All these firms are the potential entrants of the market. Let \( \omega \) denote the marginal production cost of the entrants. We assume that the goods or services offered by the entrants are incompatible with that of the incumbent monopolist, and they have not yet formed their own consumers network. In the competitive outside market, each firm’s unit charge equals its marginal cost \( \omega \), so the representative consumer’s utility derived from consuming the entrants’ goods is 
\[
\theta V(q) - \omega q.
\]

Let 
\[
G(\theta) = \max_q [\theta V(q) - \omega q], \quad \underline{G} = G(\bar{\theta}), \quad \overline{G} = G(\bar{\theta}),
\]
and \( \Delta G = \overline{G} - \underline{G} \).

Throughout this section, we assume that the network is congestible. The entry threat gives the consumer non-zero type-dependent reservation utilities, and thus the problem of the incumbent network supplier can be represented as

\[
\begin{align*}
\max_{\{U,q\} : (U,q)} & \quad v[\theta V(q) - cq] + (1 - v)[\theta V(q) - c\bar{q}] + \Psi(Q) - [vU + (1 - v)\overline{U}] \\
\text{s.t.} & \quad IR(\theta) : U \geq \underline{G}, \\
& \quad IC(\theta) : U \geq U - \Delta \theta V(q), \\
& \quad IC(\theta) : U \geq U + \Delta \theta V(q).
\end{align*}
\]

(P5)

Note that (P5) is the same as (P3) except for the non-zero type-dependent reservation utilities \( \underline{G} \) and \( \overline{G} \). The following proposition characterizes the optimal entry-deterring contract.

**Proposition 16.7.2** The optimal entry-deterring contract depends on the marginal cost of potential entrant. Specifically, there exist positive values \( \omega_1 < \omega_2 < \omega_3 < \omega_4 \) such that:

1. If \( \omega > \omega_4 \), then \( \Delta G < \Delta \theta V(q^{SB}) \), and consequently the pricing contract is: \( q = q^{SB}, \bar{q} = \bar{q}^{SB}, U = \underline{G}, \) and \( \overline{U} = \underline{G} + \Delta \theta V(q^{SB}) \).

2. If \( \omega_3 \leq \omega \leq \omega_4 \), then \( \Delta \theta V(q^{SB}) \leq \Delta G \leq \Delta \theta V(q^{FB}) \), and consequently the optimal consumption level \( \bar{q} \) and \( \bar{\bar{q}} \) are determined by

\[
\begin{align*}
\bar{q} &= V^{-1} \left( \frac{\Delta G}{\Delta \theta} \right), \\
\bar{\bar{q}} &= V^{-1} \left( \frac{\Delta \theta V(q^{FB})}{\Delta \theta} \right) + \Psi' \left( vq + (1 - v)\bar{q} \right) = c,
\end{align*}
\]

(16.7.80)

where \( q \in [q^{SB}, q^{FB}] \) and \( \bar{q} \in [\bar{q}^{FB}, \bar{q}^{SB}] \). The consumers’ information rents are \( \delta \bar{U} = \underline{G} \) and \( \delta \overline{U} = \overline{G} \).

\(^2\)Otherwise, the entrants can share the present network with the incumbent monopolist.
3. If \( \omega_2 < \omega < \omega_3 \), then \( \Delta \theta V(q^{FB}) < \Delta G < \Delta \theta V(\bar{q}^{FB}) \), and consequently the pricing contract is \( q = q^{FB}, \bar{q} = \bar{q}^{FB}, U = G \), and \( \bar{U} = \bar{G} \).

4. If \( \omega_1 \leq \omega \leq \omega_2 \), then \( \Delta \theta V(\bar{q}^{FB}) \leq \Delta G \leq \Delta \theta V(\bar{q}^{CI}) \), and consequently the optimal consumption level \( q \) and \( \bar{q} \) are given by

\[
\begin{align*}
\bar{q} &= V^{-1} \left( \frac{\Delta G}{\Delta \theta} \right), \\
\theta V'(q) + \Psi' \left( vq + (1-v)\bar{q} \right) &= c, \\
\end{align*}
\]

(16.7.81)

where \( q \in [q^{CI}, q^{FB}] \) and \( \bar{q} \in [\bar{q}^{FB}, \bar{q}^{CI}] \), “CI” denotes “countervailing incentives.” The consumers’ information rents are \( U = G \) and \( \bar{U} = \bar{G} \).

5. If \( 0 < \omega < \omega_1 \), then \( \Delta G > \Delta \theta V(\bar{q}^{CI}) \), and consequently the optimal contract is \( q = q^{CI}, \bar{q} = \bar{q}^{CI}, U = \bar{G} - \Delta \theta V(\bar{q}^{CI}), \)

\( \bar{U} = G, q^{CI} \) and \( \bar{q}^{CI} \) are given by:

\[
\begin{align*}
\theta V'(q^{CI}) + \Psi' \left( vq^{CI} + (1-v)\bar{q}^{CI} \right) &= c, \\
\left( \bar{\theta} + \frac{v}{1-v} \Delta \theta \right) V'(\bar{q}^{CI}) + \Psi' \left( v\bar{q}^{CI} + (1-v)\bar{q}^{CI} \right) &= c. \\
\end{align*}
\]

(16.7.82)

**Proof.** In (P5), we have as many regimes as combinations of binding constraints among \( IR(\theta), IR(\bar{\theta}), IC(\theta) \) and \( IC(\bar{\theta}) \). To reduce the number of possible cases, we first give the following lemmas.

**Lemma 16.7.1** A pooling contract with \( q = \bar{q} \) and \( t = \bar{t} \) can never be optimal.

**Proof.** Suppose that the optimal contract is pooling with \( q = \bar{q} = q \) and \( t = \bar{t} = t \). There are two cases to be considered.

(i) \( \bar{\theta} V'(q) > c \). Then, increase \( \bar{q} \) by \( \varepsilon \) and the transfer by \( \bar{\theta} V'(q) \varepsilon \), the \( \bar{\theta} \)-type can remain indifferent. Since at \( (q,t) \) the marginal rate of substitution between \( q \) and \( t \) is higher for \( \bar{\theta} \)-type, this new allocation is incentive compatible. This raises the firm’s revenue by \( (1-v)[\bar{\theta} V'(q) - c] \varepsilon \).

(ii) \( \bar{\theta} V'(q) \leq c \) and \( \bar{\theta} V'(q) < c \). Then, decrease \( q \) by \( \varepsilon \) and adjust \( t \) so that \( \bar{\theta} \)-type can remain on the same indifference curve. Then the firm’s total charge will be increased by \( |c - \bar{\theta} V'(q)| \varepsilon \).

Thus, in both cases, it contradicts with the fact \( (q,t) \) is optimal contract.

\( \Box \)

**Lemma 16.7.2** If the two types are offered two different contracts, the two incentive constraints cannot be simultaneously bindings.

**Proof.** Suppose by way of contradiction that both \( ICs \) are binding. From \( \theta V(q) - t + \Psi(Q) = \bar{\theta} V(\bar{q}) - \bar{t} + \Psi(Q) \) and \( \theta V(\bar{q}) - \bar{t} + \Psi(Q) = \bar{\theta} V(q) - t + \Psi(Q) \), we have \( q = \bar{q} \) and \( t = \bar{t} \). But this is impossible by Lemma 1.

\( \Box \)
**Lemma 16.7.3** The IC and IR constraints of the same type cannot be simultaneously slack.

**Proof.** If $IR(\theta)$ and $IC(\theta)$ are both slack, increase $t(\theta)$ by a tiny increment will not violate all the constraints, but the firm’s charge will be increased.

Applying the above three lemmas, only five possible regimes are needed to be considered, which are summarized in the following table.

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Regime 1</th>
<th>Regime 2</th>
<th>Regime 3</th>
<th>Regime 4</th>
<th>Regime 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IR(\bar{q})$</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>S</td>
</tr>
<tr>
<td>$IR(\theta)$</td>
<td>S</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>$IC(\bar{q})$</td>
<td>S</td>
<td>B</td>
<td>S</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>$IC(\theta)$</td>
<td>B</td>
<td>B</td>
<td>S</td>
<td>S</td>
<td>S</td>
</tr>
</tbody>
</table>

Here “B” denotes “binding” and “S” denotes “slack”.

The regimes are ordered from 1 to 5 when $\Delta G$ increases, and $\Delta G$ itself is determined by the entrants’ marginal cost $\omega$. To find how $\omega$ affects the nonlinear pricing contract of the incumbent firm, we give the following two lemmas. Lemma 16.7.4 states the change of $\Delta G$ in different regimes, and Lemma 16.7.5 shows how $\Delta G$ is affected by $\omega$.

**Lemma 16.7.4** The optimal pricing contracts and the utility difference $\Delta G$ in different regimes are:

1. In regime 1, the optimal solution to (P5) is $q = \bar{q}^{SB}$, $\bar{q} = \bar{q}^{SB}$, $U = G$, and $\bar{U} = G + \Delta \theta V(q^{SB})$. The value of utility difference satisfies $\Delta G < \Delta \theta V(q^{SB})$.

2. In regime 2, the optimal consumption level $q$ and $\bar{q}$ are determined by

$$
\begin{align*}
q &= V^{-1} \left( \frac{\Delta G}{\Delta \theta} \right), \\
\bar{q} &= V'(\bar{q}) + \Psi' \left( vq + (1 - v)\bar{q} \right) = c,
\end{align*}
\tag{16.7.83}
$$

where $q \in [q^{SB}, q^{FB}]$ and $\bar{q} \in [q^{FB}, q^{SB}]$. The consumers’ information rents are $U = G$ and $\bar{U} = \bar{G}$. The utility difference satisfies $\Delta \theta V(q^{SB}) \leq \Delta G \leq \Delta \theta V(q^{FB})$.

3. In regime 3, the optimal solution to (P5) is $q = q^{FB}$, $\bar{q} = \bar{q}^{FB}$, $U = G$, $\bar{U} = \bar{G}$, and $\Delta \theta V(q^{FB}) < \Delta G < \theta V(q^{FB})$.  


4. In regime 4, the optimal consumption level \( q \) and \( \bar{q} \) are determined by

\[
\begin{aligned}
q &= V^{-1} \left( \frac{\Delta G}{\Delta \theta} \right), \\
\theta V'(q) + \Psi' \left( vq + (1 - v)\bar{q} \right) &= c,
\end{aligned}
\]

where \( q \in [q^{CI}, q^{FB}] \) and \( \bar{q} \in [\bar{q}_F^{FB}, \bar{q}_C^{CI}] \). The consumers’ information rents are \( U' = \bar{U} = \bar{G} \). The utility difference satisfies \( \Delta \theta V(\bar{q}^{FB}) \leq \Delta G \leq \Delta \theta V(\bar{q}^{CI}) \).

5. In regime 5, the optimal contract is \( q = q^{CI}, \quad \bar{q} = \bar{q}^{CI}, \quad U = \bar{U} = \bar{G} - \Delta \theta V(\bar{q}^{CI}), \) and \( \bar{U} = \bar{G} \). The utility difference \( \Delta G > \theta V(\bar{q}^{CI}) \).

\[ q^{CI} \text{ and } \bar{q}^{CI} \text{ are determined by:} \]

\[
\begin{aligned}
\theta V'(q^{CI}) + \Psi' \left( vq^{CI} + (1 - v)\bar{q}^{CI} \right) &= c, \\
\left( \theta + \frac{v}{1 - v} \Delta \theta \right) V'(\bar{q}^{CI}) + \Psi' \left( v\bar{q}^{CI} + (1 - v)\bar{q}^{CI} \right) &= c.
\end{aligned}
\]

**Proof.**

- In regime 1, the constraints \( IR(\theta) \) and \( IC(\bar{\theta}) \) are binding. Solving (P5) in the same way as (P3), we get the second-best solution

\[ \{ q = q^{SB}, \bar{q} = \bar{q}^{SB}; U = \bar{G}, \bar{U} = \bar{G} + \Delta \theta V(q^{SB}) \}, \]

with \( \Delta G < \Delta U = \Delta \theta V(q^{SB}) \).

- In regime 2, the constraints \( IR(\theta), IR(\bar{\theta}) \) and \( IC(\bar{\theta}) \) are binding. The optimal contract set is thus given by

\[ \{ (q, \bar{q}, U, \bar{U}) : \Delta \theta V(q) = \Delta G, \bar{V}(\bar{q}) + \Psi'(Q) = c; U = \bar{G}, \bar{U} = \bar{G} \}. \]

Substituting \( IR(\theta) \) and \( IR(\bar{\theta}) \) into the objective function, the Lagrange function of is constructed as

\[
L(q, \bar{q}) = v \left[ \theta V(q) - cq \right] + (1 - v) \left[ \bar{\theta} V(\bar{q}) - c\bar{q} \right] + \Psi(Q) - \left[ v\bar{G} + (1 - v)G \right] + \lambda \left[ \Delta G - \Delta \theta V(q) \right],
\]

where \( \lambda > 0 \) is the Lagrange multiplier of the binding constraint \( IC(\bar{\theta}) \). Then \( q \) and \( \bar{q} \) are determined by:

\[
\begin{aligned}
\left( \theta - \lambda \frac{\Delta \theta}{v} \right) V'(q) + \Psi' \left( vq + (1 - v)\bar{q} \right) &= c, \\
\bar{\theta} V'(\bar{q}) + \Psi' \left( v\bar{q} + (1 - v)\bar{q} \right) &= c.
\end{aligned}
\]
Because $\theta - \frac{\lambda}{v} \Delta \theta < \theta$, from formula (16.7.77) it is easy to verify that $q < q^{FB}$ and $\bar{q} > q^{FB}$. Substituting $IR(\bar{\theta})$ and $IC(\bar{\theta})$ into the objective function of (P5) and letting $\delta > 0$ be the Lagrange multiplier associate with the binding constraint $IR(\bar{\theta})$, we obtain the following Lagrange function:

$$L(q, \bar{q}) = v \left[ \theta V(q) - cq \right] + (1 - v) \left[ \bar{\theta} V(\bar{q}) - c\bar{q} \right] + \Psi(Q) - \left[ vG + (1 - v)(\bar{G} + \Delta \theta V(q)) \right] + \delta \left[ \Delta \theta V(q) - \Delta G \right].$$

The optimal consumptions $q$ and $\bar{q}$ are determined by:

$$\begin{cases} 
\theta V'(q) + \Psi'(vq + (1 - v)\bar{q}) = c, \\
\bar{\theta} V'(\bar{q}) + \Psi'(vq + (1 - v)\bar{q}) = c.
\end{cases} \quad (16.7.87)$$

Because $\theta - \frac{1 - v - \delta}{v} \Delta \theta > \theta - \frac{1 - v}{v} \Delta \theta$, from expression (16.7.77) we have $q > q^{SB}$ and $\bar{q} < q^{SB}$. It then suffice to show that $\Delta \theta V(q^{SB}) < \Delta G < \Delta \theta V(q^{FB})$.

- In regime 3, $IR(\bar{\theta})$ and $IR(\bar{\theta})$ are binding. Then the optimal contract is given by

$$\left\{ q = q^{FB}, \bar{q} = q^{SB}; U = \bar{G}, \bar{U} = \bar{G} \right\}.$$

From the slack ICs, we can verify that $\Delta G$ satisfies $\Delta \theta V(q^{FB}) < \Delta G < \Delta \theta V(q^{FB})$.

- In regime 4, $IR(\bar{\theta})$, $IR(\bar{\theta})$ and $IC(\bar{\theta})$ are binding. The optimal contract is:

$$\left\{ (q, \bar{q}, U, \bar{U}) : \Delta \theta V(\bar{q}) = \Delta G, \theta V(q) + \Psi'(Q) = c; U = \bar{G}, \bar{U} = \bar{G} \right\}.$$

Substituting $IR(\bar{\theta})$ and $IR(\bar{\theta})$ into the objective function, and letting $\mu > 0$ the multiplier associate with the binding constraint $IC(\bar{\theta})$, we have the following Lagrange function:

$$L(q, \bar{q}) = v \left[ \theta V(q) - cq \right] + (1 - v) \left[ \bar{\theta} V(\bar{q}) - c\bar{q} \right] + \Psi(Q) - \left[ vG + (1 - v)\bar{G} \right] + \mu \left[ \Delta \theta V(q) - \Delta G \right].$$

Thus, $q$ and $\bar{q}$ are determined by

$$\begin{cases} 
\theta V'(q) + \Psi'(vq + (1 - v)\bar{q}) = c, \\
\left( \bar{\theta} + \frac{\mu}{1 - v} \Delta \theta \right) V'(\bar{q}) + \Psi'(vq + (1 - v)\bar{q}) = c.
\end{cases} \quad (16.7.88)$$
16.7. EXTENSIONS OF THE CLASSICAL MODEL

Substituting $IR(\bar{\theta})$ and $IC(\bar{\theta})$ into the objective function, and letting $\eta > 0$ be the multiplier associate with the binding constraint $IR(\bar{\theta})$, we have the Lagrange function:

$$L(q, \bar{\eta}) = v \left[ \theta V(q) - cq \right] + (1 - v) \left[ \overline{\theta} V(\bar{\eta}) - c\bar{\eta} \right] + \Psi(Q)$$

$$- \left[ v(\overline{\theta} - \Delta \theta V(\bar{\eta})) + (1 - v)\overline{\theta} \right] + \eta [\Delta G - \Delta \theta V(\bar{\eta})].$$

Then the $q$ and $\bar{\eta}$ are determined by:

$$\begin{align*}
\theta V'(q) + \Psi'(vq + (1 - v)\bar{\eta}) &= c, \\
\left( \bar{\theta} + \frac{\eta}{1 - v} \right) V'(\bar{\eta}) + \Psi'(vq + (1 - v)\bar{\eta}) &= c. 
\end{align*}$$

(16.7.89)

To compare the different consumption levels, we make some comparative static analysis. To do so, let

$$\begin{align*}
\theta V'(q) + \Psi'(vq + (1 - v)\bar{\eta}) &= c, \\
\beta V'(\bar{\eta}) + \Psi'(vq + (1 - v)\bar{\eta}) &= c. 
\end{align*}$$

(16.7.90)

If $\beta = \overline{\theta}$, it’s the expression of $q^{FB}$ and $\bar{\eta}^{FB}$; if $\beta = \overline{\theta} + \frac{v}{1 - v} \Delta \theta$, it coincides with the countervailing incentives solution $q^{CI}$ and $\bar{\eta}^{CI}$.

Differentiating these two equations with respect to $\beta$ leads to:

$$\begin{align*}
\left[ \theta V''(q) + v \Psi''(Q) \right] \frac{dq}{d\beta} + (1 - v) \Psi''(Q) \frac{d\eta}{d\beta} &= 0, \\
v \Psi''(Q) \frac{dq}{d\beta} + [\beta V''(\bar{\eta}) + (1 - v) \Psi''(Q)] \frac{d\eta}{d\beta} &= -V'(q). 
\end{align*}$$

(16.7.91)

Thus, when $\Psi''(Q) < 0$ we have

$$\begin{align*}
\frac{dq}{d\beta} &= \frac{(1 - v)V'(q)\Psi''(Q)}{\beta V''(q)[\theta V''(q) + v \Psi''(Q)] + (1 - v)\theta V''(q)\Psi''(Q)} < 0, \\
\frac{d\eta}{d\beta} &= \frac{V'(q)[\theta V''(q) + v \Psi''(Q)]}{\beta V''(q)[\theta V''(q) + v \Psi''(Q)] + (1 - v)\theta V''(q)\Psi''(Q)} > 0. 
\end{align*}$$

(16.7.92)

Because $\overline{\theta} + \frac{\Delta \theta}{1 - v} > \overline{\beta}$ and $\overline{\theta} + \frac{\Delta \theta}{1 - v} \Delta \theta < \overline{\theta} + \frac{\Delta \theta}{1 - v}$, from formula (16.7.92), it can be verified that $\overline{\theta} > \overline{q}^{FB}$, $\overline{\eta} < \overline{q}^{FB}$, $\overline{\theta} < \overline{q}^{CI}$, and $\overline{\eta} > \overline{q}^{CI}$. Thus, $\Delta G = \Delta \theta V(\overline{\eta}) \in [\Delta \theta V(\overline{q}^{FB}), \Delta \theta V(\overline{q}^{CI})]$.

- In regime 5, $IR(\overline{\theta})$ and $IC(\overline{\theta})$ are binding constraints. Substituting $U = \overline{G}$ and $u = \overline{G} - \Delta \theta V(\overline{\eta})$ into the objective function, we obtain
the following first order conditions:
\[
\begin{cases}
\theta V'(q) + \Psi' \left( vq + (1 - v)q \right) = c, \\
(\bar{\theta} + \frac{v}{1 - v} \Delta \theta) V'(\bar{q}) + \Psi' \left( v\bar{q} + (1 - v)\bar{q} \right) = c.
\end{cases}
\] (16.7.93)

It is the countervailing incentives consumption level. Note that \(\bar{\theta} + \frac{v}{1 - v} \Delta \theta > \bar{\theta} = G\), and thus, from (16.7.92) \(q^{FB} > q^{CI}, \ q^{FB} < q^{CI}\). The difference of reservation utility satisfies \(\Delta G > \Delta U = \Delta \theta V'(q^{CI})\).

\[\square\]

**Lemma 16.7.5** Suppose that \(V(0) = 0, V'(\cdot) > 0, V''(\cdot) < 0, \) and \(V(\cdot)\) satisfies the standard Inada conditions: \(\lim_{q \to +\infty} V'(q) = 0, \) and \(\lim_{q \to 0} V'(q) = +\infty\). Then the utility difference across different states \(\Delta G = \bar{G} - \bar{G}\) is a decreasing function of the marginal cost \(\omega\), \(\lim_{\omega \to 0} \Delta G = +\infty\), and \(\lim_{\omega \to +\infty} \Delta G = 0\).

**Proof.** The first order condition of \(G(\theta) = \max_q [\theta V(q) - \omega q] \) is given by \(\theta V'(q^*) = \omega\). So the maximized utility derived from network bypassing is: \(G(\theta) = \theta [V(q^*(\theta)) - q^*(\theta) V'(q^*(\theta))]\). Let \(\Phi(q) = V(q) - q V'(q)\). Then \(\Delta G = G(\bar{\theta}) - G(\theta) = \bar{\theta} \Phi(q^*) - \theta \Phi(q^*)\), and its derivative with respect to marginal cost \(\omega\) is:
\[
\frac{d\Delta G}{d\omega} = \bar{\theta} \Phi'(q^*) \frac{dq^*}{d\omega} - \theta \Phi'(q^*) \frac{dq^*}{d\omega} = -\bar{\theta} q^* V''(q^*) \frac{dq^*}{d\omega} + \theta q^* V''(q^*) \frac{dq^*}{d\omega} = -q^* + q^* < 0.
\]

It is easy to verify that when the conditions \(V(0) = 0, V'(\cdot) > 0, V''(\cdot) < 0, \) \(\lim_{q \to +\infty} V'(q) = 0, \) and \(\lim_{q \to 0} V'(q) = +\infty\) are satisfied, \(\lim_{\omega \to 0} \Delta G = +\infty\) and \(\lim_{\omega \to +\infty} \Delta G = 0\). Figure 6 depicts the relationship of \(\Delta G\) and \(\omega\). Figure 16.9 depicts the relationship between \(\Delta G\) and \(\omega\).

From the above lemmas, one can see that if the potential entrants’ competitiveness increases, the utility differences will increase from zero to infinity. Thus, there exist positive values \(\omega_i, i = 1, 2, 3, 4\), such that \(\omega_1 < \omega_2 < \omega_3 < \omega_4\) corresponding to \(\Delta \theta V'(q^{CI}), \Delta \theta V'(q^{FB}), \Delta \theta V'(q^{FB})\), and \(\Delta \theta V'(q^{SB})\), respectively, where \(q^{CI}, q^{FB}, q^{FB}\), and \(q^{SB}\) are given in expressions (16.7.82), (16.7.71) and (16.7.72).

Thus, combining Lemmas 16.7.4 and 16.7.5, we proved Proposition 16.7.2. The proof is completed. \[\square\]

**Remark 16.7.3** When \(\omega > \omega_4\), the second-best contract is also entry deterring. It means that when the outside competitors are not efficient enough
Figure 16.9: The impact of $\omega$ on $\Delta G$

to give high demand consumers enough utility exceeding their information rents acquired from the present network, the outside market is only attractive to low demand consumers. The incumbent firm need not to change its pricing contract when facing the entry threat of a firm with low competitiveness.

When $\omega_3 \leq \omega \leq \omega_4$, we have $q \in \left[ q^{SB}, q^{FB} \right]$ and $\bar{\pi} \in \left[ \pi^{FB}, \pi^{SB} \right]$. That means when the marginal cost $\omega$ decreases to the extent that the utility difference $\Delta G$ is large enough to attract high demand consumers bypassing the present network, the monopolist must give up more information rent to him by increasing the consumption level of low demand consumers. The consumption level of high demand consumers themselves should also be lowered accordingly because of network effects. In this case, the sharper competitiveness of outside competitors makes the allocations less distorted.

When $\omega_2 < \omega < \omega_3$, asymmetric information imposes no distortion on both types’ allocation. As $\omega$ decreases and $\Delta G$ increases further, $q$ will reach the first-best level, it is suboptimal for the monopolist to increase the high type’s information rent at the cost of distorting the consumption level of the low demand consumers upward. In this case, the main task for the firm toward the high type is to prevent them from bypassing the incumbent market instead of preventing them from misreporting, the participation constraints are more difficult to be satisfied than the incentive compatibility constraints. Thus only the IRs are binding, and the first-best allocation is attained.

When $\omega_1 \leq \omega \leq \omega_2$, we have $q \in \left[ q^{CI}, q^{FB} \right]$ and $\bar{\pi} \in \left[ \pi^{FB}, \pi^{CI} \right]$. The high difference of utilities induces the low type to pretend to be a high type, from which the countervailing incentives problem arises. The $IC(\theta), IR(\theta)$ and $IR(\theta)$ in (P5) are binding. Again, the allocations of the two types will
be distorted in opposite directions. But it is different from the distortions in cases 1 and 2. The monopolist then distorts $q$ downward to curb the rent of high demand consumers. In this case, however, information rent has to be given to low demand consumers to elicit them reporting their types truthfully. The information rent is a decreasing function of high demand consumers’ consumption $\bar{q}$, and so $\bar{q}$ has to be distorted upward to reduce the information rent gained by low demand consumers, while a certain amount of the low demand type’s consumption is “crowded” out of the network so that $\bar{q}$ is distorted downward.

When $0 < \omega < \omega_1$, the allocations remain at the countervailing incentives level: $q = q^{CI}$ and $\bar{q} = \bar{q}^{CI}$. The decrease in marginal cost $\omega$ demand further upward distortion on the consumption of high demand consumers (the consumption of low demand consumers will be distorted downward accordingly). Thus, the participation constraint of the low-type has to be slackened, which means certain amount of information rent should be given to the consumers with low willingness to pay. In this case, only $IC(\theta)$ and $IR(\bar{\theta})$ are binding constraints, the low-type get information rent $G - \Delta \theta V(q^{CI})$. The incumbent firm keep reducing the tariffs ($t$ and $\bar{t}$ keep decreasing in this case) instead of distorting allocations to prevent high demand consumers from bypassing and low demand consumers from misreporting.

Changes in consumption with marginal cost $\omega$ are summarized in the following Figure 16.10.

![Figure 16.10: Impact of potential entrant marginal cost $\omega$ on second-best consumption](image-url)
16.8 Adverse Selection in Competitive Market

In this section we mainly discuss the issue of adverse selection in a competitive market. Different from the above discussions, in the competitive market, there are many suppliers offering goods to customers. However, due to the asymmetry of information, the market is inefficient, and even the competitive equilibrium may fail to exist.

George Akerlof (1940- , for his biography, see Section 15.5.2) first introduced information asymmetry in market analysis in 1970. In a used car market, a seller knows more about the car performance than a buyer. For each possible market price, used automobiles of good quality may be withdrawn from the market, and those of poor quality will enter the market. As the price drops, the quality of used automobiles in the market will become worse and worse, which may eventually lead to market collapse. Here we use the insurance market to discuss the impact of asymmetric information on market equilibrium.

In a competitive insurance market, each insured customer has his own private information, which is not known by insurance companies. For example, in health insurance, customers have information about their own health. In property insurance, there is private information about the possibility of accidents related to behavior. We introduce a simple continuous situation.

Assume that the probability of an accident \( p \in [\underline{p}, \bar{p}] \subseteq [0, 1] \) is a private information with a density function \( f(p) \), the accidental loss is \( L \) (which is identical for all types), and each customer’s utility function \( u(w) \) for wealth is the same, which is strictly concave.

Under complete information, for customers of type \( p \), in the competitive market, the premium rate is also \( p \). The \( L \) loss premium is \( I = pL \). Since the customers are risk-averse, then \( u(w - pL) > pu(w - L) + (1 - p)u(w) \).

Under incomplete information, there is only one type of contract in the market in the Akerlof (1970) model. Consider a simplified scenario, assuming that the contract is full loss insurance, the insurance company has no customer’s private information, the premium rate is \( r \), and the cost of insurance is \( I = rL \). If market equilibrium exists, it has the following structure: there exists \( \hat{p} \) such that:

\[
\begin{align*}
    u(w - I(\hat{p})) &< pu(w - L) + (1 - p)u(w), &\text{if } p < \hat{p}, \\
    u(w - I(\hat{p})) &> pu(w - L) + (1 - p)u(w), &\text{if } p > \hat{p}, \\
    u(w - I(\hat{p})) &= \hat{p}u(w - L) + (1 - \hat{p})u(w), &\text{if } p = \hat{p},
\end{align*}
\]

where \( I(\hat{p}) = \int_{\underline{p}}^{\hat{p}} pf(p)dp \).

Let \( \tilde{p} = \int_{\underline{p}}^{\hat{p}} pf(p)dp \). If \( u(w - I(\tilde{p})) < pu(w - L) + (1 - p)u(w) \), it means \( p \in [\underline{p}, \tilde{p}] \) types cannot be covered in the insurance market, leading to mar-
ket failure. In some extreme cases, only some customers with the highest probability of accidents will be covered by the market, resulting in the phenomenon of Gresham’s Law: “bad money drives out good money.” The following example reveals the possibility of a collapse of the second-hand goods market, which is discussed in more detail in Akerlof (1970).

Example 16.8.1 Suppose that in a used car market, the seller knows the quality of the car, and it is characterized by $\theta$, which follows a uniform distribution over $[0, \bar{\theta}]$. For a $\theta$ type vehicle, the seller’s rating is $\theta$, and the buyer’s rating is $k\theta$, which satisfies $1 < k < 2$. Obviously, under complete information, all used cars (except $\theta = 0$) will be traded. If the transaction price is $p(\theta)$, after transaction, the utility of two parties is $p(\theta) - \theta$ and $k\theta - p(\theta)$, respectively, and market equilibrium will be Pareto optimal. However, if the used car quality information is asymmetric, when the market price is $p$, the used car with $\theta \in [0, p]$ will still be on the market. At this time, the average quality is $\frac{p}{2}$. If consumers purchase, their expected utility is $kp\frac{p}{2} - p < 0$, the used car market does not exist at all.

16.8.1 Screening Mechanism of Competitive Market under Asymmetric Information

When there are more than one type of contracts, different sellers may choose different contracts to cater to different consumers. In this case, market equilibrium will have new features, such as the absence of pooling equilibrium. Rothschild and Stiglitz (1976) first introduced a contract screening mechanism in the competitive market. Below, we briefly discuss the various types of contract equilibrium in the competitive market.

For the sake of simplicity, suppose there are two types of consumers in the market. The probability of accidents is $p_L < p_H$. There are many insurance companies in the market and they can provide consumers with different contracts. Assume that the distribution probability of the low accident type is $\lambda$. The contract form provided by insurance company $i$ is $(I_i, D_i)$, where $I_i$ is the premium and $D_i$ is the amount of compensation. If the consumer of type $i$ accepts the contract, his expected utility is:

$$p_iu(w - D_i - I_i) + (1 - p_i)u(w - I_i).$$

Assume that the insurance process is divided into two steps. Firstly, each insurance company provides insurance contracts; secondly, the customer chooses one of the most favorable insurance contracts. We also assume that an insurance company has the ability to fulfill its contract so that even if the insurance contract is loss-making, it will be implemented. In addition, in the competitive environment, an insurance company’s expected profit is zero, and imagine that insurance companies compete in a Bertrand-like manner.
In the equilibrium, there may be two forms of pure strategies. The first form is a pooling equilibrium, that is, all insurance companies choose the same contract which attracting all types of customers. The second one is a separating equilibrium, i.e., there are two different types of insurance contracts in the market, and different types choose different contracts.

We first discuss the pooling equilibrium. Assume that $\alpha = (I, D)$ is the insurance contract. At this point, for both types, their expected benefits are:

$$U^i \equiv p_i u(w - D - I) + (1 - p_i) u(w - I).$$

The zero profit condition implies:

$$I = [p_L \lambda + p_H (1 - \lambda)](L - D).$$

In the pooling contract, low-accident type of consumers subsidize high-accident type. Then, there are positive profit opportunities in the insurance market. New insurance companies can design new contracts $\beta = (\hat{D}, \hat{I})$ to attract low-accident type of consumers only if they meet:

$$p_H u(w - D - I) + (1 - p_H) u(w - I) > p_H u(w - \hat{D} - \hat{I}) + (1 - p_H) u(w - \hat{I}),$$

$$p_L u(w - D - I) + (1 - p_L) u(w - I) < p_L u(w - \hat{D} - \hat{I}) + (1 - p_L) u(w - \hat{I}),$$

$$\hat{I} > p_L (L - \hat{D}).$$

Let $W_1 = u(w - I)$ and $W_2 = u(w - D - I)$. Figure 16.11 shows such a new contract.

![Figure 16.11: Pooling equilibrium does not exist](image)

When $\hat{D} < D$, $\hat{I} < I$, low-accident type of consumers are less sensitive to deductibles, but are more sensitive to insurance costs; however, for high-accident type of consumers, the opposite is true. The insurance company can profit from low-accident type of customers.
For the separating equilibrium, there are two insurance contracts in the market: one is for the low-accident type $\alpha^L : (D_L, I_L = p_L(L - D_L))$, and the other is for the high-accident type $\alpha^H : (D_H, I_H = p_H(L - D_H))$. Rothschild and Stiglitz (1976) found that separating equilibrium may not exist under certain conditions.

We know that high-accident type of consumers are very sensitive to deductibles, so it is assumed that the high-accident contract is $\alpha^H = (0, I_H = p_H L)$, while the low-accident contract $\alpha^L = (D_L, I_L = p_L, L - D_L)$ satisfies:

$$U^L = p_L u(w - D_L - I_L) + (1 - p_L) u(w - I_L)$$

s.t. $u(w - I_H) \geq p_H u(w - D_L - I_L) + (1 - p_H) u(w - I_L)$.

In the separating equilibrium, the low-accident contract is not related to the distribution of the two types of participants in the market, but is greatly affected by $p_H - p_L$, because it is necessary to avoid the high accident type choosing the low-accident contract. This however makes low-accident type of customers face high risks. If the probability of the low accident type in the market is high, there will be a positive profit opportunity in the market, which will allow the new insurance company to provide a pooling contract that caters to both types of customers (the contract $\gamma$ in Figure 16.12). Figure 16.12 depicts such a new contract.

![Figure 16.12: Separating equilibrium does not exist](image-url)

For the existence of an equilibrium, Wilson (1977) introduced the “anticipatory equilibrium.” Under this concept of equilibrium, if a new contract will result in a loss after the old contract is withdrawn, then the new contract will not be introduced. Wilson proved that under such constraint, a
market equilibrium exists, and a pooling equilibrium may also exist. Riley (1979) introduced “reactive equilibrium.” Under this concept of equilibrium, if a series of chain reactions will be triggered after the introduction of a new contract and the original new contract loses money, the new contract will not be introduced. Similarly, under such constraint, a market equilibrium also exists.

Hellwig (1987) and Engers and Fernandez (1987) provided the game theoretic foundations for the anticipatory equilibrium and reactive equilibrium, respectively. For the new equilibrium’s ability to interpret real-world problems, Kreps (1990, pp. 645) believed that this is closely related to the market regulatory environment. If the regulatory agency does not allow the initial contract (even if it loses money) to exit, the reactive equilibrium will be closer to reality. If the regulatory agency allows the original contract to withdraw from the market, then the anticipatory equilibrium will be closer to reality.

16.8.2 Signaling under Asymmetric Information

We discussed the signal mechanism of education in Part 3 on game theory. Different employees reveal the ability information through diplomas. Below we use an example (Tirole, 1988) to discuss how producers can show their quality information through some mechanism in the presence of asymmetric information.

Let $s$ be the quality of the product, and the utility level of the consumer at price $p$ be $\theta_s - p$. Assume $s \in \{0, 1\}$ and the production cost per unit of enterprise is $c_s$ ($c_0 < c_1$). We mainly discuss the monopoly situation. There are two periods repeated purchases of the product. Assume that the quality of the product cannot be changed and consumers will understand the quality of the product after using the product. Let $\delta$ be the time discount rate. The monopolist can display quality information by pricing, and $p_1$ shows high-quality price information. The profit of high-quality type is:

$$\pi_1 = (p_1 - c_1) + \delta(\theta - c_1).$$

In order to avoid the imitation of low-quality type, it is required that:

$$p_1 \leq c_0,$$

which implies:

$$\pi_1 \leq \delta(\theta - c_1) - (c_1 - c_0).$$

If $\delta(\theta - c_1) > (c_1 - c_0)$, there is a high-quality revelation equilibrium. The monopolist asks $p_1 = c_0$ for overall profit by obtaining a monopoly profit in the second stage to make up for the first-stage profit loss. If $\delta(\theta - c_1) \leq (c_1 - c_0)$, there is no separating equilibrium of high-quality goods. $\delta(\theta - c_1)$
is the benefit obtained by high-quality type in revelation equilibrium, and $c_1 - c_0$ is the cost of showing high quality. For this reason, a separating equilibrium can exist if and only if high-quality type shows that its benefits outweigh its costs.

In the separating equilibrium, we found that the monopolist established his reputation through initial price concessions. In real life, some firms choose low prices to attract customers when they open a business, the logic behind is related to this. However, there are many ways for a company to display signals. For example, a company can display its type through extravagant advertising expenditures.

Let $A$ be an extravagant advertisement expenditure. Assume that it is a signal of quality. When $A = \theta - c_0$, we find that low-quality companies have no incentive to imitate because

$$\pi_0 \leq (\theta - c_0) - (\theta - c_0) = 0,$$

while the profit of a high-quality type under advertising investment is:

$$\pi_1 = (\theta - c_1) + \delta(\theta - c_1) - (\theta - c_0) = \delta(\theta - c_1) - (c_1 - c_0).$$

Therefore, when $\delta(\theta - c_1) > (c_1 - c_0)$, the advertisement becomes a signal for displaying the quality information of the company.

In addition, the company can also display the quality information through some warrants, after-sales services, etc, the mechanism behind which is that high-quality companies have higher returns than low-quality companies in terms of displaying signals, such as those achieved by repeated purchases in the future.

16.9 Further Extensions

The main theme of this chapter was to determine how the fundamental conflict between rent extraction and allocative efficiency could be solved in a principal-agent relationship with adverse selection. In the model discussed in this chapter, because there is only one binding incentive constraint and one binding participation constraint, this conflict is relatively easy to understand. Here we mention several possible directions for extension. We can consider two-dimensional adverse selection models, models that involve random participation constraints, limited liquidity constraints, or models of supervision of the agent by the principal. For a detailed discussion of these topics and their applications, interested readers can refer to Laffont and Martimort (2002).
16.10 Biographies

16.10.1 Oskar Lange

Oskar Ryszard Lange (1904-1965), a Polish economist, politician and diplomat. He earned his doctorate in law in 1928 and taught statistics and economics at the University of Kraków, Poland. He taught statistics and economics at the University of Michigan and the University of Chicago from 1938 to 1945. From 1945 to 1946, he was the Polish ambassador to the United States. From 1946 to 1947, he also served as Poland’s delegate to the United Nations Security Council. After 1948, he was a professor who taught statistics and economics at the University of Warsaw, Poland. When the Polish Workers’ Party and the Socialist Party merged into the Polish United Workers Party, he was elected as a member of the Central Committee.


He is best known for advocating the use of market pricing tools in socialist systems and providing a model of market socialism. Lange proposed the decentralized model of the socialist economy which is the famous “Lange model” for the first time in the controversy with Mises (1881-1973) and Hayek (1899-1992) in the mid 1930s. Through analysis of the model, he believed that in the socialist economy, prices are not arbitrarily established, but are as objective as the market prices in the free competition system. It maintains that although the means of production are nationalized, the prices of consumer goods and labor are still priced through the market, while the price of the means of production is simulated by the planning agency, following the same “trial and error method” as the competitive market mechanism. Lange introduced the role of the market mechanism into the socialist economy and created a precedent for the analysis of the operation of the market mechanism in the socialist economy.

During the World War II, Lange studied the issue of price and employment in the capitalist economy. He started from the general equilibrium theory and analyzed the currency effect. He pointed out that price elasticity can lead to the automatic maintenance or restoration of supply and demand balance of production factors only under special conditions. However, he believed that the possibility of reaching this special condition in the capitalist economy since the beginning of World War I in 1914 is very small.
Lange also did a lot of pioneering work in applying econometrics to plan the socialist national economy and applying the cybernetics method to economic research. He regarded the scientificization of the plan as the theme of the theory of socialist economic operation which formed the modern scientific planning theory.

16.10.2 George Akerlof

George A. Akerlof (1940– ) was born in New Haven, United States. He received a Ph.D. degree at the Massachusetts Institute of Technology in 1966 and has been Koshland Professor of Economics Emeritus at the University of California, Berkeley since 1980. In 2001, he was awarded the Nobel Memorial Prize in Economic Sciences with two other American economists, Michael Spence and Joseph Stiglitz, for their “analyses of markets with asymmetric information”.

Akerlof’s paper “The Market for ‘Lemons’: Quality Uncertainty and the Market Mechanism” published in 1970 was one of the most important research in the information economics literature. It proposed a simple and profound generalization idea, which had huge impact due to its wide applications. In this paper, he introduced a well-known model in the study of information economics—“The Market for Lemons”, which was mainly used to describe a situation where a seller had more information on the quality of goods than a buyer, and poor quality goods would expel good quality goods, resulting in a reduction in the average quality of goods on the market. Akerlof’s theory is widely used in quite different fields such as health insurance, financial markets, and employment contracts.

Today, “lemon”, the colloquial expression of the defective used car, has become a famous metaphor in the economist’s theoretical vocabulary. He believed that the problem of information asymmetry may lead to the collapse of the entire market, or contraction of the market, so that the market is flooded with only inferior goods. Akerlof also pointed out that similar information asymmetries were particularly common in developing countries and had a significant impact. He used India’s credit market in the 1960s as an example to illustrate the issue of adverse selection. The interest rate claimed by lenders in small places in India was twice the interest rate in big cities. A lender who borrowed in towns and then lent out in rural areas did not know the borrower’s reputation and was therefore vulnerable to heavy losses. A key insight in the “Lemon thesis” is that economic agents have strong incentives to offset the adverse effects of information problems on market efficiency. Akerlof thought that many market institutions could be seen as ways to solve asymmetric information problems.

Akerlof believes that economic theorists, like French chefs in regard to food, and it is necessary to develop styled models whose ingredients are limited by unwritten rules. Just as traditional French cooking does not use...
raw fish or seaweed, so neoclassical economic models do not make assumptions from psychology, anthropology, or sociology. He disagrees with any rules that limit the nature of the ingredients in economic models. Therefore, in addition to the research on asymmetric information, Akerlof also developed economic theory that borrows from sociology and social anthropology. The most noteworthy contribution in this regard was his concern about the efficiency of the labor market.

16.11 Exercises

Exercise 16.1 Consider the principal-agent model with unobserved cost function of agent. Different cost functions \( C(q, \theta) \) represent different types. Assume \( C_q > 0, C_\theta > 0, C_{qq} > 0 \) and \( C_{q\theta} > 0 \), where \( q \) is the perfectly observed output of the agent. Assume two types of agent, \( \theta \in \{ \theta_L, \theta_H \} \), with the probability \( \nu \) and \( 1 - \nu \) and \( \Delta \theta \equiv \theta_H - \theta_L > 0 \). Output is evaluated at \( S(q) \) by the principal, where \( S' > 0, S'' < 0 \) and \( S(0) = 0 \). Let \( T \) denote the transfer from the principal to the agent. The payoff function of the agent is \( T - C(q, \theta) \). The principal and the risk-neutral agent signed the contract at the interim stage.

1. Write out the principal’s optimization problem satisfying incentive feasible constraints.
2. What is the optimal payment of \( T_H \) and \( T_L \) and economic rent for each type?
3. Find out the first-order conditions for the problem of the principal’s incentive compatibility. What do they mean?
4. Let \( S(q) = q, C(q, \theta) = \theta q^2 / 2, \theta_L = 1, \theta_H = 2, \nu = 2/3 \). Solve for the optimal contract.

Exercise 16.2 (Risk Averse Principal) Under the same setup as the previous question, suppose that the principal is risk-averse and define the monetary benefits of the transaction, namely \( S(q) - t \), of vNM utility function \( v(\cdot) \), satisfying \( v' > 0, v'' < 0 \) and \( v(0) = 0 \).

1. Write down the principal’s optimization problem satisfying incentive compatibility constraints.
2. What is the optimal payment of \( T_H \) and \( T_L \) and economic rent for each type?
3. Find the first-order conditions of the principal’s problem. Comparing the output distortions of the risk-averse principal and the risk-neutral principal, which distortion of the output is greater? why?
4. Suppose that \( v(x) = \frac{1-e^{-rx}}{r} \). Find out the first order condition for the problem. What happens when \( r \to 0 \)? Does the solution converge to the distortion of the situation with a risk-neutral principal and an interim participation constraint? What happens when \( r \) approaches infinity? Can the first-best output be achieved?

**Exercise 16.3 (Shutdown Contract)** Under the setting of Exercise 16.1, assume that the cost function is \( C(q, \theta) = \theta q \). Solve for the optimal contract with shutdown property.

1. Under what conditions, the optimal contract will shut down the inefficient participant.

2. Assume that the reservation utility is \( U_0 \). Prove that the conditions for the shutdown contract are \( \frac{v}{1-r} \Delta \theta \eta q^{SB} + U_0 \geq S(q^{SB}) - \theta q^{SB} \).

**Exercise 16.4 (State-dependent fixed costs)** Under the same settings as Exercise 16.1, assume that the cost function is given by \( C(q, \theta) = \theta q + F(q) \), and the fixed cost satisfies \( F(\theta_L) > F(\theta_H) \), where higher marginal costs are associated with lower fixed costs and vice versa.

1. Write down the principal’s optimization problem satisfying incentive feasible constraints.

2. What is the optimal payment of \( T_H \) and \( T_L \) and economic rent for both types?

3. Find the first-order conditions of the principal’s incentive compatibility problem.

4. Discuss the different intervals of the solution, based on the different positions of \( F(\theta_L) - F(\theta_H), \Delta \theta q_H, \) and \( \Delta \theta q_L \).

**Exercise 16.5 (Three types of agent)** Under the setting of Exercise 16.1, assume there are three possible types \( \theta \) of agent, \( \{\theta_1, \theta_2, \theta_3\} \), and \( \theta_3 - \theta_2 = \theta_2 - \theta_1 = \Delta \theta \), with probability \( v_1, v_2, v_3 \), respectively. The direct revelation mechanisms is \( \{(t_1, q_1), (t_2, q_2), (t_3, q_3)\} \). Other setting is the same as Exercise 16.1. The cost function is \( C(q, \theta) = \theta q \).

1. Write down the economic rents, participation constraints, and incentive compatibility constraints for the three types.

2. Simplify participation constraints and incentive compatibility constraints.

3. If \( v_2 > v_1 v_3 \), prove that the monotonicity condition is binding and solve for the optimal solution.
4. If $v_2 \leq v_1 v_3$, solve for the optimal contract.

**Exercise 16.6 (Agent with Any Number of types)** Assume that there are $n$ types of agent, $\theta_n > \cdots > \theta_2 > \theta_1$. The probability of each type $\theta_i$ is $\beta_i$, one can refer to the multi-type question in section 16.5 of the textbook.

1. Give the Spence-Mirrlees single crossing condition.

2. Write down the local downward incentive compatibility constraints.

3. Prove that the single-crossing condition implies the monotonicity and local incentive compatibility constraints.

4. Prove that at the optimal solution, all local downward incentive compatibility constraints (LDICs) are binding.

5. Prove that all LDICs are binding, plus the monotonicity condition, guarantee that all local upward incentive compatibility constraints (LUICs) are satisfied. Also prove global incentive compatibility constraints hold.

6. Solve the simplified optimal contract problem.

**Exercise 16.7 (Continuum Types of Agent)** In the above question, the number of types is finite. It is now assumed that there is a continuum of types, that is, $\theta$ no longer takes a finite number of values, but is defined on the interval $[\underline{\theta}, \bar{\theta}]$, and the density function is $f(\theta)$. Due to the revelation principle, we only consider the direct revelation mechanism $\{q(\theta), T(\theta)\}$.


2. Write down the local downward incentive compatibility constraint.

3. Refer to the content of Section 16.5.2, write down monotonicity condition, local incentive compatibility constraints, and Hamiltonian function of the optimal contract problem.

4. Solve the optimization problem and explain the results. Compare the optimal contract problem with the continuous type to the two-types case in Exercise 16.1.

**Exercise 16.8** Consider a screening model. The seller’s cost function is $C(q) = q^{1/2}$. If a buyer consumes $q$ units of goods, his utility function is $\theta \ln q$. The buyer’s type $\theta$ is private information and follows a uniform distribution on $[0, 1]$.

1. Wrote down a monopolist’s optimization problem.

2. Solve for the optimal contract $(q(\theta), t(\theta))$. 
3. Find the optimal non-linear pricing rule, $t(\theta)$, that implements the direct revelation mechanism.

**Exercise 16.9** A monopolist wants to sell a single item to a consumer. The latter’s willingness to pay for this item is $t_1$ or $t_2 (> t_1)$. The seller knows that the buyer is subject to liquidity constraint and the maximum payment for the item is $t \in \{t_1, t_2\}$. The buyer’s willingness to pay is private information, and the seller only knows the probability $\mu(t_i)$ of type $t_i$. This item is worth $c < t$ to the seller. Suppose that both the buyer and the seller are risk-neutral. Let $v$ denote the probability that the transaction occurs, $h$ denote the payment to the seller, and $t_i$ be the value of the buyer. The buyer’s revenue function is:

$$vt_i - h,$$

and the seller’s revenue function is:

$$h - vc.$$

In addition, regardless of the type, the buyer’s reservation utility is 0.

1. Give the participation constraints, incentive compatibility constraints and liquidity constraints required by a direct mechanism.

2. Find the direct mechanism for maximizing the expected return of the seller.

**Exercise 16.10 (Bunching and Ironing)** In the agency problem with a continuum of types, it is generally assumed that the hazard rate condition, that is, the hazard rate $h(\theta) = \frac{f(\theta)}{1-F(\theta)}$ is increasing in $\theta$.

1. Prove that if the hazard rate condition holds, the monotonicity condition is satisfied in the implementation of the optimal contract.

2. If the monotonicity condition does not hold, then in most cases we need to re-solve the optimization problem. One can refer to the content of Section 16.5.3 of the textbook, and need to redefine the Hamiltonian function and the corresponding constraints.

3. Use Pontryagin maximum principle to re-solve the optimal contract problem.

**Exercise 16.11** Suppose that two persons consider trading some kind of asset at price of $p$. This asset can only be used as a store of wealth. Person 1 currently owns this asset. Everyone’s evaluation of this asset is only known to himself, and each person only cares about the expected value of the asset after one year. Assume that only when both sides think that the transaction can make their situation better will they trade at the price of $p$. Prove the probability of a transaction is zero.
Exercise 16.12 Consider the following process. First, the nature determines the type of worker. It is continuously distributed over the interval \([\theta, \bar{\theta}]\). Once the worker knows his type, he can choose whether to participate in an exam without cost. The exam accurately reflects his ability. Finally, after observing whether the worker took the test and the performance of the worker who took the test, the two companies started bidding on the service of the worker. Prove that in any subgame perfect Nash equilibrium of this model, all worker types take the exam, and the company does not provide any salary greater than \(\bar{\theta}\) for any worker who does not take the exam.

Exercise 16.13 (Lending with Adverse Selection) Suppose that there is a continuum of risk-neutral borrowers with personal wealth and limited liability. A proportion \(\nu\) of borrowers (called type 1) have risk-free projects with return \(h\) for an investment of 1. A proportion \(1 - \nu\) of borrowers (called type 2) have stochastically independent projects with return \(h\) with probability \(\theta\) belonging to \((0, l)\) and return 0 with probability \(1 - \theta\), for an investment of 1. If he does not apply for a loan the borrower has an outside opportunity that generates a utility of \(u\). There is a single risk-neutral bank available for loans which has a financing cost of \(r\). The bank offers contracts to maximize its expected profit. For simplicity, we assume that all projects are socially valuable, i.e.,

\[\theta h > r + u.\]

1. Explain why there is no loss of generality in considering the menu of contracts, \((r_1, P_1)\) and \((r_2, P_2)\), where \(P_i\) is the probability of obtaining a loan and \(r_i\) is the repayment to the bank when the investment succeeds, whenever the borrower announces that he is of type \(i\).

2. Write down the maximization program of the bank which chooses the menu, \((r_1, P_1)\) and \((r_2, P_2)\), to maximize its expected profit under the borrower’s participation and incentive compatibility constraints (for simplicity assume that if a borrower applies for a loan he loses his outside opportunity).

3. Show that the optimal contract entails a non random allocation of loans, i.e., \(P_i\) is either 0 or 1, for \(i = 1, 2\). Characterize the optimal contract and discuss.

Exercise 16.14 (Bribery Game) Consider a regulatory agency to provide a service to citizens with a fixed period of delay. Under the normal operation of the regulatory agency, citizens can get the utility of \(u_0\) (depending on their valuation of time). Officials who devote extra effort can shorten the delay period. Officials can shorten the delay period of \(q\) at the cost of \(\frac{(q - Q)^2}{2}\), where \(Q\) is a constant. Assume that there is \(\nu\) (or \(1 - \nu\)) of type 1
(type 2) citizens rating \( q \) as \( \theta_L q (\theta_H q) \). Citizens are willing to bribe officials to shorten the delay period. Write down the optimal bribe contract that officials provide to citizens.

**Exercise 16.15**

Monopolists can produce a product at different quality levels. The cost required to produce a unit with a mass of \( s \) is \( s^2 \). Consumers buy up to one unit of product. If it consumes a product with unit mass \( s \), its utility function is \( u(s|\theta) = s\theta \). The monopolist can determine the price and quality of the product. Consumers observe the quality and price of the product and decide what quality of product to buy.

1. Solve for the optimal solution.
2. Suppose that seller cannot observe \( \theta \), and prob \((\theta = \theta_H) = 1 - \beta \), prob \((\theta = \theta_L) = \beta \), where \( \theta_H > \theta_L > 0 \). Solve for the second-best outcome and information rent of consumer.
3. Suppose that \( \theta \) follows a uniform distribution on the interval \([0, 1]\). Find the second-best contract.

**Exercise 16.16 (Labor Contract)**

Consider the following setting. A firm faces a worker. The worker’s utility function is \( U^A = u(c) - \theta l \), where \( c \) is consumption and \( l \) is labor supply. \( \theta \in \{\theta_L, \theta_H\} \) is a parameter whose value is privately known by the worker. \( \theta_L < \theta_H \), and \( u(\cdot) \) is an increasing and concave function. The fraction of worker who has a lower disutility \((\theta = \theta_L)\) is \( \nu \). The agent’s optimal choice must satisfy the budget constraint \( c \leq T \), where \( T \) is the payment he received from the employer. The utility function of the employer is \( U^P = f(l) - T \), where \( f(l) \) is a production function with decreasing returns to scale.

1. Suppose that the employer knows \( \theta \). Give the solution for the employer to maximize his utility subject to the worker’s participation constraint. This solution is called as the first-best solution.
2. Suppose that the employer can observe and verify the labor supply but cannot observe \( U \) and \( \theta \). Prove that the first-best solution can not be achieved. The employer can now provide the contract menu, \((T_L, l_L) \) and \((T_H, l_H) \), where \((T_H, l_H) \) is the contract selected by type \( \theta_H \) and \((T_L, l_L) \) is the contract selected by type \( T_L \). Derive the optimal contract and compare the second-best solution with the first-best solution.
3. Suppose that the worker with a lower disutility of the effort has an outside opportunity that can bring him the utility of \( V \). Compare the first-best solution with the second-best solution in this case. Notice that the solution will depend on \( V \). Consider respectively \( l^{SB}_H \Delta \theta \geq \)
Exercise 16.17 (Labor Contract with Adverse Selection) Consider a principal-agent relationship in which the principal is the employer and the agent is the worker. For worker of type \( \theta \in \{ \theta_L, \theta_H \} \), the work disutility for production \( y \) is \( \psi(\theta y) \). In other words, the worker of type \( \theta \) must work on \( l \) units (\( l = \theta y \)), and the resulting disutility is \( \psi(l) \). If he gets a compensation of \( t \) from his employer, the net utility is \( U = t - \psi(\theta y) \), and the employer’s utility function is \( V = y - t \).

1. Refer to revelation principle, and write down the characteristic of direct mechanism.

2. Suppose that \( \nu \) (or \( 1 - \nu \)) is the probability that the worker is of type \( \theta_L \) (or \( \theta_H \)). Find the contract maximizing the expected utility of the employer while satisfying the workers’ incentive compatibility and participation constraints.

Exercise 16.18 (Information and Incentive) An agent (natural monopoly manufacturer) produces \( q \) units of output with the variable cost function \( \theta q \) (\( \theta \in \{ \theta_L, \theta_H \} \), \( \Delta \theta = \theta_H - \theta_L \)). The utility obtained by the principal from production is \( S(q) \) (\( S' > 0 \) and \( S'' < 0 \)) and the transfer to the agent is \( T \). The utility function of the principal is \( V = S(q) - T \), and the agent’s utility function is \( U = T - \theta q \). In addition, the agent’s current utility is normalized to 0.

1. When the principal has complete information on \( \theta \), write down the principal’s first-best contract.

2. Suppose that \( \theta \) is the private information of the agent and \( \nu = Pr(\theta = \theta_L) \). Write down the optimal contract for the principal that satisfies the agent’s participation constraints at the interim stage (Suppose that the value of the project is large enough, so the principal is always willing to have a positive output.)

3. Suppose that with information technology the principal can obtain signals \( \sigma \in \{ \sigma_L, \sigma_H \} \) with

\[
\nu = Pr(\sigma = \sigma_L|\theta = \theta_L) = Pr(\sigma = \sigma_H|\theta = \theta_H) \geq \frac{1}{2} q.
\]

Derive the principal’s updated beliefs about the agent’s efficiency, namely \( \nu = Pr(\theta = \theta_L|\sigma = \sigma_L) \) and \( \nu = Pr(\theta = \theta_H|\sigma = \sigma_H) \). Solve for the optimal contract for each \( \sigma \).
4. Prove that an increase in $\mu$ will lead to an increase in the expected
utility of the principal.

**Exercise 16.19** A government agency signs a procurement contract with a
firm. The marginal cost of the firm producing $q$ units is $c$, so its profit is
$P - cq$. The cost of the firm is private information, which may be high cost
or low cost ($0 < c_L < c_H$). The government’s prior belief about the cost is
$Prob(c = c_L) = \beta$, and it makes a take-it-or-leave-it offer to the firm whose
reservation profit is zero.

1. Let $B(q)$ represent the revenue function of the government when it
obtains $q$ units. What is the government’s second-best contract?

2. Compare the second-best contract with the first-best contract.

3. Suppose that $c$ follows a uniform distribution over $[0, 1]$. Solve for the
first-best contract and the second-best contract.

**Exercise 16.20 (Lemon Market)** Consider a used car market. There are many
sellers in the market. Each seller has a used car ready to sell. It will be sold
with the car’s quality $\theta \in [0, 1]$ that follows a uniform distribution over
$[0, 1]$. If a seller of type $\theta$ sells his car at price $p$, then his utility is $u_s(p, \theta)$;
if he does not sell his car, the utility obtained is 0. If a buyer buys a used
car, his utility function is $\theta - p$; if not, it is 0. The quality of the used car is
the seller’s private information and the buyer does not know beforehand.
Assume that there are not enough used cars in the market for all possible
buyers.

1. Prove that for competitive equilibrium under asymmetric informa-
tion, $E(\theta|p) = p$.

2. Prove that if $u_s(p, \theta) = p - \frac{\theta}{2}$, then any $p \in (0, 1/2)$ is an equilibrium
price.

3. If $u_s(p, \theta) = p - \sqrt{\theta}$, find the equilibrium price and indicate which
kind of used cars will be traded in equilibrium.

4. Suppose that $u_s(p, \theta) = p - \theta^3$. Solve for the equilibrium price, and
find how many equilibria there are in this case.

5. Are the above outcomes Pareto efficient? If possible, propose a Pareto
improvement scheme.

**Exercise 16.21 (Insurance Market)** Consider the following insurance mar-
ket. There are two types of consumers: high risk and low risk. The initial
wealth of each consumer is $W$, but it is possible to lose $L$ due to fire. For
high-risk and low-risk consumers, probabilities of fire occurrences are $p_H$
and $p_L$, respectively, where $p_H > p_L$. Both types are pursuing the largest expected utility, and the utility function is $u(W)$, which satisfies neoclassical properties (such as differentiability, concavity, monotonicity, etc.). There are two insurance companies and both are risk-neutral. Any insurance contract consists of a premium of $M$ and a compensation of $R$ insurance company pays according to claims.

1. Suppose that each consumer purchases up to one insurance contract. Prove that the insurance contract specifies the wealth of the insured in the “no loss” state and the “in loss” state.

2. Suppose that each insurance company provides contract at the same time, and each company can provide any finite number of insurance contracts. What is the subgame perfect equilibrium for this problem? Does the equilibrium necessarily exist?

Exercise 16.22 (Pollution Regulation) Consider a manufacturer whose revenue is $R$. The manufacturer generates $x$ units of pollution in production. The damage caused by the pollution is $D(x)$, with $D'(x) > 0$ and $D''(x) \geq 0$. The cost function of the manufacturer is $C(x, \theta)$, satisfying $C_x < 0$ and $C_{xx} > 0$. $\theta$ is the parameter that the manufacturer privately observes, for $\theta \in \{\theta_L, \theta_H\}$, while $\nu = Pr(\theta = \theta_L)$ is public knowledge.

1. The first-best pollution $x^*(\theta)$ under complete information is given by:
   \[ D'(x) + C_x(x, \theta) = 0. \]
   Prove that if the regulator does not have to meet the manufacturer’s participation constraint, he can implement $x^*(\theta)$ by giving the manufacturer a transfer payment equal to the damage of the pollution.

2. Suppose that manufacturer can now refuse to accept regulation (in this case the manufacturer utility is 0), and that the regulator’s objective function is as follows:
   \[ W = -D(x) - (1 + \lambda)T + T - C(x, \theta), \]
   where $T$ is the transfer payment from the regulator to the manufacturer, and $(1 + \lambda)$ is the opportunity cost of social fund spending. For any $\theta$, the regulator must satisfy the manufacturer’s participation constraint:
   \[ T - C(x, \theta) \geq q_0. \]
   Write down the decision rule $\hat{x}(\theta), \theta \in \{\theta_L, \theta_H\}$ that maximizes $W$ under complete information, and compare with problem 1.

3. Suppose that $C_\theta < 0$ and $C_{x\theta} < 0$. Find the menu of contracts $(T_L, x_L)$ and $(T_H, x_H)$ that maximizes the expected $W$ under participation and incentive compatibility constraints.
4. Suppose that $\theta$ is distributed in the interval $[\theta_L, \theta_H]$ according to the distribution function $F(\theta)$ and the density function $f(\theta)$, satisfying:

$$\frac{d}{d\theta} \left( \frac{1 - F(\theta)}{f(\theta)} \right) < 0,$$

and $C_{\theta \theta x} \leq q_0$. Answer question 3 under these new conditions.

**Exercise 16.23 (Life Insurance Demand and Adverse Selection)** A group of consumers have income of $y_0$ in period 1 and no income in period 2. Each consumer knows his/her probability of death, $\pi_i$. Insurance companies can neither observe the probability of death of consumers nor observe the behavior of consumers. Insurers offer insurance at a price of $p$. If a consumer purchases $x$ units of insurance, the insurance company will receive $px$ of income for the period and pay $x$ when the consumer dies. The consumer can choose to purchase the insurance amount of $x$ and save the amount of $s$. Savings of the current period are added to the income of the consumer when he is alive in the next period. Assume there are many consumers. The goal of consumers is to maximize the income of each period to increase the consumption of family members. Consumer’s utility function is of Bernoulli form.

1. If a consumer’s utility function is:

$$u(c_1, c_2a, c_2d) = \log c_1 + (1 - \pi_i) \log c_2a + \pi_i \log c_2d.$$ 

Calculate the demand price elasticity of life insurance.

2. Suppose that the competition between insurance companies means that consumers’ expectation payments are equal to insurance premiums. Prove that $p$ is greater than the average of $\pi_i$.

**Exercise 16.24 (Self-Managing Manufacturer)** A manufacturer’s production function is $y = \theta l^{1/2}$, where $l$ is the number of workers (here considered as a continuous variable) and $\theta > 0$ is the parameter only known to the manufacturer. The fixed cost of production is $A$, so $p$ represents the price of the manufacturer’s product in the competitive market.

1. Suppose that the manufacturer is managed by the worker, and the objective function is:

$$U^{SM} = \frac{py - A}{l}.$$ 

Find the optimal number of workers for this worker self-managing manufacturer.
2. Let $w$ represent the wage rate under the perfectly competitive labor market, i.e., $w$ is the opportunity cost of labor in this economy. What is the optimal allocation of labor? What happens if $w$ is too large? Why is the scale of self-managed manufacturer generally not optimal?

3. Suppose that the government knows $\theta$. Investigate the situation where $w$ is small enough to make the self-managed company relatively reasonable. Find the unit product tax $\tau$ that can restore the first-best labor allocation. Prove that the same goal can be achieved by imposing a fixed tax of $T$ on the manufacturer (assuming that the size of the manufacturer is negligible compared to the entire economy).

4. Suppose that the government does not know $\theta$, only knows that $\theta$ may be $\theta_L$ or $\theta_H$, $\Delta \theta = \theta_H - \theta_L > 0$. The government uses a mechanism $(l(\tilde{\theta}), t(\tilde{\theta}))$ specifying a labor input of $l(\tilde{\theta})$ and a transfer payment of $t(\tilde{\theta})$ for the type of manufacturer report $\tilde{\theta}$. The manufacturer’s objective function is now:

$$U^{SM} = \frac{p\tilde{\theta}(l(\tilde{\theta}))^{1/2} + t(\tilde{\theta}) - A}{l(\tilde{\theta})}.$$

Write down a regulatory mechanism that leads to truthful type reporting.

5. Suppose that $\nu = Pr(\theta = \theta_L)$. Government wants to maximize

$$U^G = p\theta^{1/2} - wl.$$

Prove that even if the transfer payment is costless to the government, the first-best outcome cannot be implemented. At this point, suppose the opportunity cost of the manufacturer is 0. What is the optimal regulatory mechanism?

**Exercise 16.25 (Insurance Contract)** In a continuum economy, the economic individual’s production function is $q = \theta l$, where $\theta$ is the productivity parameter and the probability density function is $f(\theta), \theta \in [\underline{\theta}, \overline{\theta}]$. The utility function of the economic individual is $u = u(c) - l$, which is a concave function.

1. If it is an autarky economic environment, solve for the distribution of output and consumption in the economy.

2. Suppose that the insurance contract is signed in advance, that is, all economic individuals sign the insurance contract before they know their productivity parameter $\theta$. If $\theta$ and $l$ can be observed afterwards, solve for the optimal insurance contract. If only $\theta$ can be observed afterwards, solve for the optimal insurance contract.
3. In the previous question, if \( \theta \) and \( l \) are not observable afterwards, and \( f(\theta)/[1-F(\theta)] \) monotonically increases, what is the optimal insurance contract?

**Exercise 16.26** Consider an educational investment signaling model. There are two possible types of employee: \( \theta \in \{ \theta_H, \theta_L \} \) with \( \theta_H > \theta_L \). Given \( i \in \{ H, L \} \), the ex ante probabilities of this employee’s type \( \theta_i \) is \( \beta_i \). The reservation utility for all employees is \( \bar{u} = 0 \). Each type \( \theta \) can produce \( \theta \) for the firm. The firm is willing to hire an employee at the salary level of \( w \) if and only if that employee’s expected productivity can at least offset wages. Type \( \theta \) can get \( e \) year of education at a cost of \( c(e, \theta) = \frac{e}{\theta} \). The educational investment cost function \( c(e, \theta) \) satisfies the single crossover property for \( (e, \theta) \), that is, if \( e > e' \), then \( c(e, \theta_L) - c(e', \theta_L) > c(e, \theta_H) - c(e', \theta_H) \). Given a salary level of \( w \) and an education level of \( e \), type \( \theta \) has a reward function of \( u(w, e|\theta) = w - c(e, \theta) \). Consider the following sequential actions:

- The employee observe his own type that is his private information;
- The employee chooses education investment level;
- The firm observes the level of the employee’s education, but cannot observe his type;
- The employee proposes a salary offer to the firm;
- The firm either rejects the offer or accepts the offer and employs the employee at the salary level.

It is assumed that education investment can promote low-type employee to high-type employee. Specifically, suppose that the probability of type \( \theta_L \) investing in an \( e \) year education to become type \( \theta_H \) is \( p(e) \), which satisfies the following properties: \( p(0) = 0 \), \( \lim_{e \to \infty} p(e) = 1 \), \( p' > 0 \), \( p'(0) = \infty \), \( \lim_{e \to \infty} p'(e) = 0 \) and \( p'' < 0 \). Once type \( \theta_L \) is invested in education and converted to type \( \theta_H \), he can observe his type change before entering the labor market. First, it is assumed that there is no asymmetric information between the firm and its employees. Answer the following questions:

1. Write down the optimization problem that can find the optimal salary and the optimal education investment level.
2. Solve this problem. In particular, prove that the educational investment level of type \( \theta_H \) is 0, while the education investment level of the low-type employee is strictly positive.
3. Suppose that investment in education now becomes more efficient, that is, the probability that type \( \theta_L \) invests \( e \) in education and becomes type \( \theta_H \) with the probability \( q(e) \). It satisfies \( q(e) > p(e) \) for any \( e > 0 \). What is the optimal education investment level at this time? What is the salary of type \( \theta_L \) at this time? And give an intuitive explanation.
Exercise 16.27 (Educational Investment Signaling Model 1) Assume that there is asymmetric information between a firm and its employees, and we adopt a refined Bayesian pure strategic equilibrium solution concept. Consider separating equilibrium to satisfy $e_H \neq e_L$. Answer the following questions:

1. In any segregated equilibrium, the type $\theta_L$ employee selects the education level $e_L = e_{FB}$, where FB represents the first-best.

2. Describe the education investment level of type $\theta_H$, which satisfies $e_H \neq e_{FB}$.

3. Explain whether separation equilibrium always exists.

4. Suppose that investment in education is now becoming more efficient. That is, the probability that a worker of type $\theta_L$ invests in $e$ year education and becomes type $\theta_H$ is $q(e)$. It satisfies $q(e) > p(e)$ for any $e > 0$. Does the employee of type $\theta_H$ change the level of education investment in equilibrium as described above? Explain your conclusions.

Exercise 16.28 (Educational Investment Signaling Model 2) Consider the mixed equilibrium. It is satisfied that $e_H = e_L = e^*$. Answer the following questions.

1. Given educational investment $e^*$ under mixed equilibrium, what is the salary of employee?

2. Describe the mixed equilibrium $e^*$.

3. Does the mixed equilibrium $e^* = 0$ always exist? If $e^* = e_{FB}$, does the mixed equilibrium always exist?

4. Suppose that investment in education now becomes more efficient. That is, the probability that a worker of type $\theta_L$ invests in $e$ year education and becomes type $\theta_H$ is $q(e)$. It satisfies $q(e) > p(e)$ for any $e > 0$. Describe the influence of the change on the mixed equilibrium $e^*$. Explain your conclusions.

5. Given the assumption of effective education investment, is there mixed equilibrium satisfying intuitive criterion?

Exercise 16.29 (Supervision Cost) There are two types of economic individuals in the economy, banks and firms respectively, indexed by $i = 1, 2, 3, \ldots$. Each bank receives one unit of investment endowment on every period, which can be invested or lent to firms. If a bank makes investment, it can get $t_i$ units of certain benefit. A firm does not have any endowment on
every period. But, every firm has an investment project, which can get a random benefit of \( w_i \in [0, \bar{w}] \) with one unit investment, where \( w_i \) is independently identically distributed. The density function is \( f(w) \) and the distribution function is \( F(w) \). \( f \) is continuous and differentiable, which is common knowledge. For every firm \( i \), \( w_i \) can be observed without cost. The bank need to pay \( \gamma \) units of effort cost to observe the investment benefit of a firm. The firm and bank sign loan contract ex ante, the interest rate being \( x \). The firm makes signal to the bank \( w_s \in [0, \bar{w}] \) after observing the return of \( w \). If \( w_s \in S \subseteq [0, \bar{w}] \), the bank make supervision; if \( w_s \notin S \), there is no supervision. According to the contract, the bank lend one unit of investment to the firm, the firm should return \( R(w) \) unit if \( w_s \in S \), where \( 0 \leq R(w) \leq w \). If \( w_s \notin S \), the firm should return \( K(w) \) unit. In order to guarantee that all borrowing demand is satisfied, assume the ratio of bank is \( 1/2 < \alpha < 1 \).

1. Prove that \( R(w) = w \) and \( K(w) = x \) are the optimal payment plan.

2. For the optimal payment plan, rewrite the profit of bank and firm (a function of \( x \)), and prove that the profit of firm is monotonously decreasing with \( x \).

3. Write down the optimization problem of firm, including the participation constraint of bank.

4. Suppose that the profit function of bank is concave, prove there exist Nash equilibrium in the credit allocation of the economy.

5. Compared with Stiglitz and Weiss (1981) where all firms are homogenous ex ante, why there still exists credit allocation?

Exercise 16.30 (State Verification with Cost) Suppose that there exist two types of agent, with different cost function \( C(q, \theta) \). Assume \( C_q > 0, C_\theta > 0, C_{qq} \geq 0 \) and \( C_{q\theta} > 0 \), where \( q \) is the output observed perfectly by the principal. \( \theta \in \{\theta_L, \theta_H\} \), with the probabilities of \( v \) and \( 1 - v \), and \( \Delta \theta = \theta_H - \theta_L > 0 \). The output \( q \) is valued via \( S(q) \) by the principal, where \( S' > 0, S'' < 0 \) and \( S(0) = 0 \). Let \( t \) denote the transfer payment from principal to agent. The benefit function of agent is \( t - C(q, \theta) \) and the cost function is \( C(q, \theta) = \theta q \).

The principal has an auditing technology. The principal can observe the true type of agent with the probability of \( p \) by paying a cost of \( c(p) \). For the audit cost, assume \( c(0) = 0, c' > 0, c'' > 0 \). To insure the existence of interior solution, suppose that the cost function satisfies Inada conditions. The incentive mechanism include four variables: payment \( t(\tilde{\theta}) \), output \( q(\tilde{\theta}) \), the probability of observing true type by auditing \( p(\tilde{\theta}) \), and if \( \tilde{\theta} \) that the agent reported is different from the true \( \theta \), the principal can exert
16.11. EXERCISES

a punishment function $P(\theta, \tilde{\theta})$ on the agent. In the equilibrium, revelation principle indicates that the report is true.

1. Write down the participation and incentive compatibility constraints.

2. Suppose that the punishment is exogenous, $P(\theta, \tilde{\theta}) \geq l$, and $P(\tilde{\theta}, \tilde{\theta}) \geq l$. Solve for the optimal contract.

3. Suppose that the punishment is endogenous, $P(\theta, \tilde{\theta}) \geq l - \theta q$ and $P(\tilde{\theta}, \tilde{\theta}) \geq l - \tilde{\theta} q$. Solve for the optimal contract.

4. Compare the difference of the above two results.

Exercise 16.31 (Financial Contract) Suppose that there are two types of participants, with bank acting as principal and firm acting as agent. The profit of the investment project operated by the firm is $\theta$. Among them, a high profit of $\theta$ is obtained with probability of $v$, and a low profit of $\tilde{\theta}$ is obtained with probability of $1 - v$. The principal has a random auditing technique with a probability of $p(\theta)$. The cost of using this technique is $c(p)$.

1. Prove that if the type of profit reported by the agent is a high type, the optimal probability of auditing is 0.

2. Write down the incentive compatibility and participation constraints for this problem, as well as the principal’s objective function.

3. Solve for the optimal contract.

4. Under the optimal contract, will there be credit rationing in the economy?

Exercise 16.32 (Termination Threat) Suppose that there are two types of participants, with bank acting as principal and firm acting as agent. The profit of the investment project operated by the firm is $\theta$, where a high profit of $\theta$ is obtained with probability of $v$ and a low profit of $\tilde{\theta}$ is obtained with probability of $1 - v$. The agent does not have any endowment at the beginning. If he needs to invest, he must invest $I$ units. Suppose that $v \theta + (1 - v) \tilde{\theta} > I$ and $\theta > I$. Assume that the relationship of the contract lasts for two periods, and the profit type $\theta$ in both periods is independent and the time discount is 1. After knowing the type of agent’s report in the first period, the principal can terminate the contract relationship with the agent at the end of the first period. The probability is $\pi$ and $\tilde{\pi}$, respectively, for these two types.

1. Write down the incentive compatibility and participation constraints for this two-stage problem.
2. Suppose that the agent has limited liability, \( t \leq \theta \) and \( t \leq \theta \). Solve for the optimal contract.

3. Compare the difference between the contract and the one with the random auditing technique.

### 16.12 References

**Books and Monographs:**


**Papers:**


CHAPTER 16. PRINCIPAL-AGENT THEORY: HIDDEN INFORMATION


16.12. REFERENCES


Chapter 17

Principal-Agent Theory: Moral Hazard

17.1 Introduction

In the previous chapter, we stressed that the delegation of tasks creates an information gap between the principal and his agent when the latter has some private information relevant to determining the efficient level of trade. The principal thus needs to design appropriate incentive contracts to induce the agent to tell the truth. However, it is costly to provide incentives to the agent such that he truthfully reveals his characteristic. As such, the first-best outcome in general cannot be achieved and only the second-best outcome is possible. The optimal contract obtained under asymmetric information is different from that under complete information. The basic tradeoff in adverse selection is between the extraction of information rent and the efficiency of allocation.

Adverse selection is not the only problem under asymmetric information. In many situations, the principal usually has no control over the agent’s actions and the agent’s behavior is also unobservable so he can always act in a way that is beneficial to himself and unfavorable to the principal. We call this type of asymmetric information as hidden action or moral hazard.

Here are some examples.

1. The bank does not know the credit rating of the borrowing company, nor does it know whether the company will abuse the money after obtaining the loan.
2. The employer does not know the employee’s ability to work, nor does she know if the employee is lazy and does not care for machines and equipments.
3. A driver may not be particularly careful after buying a full
car insurance.

(4) The public sector is not particularly concerned with job performance and innovation. The profit or loss belongs to the public, but the risk is his own and he would not take risks to innovate.

(5) Duplicity: say yes and mean no. Saying one thing while acting differently.

(6) Do not take good care of public property as treating their private ones.

(7) Many government officials are not accountable and are reluctant to take risks. In general, more work implies more mistakes on decisions, less work implies fewer mistakes and no work implies no mistakes. Why would they get work done?

(8) The absolute equalization before the reform and opening-up in China made workers and peasants have no enthusiasm to work hard.

(9) The teacher does not know whether the student is listening or not. Even if the student is staring at the teacher. Maybe he is thinking about other things.

(10) If there is no assessment or pressure on students, they may not work hard.

The leading candidates for such moral hazard actions are effort variables. In this chapter, we present moral hazard with the agent’s effort. The basic analysis of moral hazard issues is as follows. The agent’s level of effort will affect the principal’s revenue, but the level of effort is not observable. As the effort will bring a disutility (effort cost) to the agent, it may cause the agent not to work hard. At the same time, the outcomes of effort are uncertain. Effort may bring good outcome (such as high output) and may also bring bad outcome (such as low output). Therefore, the outcomes (such as output level) is a random variable. But the level of effort will affect the probabilities of outcomes. For instance, the output of a field depends on the amount of time that the tenant has spent selecting the best crops or the quality of their harvesting. Similarly, the probability that a driver has a car accident depends on how safely he drives, which also affects his demand for insurance. Also, a regulated firm may have to perform a costly and unobservable investment to reduce its cost of producing a socially valuable good.

Therefore, the principal needs to design an appropriate contract to give the agent incentive to work hard instead of being lazy. What we want to examine is what kind of mechanism will make the agent work hard. In
the case of complete information, it is easy to make the agent work hard through a strict reward and punishment system. In the case of incomplete information, incentives are needed and the principal needs to pay a certain amount of information rent to induce the agent to work hard. For example, the benefits, bonus and other benefits that companies provide in order to make an employee work hard are information rents. However, it is obviously impossible for such an incentive contract to be based on unobservable efforts, but only on the performance of the agent.\(^1\) These performances are affected by the level of efforts of the agent and they are also disturbed by some random factors.\(^2\)

What the principal needs to do is to design a wage contract based on observable performance so as to induce the agent’s optimal effort level to maximize their own gains. In this process, the principal needs to face the tradeoff between incentives and risk or equivalently the tradeoff between efficiency and insurance. On the one hand, a high-intensity incentive contract can induce agents to work hard and thus increase the benefits of the principal. This is called the incentive effect. On the other hand, since the performance indicators used include noise factors, the performance-based incentives will be amplified by high incentives. The uncertainty caused by noise increases the risk that an agent needs to take. In this way, the risk-neutral principal must provide insurance for a risk-averse agent by paying more risk premium, which is a risk effect.\(^3\) What the principal needs to do is to make an efficiency-insurance or incentive-risk tradeoff to determine the optimal incentive intensity.

It should be emphasized that one of the differences between adverse selection and moral hazard is that for the principal, the uncertainty in the case of adverse selection is exogenous, but in the case of moral hazard it is endogenous. Therefore, the probability of moral hazard is different from the probability of nature state. This is because the probability of moral hazard is partially determined by the degree of effort of the agent. For instance, careful driving will reduce the probability of car accidents. This uncertainty is critical to understanding the contractual issues of the moral hazard. If the correspondence between the agent’s effort and performance is completely determined, there is no difficulty for the principal and the court to infer the agent’s level of effort based on the observed outcomes. Even if the agent’s effort cannot be directly observed, it may also be indirectly stipulated by the contract because the outcome itself is observable and verifiable.

---

\(^1\)For instance, the research achievements of researchers, the number of products produced by workers on the production line, etc.

\(^2\)For instance, agricultural outputs depend not only on the efforts of farmers, but also on climate, hydrology and many other uncontrollable human factors.

\(^3\)If the relationship between a performance indicator and effort is weak and stochastic, it is undoubtedly to guide agents to engage in a gambling activity with high-intensity incentives on such performance indicators.
We will discuss the properties of incentive schemes that induce a costly effort. Such schemes must thus satisfy an incentive compatibility constraint and the agent’s participation constraint. Among such schemes, the principal prefers the one that implements the positive level of effort at minimal cost. By minimizing the cost of the incentive scheme, it is possible to derive the second-best cost for the implementation of the level of effort. In general, the second-best cost is greater than the first-best cost at the observable level of effort. An allocative inefficiency emerges as a result of the conflict of interests between the principal and the agent. As such, the best outcome one can obtain is only second-best but not first-best.

\section*{17.2 Basic Model}

In this section, we will discuss the interaction between principal and agent by establishing the simplest form of moral hazard. The form of interaction is a contract in which the principal does not know the actual action of the agent. In the interaction, the principal has a task delegated to the agent and the agent needs to devote energy and bear the cost of the effort. After the task is completed, the agent’s payoff comes from the principal’s payment to the agent. There may be a potential conflict of interest between the principal and the agent (the principal wants the agent to work hard and the agent does not want to). However, the principal can use the design of the contract to ease their differences of interest and motivate the agent to work hard. Encouraging the agent to work hard is the core issue to be solved.

\subsection*{17.2.1 Model Setting}

Consider a simplest case where the completion of a task has two possible outcomes: $q = 1$ on success and $q = 0$ on failure. The returns to the principal are $\bar{S}$ and $S$, respectively, where $\bar{S} > S$. The agent has two possible actions $e$ which are working and shirking, corresponding to $e = 1$ or 0, respectively. In general, the outcomes of the task depends on other external factors in addition to the effort. Therefore, we assume that the agent’s action does not directly affect the outcomes, but only affects the probabilities of the outcomes.

Let $\pi_e$ denote the probability that the agent succeeds when the agent chooses action $e$. The harder the agent works, the higher the probability of successful task is (i.e. $\pi_1 > \pi_0$). Let $\Delta\pi = \pi_1 - \pi_0$ denote the increase in the probability that the agent will choose to work harder to succeed.

Note that effort improves production in the sense of first-order stochastic dominance, namely, the probability $\Pr(\tilde{q} \leq q^*|e)$ is decreasing with $e$ for any given production $q^*$, or the probability $\Pr(\tilde{q} \geq q^*|e)$ is increasing with $e$ for any given production $q^*$. Indeed, we have $\Pr(\tilde{q} \leq q|e = 1) = 1 - \pi_1 <
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In addition, the cost of the agent’s choice for effort is denoted as \( c(1) = \psi > 0 \) and \( c(0) = 0 \). The cost is zero when no effort is made. The principal cannot directly observe the agent’s action. The compensation to the agent can only rely on the outcome \( q \). For this reason, the compensation contract is in the form of \( t(q) \). \( \bar{t} = t(1) \) and \( t = t(0) \) are the compensations for success and failure, respectively.

Assume that the principal is risk-neutral and his utility function is \( V(y) = y \). The agent’s utility function for compensation and effort is separable:

\[
U(t, e) = u(t) - c(e),
\]

where \( u’ > 0, u'' \leq 0 \). \( h = u^{-1} \), which will be used to find the solution later. Obviously, \( h \) is a strictly increasing and convex function: \( h’ > 0, h'' \geq 0 \). Under such a utility function, \( h(\psi) \) is the first-best cost \( C_{FB} \) of implementing the positive effort level.

If the agent’s effort level is \( e = 1 \), the principal’s expected utility function is:

\[
\pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(S - t) \tag{17.2.1}
\]

If the agent’s effort level is \( e = 0 \), the principal’s expected utility function is:

\[
\pi_0(\bar{S} - \bar{t}) + (1 - \pi_0)(S - t) \tag{17.2.2}
\]

If the agent chooses \( e = 1 \), his expected utility is:

\[
\pi_1 u(\bar{t}) + (1 - \pi_1)u(t) - \psi \tag{17.2.3}
\]

In the following, the question we have to answer is that in the presence of moral hazard, if the principal wants the agent to make the effort, can she always choose an appropriate contract so that the agent’s choice is consistent with the principal’s goal?

The timing structure of moral hazard game is as follows:

- \( t = 0 \): the principle offers \( \{\bar{t}, \bar{t}\} \);
- \( t = 1 \): the agent accepts or rejects the contract;
- \( t = 2 \): the agent decides on effort input;
- \( t = 3 \): realize output \( q \);
- \( t = 4 \): implement the contract.

17.2.2 Benchmark Case

In order to study the optimal contract under incomplete information, we first consider the benchmark case of the optimal contract under complete information. In this case, the agent’s actions can be put into the contract and can be executed by a third party, such as a legal system. The problem
of the principal is to design an optimal contract subject to the participant constraint. That is, the contract sets the action of the agent and the expected utility obtained by the agent that is not less than the reservation utility. For simplicity, let reservation utility be 0. Under condition (17.2.6), we want \( e = 1 \) to be optimal. The contract is the solution to the following problem:

\[
\max_{\{\bar{t}, t\}} \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\bar{S} - t)
\]

s.t. \( \pi_1u(\bar{t}) + (1 - \pi_1)u(t) - \psi \geq 0 \).

Since the action can be determined by the contract, the optimal contract just needs to impose that if the agent does not choose \( e = 1 \), he will be given a serious punishment. Then the agent will abide by the contract. The above inequality constraint is called participation constraint. If the agent accepts the contract, his expected utility cannot be lower than the reservation utility. Obviously in this optimization problem, this constraint must be an equality constraint (sometimes called a binding constraint). Let \( \lambda \) be the Lagrangian multiplier involved in the constraint, we have

\[
L(\lambda, \bar{t}, \bar{t}) = \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\bar{S} - t) + \lambda[\pi_1u(\bar{t}) + (1 - \pi_1)u(t) - \psi].
\]

The first-order conditions for \( \bar{t} \) and \( t \) are:

\[
-\pi_1 + \lambda \pi_1u'(\bar{t}^*) = 0, \quad (17.2.4)
\]

\[
-(1 - \pi_1) + \lambda(1 - \pi_1)u'(t^*) = 0. \quad (17.2.5)
\]

From the formula (17.2.4) and the formula (17.2.5), it can be immediately derived:

\[ t^* = \bar{t}^* \]

Thus the agent can avoid all risks in the optimal contract. Since the participation constraint must be satisfied and the first-best contract is \( t^* = \bar{t}^* = h(\psi) \). That is, the transfer payment to the agent is equal to the cost of effort. In this case, as long as the agent works hard, whether the outcome is good or bad, the principal should pay the same amount to the agent since the difference is caused by nature and it is not the agent's fault. For the principal, the expected return is:

\[ \pi_1\bar{S} + (1 - \pi_1)\bar{S} - h(\psi). \]

If the principal wants the agent to choose \( e = 0 \), then the contract at this time is a solution to the following optimization problem:

\[
\max_{\{\bar{t}, \bar{t}\}} \pi_0(\bar{S} - \bar{t}) + (1 - \pi_0)(\bar{S} - \bar{t})
\]

s.t. \( \pi_0u(\bar{t}) + (1 - \pi_0)u(\bar{t}) \geq 0 \).
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Similar to the reasoning above, we get the contract $\bar{t}' = \bar{t} = 0$ and the agent’s expected return is

$$\pi_0 \bar{S} + (1 - \pi_0) \underline{S}.$$  

Thus, if inducing effort is optimal to the principal, it requires that $\pi_1 \bar{S} + (1 - \pi_1) \underline{S} - h(\psi) \geq \pi_0 \bar{S} + (1 - \pi_0) \underline{S}$, or

$$\Delta \pi (\bar{S} - \underline{S}) \geq h(\psi).$$ \hspace{1cm} (17.2.6)

Equation (17.2.6) means that the gain of increasing effort from $e = 0$ to $e = 1$ is greater than or equal to the first best cost of inducing effort.

Under condition (17.2.6), it is clear that the principal stipulates that the agent’s action in the first-best contract is $e = 1$. Therefore, the contract is the first-best in the case of complete information. Figure 17.1 depicts the level of effort under the first-best contract.

Figure 17.1: The level of effort under the first-best contract

17.2.3 Incentive Feasible Contracts

If the principal cannot observe the agent’s action, the principal and the agent are unable to agree on the agent’s action in the contract. If there is only one interaction between the principal and the agent, the agreed action in the contract is still unenforceable as long as the agent’s action cannot be verified by a third party even if the principal can observe the agent’s action.

In this situation, the contract can only be based on verifiable outcomes, that is, whether the task is successful. At this point, the agent’s contractual form of avoiding all risks in the incomplete information situation is not implementable. Because if $t^* = \bar{t}^*$, the agent’s compensation does not depend on the outcome. The agent has no incentive to choose the action $e = 1$. In fact, the intuition is very simple. It is unknown whether the difference in outcomes is caused by hard work or nature (uncertainty) when the action is not observable. As such, the principal should not give the agent the same amount of compensation.

Thus, an implementable contract needs to introduce a new constraint:

$$\pi_1 u(\bar{t}) + (1 - \pi_1) u(\underline{t}) - \psi \geq \pi_0 u(\bar{t}) + (1 - \pi_0) u(\underline{t}).$$ \hspace{1cm} (17.2.7)
We call the expression (17.2.7) the *incentive compatibility constraint*. Under this condition, even if the principal can not specify the agent’s actions, the agent will have the incentive to choose the principal’s desired action. Note that the complete information contract \( \bar{\tau} = \bar{\iota} \) does not satisfy the incentive compatibility constraint (17.2.7).

In addition, the participation constraint must be met:

\[
\pi_1 \bar{\tau} + (1 - \pi_1) \bar{\iota} - \psi \geq 0. 
\] (17.2.8)

Note that the agent’s participation constraint must be established before the output variations are realized.

The principal hopes that each level of effort taken by the agent corresponds to a contract that guarantees that the agent’s moral hazard incentive compatibility constraint and participation constraint be established.

**Definition 17.2.1 (Incentive Feasible Contract)** A contract is said to be *incentive feasible* if it satisfies the incentive compatibility and participation constraints (17.2.7) and (17.2.8).

### 17.2.4 Risk Neutrality and First-Best Implementation

If the agent is risk-neutral, we have (up to an affine transformation) \( u(t) = t \) for all \( t \) and \( h(u) = u \) for all \( u \). The principal who wants to induce effort must thus choose the contract to solve the following optimization problem:

\[
\max_{\{\bar{\tau}, \bar{\iota}\}} \pi_1 (\bar{S} - \bar{\tau}) + (1 - \pi_1) (\bar{S} - \bar{\iota}) 
\] (17.2.9)

s. t. \[
\begin{align*}
\pi_1 \bar{\tau} + (1 - \pi_1) \bar{\iota} - \psi & \geq \pi_0 \bar{\tau} + (1 - \pi_0) \bar{\iota}; \\
\pi_1 \bar{\tau} + (1 - \pi_1) \bar{\iota} - \psi & \geq 0.
\end{align*}
\] (17.2.10) (17.2.11)

Incentive compatibility constraint (17.2.10) can be written as:

\[
\Delta \pi_1 \bar{\iota} \geq \Delta \pi_1 \bar{\tau} + \psi. 
\] (17.2.12)

First, we can verify that in the above optimization problem, (17.2.11) must be binding. Because one of (17.2.10) and (17.2.11) must be binding, otherwise we can reduce \( \bar{\tau} \) to increase the principal’s expected utility. Secondly, if (17.2.11) is not binding, we can reduce \( \bar{\tau} \) and \( \bar{\iota} \) by the same small amount. Under the above two constraints, the principal’s expected utility can be increased. Of course, (17.2.10) may not be binding. The shaded area in Figure 17.2 depicts the incentive feasible transfers. The optimal contracts that are the first-best correspond to the portion of the participation constraint line that also satisfies the incentive compatibility constraint in Figure 17.2. This is because the objective function (17.2.9) can be written as:

\[
\max_{\{\{\bar{\tau}, \bar{\iota}\}\}} \pi_1 (\bar{S} - \bar{\tau}) + (1 - \pi_1) [\bar{S} - \pi_1 \bar{\tau} + (1 - \pi_1) \bar{\iota}],
\]
namely:
\[
\min_{\{\ell, \bar{\ell}\}} \pi_1 \ell + (1 - \pi_1)\bar{\ell}.
\]

Among all optimal contracts, the one satisfies both the incentive compatibility constraint (17.2.10) and participation constraint (17.3.17) is given by
\[
\ell_{SB} = -\frac{\pi_0}{\Delta \pi} \psi, \quad (17.2.13)
\]
\[
\bar{\ell}_{SB} = 1 - \frac{\pi_0}{\Delta \pi} \psi. \quad (17.2.14)
\]

Figure 17.2: Optimal contract when the agent is risk-neutral

The principal’s expected payoff is \(\pi_1 \bar{S} + (1 - \pi_1)S - \psi\). This result is consistent with the complete information situation and the expected return of the agent is 0 so that the optimal contract is first-best contract. Even so, the reward for different outcomes is not the same. This is different from the pooling contract under complete information although the expectation is equal to 0. So we have the following proposition:

**Proposition 17.2.1** Moral hazard is not an issue with a risk-neutral agent despite the unobservability of effort. The first-best level of effort is still implementable.

**Remark 17.2.1** One may find the similarity of these results with those in the previous chapter. In both cases, when contracting takes place ex ante, the incentive-compatibility constraint, under either adverse selection or moral hazard, does not conflict with the ex ante participation constraint with a risk-neutral agent, and the first-best outcome is still implementable.

**Remark 17.2.2** Inefficiencies in effort provision due to moral hazard will arise when the agent is no longer risk-neutral. There are two alternative ways to model these transaction costs. One is to maintain risk-neutrality for positive income levels but to impose a limited liability constraint, which requires transfers not to be too negative. The other is to let the agent be strictly risk-averse. In the following, we analyze these two contractual environments and the different trade-offs they imply.
17.3 Optimal Contract under Limited Liability and Risk Aversion

17.3.1 Second-Best Contract under Limited Liability

In the previous section, we found that the imposition of the incentive compatibility constraint still results in the first-best outcome of the contract or does not change the agent’s behavior (as opposed to the complete information condition) if the agent is risk-neutral. However, the incentive compatibility constraint requires that the agent’s payment must be different between the success and failure of the task: \( t - \bar{t} \geq \frac{\psi}{\Delta \pi} \). This means that the agent will be punished for failure.

In the optimal contract shown in Figure 17.2, the agent’s loss is bounded by \( t - \bar{t} \in [-l, \infty) \). These constraints together with the incentive compatibility and participation constraints may prevent the principal from implementing the first-best level of effort even if the agent is still risk-neutral. Indeed, when the principal wants to induce a high effort, her program can be written as

\[
\begin{align*}
\max_{\{t, \bar{t}\}} & \quad \pi_1 (\bar{S} - \bar{t}) + (1 - \pi_1)(S - t) \\
\text{s. t.} & \quad \pi_1 \bar{t} + (1 - \pi_1) \bar{t} - \psi \geq \pi_0 \bar{t} + (1 - \pi_0)t; \\
& \quad \pi_1 t + (1 - \pi_1) \bar{t} - \psi \geq 0; \\
& \quad \bar{t} \geq -l; \\
& \quad t \geq -l.
\end{align*}
\]

The second-best contract can be analyzed with the following figures. There are two cases to be considered.

Case 1. \( -\frac{\pi_0}{\Delta \pi} \psi \geq -l \), i.e., the transfer obtained without limited liability under the failure outcome does not exceed the upper bond of penalty, then the limited liability constraint is not binding, so that we still have the first-best outcome although the set of first-best outcomes is shrinking. This situation can be illustrated by Figure 17.3: We have seen that because of the existence of the upper bound of penalty, the set of incentive-feasible contracts has been restricted. But, since the agent’s participation constraints are still satisfied, it results in the first-best outcomes in the range of optimal contracts.
Case 2. $-\frac{\pi_0}{\Delta \pi} \psi < -l$. In this case, the transfer obtained without limited liability under the failure outcome exceeds the upper bound of penalty, the limited liability constraint is binding, and we can only have the second-best contract. We graphically show them in Figure 17.4, in which the region of incentive feasible contracts is restricted. In this new region, the agent’s participation constraint (17.2.11) becomes inequality constraint and the principal’s optimal decision is to minimize the expected information rent in the viable area, i.e.,

$$\min_{(\tilde{t}, t)} \pi_1 \tilde{t} + (1 - \pi_1) t$$

subject to the limited liability constraint and incentive compatibility constraint.

As such, $t^{SB} = -l$, at which the limited liability affects the transfer (penalty) for failure and it is equal to the upper bound of the penalty. We have $\tilde{t}^{SB} = -l + \frac{\psi}{\Delta \pi}$, and thus the expected utility of the principal is $\pi_1 \tilde{S} + (1 - \pi_1) S - (\psi \frac{\pi_0}{\Delta \pi} - l)$ and the agent’s expected utility is $\pi_1 / \Delta \pi \psi - l > 0$. We can further obtain that the principal will provide incentives for the agent to choose $e = 1$ if and only if $\Delta \pi \Delta S \geq \psi + \pi_0 / \Delta \pi \psi - l$.

We see that only when the failure state occurs, the limited liability (upper bound of punishment) constraint may be binding. Because exerting effort requires a difference between $\tilde{t}$ and $t_f$ there must be $\tilde{t} > -l$ when $\tilde{t} \geq -l$. When the upper bound of penalty is smaller than the payment under the original failure, the principal’s punishment for the agent is limited. In this case, the agent can only pay a penalty of $l$. When the task is successful, the agent gets a reward of $-l + \psi / \Delta \pi$. Therefore, the agent obtains a non-negative expected rent $EU^{SB} > 0$, which means the optimal contract is the second-best, but not the first-best. This rent results from the additional payments made by the principal to the agent due to the combined
Figure 17.4: Only the second-best outcomes prevail when $0 < l < \frac{\pi_0 \psi}{\Delta \pi}$

effect of moral hazard and limited liability. With the continuous increase of $l$, the degree of protection of the agent with limited liability is getting smaller and smaller. The tension between moral hazard and limited liability constraints is getting smaller. When $l > \frac{\pi_0 \psi}{\Delta \pi}$, there will be no tension between them.

Summarizing the above discussion, we have the following proposition.

**Proposition 17.3.1** With limited liability, the optimal contract inducing effort from the agent entails:

1. For $l > \frac{\pi_0 \psi}{\Delta \pi}$, the participation constraint is binding so that the set of the first-best transfers is given in Figure 17.3, and one of them that satisfies the incentive compatibility and participant equality constraints is given by $(t^*, \bar{t}^*) = \left( -\frac{\pi_0}{\Delta \pi \psi}, \frac{(1 - \pi_0) \psi}{\Delta \pi} \right)$. The agent has no expected limited liability rent, i.e., $EU_{SB} = 0$.

2. For $0 < l < \frac{\pi_0 \psi}{\Delta \pi}$, the limited liability and incentive compatibility constraints are binding. The second-best transfers are given by

   $t_{SB}^{SB} = -l$, \hspace{1cm} (17.3.20)

   $\bar{t}_{SB}^{SB} = -l + \frac{\psi}{\Delta \pi}$. \hspace{1cm} (17.3.21)

Moreover, the agent’s expected limited liability rent $EU_{SB}$ is non-negative:

$EU_{SB} = \pi_1 \bar{t}_{SB}^{SB} + (1 - \pi_1) t_{SB}^{SB} - \psi = -l + \frac{\pi_0 \psi}{\Delta \pi} \geq 0$. \hspace{1cm} (17.3.22)
17.3.2 Second-Best Contract under Risk Aversion

We now discuss the optimal contract when the agent is risk-averse. In this case, there is a fundamental trade-off: efficiency and insurance. We have seen that in the case of incomplete information, if the agent does not bear any risk, the compensation that is independent of the outcome will not satisfy the incentive compatibility constraints. In this way, the agent needs to take a certain risk in order to satisfy incentive compatibility constraints. However, for a risk-averse agent, taking risks will reduce his expected payoff. Since the agent’s compensation must also satisfy the participation constraints, the cost for the risk will eventually be transferred to the principal. The problem of the principal is then to balance the efficiency from incentives and the cost of risk from providing incentives.

The principal’s program is then written as:

\[
\max \pi_1 (\bar{S} - \bar{t}) + (1 - \pi_1)(S - t), \tag{17.3.23}
\]

s.t.

\[
\pi_1 u(\bar{t}) + (1 - \pi_1)u(t) - \psi \geq \pi_0 u(\bar{t}) + (1 - \pi_0)u(t), \tag{17.3.24}
\]

\[
\pi_1 u(\bar{t}) + (1 - \pi_1)u(t) - \psi \geq 0. \tag{17.3.25}
\]

If this is a concave plan, the first-order Kuhn-Tucker condition is the necessary and sufficient condition for the optimal solution. But this is not the case since the concave function \(u(\cdot)\) appears on both sides of the incentive constraint (17.3.24). However, the following change of variables makes the new program a concave plan.

Define \(\bar{u} = u(\bar{t})\) and \(u = u(t)\), so \(\bar{t} = h(\bar{u})\) and \(t = h(u)\). In this way, the principal’s problem becomes:

\[
\max \{\bar{u}, u\} \pi_1 (\bar{S} - h(\bar{u})) + (1 - \pi_1)(S - h(u)) \tag{17.3.26}
\]

s.t.

\[
\pi_1 \bar{u} + (1 - \pi_1)u - \psi \geq \pi_0 \bar{u} + (1 - \pi_0)u, \tag{17.3.27}
\]

\[
\pi_1 \bar{u} + (1 - \pi_1)u - \psi \geq 0. \tag{17.3.28}
\]

Since the objective function of the above problem is strictly concave in \(\bar{u}\) and \(u\) since \(h(\cdot)\) is strictly convex, and the constraints given by (17.3.27) and (17.3.28) are linear. The first-order condition of the Lagrangian function is the sufficient condition for the constrained optimal solution.

Let \(\lambda\) and \(\mu\) be the Lagrange multipliers for the constraints (17.3.27) and (17.3.28), respectively. The Lagrangian function is

\[
L(\bar{u}, u; \lambda, \mu) = \pi_1 (\bar{S} - h(\bar{u})) + (1 - \pi_1)(S - h(u)) + \lambda[\pi_1 \bar{u} + (1 - \pi_1)u - \psi - \pi_0 \bar{u} - (1 - \pi_0)u] + \mu[\pi_1 \bar{u} + (1 - \pi_1)u - \psi].
\]
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Thus, the first-order conditions for $\bar{u}$ and $\underline{u}$ are given by

$$\frac{1}{u'(\bar{t}^{SB})} = \mu + \lambda \frac{\Delta \pi}{\pi_1},$$

(17.3.29)

$$\frac{1}{u'(\underline{t}^{SB})} = \mu - \lambda \frac{\Delta \pi}{1 - \pi_1},$$

(17.3.30)

where $\bar{t}^{SB}$ and $\underline{t}^{SB}$ are the transfers.

Multiplying (17.3.29) by $\pi_1$ and (17.3.30) by $(1 - \pi_1)$, and making the summation yields

$$\mu = \pi_1 u'(\bar{t}^{SB}) + (1 - \pi_1) u'(\underline{t}^{SB}) = E \left( \frac{1}{u'(\bar{t}^{SB})} \right) > 0,$$

and thus the participation constraint (17.3.28) is binding.

Multiplying (17.3.29) by $\pi_1 u(\bar{t}^{SB})$ and (17.3.30) by $(1 - \pi_1) u(\underline{t}^{SB})$, and making the summation, taking into account the expression of $\mu$ obtained above yields

$$\lambda \Delta \pi \left( u(\bar{t}^{SB}) + u(\underline{t}^{SB}) \right) = E_q \left( u(\bar{t}^{SB}) \left( \frac{1}{u'(\bar{t}^{SB})} \right) - E \left( \frac{1}{u'(\bar{t}^{SB})} \right) \right).$$

(17.3.31)

Using the slackness condition $\lambda \left( \Delta \pi (u(\bar{t}^{SB}) + u(\underline{t}^{SB})) - \psi \right) = 0$ to simplify the left-hand side of (17.3.31), we get

$$\lambda \psi = \text{cov} \left( u(\bar{t}^{SB}), \frac{1}{u'(\bar{t}^{SB})} \right).$$

(17.3.32)

By assumption, $u(\cdot)$ and $u'(\cdot)$ co-vary in opposite directions. Moreover, a constant wage $\bar{t}^{SB} = \underline{t}^{SB}$ does not satisfy the incentive compatibility constraint, and thus $\bar{t}^{SB} \neq \underline{t}^{SB}$. Hence, the right-hand side of (17.3.32) is positive. Thus we have $\lambda > 0$, and then the incentive-compatibility constraint (17.3.27) is binding.

Also, by the Kuhn-Tucker theorem, since $\lambda > 0, \mu > 0$ and $u$ is a concave function, we have $\bar{t}^{SB} > \underline{t}^{SB}$. The compensation when the task is successful is greater than the compensation when it fails.

Since the incentive compatibility constraint (17.3.27) and the participation constraint (17.3.28) are all equality constraints, we obtain the following equation by solving these two equations:

$$\bar{u} = \psi + (1 - \pi_1) \frac{\psi}{\Delta \pi},$$

(17.3.33)

$$\underline{u} = \psi - \pi_1 \frac{\psi}{\Delta \pi},$$

(17.3.34)

and thus we have

$$\bar{t}^{SB} = h(\psi + (1 - \pi_1) \frac{\psi}{\Delta \pi}),$$

(17.3.35)

$$\underline{t}^{SB} = h(\psi - \pi_1 \frac{\psi}{\Delta \pi}).$$

(17.3.36)

We then have the following proposition.
Proposition 17.3.2 When the agent is strictly risk-averse, the second-best optimal contract that induces effort makes both the agent’s participation and incentive-compatibility constraints binding. This contract does not provide full insurance. Moreover, the second-best transfers are given by

\[ t_{SB}^{\Delta} = h\left(\psi + \left(1 - \pi_1\right)\frac{\psi}{\Delta \pi}\right) = h\left(\frac{1 - \pi_0}{\Delta \pi}\psi\right) \]  
(17.3.37)

and

\[ t_{SB}^{\Delta} = h\left(\psi - \pi_1\frac{\psi}{\Delta \pi}\right) = h\left(-\frac{\pi_0}{\Delta \pi}\psi\right). \]  
(17.3.38)

17.3.3 Basic Trade-off: Insurance and Efficiency

The second-best cost of inducing a high effort \( e = 1 \) is:

\[ C_{SB} = \pi_1 h\left(\psi + \left(1 - \pi_1\right)\frac{\psi}{\Delta \pi}\right) + \left(1 - \pi_1\right) h\left(\psi - \pi_1\frac{\psi}{\Delta \pi}\right) \]

\[ = \pi_1 h\left(\frac{1 - \pi_0}{\Delta \pi}\psi\right) + \left(1 - \pi_1\right) h\left(-\frac{\pi_0}{\Delta \pi}\psi\right). \]  
(17.3.39)

Since \( h(\cdot) \) is strictly convex, we have \( C_{SB} > h(\psi) = C_{FB} \) according to the Jensen’s inequality. The principal’s return when she induces the agent to exert \( e = 1 \) is still

\[ B = \Delta \pi \Delta S. \]

Thus, in the risk-averse situation, the principal will induce the agent to choose \( e = 1 \) if and only if \( B = \Delta \pi \Delta S \geq C_{SB} > h(\psi) = C_{FB} \). Under moral hazard, inducing high-level efforts is more difficult than under complete information. Figure ?? compares the first-best contract and the second-best contract under the complete and incomplete information.

Figure 17.5 shows the second-best solution under risk-averse agents.

![Figure 17.5: The second-best solution under risk-aversion](image)

When \( B \) belongs to the interval \( [C_{FB}, C_{SB}] \), the second-best effort level is zero and therefore it is strictly less than the first-best effort level. Due to the combined effect of moral hazard and agent’s risk-aversion, the agent’s efforts are distorted downward.
CHAPTER 17. PRINCIPAL-AGENT THEORY: MORAL HAZARD

The following proposition summarizes the basic trade-off under moral hazard.

**Proposition 17.3.3** For the optimal contract with moral hazard and risk-aversion, there is a trade-off between inducing effort and providing insurance to the agent (avoiding risk), and the principal induces a positive effort from the agent less often than when effort is observable.

17.4 Contract Theory at Work

This section elaborates on the basic moral hazard model discussed above in a number of settings that have been discussed extensively in the contracting literature.

17.4.1 Efficiency Wage

In the labor market, a common observation is that in some firms, the wages of employees are often higher than the competitive wages in the labor market. This wage is called efficiency wage. Here, the exceeding part refers to the deduction of possible human capital factors. In this regard, a macroeconomic question is what prevents firms from adjusting their wages to market equilibrium wages.

Shapiro and Stiglitz (AER, 1984) first discussed the microeconomic logic behind efficient wages. Their research framework is to consider this problem in a dynamic environment. Here we consider a simplified static version where we refer to the Laffont and Martimort (2002) model settings.

Suppose that a firm employs a worker to perform a task, and both the owner and the employee are risk-neutral. We normalize the equilibrium wages outside the market to 0. The firm wants the worker to work hard. Assume that the level of effort is discrete, i.e. \( e \in \{0, 1\} \), the level of effort determines the outcomes of the task, and there are only two outcomes, \( y \in \{0, 1\} \). When the task succeeds, \( y = 1 \), the firm’s added value is \( \bar{V} \); when it fails, the added value is \( V \) with \( 0 \leq V \), denoting \( \Delta V = \bar{V} - V \). The probability of success of a task depends on the efforts of the employee, and let \( \pi_e \) be the probability of succeed when the effort is \( e \) with \( 1 > \pi_1 > \pi_0 > 0 \), denoting \( \Delta \Pi = \pi_1 - \pi_0 \). Assume the effort function is \( \psi(1) = \psi \) and \( \psi(0) = 0 \), which can not be directly observed by the firm. The firm wants the worker to work hard, but the most likely punishment is dismissal. At this point, the agent can only be rewarded for a good performance and cannot be punished for a bad outcome, since they are protected by limited liability.

To induce effort, the principal (firm) must find an optimal compensation
scheme \( \{(t, \bar{t})\} \) that is the solution to the program below:

\[
\begin{align*}
\max_{(t, \bar{t})} & \quad \pi_1(\bar{V} - \bar{t}) + (1 - \pi_1)(V - t) \\
\text{s.t.} & \quad \pi_1 \bar{t} + (1 - \pi_1)t - \psi \geq \pi_0 \bar{t} + (1 - \pi_0)t, \\
& \quad \pi_1 \bar{t} + (1 - \pi_1)t - \psi \geq 0, \\
& \quad \bar{t} \geq 0.
\end{align*}
\] (17.4.40)

The problem is isomorphic to the one analyzed earlier. The limited liability constraint is binding at the optimum, and the firm chooses to induce a high effort when \( \Delta \pi \Delta V \geq \frac{\pi_1 \psi}{\Delta \pi} \). As such, we obtain a second-best optimal contract, at which \( \bar{t}^{SB} = 0 \) and \( \bar{t}^{SB} = \frac{\psi}{\Delta \pi} > 0 \). At this point, the expected wage is greater than the market equilibrium wage, and the reason why the efficiency wage is greater than the market equilibrium wage is that the firm needs to give the worker an incentive to work hard. The expected utility of the firm is \( \pi_1 \bar{t} + (1 - \pi_1)t - \psi = \pi_0 \frac{\psi}{\Delta \pi} > 0 \). The incentive cost of the firm is \( \frac{\pi_1 \psi}{\Delta \pi} \), and only when \( \Delta \pi \Delta V > \frac{\pi_1 \psi}{\Delta \pi} \), the firm has incentive to induce the worker to work hard.

The positive wage \( \bar{t}^{SB} = \frac{\psi}{\Delta \pi} \) is often called an efficiency wage because it induces the agent to exert a high (efficient) level of effort. To induce production, the principal must give up a positive share of the firm’s profit to the agent.

### 17.4.2 Sharecropping

The theoretical framework of moral hazard is also widely applied to development economics. The sharing of rent in the agricultural economy is an incentive mechanism. If the landlord employs a tenant to work on the farm, the tenant with the fixed income does not have the incentive to do best. If the landlord has full control over the work situation of the tenant, he can naturally command him accordingly. However, to obtain such sufficient information, the landlord will have to spend much effort to supervise the monitoring personally, which may not be worth for landlord to do it. Another approach is to rent the land at a fixed amount which can give the tenant a very large incentive. However, agriculture is a high-risk production. The tenant may not be able to bear this uncertainty at all. The sharecropping arrangement then arises. This approach attenuates the incentives for work rather than eliminates them and bears some of the risk. Similar ideas has also been applied to health insurance. Most health insurance policies require patients make a copayment.

The idea of sharecropping was originated from Steven Cheung (1969) who studied the rent sharing contract. Stiglitz (RES, 1974) extends Cheung’s idea for a general setup of moral hazard problem. In the sharecropping model examined by Stiglitz (1974), the principal is the landlord and
the agent is the tenant. The tenant has only limited responsibilities. Thus the optimal contract provided by Stiglitz is the second-best, while the rent sharing contract that has widely used in real world in general is not a second-best contract and more beneficial to tenants but not to landlords.

We will discuss a simplified version of the Stiglitz (1974) model. By exerting an effort \( e \) in \( \{0, 1\} \), the tenant increases (decreases) the probability \( \pi(e) \) (resp. \( 1 - \pi(e) \)) that a large \( \bar{q} \) (resp. small \( q \)) quantity of an agricultural product is produced. The price of this good is normalized to one so that the principal’s stochastic return on the activity is also \( \bar{q} \) or \( q \), depending on the state of nature.

We assume that the agent is risk-neutral and protected by limited liability. The principal’s optimal contract must solve

\[
\max_{\{t, \bar{t}\}} \left\{ \pi_1 (\bar{q} - \bar{t}) + (1 - \pi_1) (q - t) \right\}
\]

s.t.

\[
\begin{align*}
\pi_1 \bar{t} + (1 - \pi_1) t - \psi & \geq \pi_0 \bar{t} + (1 - \pi_0) t, \quad (17.4.45) \\
\pi_1 \bar{t} + (1 - \pi_1) t - \psi & \geq 0, \quad (17.4.46) \\
t & \geq 0. \quad (17.4.47)
\end{align*}
\]

The optimal contract therefore satisfies \( t_{SB} = 0 \) and \( \bar{t}_{SB} = \frac{\psi}{\Delta \pi} \). This is again akin to an efficiency wage. The expected utilities obtained respectively by the principal and the agent are given by

\[
EV_{SB} = \pi_1 \bar{q} + (1 - \pi_1) q - \frac{\pi_1 \psi}{\Delta \pi}, \quad (17.4.48)
\]

and

\[
EU_{SB} = \frac{\pi_0 \psi}{\Delta \pi}. \quad (17.4.49)
\]

The flexible second-best optimal contract described above has sometimes been criticized as not corresponding to the contractual arrangements observed in most agrarian economies. Contracts often take the form of simple linear schedules linking the tenant’s production to his compensation. In particular, in reality the tenant’s contract usually has the form of sharing, \( t = \alpha q \), where \( \alpha \) is the share of tenants’ harvest. Next we will discuss the optimal share ratio under the sharecropping. Since \( \bar{q} > 0 \) and \( \bar{t} = \alpha \bar{q} > 0 \), the limited liability constraint is naturally satisfied in the following discussion.

The optimal rent sharing contract is to solve the problem:

\[
\max_{\alpha} (1 - \alpha)(\pi_1 \bar{q} + (1 - \pi_1) \bar{q})
\]

s. t.

\[
\begin{align*}
\alpha (\pi_1 \bar{q} + (1 - \pi_1) q) - \psi & \geq \alpha (\pi_0 \bar{q} + (1 - \pi_0) q), \quad (17.4.51) \\
\alpha (\pi_1 \bar{q} + (1 - \pi_1) q) - \psi & \geq 0, \quad (17.4.52)
\end{align*}
\]

The (17.4.51) and (17.4.52) are incentive compatibility constraint and participation constraints, respectively, when the incentive tenant chooses to
take effort. Since \( \pi_0 > 0, \bar{q} > 0 \) and \( q > 0 \). If (17.4.51) holds, (17.4.52) must also hold. Therefore, only the incentive compatibility constraint (17.4.51) is binding. From this we get that the optimal contract for sharecropping is \( \alpha^{SB} = \frac{\psi}{\Delta \pi \Delta q} \). Since the condition for selecting an incentive compatibility contract is \( \Delta q \Delta \pi > \psi \), the optimal sharing ratio needs to satisfy \( \alpha^{SB} < 1 \).

This sharing rule yields the following expected utilities to the principal and the agent, respectively,

\[
EV_\alpha = \pi_1 \bar{q} + (1 - \pi_1)q - \left( \frac{\pi_1 \bar{q} + (1 - \pi_1)q}{\Delta q} \right) \frac{\psi}{\Delta \pi},
\]

and

\[
EU_\alpha = \left( \frac{\pi_1 \bar{q} + (1 - \pi_1)q}{\Delta q} \right) \frac{\psi}{\Delta \pi}.
\]

Since there is no need to satisfy the limited liability constraints, this optimal sharecropping contract is different from the previous second-best contract for the principal. Indeed, comparing (17.4.48) and (17.4.53) on the one hand and (17.4.49) and (17.4.54) on the other hand, we observe that the constant sharing rule benefits the agent but not the principal since the second-best contract brings more expected utility to the principal than the constant sharecropping contract. A linear contract is less powerful than the optimal second-best contract.

Thus, the constant sharecropping contract is an inefficient way to extract rent from the agent even if it still provides sufficient incentives to exert effort. With a linear sharing rule, the agent always benefits from a positive return on his production, even in the worst state of nature. This positive return yields to the agent more than what is requested by the second-best optimal contract in the worst state of nature, namely zero. Punishing the agent for a bad performance is thus found to be rather difficult with a linear sharing rule.

### 17.4.3 Financial Contracts: Credit Rationing

Information asymmetry in financial markets has two important issues, one is the adverse selection discussed in the previous chapter, the other is the moral hazard of borrowers to be discussed here. The discussion is about the credit rationing in financial market which was studied by Holmstrom and Tirole (1994) on credit rationing. It is also referred to the discussion of Laffont and Martimort (2002).

Moral hazard is an important issue in financial markets. In Holmstrom and Tirole (AER, 1994), it is assumed that a risk-averse entrepreneur wants to start a project that requires an initial investment worth an amount \( I \). The entrepreneur has no cash of his own and must raise money from a bank or any other financial intermediary. The return on the project is random and
equal to $\bar{V}$ (resp. $V$) with probability $\pi(e)$ (resp. $1 - \pi(e)$), where the effort exerted by the entrepreneur $e$ belongs to $\{0, 1\}$. We denote the spread of profits by $\Delta V = \bar{V} - V > 0$. The financial contract consists of repayments $\{(\bar{z}, \bar{\bar{z}})\}$, depending upon whether the project is successful or not.

To induce effort from the borrower, the risk-neutral lender’s program is written as

$$\max_{(\bar{z}, \bar{\bar{z}})} \pi_1 \bar{z} + (1 - \pi_1)\bar{\bar{z}} - I \quad(17.4.55)$$

subject to

$$\pi_1 u(\bar{V} - \bar{z}) + (1 - \pi_1)u(V - \bar{\bar{z}}) - \psi \geq 0, \quad(17.4.56)$$

$$\pi_1 u(\bar{V} - \bar{\bar{z}}) + (1 - \pi_1)u(V - \bar{\bar{z}}) - \psi \geq 0. \quad(17.4.57)$$

Note that the project is a valuable venture if it provides the bank with a positive expected profit.

With the change of variables, $\tilde{t} = \bar{V} - \bar{z}$ and $t = V - \bar{\bar{z}}$, the principal’s program takes its usual form. Then, the second-best cost of implementing the positive effort is given by

$$C_{SB} = \pi_1 h(\psi + (1 - \pi_1)\frac{\psi}{\Delta \pi}) + (1 - \pi_1)h(\psi - \pi_1\frac{\psi}{\Delta \pi}),$$

where $h = u^{-1}$. At the same time, we have $\Delta \pi \Delta V \geq C_{SB}$ so that the lender wants to induce the positive effort level even in a second-best environment.

The lender’s expected profit is worth:

$$V_1 = \pi_1 \bar{V} + (1 - \pi_1)V - C_{SB} - I. \quad(17.4.58)$$

At this point, the initial investment of $I$ has an important impact on whether the principal will lend to the firm, because the principal will finance the project only if the expected profit from the project is positive: $V_1 > 0$. From (17.4.58), this requires the investment to be low enough, and typically we must have

$$I < I_{SB} = \pi_1 \bar{V} + (1 - \pi_1)V - C_{SB}. \quad(17.4.59)$$

But, under complete information and no moral hazard, the project would instead be financed as soon as

$$I < I^* = \pi_1 \bar{V} + (1 - \pi_1)V. \quad(17.4.60)$$

Comparing (17.4.59) with (17.4.60), we obtain $I_{SB} < I^*$. This implies that in an asymmetric environment, the borrower’s moral hazard implies that the borrower’s credit scale is smaller than that in a complete information environment, i.e., for $I$ in $[I_{SB}, I^*]$, some projects are financed under complete information but no longer under moral hazard. This result can be seen as similar to the form of creditrationing.
Note that the optimal financial contract offered to the risk-averse and cashless entrepreneur does not satisfy the limited liability constraint \( t \geq 0 \). Indeed, we have \( t^{SB} = h\left(\psi - \frac{\pi_1 \psi}{\Delta \pi}\right) < 0 \). To be induced to make an effort, the agent must bear some risk, which implies a negative payoff in the bad state of nature. Adding the limited liability constraint, the optimal contract would instead entail \( t^{LL} = 0 \) and \( \bar{t}^{LL} = h\left(\frac{\psi}{\Delta \pi}\right) \). This contract has sometimes been interpreted in the corporate finance literature as a debt contract, with no money being left to the borrower in the bad state of nature and the residual being pocketed by the lender in the good state of nature.

Finally, since \( h(\cdot) \) is strictly convex and \( h(0) = 0 \), we have

\[
\bar{t}^{LL} - t^{LL} = h\left(\frac{\psi}{\Delta \pi}\right) > t^{SB} - t^{SB} = h\left(\psi + (1 - \pi_1) \frac{\psi}{\Delta \pi}\right) - h\left(\psi - \frac{\pi_1 \psi}{\Delta \pi}\right).
\]

This inequality shows that the debt contract has worse incentive effect than the optimal incentive contract. Under the debt contract, in order to encourage the agent to choose to work hard, it is necessary to widen the transfer gap of the agent under different states, which increases the risk faced by the agent, but will undoubtedly bring greater agent cost to the principal.

We have discussed above the case in which entrepreneurs do not own any amount of initial capital. In reality, however, an entrepreneur shall not have access to external finance, such as bank loans, if he does not own some amount of capital as collateral. Indeed, an entrepreneur with a larger amount of collateral capital can have access to more bank loans, which is also called credit rationing. Holmstrom and Tirole (1997) and Chapter 3 of Tirole (2006) discuss this issue. We briefly discuss it here.

Assume there is an entrepreneur who starts out with amounts of initial capital (cash) \( A \). There is an economically viable project which costs \( I > A \). Thus, the entrepreneur needs at least \( I - A \) in external funds (say from a bank) to be able to invest. Assume that the investment generates a financial return equaling either 0 (failure) or \( R \) (success). In the absence of proper incentives or outside monitoring, the entrepreneur may deliberately reduce the probability of success in order to obtain a private benefit. If the entrepreneur is diligent, the probability of success is \( p_H \); otherwise, it is \( p_L \) with a private benefit equaling \( B \). Assume that \( \Delta p = p_H - p_L \). For simplicity, the entrepreneur and the bank are assumed to be risk neutral. Due to limited liability, the income of the entrepreneur is no less than 0.

\[\text{It can be easily seen that, when } h \text{ is differentiable, by the mean-value theorem, } h\left(\frac{\psi}{\Delta \pi}\right) - h\left(\psi + (1 - \pi_1) \frac{\psi}{\Delta \pi}\right) = h'(\delta_1)(\frac{\psi}{\Delta \pi} - \psi) > h'(\delta_2)(\frac{\psi}{\Delta \pi} - \psi) = h(0) - h\left(\psi - \frac{\pi_1 \psi}{\Delta \pi}\right) \text{ by noting that } h'(\cdot) \text{ is monotonically increasing and } \delta_1 > \psi + (1 - \pi_1) \frac{\psi}{\Delta \pi} > 0 > \delta_2. \text{ It may be remarked that inequality becomes equality when } h(\cdot) \text{ is linear, i.e., the entrepreneur is risk-neutral.}\]
Assuming a competitive market for banks, the equilibrium market interest rate equals 0. In the debt finance contract, if the project succeeds, the bank is paid $R_I$, and an entrepreneur is paid $R_B = R - R_I$.

For a perfectly competitive bank, zero-profit condition implies that $p_H R_I = I - A$. Further assume that the project generates positive returns only when the entrepreneur is diligent, namely that $p_H R > I > p_L R + B$.

A necessary condition for the entrepreneur to be diligent is that $p_H R_B = p_L R_B + B$, or $R_B \geq B/\Delta p$. This thus leaves at most $R - (B/\Delta p)$ to compensate the bank, so the pledgable expected income for the bank equals $p = p_H [R - (B/\Delta p)]$.

Thus, an entrepreneur shall have access to direct finance only when $p = p_H [R - (B/\Delta p)] \geq I - A$ is fulfilled. Define $\bar{A} = p_H (B/\Delta p) - (p_H R - I)$. Then only entrepreneurs with initial amounts of capital $A \geq \bar{A}$ can invest using direct finance. As is obvious, $\bar{A} > 0$ when $p_H R - I < p_H (B/\Delta p)$. Also, the lower bound requirement for external financing, $\bar{A}$, increases as either $B$ or $I$ increases. As such, entrepreneurs with small amounts of initial capital, namely small $A$, may not have access to direct finance. Moreover, such a credit rationing issue will be more serious if the project scale $I$ increases or the private benefit $B$ increases (i.e., the moral hazard issue becomes worse).

17.5 Extensions of the Basic Model

In this section, we will extend the basic model. In the basic model, there are two output outcomes and two agent’s actions. If the outcomes or actions are finite or continuous, what will the new features of the second-best contracts be? We mainly focus on situations where agents are risk-averse.

17.5.1 More than Two Outcomes and Two Actions

We now extend our previous $2 \times 2$ model to allow for more than two levels of performance. We consider a production process where $n$ possible outcomes can be realized. Those performances can be ordered so that $q_1 < q_2 < \cdots < q_i < \cdots < q_n$. We denote the principal’s return in each of those states of nature by $S_i = S(q_i)$. In this context, a contract is a $n$-tuple of payments $\{(t_1, \ldots, t_n)\}$. Also, let $\pi_{ik}$ be the probability that production level is $q_i$ when the effort level is $e_k$ so that for all pairs $(i, k)$, we have $\sum_{i=1}^{n} \pi_{ik} = 1$. We still assume that only two levels of effort are feasible. i.e., $e_k$ in $\{0, 1\}$, and let $\Delta \pi_i = \pi_{i1} - \pi_{i0}$.

We want the agent’s effort $e = 1$ to result in a better performance (in the random sense). So we may use the maximum likelihood ratio criterion to reward the agent.

**Definition 17.5.1** The probabilities of success satisfy the monotone likelihood ratio property (MLRP) if $\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}}$ is nondecreasing in $i$. 
17.5. EXTENSIONS OF THE BASIC MODEL

Suppose now that the agent is strictly risk-averse. The optimal contract that induces effort must solve the program below:

$$\max_{\{t_1, \ldots, t_n\}} \sum_{i=1}^{n} \pi_{i1}(S_i - t_i) \quad (17.5.62)$$

subject to

$$\sum_{i=1}^{n} \pi_{i1}u(t_i) - \psi \geq \sum_{i=1}^{n} \pi_{i0}u(t_i) \quad (17.5.63)$$

and

$$\sum_{i=1}^{n} \pi_{i1}u(t_i) - \psi \geq 0, \quad (17.5.64)$$

where (17.5.63) and (17.5.64) are the agent’s incentive compatibility and participation constraints, respectively.

Using the same change of variables as before, it should be clear that the program is again a concave problem with respect to the new variables $u_i = u(t_i)$ so that $h_i(\cdot) = u_i^{-1}$. The optimization problem then becomes:

$$\max_{\{u_1, \ldots, u_n\}} \sum_{i=1}^{n} \pi_{i1}(S_i - h(u_i)) \quad (17.5.65)$$

s. t. $$\sum_{i=1}^{n} \pi_{i1}u_i - \psi \geq \sum_{i=1}^{n} \pi_{i0}u_i, \quad (17.5.66)$$

and $$\sum_{i=1}^{n} \pi_{i1}u_i - \psi \geq 0. \quad (17.5.67)$$

Let $\mu$ and $\lambda$ be the Lagrange multipliers of the participate constraint (17.5.67) and the incentive compatibility constraint (17.5.66), then the first-order conditions of the principal’s program are written as:

$$\frac{1}{u'(t_{SB}^i)} = \mu + \lambda \left( \frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}} \right) \quad \forall \ i \in \{1, \ldots, n\}. \quad (17.5.68)$$

Multiplying each of these equations by $\pi_{i1}$ and summing over $i$ yields $\mu = E_q \left( \frac{1}{u'(t_{SB}^i)} \right) > 0$, where $E_q$ denotes the expectation operator with respect to the distribution of outputs induced by effort $e = 1$.

Multiplying (17.5.68) by $\pi_{i1}u(t_{SB}^i)$, summing all these equations over $i$, and taking into account the expression of $\mu$ obtained above lead to

$$\lambda \left( \sum_{i=1}^{n} (\pi_{i1} - \pi_{i0})u(t_{SB}^i) \right) = E_q \left( u(t_{SB}^i) \left( \frac{1}{u'(t_{SB}^i)} - E \left( \frac{1}{u'(t_{SB}^i)} \right) \right) \right). \quad (17.5.69)$$
Using the slackness condition \( \lambda \left( \sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u(t_{i}^{SB}) - \psi \right) = 0 \) to simplify the left-hand side of (17.5.69), we finally get

\[
\lambda \psi = \text{cov} \left(u\left(\tilde{t}_{SB}\right), \frac{1}{u'(\tilde{t}_{SB})}\right), \tag{17.5.70}
\]

By assumption, \( u(\cdot) \) and \( u'(\cdot) \) co-vary in opposite directions. Moreover, a constant wage \( t_{i}^{SB} = t_{SB} \) for all \( i \) does not satisfy the incentive-compatibility constraint, and thus \( t_{i}^{SB} \) cannot be constant everywhere. Hence, the right-hand side of (17.5.70) is necessarily strictly positive. Thus we have \( \lambda > 0 \), and the incentive-compatibility constraint is binding.

Coming back to (17.5.68), we observe that the left-hand side is increasing in \( t_{i}^{SB} \) since \( u(\cdot) \) is concave. For \( t_{i}^{SB} \) to be nondecreasing with \( i \), MLRP must hold. Then higher outputs are also those that are the more informative ones about the realization of a high effort. Hence, the agent should be more rewarded as output increases.

When the level of performance is continuum, we can similarly analyze the optimal contract. In this case, the level of performance \( \tilde{q} \) is drawn from a continuous distribution with a cumulative function \( F(\cdot|e) \) and the density function \( f(\cdot|e) \) on the support \([q, \tilde{q}]\). This distribution is conditional on the agent’s level of effort, which still takes two possible values in \( \{0, 1\} \). By the same procedure as above for the discrete case, we can find the optimal contract \( t_{i}^{SB}(\pi) \) that is monotonically increasing in \( \pi \) when the monotone likelihood property

\[
dq \left( \frac{f(q=1|a)}{f(q=0|a)} \right) \geq 0
\]

is satisfied.

17.5.2 Two Levels of Performance and Continuous Actions

We now discuss the optimal contract with continuous actions and two levels of performance. Assume that the set of agent’s efforts is continuum, \( a \in [0, \infty) \). There are two possible outcomes, \( q \in \{0, 1\} \). \( q = 1 \) represents a good performance, otherwise it is a bad performance. Performance are contingent on efforts and \( p(a) \equiv \text{prob}(q = 1|a) \in (0, 1) \) indicates the probability of a good performance occurring when the effort is \( a \). Suppose that there is \( p'(a) > 0 \) which means that if the agent chooses a higher level of effort, the probability of a good performance is also higher. To simplify the discussion, we assume that the agent’s effort to select \( a = \Psi(a) \). Let \( w \) be the principal’s compensation to the agent. Since actions are not observable, the transfers to the agent is based on performance \( q \).

The utility function of the principal is \( V(q - w) = q - w \), that is, the principal is risk-neutral. The agent’s utility function is \( u(w) - \Psi(a) \), which satisfies \( u'(\cdot) > 0 \) and \( u''(\cdot) \leq 0 \). So the agent is risk-averse.

In the following, we consider the choice of the principal’s contract when the information is complete and incomplete.
If the agent’s actions can be observed by the principal and can be verified by a third party, then the principal’s optimal contract is to solve the following maximization problem:

\[
\max_{a, w_0, w_1} p(a)(1 - w_1) + (1 - p(a))(-w_0)
\]

\[
\text{s.t. } p(a)u(w_1) + (1 - p(a))u(w_0) - a \geq 0,
\]

where (17.5.72) is the agent’s participation constraint. Let \( \lambda \) be the Lagrangian multiplier of the participation constraint. We have the first-order condition:

\[
\frac{1}{u'(w_1)} = \lambda = \frac{1}{u'(w_0)}
\]

so that \( w_1 = w_0 \). The agent is then fully insured and the principal bears all risks.

We now discuss the second-best contract when information is incomplete. Since the agent’s actions cannot be observed by the principal, the agent’s actions must satisfy the incentive compatibility condition. Then the principal’s maximization problem is:

\[
\max_{a, w_0, w_1} p(a)(1 - w_1) + (1 - p(a))(-w_0)
\]

\[
\text{s.t. } p(a)u(w_1) + (1 - p(a))u(w_0) - a \geq 0,
\]

\[
a = \arg\max_a p(a')u(w_1) + (1 - p(a'))u(w_0) - a',
\]

where (17.5.76) is the incentive compatibility condition in a continuous situation.

From the (17.5.76), we obtain:

\[
p'(a)(u(w_1) - u(w_0)) = 1.
\]

Let \( \mu \) and \( \lambda \) be the Lagrange multipliers of the formula (17.5.72) and (17.5.77) respectively. From the first-order conditions of \( w_0 \) and \( w_1 \), we get:

\[
\frac{1}{u'(w_1)} = \mu + \lambda \frac{p'(a)}{p(a)},
\]

\[
\frac{1}{u'(w_0)} = \mu - \lambda \frac{p'(a)}{1 - p(a)}.
\]

When the incentive compatibility constraint (17.5.77) is binding, we get \( w_1 > w_0 \) which means that the agent’s greater effort will receive higher compensation and the agent bears some risk under the second-best optimal contract.
17.5.3 Continuous Performance and Continuous Actions

We now discuss the case where performance and actions are both continuous. In this case, it is more clearly to see the trade-off between efficiency and insurance in a second-best contract.

We assume that the agent's effort \( a \in [0, \infty) \) (in the probability sense) determines the principal's outcome \( q = a + \epsilon \) where \( \epsilon \in (-\infty, \infty) \) obeys the normal distribution. It can be understood as an external factor that influences outcome with an expectation of 0 and a variance of \( \sigma^2 \). The principal is risk-averse and the agent’s utility function for income is \( u(w) = -e^{-rw} \) which is the constant-coefficient absolute risk-aversion utility function. Assume that the principal’s compensation for the agent is a linear function of the outcome: \( w(q) = \alpha + \beta q \). We examine how \( \alpha \) and \( \beta \) are chosen in the optimal contract. In addition, we assume that the cost of choosing \( a \) for the agent is \( c(a) \).

Under the linear contract, the agent’s compensation also obeys a normal distribution. The expectation is \( \bar{w} = \alpha + \beta a \) and the variance is \( \sigma^2(w) = \beta^2 \sigma^2 \). In the constant-coefficient absolute risk-aversion utility function, the expected utility is \( -e^{-(\bar{w} - r\sigma^2(w)/2)} \).

Then, under the linear compensation contract, the agent’s expected utility for compensation is:

\[
-e^{-(\alpha + \beta a - r\beta^2 \sigma^2/2)}.
\]  

(17.5.80)

The agent’s incentive compatibility constraint is

\[
a = \arg\max_a \alpha + \beta a' - r\beta^2 \sigma^2/2 - c(a'),
\]

which leads to the first-order condition:

\[
c'(a) = \beta.
\]  

(17.5.81)

Thus, the principal’s optimization problem is:

\[
\max_{a,\alpha,\beta} (1 - \beta)a - \alpha
\]

subject to

\[
\alpha + \beta a - r\beta^2 \sigma^2/2 - c(a) \geq \bar{u},
\]

\[
c'(a) = \beta.
\]  

(17.5.83)

(17.5.84)

Here, \( -e^{(-\bar{u})} \) that corresponds to \( -\bar{u} \) is the agent’s reservation utility. Obviously, the participation constraint (17.5.83) must be binding, otherwise a higher profit can be obtained by lowering \( \alpha \). Substituting participation constraint and incentive compatibility constraint into the objective
function, the above optimization problem becomes:

$$\max_a a - \frac{re'(a)^2 \sigma^2}{2} - c(a),$$

from which we have the first-order condition:

$$1 - re'(a)c''(a)\sigma^2 - c'(a) = 0,$$

and thus

$$\beta = c'(a) = \frac{1}{1 + re''(a)\sigma^2},$$

where $\beta$ is the marginal payment of the agent’s effort that has a direct influence on the agent’s incentive for the effort. So this coefficient is called incentive intensity, which depends on the risk metric $\sigma^2$ and the degree of risk-aversion of the agent. We can regard $r \to 0$ as a risk-neutral case. According to (17.5.80), if $r = 0$, the utility of the consumer depends only on the expectation of income. When the agent is risk-neutral or the outcome has no risk, the incentive intensity is 1. As the agent’s risk-aversion increases and the risk of the outcome increases, the incentive intensity becomes smaller and smaller. And this is precisely a fundamental trade off between efficiency and insurance under moral hazard.

For general continuous performance and continuous actions, optimal contracts do not necessarily have linear features. Holmstrom and Milgrom (1987) give the conditions that ensure the linear contracts to be optimal.

In reality, incentive contracts usually contain more factors. We use the results from the above continuous case to discuss an incentive contract with multiple factors. Consider the following case: A manager who is trying to invest $a$ will affect the firm’s current profit and stock price as well. These two ways of influences are different because the factors affecting the stock price exceed the corporate accounting statements. These factors affect the future profitability of the firm.

Let $q = a + \epsilon_q$ be the current profit level of the firm where $\epsilon_q \sim N(0, \sigma^2_q)$. $p = a + \epsilon_p$ is the stock price of the firm, where $\epsilon_p \sim N(0, \sigma^2_p)$. $\text{Cov}(\epsilon_q, \epsilon_p) = \sigma_{pq}$ is the covariance between stock price and current profit (external) influence factor. If $\sigma_{pq} = 0$, it means that these two factors are independent.

The agent’s utility function for income is $u(w) = -e^{-rw}$ and the principal is risk-neutral. We consider a linear contract for manager compensation $w = \alpha + p\beta_p + q\beta_q$. We focus on how the manager’s compensation is dependent on the current profits and the firm’s stock price in the optimal contract under incomplete information.

To simplify the discussion, we assume that the private cost of the manager’s effort is $c(a) = \frac{ca^2}{2}$. The shareholder is the principal and her problem
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is:

$$\max_{a, \alpha, \beta_p, \beta_q} q - (\alpha + p\beta_p + q\beta_q)$$

s. t. $E[-e^{-(\alpha + p\beta_p + q\beta_q - c(a))}] \geq -e^{-\bar{u}}$,

$$a \in \arg\max_a E[-e^{-(\alpha + p(a')\beta_p + q(a')\beta_q - c(a'))}],$$

where $E[\cdot]$ is the expected value for $\epsilon_p$ and $\epsilon_q$. Use the certainty equivalence principle of income, the above optimization problem can be rewritten as:

$$\max_{a, \alpha, \beta_p, \beta_q} \left(1 - \beta_p - \beta_q\right)a - \alpha$$  (17.5.85)

s. t. $$(\beta_p + \beta_q)a - \frac{r(\beta_p^2\sigma_p^2 + \beta_q^2\sigma_q^2 + 2\beta_p\beta_q\sigma_{pq})}{2} - \frac{ca^2}{2} \geq \bar{u};$$  (17.5.86)

$$a \in \arg\max_a (\beta_p + \beta_q)a' - \frac{r(\beta_p^2\sigma_p^2 + \beta_q^2\sigma_q^2 + 2\beta_p\beta_q\sigma_{pq})}{2} - \frac{ca'^2}{2}. \quad (17.5.87)$$

Solving the incentive compatibility constraint (17.5.87), we get:

$$a = \frac{\beta_p + \beta_q}{c}.$$

Substituting it into the objective function (17.5.85) and noting that the participation constraint is binding, we can simplify the principal’s optimization problem as:

$$\max_a (1 - \beta_p - \beta_q)\frac{\beta_p + \beta_q}{c} - \alpha$$  (17.5.88)

subject to

$$\left(\beta_p + \beta_q\right)\frac{\beta_p + \beta_q}{c} - \frac{r\left(\beta_p^2\sigma_p^2 + \beta_q^2\sigma_q^2 + 2\beta_p\beta_q\sigma_{pq}\right)}{2} - \left(\beta_p + \beta_q\right)\frac{\beta_p + \beta_q}{c} = \bar{u}. \quad (17.5.89)$$

Simplifying the first-order conditions of $\beta_p$ and $\beta_q$, we get:

$$\beta_p^* = \frac{\sigma_q^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2\sigma_{pq} + \frac{1}{1 + r\Omega}},$$

$$\beta_q^* = \frac{\sigma_p^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2\sigma_{pq} + \frac{1}{1 + r\Omega}},$$

where $\Omega = \frac{\sigma_p^2\sigma_q^2 - \sigma_{pq}^2}{\sigma_q^2 + \sigma_p^2 + 2\sigma_{pq}}$. The $\beta^*_p$ and $\beta^*_q$ obtained above are the marginal compensations for the managers of share price and current profit in the optimal contract.
When $\sigma_{pq} = 0$, $\beta^*_p$ and $\beta^*_q$ in the optimal contract become:

$$
\beta^*_p = \frac{\sigma_q^2}{\sigma_q^2 + \sigma_p^2 + r c 2 \sigma_q^2 \sigma_p^2},
$$
$$
\beta^*_q = \frac{\sigma_p^2}{\sigma_q^2 + \sigma_p^2 + r c 2 \sigma_q^2 \sigma_p^2}.
$$

When the agent is risk-neutral (i.e. $r = 0$), $\beta^*_p$ and $\beta^*_q$ in the optimal contract become:

$$
\beta^*_p = \frac{\sigma_q^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2 \sigma_{pq}},
$$
$$
\beta^*_q = \frac{\sigma_p^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2 \sigma_{pq}},
$$

and thus $\beta^*_p + \beta^*_q = 1$. Along with $\sigma_q^2$ ($\sigma_p^2$) increases, $\beta^*_q$ becomes smaller (larger) and $\beta^*_p$ becomes larger (smaller). That is, if the profit risk is greater, the compensation to the manager reduces the sensitivity to profit and increases sensitivity to stocks, and vice versa.

When $\epsilon_p = \epsilon_q + \zeta$ with $\zeta \sim N(0, \sigma^2_{\zeta})$, and $\zeta$ and $\epsilon_q$ are independent so that $\sigma_{pq} = \sigma^2_q$, we then have $\beta^*_p = 0$ and $\beta^*_q = \frac{1}{1 + r c \sigma_q^2}$. Thus the stock will not enter the manager’s compensation contract because the profit is a sufficient statistic of the stock. Introducing the stock will not increase the manager’s incentives, but it will bring greater risks. Holmstrom (1979) proved the conclusion of sufficient statistics in similar incentive contracts.

### 17.6 Relative Performance Incentive of Multi-Agents

In reality, there are usually multiple agents inside an organization which will bring about some new incentive problems. The compensation of an agent depends not only on its own performance but also on the performance of others which is often referred to as relative performance incentives in the literature. There are many studies that use incentive theory to analyze incentives for government-level institutions. In addition to fiscal incentives, local officials are also concerned about their promotion. And an official’s promotion depends on his performance relative to other officials at the same level, which is a common incentive method in hierarchical organizations or nations such as in China.

In this section we discuss the logic of promotion or tournaments in organizational incentives and compare the differences between it and the incentive of sharing.
17.6.1 Tournament Model

We first introduce the tournament model of Lazear and Rosen (1981). They first considered a simple case: the principal allows two risk-neutral agents to complete their tasks. The performance of the task depends on their respective level of effort and some external factors. Lazear and Rosen revealed that in a risk-neutral situation, tournament can result in the first-best compensation contract.

Suppose that the performance function of the two agents is

\[ q_i = a_i + \epsilon_i, \quad i = 1, 2, \]

where \( a_i \) is effort and \( \epsilon_i \) is i.i.d. The distribution function is \( F(\cdot) \) with mean 0 and variance \( \sigma^2 \). Assume that each agent’s cost function of effort is \( c(a_i) = \frac{ca_i^2}{2} \).

Optimal effort \( a^* \) then satisfies:

\[ a^* = \frac{1}{c}. \]

Consider a tournament: if an agent’s performance is higher than that of another agent, he will get a high compensation; otherwise, he will get a low compensation. Let \( W_1 \) and \( W_2 \) with \( W_1 > W_2 \) be the high and low compensations, respectively, and \( \Delta W = W_1 - W_2 \). Assume that the two agents are symmetric, that is, they have the same cost and the same reservation utility. Their participation constraints are:

\[ W_1 P_i + W_2 (1 - P_i) - \frac{ca_i^2}{2} \geq \bar{u}, \]

where \( P_i \) is the probability that the participant \( i \) wins, defined as:

\[ P_i = \text{prob}(q_i > q_j) = \text{prob}(\epsilon_j - \epsilon_i < a_i - a_j) = H(a_i - a_j), \]

where \( H(\cdot) \) is the distribution function of \( \epsilon_j - \epsilon_i \) with a mean 0, a variance \( 2\sigma^2 \), and its density function is denoted by \( h(\cdot) \). Under such a contract, the first-best choice for the agent \( i \) is given by:

\[ \Delta W \frac{\partial P_i}{\partial a_i} - ca_i = \Delta W h(a_i - a_j) - ca_i = 0. \]

In symmetric equilibrium \( a^*_i = a^* \), we have

\[ a^* = \frac{\Delta W h(0)}{c}. \]

Thus, when

\[ \Delta W = \frac{1}{h(0)}, \quad (17.6.90) \]

we have \( a^* = a^* \). At the same time, the agent’s participation constraint is:

\[ W_1 \frac{1}{2} + W_2 \frac{1}{2} - \frac{1}{2c} \geq \bar{u}, \quad (17.6.91) \]
which is binding at optimal.

Then, solving (17.6.90) and (17.6.91) for \( W_1 \) and \( W_2 \), we have

\[
W_1 = \frac{h(0) + c}{2ch(0)} + \bar{u}
\]

and

\[
W_2 = \frac{h(0) - c}{2ch(0)} + \bar{u},
\]

at which the tournament contract achieves the outcomes of the optimal contract, although the form of the contract was very different.

### 17.6.2 Tournament Contract under Risk Aversion

Now consider a tournament contract when the agents are risk-averse and compare it to a piece-rate contract or an individual performance contract. The following discussion is from Green and Stockey (1983), and readers can refer to the discussion of Bolton and Dewatripont (2005).

There are two agents and they need to undertake two tasks. There are only two outcomes, namely \( q_i \in \{0, 1\} \), corresponding to “failure” and “success”. Assume that the two tasks are similar and their outcomes depend on a common state. There are two states. One has no external shock. The probability of success of the task depends on the agent’s effort level of \( a_i \). The other has an external shock. Once it occurs, the task will fail regardless of the agent’s effort. Assume that the principal knows whether there is an external shock with the probability \( 1 - \xi \). In the absence of an external shock, the agent’s effort \( a_i \) has the following relationship to the probability of success of the task:

\[
\Pr(q_i = 1|a_i) = \xi a_i.
\]

The agent’s effort cost function is:

\[
c(a_i) = \frac{ca_i^2}{2}.
\]

The agents’ utility functions are the same:

\[
u(w) - c(a),
\]

with \( u' > 0 \) and \( u'' < 0 \). The principal is risk-neutral.

Let us discuss the tournament first. Since the outcome of the task and the common shock are observable, the feasible contract contains five possible compensations \( w = (w_{11}, w_{10}, w_{01}, w_{00}, w_c) \) where \( w_{ij} \) is the compensation for the agent \( i \) when his outcome is \( q_i \) and the outcome of his opponent \( j \) is \( q_j \). \( w_c \) is the compensation obtained by the agent when an external shock occurs. In the following we discuss the optimal incentive contract for agent \( i \).
The incentive constraint for agent $i$ is:

$$a_i = \arg\max_a \max \{ \xi[a(1 - a_j)u(w_{10}) + aa_ju(w_{11}) + (1 - a)(1 - a_j)u(w_{00}) + (1 - a)a_ju(w_{01})] + (1 - \xi)u(w_c) - \frac{1}{2}ca_i^2 \}. \quad (17.6.92)$$

The first-order condition for the above incentive-compatibility constraint (17.6.92) can be written as:

$$\xi[(1 - a_j)u(w_{10}) + a_ju(w_{11}) - (1 - a_j)u(w_{00}) - a_ju(w_{01})] = ca_i. \quad (17.6.93)$$

If the principal wants the agent $i$ to choose $a_i$, his problem is to minimize the implementation cost:

$$\min_w \xi[a_i(1 - a_j)w_{10} + a_ia_jw_{11} + (1 - a_i)(1 - a_j)w_{00} + (1 - a_i)a_jw_{01}] + (1 - \xi)w_c$$

s.t. $\xi[a_i(1 - a_j)u(w_{10}) + a_ia_ju(w_{11}) + (1 - a_i)(1 - a_j)u(w_{00}) + (1 - a_i)a_ju(w_{01})] + (1 - \xi)u(w_c) - \frac{1}{2}ca_i^2 \geq \bar{u}, \quad (17.6.94)$

$$\xi[(1 - a_j)u(w_{10}) + a_ju(w_{11}) - (1 - a_j)u(w_{00}) - a_ju(w_{01})] = ca_i. \quad (17.6.95)$$

Let $\mu$ and $\lambda$ be the Lagrange multipliers of the participation constraint (17.6.94) and the incentive compatibility constraint (17.6.95). The first-order conditions for $w_{10}$ and $w_{11}$ are respectively given by:

$$\xi a_i(1 - a_j) = \xi(1 - a_j)[\mu a_i + \lambda]u'(w_{10}), \quad (17.6.96)$$

$$\xi a_i a_j = \xi a_j[\mu a_i + \lambda]u'(w_{11}). \quad (17.6.97)$$

From (17.6.96) and (17.6.97), we obtain $w_{10} = w_{11}$.

Similarly, we can get the first-order conditions for $w_{00}$ and $w_{01}$:

$$\xi (1 - a_i)(1 - a_j) = \xi(1 - a_j)[\lambda (1 - a_i) - \lambda]u'(w_{00}), \quad (17.6.98)$$

$$\xi (1 - a_i) a_j = \xi a_j[\mu(1 - a_i) - \lambda]u'(w_{01}). \quad (17.6.99)$$

By (17.6.98) and (17.6.99), we get $w_{00} = w_{01}$.

The first-order condition for $w_c$ is:

$$(1 - \xi) = (1 - \xi)u'(w_c). \quad (17.6.100)$$

From (17.6.100) and (17.6.98), due to $\lambda > 0$, we have $w_c \neq w_{00}$.

Comparing (17.6.98) with (17.6.96) and (17.6.100) with (17.6.96), we have $w_{10} \neq w_{00}$ and $w_c \neq w_{10}$. 


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17.6.3 Comparison of Tournament and Individual Performance Contract

Let us compare tournament and individual performance contract. For the tournament, it is mainly to compare the performances of agent $i$ and agent $j$, namely $q_i$ and $q_j$. If $q_i > q_j$, agent $i$’s reward is $W$; if $q_i < q_j$, agent $i$’s reward is $L$; if $q_i = q_j$, agent $i$’s reward is $T$. According to the previous discussion of the optimal contract, this means $W = w_{10}$, $L = w_{01}$, $T = w_{00}$, $T = w_{01}$, $T = w_{11}$, $T = w_c$.

For the individual performance contract, since this compensation method only depends on its own performance, $W_1 = w_{10} = w_{11}$, $W_0 = w_{01} = w_{00} = w_c$ according to the previous optimal contract.

We then obtain the following conclusions:

Conclusion 1: When the common shock can be separated (i.e. $w_c \neq w_{ij}$) and there is common shock (i.e. $\xi = 1$), then for any relative performance incentive, we have $w_{10} \neq w_{11}$, $w_{00} \neq w_{01}$ or both. Moreover, the relative performance and tournament contract are the second-best while the individual performance contract is the first-best for the principal.

Conclusion 2: When $\xi < 1$, there is a common shock such that the individual performance contract is the second-best for the principal. This is because the individual performance contract requires $w_c = w_{00}$, but the first-best contract requires $w_c \neq w_{00}$.

Conclusion 3: When $\xi < 1$, the tournament dominates the individual performance contract for the principal.

Conclusions 1 and 2 above can be derived from the characteristics of tournament and individual performance contract.

In the following, we discuss the possibility of Conclusion 3. Assume that the best situation to the principal is to make both agents choose $a_1 = a_2 = 1$. In the tournament, given agent $j$ chooses $a_j = 1$, the cost for the principal to induce the agent $i$ to choose $a_i = 1$ is given by

$$\min_{w} \xi[a_i T + (1 - a_i)L] + (1 - \xi)T$$

s.t. $\xi[a_i u(T) + (1 - a_i)u(L)] + (1 - \xi)u(T) - \frac{1}{2} ca_i^2 \geq \bar{u}$, $\xi[u(T) - (1 - a_j)u(L)] = ca_i$.

To induce the agent $i$ to choose $a_i = 1$, the principal sets:

$$u(T) = u(L) + \frac{c}{\xi}.$$
When \( a_i = a_j = 1 \), agents \( i \) and \( j \) will receive a payment of \( T \). In this case, the agent completely avoids the risk and there is no insurance cost.

However, for individual performance contract, if the principal wants to induce the agents to choose \( a_1 = a_2 = 1 \), the agents’ compensations must be: \( W_1 = w_{11} = w_{10} > W_0 = w_{01} = w_{00} \). Otherwise the agents will not have the incentive to choose \( a_i = 1 \) which means that the agents’ payment is risky. Therefore, under a common shock, the tournament contract dominates the individual performance contract for principal.

The above is just a comparison of the utility of the two contract forms through a simple case. For more general situations, see Lazear and Rosen (1981) and Green and Stockey (1983).

17.7 Performance Incentive under Multi-Task

In many organizations, agents often perform multiple tasks at the same time. For example, for teachers, not only do they need to impart knowledge to students, they also need to cultivate students’ imagination and creativity. But these different aspects of the task are very different in terms of measurement. The imparting of knowledge can be reflected in students’ achievement. However, students’ imagination and creativity are not well recognized. How do we form effective incentives for teachers in these two tasks? In the literature, the study of this problem is called multi-task principal-agent. Holmstrom and Milgrom (1991) conducted a deep study of this issue. We will discuss the incentive problem with multiple tasks through a simple model.

Assume that one agent is engaged in two tasks and the outcome of each task depends on respective effort:

\[ q_i = a_i + \epsilon_i, i = 1, 2. \]

To simplify the discussion, assume \( \epsilon_i \sim N(0, \sigma_i^2) \), \( \text{cov}(\epsilon_1, \epsilon_2) = 0 \), where \( \epsilon_i \) is understood as a measurement error.

Assume that the cost of efforts exerted by the agent is:

\[ \Psi(a_1, a_2) = \frac{1}{2}(c_1 a_1^2 + c_2 a_2^2) + \delta a_1 a_2, \]

where \( 0 \leq \delta \leq \sqrt{c_1 c_2} \). If \( \delta = 0 \), this means that the two efforts are completely independent. If \( \delta = \sqrt{c_1 c_2} \), the two efforts can be substituted.

Assume that the principal is risk-neutral and the agent has constant-coefficient absolute risk-aversion utility function:

\[ u(w; a_1, a_2) = -e^{-r|w - \Psi(a_1, a_2)|}, \]

where \( w \) is the principal’s monetary compensation to the agent.
Since the principal can only observe the two outcomes of the agent, the principal formulates a linear incentive contract for the outcome:

\[ w = \alpha + \beta_1 q_1 + \beta_2 q_2. \]

By the agent’s deterministic equivalence principle, the agent’s deterministic equivalent net income is:

\[
\alpha + \beta_1 a_1 + \beta_2 a_2 - \frac{r}{2}(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2) - \frac{1}{2}(c_1 a_1^2 + c_2 a_2^2) - \delta a_1 a_2.
\]

From the incentive compatibility condition of the agents above, the first-order condition of \( a_i \) is obtained:

\[
\beta_i = c_i a_i + \delta a_j, i \neq j, i, j = 1, 2,
\]

and thus

\[
a_i = \frac{\beta_i c_j - \delta \beta_j}{c_1 c_2 - \delta^2}.
\]

The principal’s problem is to choose \( \alpha, \beta_1, \beta_2 \) to implement the following optimization problem:

\[
\max a_1(1 - \beta_1) + a_2(1 - \beta_2) - \alpha \quad (17.7.101)
\]

s.t. \( a_i = \frac{\beta_i c_j - \delta \beta_j}{c_1 c_2 - \delta^2}, i \neq j, i, j = 1, 2, \quad (17.7.102) \)

\[
\alpha + \beta_1 a_1 + \beta_2 a_2 - \frac{r}{2}(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2)
\]

\[
- \frac{1}{2}(c_1 a_1^2 + c_2 a_2^2) - \delta a_1 a_2 \geq \bar{w}, \quad (17.7.103)
\]

where \( \bar{w} \) is the certainty equivalence income under the agent’s reservation utility. Optimization means that the participation constraint (17.7.103) must be binding. Substituting \( a_1 \) and \( a_2 \) of incentive compatibility constraint (17.7.102) and \( \alpha \) of participation constraint (17.7.103) into the objective function (17.7.101), we have the unconstrained optimization problem:

\[
\max_{(\beta_1, \beta_2)} \left( \frac{\beta_1 c_2 - \delta \beta_2 + \beta_2 c_1 - \delta \beta_1}{c_1 c_2 - \delta^2} \right) - \frac{r}{2}(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2)
\]

\[
- \frac{1}{2} c_1 \left( \frac{\beta_1 c_2 - \delta \beta_2}{c_1 c_2 - \delta^2} \right)^2 - \frac{1}{2} c_2 \left( \frac{\beta_2 c_1 - \delta \beta_1}{c_1 c_2 - \delta^2} \right)^2
\]

\[
- \delta \left( \frac{\beta_1 c_2 - \delta \beta_2}{c_1 c_2 - \delta^2} \right) \left( \frac{\beta_2 c_1 - \delta \beta_1}{c_1 c_2 - \delta^2} \right).
\]

The first-order conditions for \( \beta_1 \) and \( \beta_2 \) are then given by

\[
\beta_1 = \frac{c_2 - \delta + \delta \beta_2}{c_2 + r \sigma_1^2 (c_1 c_2 - \delta^2)}, \quad (17.7.104)
\]

\[
\beta_2 = \frac{c_1 - \delta + \delta \beta_1}{c_1 + r \sigma_2^2 (c_1 c_2 - \delta^2)}, \quad (17.7.105)
\]
from which we obtain $\beta_1^*$ and $\beta_2^*$ in the optimal contract:

$$\beta_1^* = \frac{1 + (c_2 - \delta)r\sigma_2^2}{1 + rc_2\sigma_2^2 + rc_1\sigma_1^2 + r^2\sigma_2^4\sigma_2^2(c_1c_2 - \delta^2)},$$

(17.7.106)

$$\beta_2^* = \frac{1 + (c_1 - \delta)r\sigma_1^2}{1 + rc_2\sigma_2^2 + rc_1\sigma_1^2 + r^2\sigma_2^4\sigma_2^2(c_1c_2 - \delta^2)}.$$

(17.7.107)

When the two tasks are independent (i.e. $\delta = 0$), we have $a_i = \frac{\beta_i}{\sigma_i}$, according to the agent’s incentive-compatibility condition (17.7.102). Based on the conditions (17.7.106) and (17.7.107) of the second-best contract, we get $\beta_1^* = \frac{1}{1 + rc_i\sigma_i}$. The result of this is that the principal takes a separate second-best contract for each of the agent’s tasks.

In addition, as $\sigma_2^2$ increases, the incentive intensity of the second task will decrease $\frac{\partial \beta_2^*}{\partial \sigma_2^2} < 0$. Since the derivative of $\beta_2^*$ with respect to $\sigma_2^2$ in (17.7.107) is less than 0, the incentive for the first task will also decrease. For interior solutions, we have $\frac{\partial \beta_1^*}{\partial \sigma_2^2} < 0$. Thus, if there is a substitutivity for efforts, there will be complementarity between multiple tasks. A decrease in the incentive intensity of a task means that the incentive intensity of the other task will also decrease. Holmstrom and Milgrom (1991) push this logic to the extreme and find that under certain conditions, if the principal cares about the performance of these two tasks, the principal’s optimal contract does not provide any incentive for all tasks when the measurement error of a certain task tends to be infinite ($\sigma_i^2 \to \infty$) or cannot be measured.

This shows that the intensity of incentives that the principal needs to face is the trade-off between efficiency and insurance. Incentive performance with high-powered incentives will amplify the uncertainty caused by noise and increase the risk that agents must bear. If a relationship between performance indicators and effort is weak and highly random, the high-powered incentives on such performance indicators undoubtedly means guiding agent to engage in activities that are tantamount to gambling.

### 17.8 Implicit Incentive and Career Concern

In the labor market, the incentives of workers come from explicit incentives for job performance. For example, the better the performance, the higher the compensation is. On the other hand, it also comes from the evaluation of the workers in the market, especially the market’s assessment of their abilities. It directly influences their future value in the labor market. The latter is called implicit incentive or career motivation. Holmstrom (1982, 1999) analyzed the logic of implicit incentive. We will discuss the implicit incentive in the principal-agent problem through a simple example.

Assume that there is no incentive contract between the principal and the agent. The agent pays attention to the current income and future earnings.
Consider a model of two periods. The outcome of the agent in each period depends on effort, capabilities, and external uncontrollable factors:

\[ q_t = \theta + a_t + \epsilon_t, \quad t = 1, 2, \]

where \( \theta \) is the intrinsic ability of the agent and does not change over time. Assume that the ability and effort of the agent cannot be observed. This model differs from the previous moral hazard model in that there is an information asymmetry about type. To simplify the discussion, assume that the principal and the agent are risk-neutral. The agent’s effort cost is \( \Psi(a_t) \) which satisfies \( \Psi'(a_t) > 0 \) and \( \Psi''(a_t) > 0 \).

In the future period 2, all employers in the market will observe the agent’s outcome \( q_i \). Since in period 2, the agent does not have incentive for working, which means \( a_2 = 0 \), the market wage to the agent is \( w_2(q_1) = E(\theta|q_1) \). The higher the capacity, the greater the outcome is. Therefore, on the posterior estimate of the agent’s ability, the higher the outcome of the first-stage agent, the higher the estimation of his ability in the market. In a pure-strategy equilibrium, suppose that the market expects the agent to invest \( a^* \) in period 1. The ability’s posterior belief is:

\[ w_2(q_1) = E(\theta|q_1) = q_1 - a^* = \theta + a_1 - a^*. \]

Let \( \delta \) be the agent’s discount rate for the future. The longer the future period, the greater the discounting, and so \( \delta > 1 \).

The agent’s goal for the first period is:

\[ \max_{a_1} w_1 + \delta w_2(q_1) - \Psi(a_1) = w_1 + \delta(\theta + a_1 - a^*) - \Psi(a_1). \]

Since there is no explicit incentive, \( w_1 \) is not related to \( q_1 \) and the first-order condition for \( a_1 \) is:

\[ \Psi'(a^*) = \delta. \]

The optimal labor input is determined by solving the problem:

\[ \max_{a_1} a_1 - \Psi(a_1). \]

In this way, \( \Psi'(a^{FB}) = 1 \). When \( \delta > 1 \), the agent’s effort will exceed the first-best effort even if there is no incentive contract between the principal and the agent under the implicit incentive.

17.9 Incentive Intensity and Performance Measurement Distortion Trade-Off

In the previous principal-agent problem, the incentive intensity of the agent is traded off between efficiency and insurance, which is based on an
implicit hypothesis that there is a performance measure about the principal’s interest (target). However, this assumption does not necessarily hold in reality. Although for listed companies, the stock price can be considered as an objective measure of the interests of shareholders. But for non-listed companies, government agencies, non-profit organizations and other institutions, there is usually no objective measure. Even if there is interest in an organization, it cannot be contracted and incorporated into the contract. Therefore, performance measurement is not an objective measure of the organization’s goals.

For example, for government agencies, social welfare can be regarded as the government’s goal, but it is difficult to measure. Economic outcome such as GDP is easy to measure, but it is not equal to social welfare. In this situation, how can the principal-agent problem be analyzed and discussed? Baker (1992) took the lead in introducing the analytic framework of this issue. He found that there is a trade-off between incentive intensity and performance measurement distortion.

Assume that the principal’s goal is \( V(a, \epsilon) \) where \( a \) is the agent’s effort whose cost function is assumed to be \( C(a) = \frac{\epsilon}{2} a^2 \) and \( \epsilon \) is an external influence factor. \( V(a, \epsilon) \) cannot be specified in the contract. Assume that \( P(a, \epsilon) \) is a performance measure that can be contracted. For the sake of discussion, we assume that the principal and the agent are risk-neutral. Also assume that the incentive contract provided by the principal is \( w(P) = \alpha + \beta P \). If \( P(a, \epsilon) = V(a, \epsilon) \), the incentive intensity under the optimal contract is \( \beta = 1 \) since the agent does not have a risk cost. What is the incentive intensity under the optimal contract if \( P(a, \epsilon) \neq V(a, \epsilon) \)?

First of all, we standardize performance measurement so that the effort’s marginal contribution to the principal’s goal and the performance are consistent:

\[
E[P_a(a, \epsilon)] = E[V_a(a, \epsilon)].
\]

Under the above incentive contract, the agent’s utility is:

\[
u(w, a) = w - C(a) = \alpha + \beta P(a, \epsilon) - C(a).
\]

Assume that the agent’s reservation utility is \( \bar{u} \). Then, the incentive compatibility and participation constraints of the agent are:

\[
\alpha + \beta P(a, \epsilon) - C(a) \geq \bar{u}, \quad (17.9.108)
\]

\[
\beta P_a(a^*, \epsilon) = C'(a^*). \quad (17.9.109)
\]

The principal’s goal is:

\[
\max_{\alpha, \beta} E[V(a^*, \epsilon) - \alpha + \beta P(a^*, \epsilon)]
\]
subject to the incentive compatibility and participation constraints (17.9.108) and (17.9.109), which leads to:

$$\beta^* = \frac{E[V_a \alpha^*_a]}{P_a \alpha^*_\beta},$$  

(17.9.110)

By (17.9.109), we have:

$$\alpha^*_\beta = \frac{P_a}{c - \beta P_{aa}},$$  

(17.9.111)

where $P_{aa} = \frac{\partial^2 P}{\partial a \partial a}$, namely the second-order partial derivative of $P$ in $a$. Substituting (17.9.111) into (17.9.110), we obtain:

$$\beta^* = \frac{E[V_a P_a]}{P_a^2}.$$  

(17.9.112)

Without loss of generality, suppose that $E[P_a] = E[V_a] = 1$ for $a^*$. Then (17.9.112) can be written as:

$$\beta^* = \frac{\text{cov}(V_a, P_a) + 1}{\text{var}(P_a) + 1} = \frac{\rho \sigma V_a \sigma P_a + 1}{\sigma P_a^2 + 1},$$  

(17.9.113)

where $\text{var}(P_a)$ is the variance of $P_a$, $\rho$ is the correlation coefficient of $V_a$ and $P_a$, and $\sigma V_a$ and $\sigma P_a$ are the standard deviations of $V_a$ and $P_a$, respectively.

If $V_a$ and $P_a$ are not completely correlated, then the incentive intensity mainly depends on the correlation between incentive performance $V_a$ and $P_a$. The higher the correlation, the more incentive intensity the principal gives to the agent. At the same time, if $\rho < 0$, $\beta^*$ may even be negative under certain conditions.

For a more in-depth discussion on the impact of the difference between contract performance and the value of the organization, see the discussion in Baker (1992).

17.10 A Mixed Model of Moral Hazard and Adverse Selection

So far, when we discuss asymmetric information between a principal and an agent in the previous and current chapters, we only allow either adverse selection or moral hazard presented, but not both. However, in many cases, a principle may know information neither about agent’s action (such as his efforts) nor his characteristic (such as his risk-aversion or cost). In this section, we consider the case where both action and characteristic of an agent are unknown to the principal. In doing so, we introduce a mixed model of moral hazard and adverse selection, which was studied in Meng
and Tian (GEB, 2013). This model investigates the optimal wage contract design when both efforts and risk-aversion of the agent are unobservable.

From the previous section on performance incentives under multi-task, we saw that using high-powered incentives amplify the uncertainty brought about by noise and increases the risk that agents have to take, but it is assumed that the noise parameters can be observed by the principal. In this section, we assume that the risk-aversion coefficient of an agent is his private information. The general theoretical model in this section reveals that in an innovative economy, the risk (insurance) effect should dominate the incentive effect because of the greater risk of innovative activities, so the low-powered incentive contract is reasonable and relatively optimal. When moral hazard and adverse selection coexist, the incentive intensity of the optimal contract will be further reduced. This provides a new explanation for the ubiquitous phenomenon of low-powered incentives, and proves the necessity of low-powered incentives for innovation-driven.

In the previous sections, we also assumed that the effort $e$ or the performance output $\tilde{q}$ is one-dimensional. Now we turn to a more general situation that effort and performance are both multidimensional and continuous. We will first consider the benchmark case where only agent’s efforts are unobservable, and then consider the case where either risk-aversion or cost is also unobservable.

### 17.10.1 Optimal Wage Contract with Unobservable Efforts

Consider a principal-agent relationship in which the agent controls $n$ activities that influence the principal’s payoff. The principal is risk-neutral and her gross payoff is a linear function of the agent’s effort vector $e$:

$$V(e) = \beta' e + \eta,$$

(17.10.114)

where the $n$-dimensional vector $\beta$ characterizes the marginal effect of the agent’s effort $e$ on $V(e)$, and $\eta$ is a noise term with zero mean. The agent chooses a vector of efforts $e = (e_1, \cdots, e_n)' \in \mathbb{R}_n^+$ at quadratic personal cost $\frac{e'C e}{2}$, where $C$ is a symmetric positive definite matrix. The diagonal element $C_{ii}$ reflects the agent’s task-specific productivities, while the sign of off-diagonal elements $C_{ij}$ indicates the relationship between two tasks $i$ and $j$, which are substitute (resp. complementary, independent) if $C_{ij} > 0$ (resp. $< 0$, $= 0$). The agent’s preferences are represented by the negative exponential utility function $u(x) = -e^{-rx}$, where $r$ is the agent’s absolute risk-aversion and $x$ is his compensation minus personal cost.

It is assumed that there is a linear relation between the agent’s efforts and the expected levels of the performance measures:

$$P_i(e) = b_i' e + \varepsilon_i, i = 1, \cdots, m,$$

(17.10.115)
where \( b_i \in \mathbb{R}^n \) captures the marginal effect of the agent’s effort \( e \) on the performance measure \( P_i(e) \); \( B = (b_1, \ldots, b_m)' \) is an \( m \times n \) matrix of performance parameters, and it is assumed that the matrix \( B \) has full row rank \( m \) so that every performance measure cannot be replaced by the other measures; and \( \epsilon = (\epsilon_1, \ldots, \epsilon_m)' \) is an \( m \times 1 \) vector of normally distributed variables with mean zero and variance-covariance matrix \( \Sigma \).

**Definition 17.10.1** (Orthogonality) A performance system is said to be orthogonal if and only if \( b_i' C^{-1} b_j = 0 \) and \( \text{Cov}(\epsilon_i, \epsilon_j) = 0 \), for \( i \neq j \), that is, \( B'C^{-1}B \) and \( \Sigma \) are both diagonal matrices.

**Definition 17.10.2** (Cost-adjusted Correlation) The cost-adjusted correlation between two performance measures \( i \) and \( j \) is the ratio of the cost-adjusted inner product of their vectors of sensitivities divided by the covariance of the error terms:

\[
\rho_{ij} = \frac{b_i' C^{-1} b_j}{\sigma_{ij}}
\]

which measure performance not only about the agent’s task-specific abilities and interaction among tasks, but also what degree two performance measures are aligned with each other. If \( n \) tasks are technologically independent and identical, i.e., \( C = cI \), then we use the concept correlation \( \rho_{ij} = b_i' b_j / \sigma_{ij} \) to measure the degree of alignment between two performance measures.

**Definition 17.10.3** (Cost-adjusted Congruence) The cost-adjusted congruence of a performance measure \( P_i = b_i'e + \epsilon_i \) is defined as

\[
\Gamma_i = \frac{b_i' C^{-1} \beta}{\sqrt{b_i' C^{-1} b_i \sqrt{\beta' C^{-1} \beta}}}. \tag{17.10.117}
\]

A performance measure with nonzero cost-adjusted congruence is called congruent; a performance measure with unit cost-adjusted congruence is said to be perfectly congruent. We assume that there exists at least one congruent measure, i.e., \( BC^{-1} \beta \neq 0 \).

The principal compensates the agent’s effort through a linear contract:

\[
W(e) = w_0 + w' P(e), \tag{17.10.118}
\]

where \( P(e) = (P_1(e), \ldots, P_m(e))' \), \( w_0 \) denotes the base wage, and \( w = (w_1, \ldots, w_m)' \) the performance wage. Under this linear compensation rule, the principal’s expected profit is:

\[
\Pi_p = V(e) - W(e) = \beta' e - w_0 - w' Be,
\]
and the agent’s certainty equivalent is

$$CE_a = w_0 + w'Be - \frac{1}{2}e'Ce - \frac{r}{2}w'\Sigma w.$$  \hfill (17.10.119)

The principal’s problem is to design a contract \((w_0, w)\) that maximizes her expected profit \(\Pi_p\) while ensuring the agent’s participation and eliciting his optimal effort. The optimization problem of the principal is thus formulated as:

$$\max_{\{w_0, w, e\}} \beta' e - w_0 - w'Be$$

s.t.  

$$IR: w_0 + w'Be - \frac{1}{2}e'Ce - \frac{r}{2}w'\Sigma w \geq 0,$$

$$IC: e \in \arg\max_{\tilde{e}} \left[ w_0 + w'\tilde{B}\tilde{e} - \frac{1}{2}\tilde{e}'\tilde{C}\tilde{e} - \frac{r}{2}w'\Sigma w \right].$$

The \(IR\) constraint ensures that the principal cannot force the agent into accepting the contract, and here the agent’s reservation utility is normalized to zero; the \(IC\) constraint represents the rationality of the agent’s effort choice.

We now consider the effort choosing problem of the agent for a given incentive scheme \((w_0, w)\). Since the objective is concave by noting that the second-order derivative of \(CE_a\) with respect to \(e\) is a negative definite matrix \(-C\), the maximizer can be determined by the first-order condition: \(Ce = B'w\). After replacing \(e\) with \(e^* = C^{-1}B'w\) and substituting the \(IR\) constraint written with equality into the principal’s objective function, the principal’s optimization problem simplifies to:

$$\max_{w \in \mathbb{R}^m} \left[ \beta' C^{-1}B'w - \frac{1}{2}w' \left( BC^{-1}B' + r\Sigma \right) w \right].$$

The optimal wage contract and effort to be elicited are therefore:

$$w^p = \left[ BC^{-1}B' + r\Sigma \right]^{-1} BC^{-1}\beta,$$  \hfill (17.10.120)

$$w_0^p = \frac{rw^p'\Sigma w^p - w^p'BC^{-1}B'w^p}{2},$$  \hfill (17.10.121)

$$e^p = C^{-1}B'w^p.$$  \hfill (17.10.122)

The resulting surplus of the principal is\(^5\)

$$\Pi_p = \frac{1}{2} \beta' C^{-1}B' \left[ BC^{-1}B' + r\Sigma \right]^{-1} BC^{-1}\beta.$$  \hfill (17.10.123)

If the performance evaluation system is an orthogonal system and the \(n\) tasks are technically identical and independent \((C = cI)\), then we have

$$w_i = \frac{b_i'\beta}{b_i'\beta + r\sigma_i^2} = \frac{|b_i||\beta|\cos(b_i, \beta)}{|b_i|^2 + r\sigma_i^2}.$$  

\(^5\)Superscript “\(p\)” denotes “pure moral hazard”.
which implies that the more sensitive the principal’s revenue to the agent’s effort is (the larger the norm of vector $\beta_i$), the greater the incentive intensity (i.e. $w_i$), and the greater the uncertainty of performance indicators ($\sigma_i^2$), the lower the incentive intensity (i.e. $w_i$).

Thus, a higher incentive pay could induce the agent to implement a higher effort, but it will also expose the agent to a higher risk. It therefore requires a premium to compensate the risk-averse agent for the risk he bears. The optimal power of incentive is therefore determined by the tradeoff between incentives and risk. Moreover, the results above show that in multi-task agency relationships, the degree of congruity of available performance measures and the agent’s task-specific abilities also affects the power and distortion of incentive contract.

17.10.2 Optimal Wage Contract with Unobservable Efforts and Risk Aversion

In reality, the types and behaviors of agents are often unobservable, which leads to the mixed problem of moral hazard and adverse selection. In this subsection, we will consider the unobservable agent risk-aversion based on the above moral hazard model.

The pure moral hazard incentive contract stated above relies crucially on the agent’s attitude towards risk, which is the key factor affecting the incentive intensity. In the following, we assume that risk-aversion $r$ is also private information of the agent, its cumulative distribution function $F(r)$ and density function $f(r)$ supported on $[r, \tilde{r}]$ are common knowledge to all parties. The principal then has to offer a contract menu $\{w_0(\hat{r}), w(\hat{r})\}$ contingent on the agent’s reported “type” $\hat{r}$ to maximize her expected payoff. The timing of the problem is as follows:

- only agents know their risk-aversion coefficient;
- the principal provides the agent with a wage contract, and the agent decides whether to accept the contract and whether to report his private information truthfully;
- the agent choose his efforts;
- the realization of performance of the agent and the profit of principal;
- the principal pays the agent.

A contract $\{w_0(\hat{r}), w(\hat{r})\}$ is said to be implementable if the following incentive compatibility condition is satisfied:

$$w_0(r) + \frac{1}{2} w(r)^T \left[ BC^{-1} B' - r\Sigma \right] w(r) \geq w_0(\hat{r}) + \frac{1}{2} w(\hat{r})^T \left[ BC^{-1} B' - r\Sigma \right] w(\hat{r}).$$

(17.10.124)
Let $U(r, \hat{r}) \equiv w_0(\hat{r}) + \frac{1}{2}w(\hat{r})' [BC^{-1}B' - r\Sigma] w(\hat{r})$,\footnote{Substituting $e^* = C^{-1}B'w$ into expression (17.10.119) yields $U = w_0 + \frac{1}{2}w'(BC^{-1}B' - r\Sigma)w$.} and $U(r) \equiv U(r, r)$, then the implementability condition of $\{U(r), w(r)\}$ is stated equivalently as:

$$\exists w_0 : [\bar{r}, \bar{r}] \rightarrow \mathbb{R}_+, \forall (r, \hat{r}) \in [\bar{r}, \bar{r}]^2, \text{ we have:}$$

$$U(r) = \max_{\hat{r}} \left\{ w_0(\hat{r}) + \frac{1}{2}w(\hat{r})' [BC^{-1}B' - r\Sigma] w(\hat{r}) \right\}. \quad (17.10.125)$$

By the taxation principle (see Guesnerie (1981), Hammond (1979), and Rochet (1985), the above is in turn equivalent to the following very similar condition:

$$\exists w_0 : \mathbb{R}^m \rightarrow \mathbb{R}_+, \forall r \in [\bar{r}, \bar{r}], \text{ we have:}$$

$$U(r) = \max_w \left\{ w_0(w) + \frac{1}{2}w'[BC^{-1}B' - r\Sigma]w \right\}. \quad (17.10.126)$$

Then $U(\cdot)$ is continuous, convex\footnote{One way to define the convex functions is through representing them as maximum of the affine functions, that is, $s(x)$ is convex if and only if $s(x) = \max_{a,b \in \Omega} (a \cdot x + b)$ for some $a \in \mathbb{R}^n, b \in \mathbb{R}$ and some $\Omega \subseteq \mathbb{R}^{n+1}$. In this example $a = -\frac{1}{2}w\Sigma w, b = w_0(w) + \frac{1}{2}w'BC^{-1}B'w$, and thus $U(r) = \max_{(a,b) \in \mathbb{R}_- \times \mathbb{R}_+} (ar + b)$ is a convex function in $r.$}, and satisfies the envelop condition:

$$U'(r) = -\frac{1}{2}w'\Sigma w. \quad (17.10.127)$$

Conversely, if (17.10.127) holds and $U(r)$ is convex, then

$$U(r) \geq U(\hat{r}) + (r - \hat{r})U'(\hat{r}) = U(\hat{r}) - \frac{1}{2}(r - \hat{r})w'\Sigma w,$$

which implies the incentive compatibility condition $U(r) \geq U(r, \hat{r})$. Formally, we have

**Lemma 17.10.1** The surplus function $U(r)$ and performance wage function $w(r)$ are implementable if and only if:

1. envelop condition (17.10.127) is satisfied;
2. $U(r)$ is convex in $r$.

Substituting $U(r)$ into the principal’s expected payoff, we get

$$\Pi = \int_{\bar{r}}^\mathcal{R} [\beta e^* - w_0(r) - w(r)'Be^*] f(r) \, dr$$

$$= \int_{\bar{r}}^{\mathcal{R}} \left\{ \beta C^{-1}B'w(r) - \frac{1}{2}w(r)' \left[ BC^{-1}B' + r\Sigma \right] w(r) - U(r) \right\} f(r) \, dr.$$
The principal’s optimization problem is therefore:
\[
\max_{U(r), w(r)} \Pi, \text{ s.t.: } U(r) \geq 0, U'(r) = -\frac{1}{2} w'(r) \Sigma w(r), U(r) \text{ is convex.}
\]
(17.10.128)

The following proposition summarizes the solution of the principal’s problem.

**Proposition 17.10.1** If \( \Phi(r) \) is nondecreasing\(^8\), then the optimal wage contract is given by
\[
w^h(r) = \left[ BC^{-1} B' + \Phi(r) \Sigma \right]^{-1} BC^{-1} \beta \quad (17.10.129)
\]
\[
w^h_0(r) = \frac{1}{2} \int_r^\infty w^h(\tilde{r})' \Sigma w^h(\tilde{r}) d\tilde{r} - \frac{1}{2} w^h(r)' \left[ BC^{-1} B' - r \Sigma \right] \times w^h(r),
\]
(17.10.130)

where \( \Phi(r) \equiv r + F(r) f(r) \).

**Proof.** Using the envelop condition \( U'(r) = -\frac{1}{2} w' \Sigma w \), the participation constraint \( U(r) \geq 0 \) simplifies to \( U(\tilde{r}) \geq 0 \). Incentive compatibility implies that only the participation constraint of the most risk-averse type can be binding, i.e., \( U(\tilde{r}) = 0 \). We therefore get
\[
U(r) = \int_r^{\infty} \frac{1}{2} w(\tilde{r})' \Sigma w(\tilde{r}) d\tilde{r}.
\]
(17.10.131)

The principal’s objective function becomes
\[
\Pi = \int_{\mathbb{R}} \left\{ \beta' C^{-1} B' w(r) - \frac{1}{2} w'(r)' \left[ BC^{-1} B' + r \Sigma \right] w(r) - \int_r^{\infty} \frac{1}{2} w(\tilde{r})' \Sigma w(\tilde{r}) d\tilde{r} \right\} f(r) dr
\]
which, by an integration of parts, gives
\[
\int_{\mathbb{R}} \left\{ \beta' C^{-1} B' w(r) - \frac{1}{2} w'(r)' \left[ BC^{-1} B' + \left( r + \frac{F(r)}{f(r)} \right) \Sigma \right] w(r) \right\} f(r) dr.
\]
Maximizing pointwise the above expression, we get
\[
w^h(r) = \left[ BC^{-1} B' + \Phi(r) \Sigma \right]^{-1} BC^{-1} \beta
\]
and
\[
w^h_0(r) = \frac{1}{2} \int_r^{\infty} w^h(\tilde{r})' \Sigma w^h(\tilde{r}) d\tilde{r} - \frac{1}{2} w^h(r)' \left[ BC^{-1} B' - r \Sigma \right] w^h(r).
\]

\(^8\)This condition is weaker than and could be implied by the monotone hazard rate property: \( \frac{d}{dr} \left[ \frac{F(r)}{f(r)} \right] \geq 0 \).

\(^9\)Superscript “h” denotes “hybrid model of moral hazard and adverse selection”.

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The only work left is to verify the convexity of $U(r)$. Notice that

$$U''(r) = -(D_r w^h)' \Sigma w^h = \Phi'(r) w^h(r)' \Sigma \left[ BC^{-1} B' + \Phi(r) \Sigma \right]^{-1} \Sigma w^h(r).$$

The second equality comes from the fact that the derivative of $w^h$ with respect to $r$ is\(^{10}\)

$$D_r w^h = - \left[ BC^{-1} B' + \Phi(r) \Sigma \right]^{-1} \Phi'(r) \Sigma \left[ BC^{-1} B' + \Phi(r) \Sigma \right]^{-1} BC^{-1} \beta \Sigma w^h.$$

It is clear that $U''(r) \geq 0$ because $\Phi'(r) \geq 0$ and the matrix $\Sigma \left[ BC^{-1} B' + \Phi(r) \Sigma \right]^{-1} \Sigma$ is positive definite. The proof is completed. \(\square\)

The following conditions prove to be sufficient for the emergence of low-powered incentives.

**Condition 17.10.1** \(\Sigma\) is diagonal.

**Condition 17.10.2** Matrix $BC^{-1} B'$ is diagonal.

**Condition 17.10.3** Matrices $BC^{-1} B'$ and $\Sigma$ commute: $BC^{-1} B' \Sigma = \Sigma BC^{-1} B'$,\(^{11}\)

**Condition 17.10.4** The following inequality holds:

$$2r \lambda_m^2 + \rho > 0 \quad (17.10.132)$$

where

$$\rho = \max \left\{ \min_{i=1, m} \lambda_i \mu_m \left( \sqrt{k_{\lambda + 1}} - k_\mu \left( \sqrt{k_{\lambda}} - 1 \right) \right)^2 + \min_{i=1, m} \mu_i \lambda_m \left( \sqrt{k_{\mu + 1}} - k_\lambda \left( \sqrt{k_{\mu}} - 1 \right) \right)^2 \right\}$$

$$= \begin{cases} 
\lambda_i \mu_m \left( \sqrt{k_{\lambda + 1}} - k_\mu \left( \sqrt{k_{\lambda}} - 1 \right) \right)^2 & \text{if } k_\mu \leq \sqrt{k_{\lambda}} + 1, \ k_\mu \geq k_{\lambda}; \\
\lambda_i \mu_m \left( \sqrt{k_{\mu + 1}} - k_\lambda \left( \sqrt{k_{\mu}} - 1 \right) \right)^2 & \text{if } k_\mu \leq \sqrt{k_{\mu}} + 1, \ k_\mu < k_{\lambda}; \\
\lambda_i \mu_m \left( \sqrt{k_{\lambda + 1}} - k_\mu \left( \sqrt{k_{\lambda}} - 1 \right) \right)^2 & \text{if } k_\mu > \sqrt{k_{\lambda}} + 1, \ k_\mu \geq k_{\lambda}; \\
\lambda_i \mu_m \left( \sqrt{k_{\mu + 1}} - k_\lambda \left( \sqrt{k_{\mu}} - 1 \right) \right)^2 & \text{if } k_\mu > \sqrt{k_{\mu}} + 1, \ k_\mu < k_{\lambda}. 
\end{cases}$$

Here $\lambda_i$ and $\mu_i$ are the $i$–th eigenvalues of $\Sigma$ and $BC^{-1} B'$ respectively in a descending enumeration, $k_\lambda = \frac{\lambda_1}{\lambda_m}$ and $k_\mu = \frac{\mu_1}{\mu_m}$ denote the spectral condition number of $\Sigma$ and $BC^{-1} B'$ respectively.

\(^{10}\)Let $A$ be a nonsingular, $m \times m$ matrix whose elements are functions of the scalar parameter $\alpha$, then

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1}.$$  

\(^{11}\)That is, $BC^{-1} B' \Sigma$ is symmetric.
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Condition 17.10.5 There exists a positive number \( \lambda \) such that \( BC^{-1}B' = \lambda \Sigma \).

Condition 17.10.1 requires that the error terms of performance measures are stochastically independent. It assumes off the possibility that different measures are affected by common stochastic factor. Condition (17.10.2) states that \( b_i^rC^{-1}b_j = 0 \) for all \( i \neq j \). Intuitively, it requires that different performance measures respond in distinct ways to the agent’s effort when cost is incorporated. Condition (17.10.4) holds when agent is sufficiently risk-averse or when either matrix \( BC^{-1}B' \) or \( \Sigma \) is well-conditioned\(^{12}\). Several important special cases are:

- the performance measures system is orthogonal. In this case conditions (17.10.1) to (17.10.3) are all satisfied;
- \( \Sigma \) is a scalar matrix, in which case (17.10.1), (17.10.3) and (17.10.4) are satisfied;
- \( BC^{-1}B' \) is a scalar matrix, in which case conditions (17.10.2), (17.10.3) and (17.10.4) are satisfied.

Condition (17.10.5) emphasizes that the covariance matrix \( \Sigma \) is a transformation of the measure-cost efficiency matrix \( BC^{-1}B' \). That is to say, correlation between any pair of performance measures \( i \) and \( j \) is constant: \( \rho_{ij} = \lambda \).

By comparing the wage contract obtained in the hybrid model to the benchmark pure moral hazard model, we find that the principal will reduce the power of incentives offered to the agent.

Theorem 17.10.1 We have the following conclusions:

1. Given any one of the conditions 17.10.1 to 17.10.4, there exists an \( i \in \{1 \cdots m\} \), such that \( |w_i^h(r)| < |w_i^p(r)| \) for all \( r \in (\underline{r}, \bar{r}] \);

2. If both condition 17.10.1 and condition 17.10.2 are satisfied, then \( |w_i^h(r)| < |w_i^p(r)| \) for all \( r \in (\underline{r}, \bar{r}] \) and all \( i \in \{1 \cdots m\} \).

3. Let \( \omega_i, i \in \mathcal{K} \equiv \{1, 2, \cdots, k\} \) denote \( k \) distinct generalized eigenvalues of \( BC^{-1}B' \) relative to \( \Sigma \), \( \mathcal{V}_i \equiv \mathcal{N}(BC^{-1}B' - \omega_i \Sigma) \) be the eigenspace corresponding to \( \omega_i \), and \( \mathcal{V}_{i}^\perp \) be its orthogonal complement. Suppose that \( BC^{-1}\beta \notin \bigcup_{i \in \mathcal{K}} \mathcal{V}_i^\perp \), then there exists a positive number \( k \in (0, 1) \) such that \( w^h = kw^p \) if and only if condition 17.10.5 is met.

\textbf{Proof.} Since the proof is long and complicated, it is omitted here and can be found in Meng and Tian (2013). 

\(^{12}\)Matrices with condition numbers near 1 are said to be well-conditioned, while matrices with high condition numbers are said to be ill-conditioned.
When the risk-aversion parameter is unobservable to the principal, the less risk-averse agent gains information rent by mimicking the more risk-averse one. The amount of information rent gained by an agent depends on the performance wage of the agent with larger risk aversion, and therefore the basic tradeoff between efficiency and rent extraction leads to low-powered incentive for all but the least risk-averse types. Under conditions (17.10.1) to (17.10.4), wage vector $w$ is shortened in different quadratic-form norms compared with the pure moral hazard case. Under condition (17.10.5), the wage vector that minimizes the cost of effort $e'C e = w' B C^{-1} B' w$ points in the same direction as the wage vector that minimizes the risk premium $r w' \Sigma w$. Consequently, the tradeoff between efficiency and rent-extraction alters only the overall intensity of wage vector, not its relative allocation among performance measures.

17.10.3 Optimal Wage Contract with Unobservable Efforts and Cost

In this subsection we assume that the cost parameter is private information instead of risk-aversion. To avoid the complicated multidimensional mechanism design issue, we assume that $C = cI$, that is, the tasks are technologically identical and independent. $\delta = \frac{1}{c}$ is assumed to be distributed on the support $[\delta, \tilde{\delta}]$, according to a cumulative distribution function $G(\delta)$ and density $g(\delta)$. The timeline of this problem is analogous to that in Section 17.10.2 except that the agent is now required to report $\hat{\delta}$. A contract menu $\{w_0(\delta), w(\delta)\}$ is said to be implementable if the following incentive compatibility condition is satisfied:

$$ w_0(\delta) + \frac{1}{2} w(\delta)' [\delta B B' - r \Sigma] w(\delta) \geq w_0(\hat{\delta}) + \frac{1}{2} w(\hat{\delta})' [\delta B B' - r \Sigma] w(\hat{\delta}), \forall (\delta, \hat{\delta}) \in [\delta, \tilde{\delta}]^2. $$

(17.10.133)

Let

$$ U(\delta, \hat{\delta}) \equiv w_0(\hat{\delta}) + \frac{1}{2} w(\hat{\delta})' [\delta B B' - r \Sigma] w(\hat{\delta}), $$

and

$$ U(\delta) \equiv U(\delta, \delta). $$

Then $\{U(\delta), w(\delta)\}$ is called implementable if

$$ \exists w_0 : [\delta, \tilde{\delta}] \rightarrow \mathcal{R}_+, \forall (\delta, \hat{\delta}) \in [\delta, \tilde{\delta}]^2, U(\delta) = \max_{\hat{\delta}} \left\{ w_0(\hat{\delta}) + \frac{1}{2} w(\hat{\delta})' [\delta B B' - r \Sigma] w(\hat{\delta}) \right\}. $$

(17.10.134)

or equivalently,

$$ \exists w_0 : \mathcal{R} \rightarrow \mathcal{R}_+, \forall \delta \in [\delta, \tilde{\delta}], U(\delta) = \max_{w \in \mathcal{R}^n} \left\{ w_0(w) + \frac{1}{2} w' [\delta B B' - r \Sigma] w \right\}. $$

(17.10.135)
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\( U(\delta) \) is necessarily continuous, increasing and convex in \( \delta \) and satisfies the envelop condition:

\[
U'(\delta) = \frac{1}{2} w' BB' w. \tag{17.10.136}
\]

Conversely, similar to the case with unobservable risk-aversion, the convexity of \( U(\delta) \) and envelop condition (17.10.136) implies

\[
U(\delta) \geq U(\hat{\delta}) + (\delta - \hat{\delta}) U'(\hat{\delta}) = U(\hat{\delta}) + \frac{1}{2} (\delta - \hat{\delta}) w' BB' w = U(\delta, \hat{\delta}),
\]

which in turn implies the implementability of contract. We summarize the above discussion in the following lemma.

**Lemma 17.10.2** The surplus function \( U(\delta) \) and wage function \( w(\delta) \) are implementable if and only if

\[
\begin{align*}
& (1) \quad U'(\delta) = \frac{1}{2} w' BB' w; \\
& (2) \quad U(\delta) \text{ is convex in } \delta.
\end{align*}
\]

The optimal \( \delta \)–contingent contract solves the following optimization problem:

\[
\max_{w(\delta), U(\delta)} \int_{\delta}^3 \left\{ \delta' B' w(\delta) - \frac{1}{2} w(\delta)' [\delta BB' + r \Sigma] w(\delta) - U(\delta) \right\} g(\delta) d\delta
\]

s.t. \( U(\delta) \geq 0, U'(\delta) = \frac{w' BB' w}{2}, U(\delta) \text{ is convex}.

**Proposition 17.10.2** With unobservable cost, if \( \delta H(\delta) \) is nonincreasing \(^{14}\), then the optimal wage is given by

\[
\begin{align*}
w^h(\delta) &= \left( H(\delta) BB' + \frac{r \Sigma}{\delta} \right)^{-1} B \beta \quad \tag{17.10.137} \\
\phantom{\text{Proposition 17.10.2}}
w^h_0(\delta) &= \frac{1}{2} \int_{\delta}^3 w^h(\delta)' BB' w^h(\delta) d\delta \\
& \quad - \frac{1}{2} w^h(\delta)' [\delta BB' - r \Sigma] w^h(\delta), \tag{17.10.138}
\end{align*}
\]

where \( H(\delta) \equiv 1 + \frac{1-G(\delta)}{g(\delta)}. \)

**Proof.** Using integration by parts, we get

\[
\int_{\delta}^3 U(\delta) g(\delta) = \int_{\delta}^3 \left[ \frac{1 - G(\delta)}{g(\delta)} \right] \frac{w' BB' w}{2} dG(\delta).
\]

\(^{13}\)In this case, let \( a = \frac{1}{2} w' BB' w, b = w_0(\delta) - \frac{1}{2} w' \Sigma w, \text{ then } U(\delta) = \max_{a,b} (a\delta + b) \) is convex in \( \delta \).

\(^{14}\)This assumption is a bit stronger than the usual monotone inverse hazard rate condition. It holds for any nondecreasing \( g(\cdot) \).
Substituting it into the expression of the principal’s expected surplus and maximizing it with respect to \( w \), we get the optimal performance wage \( w_h(\delta) \), and \( w_0^h(\delta) \) is also easily obtained. We now check the convexity of \( U(\delta) \). The first order derivative of \( w_h(\delta) \) is

\[
D_\delta w_h(\delta) = -\left[H(\delta)BB' + \frac{r}{\delta} \Sigma\right]^{-1} \left[H'(\delta) BB' - \frac{r \Sigma}{\delta^2}\right]^{-1} B \beta
\]

\[
= -\left[H(\delta)BB' + \frac{r \Sigma}{\delta}\right]^{-1} \left[H'(\delta) BB' - \frac{r \Sigma}{\delta^2}\right] w_h(\delta)
\]

\[
= -\left[H(\delta)BB' + \frac{r \Sigma}{\delta}\right]^{-1} \left(-\frac{H(\delta)}{\delta} BB' - \frac{r \Sigma}{\delta^2} + \left[H(\delta) + \delta H'(\delta)\right] BB'\right) w_h(\delta)
\]

\[
= \frac{1}{\delta} \left((BB')^{-1} - \left[H(\delta)BB' + \frac{r \Sigma}{\delta}\right]^{-1} \left[H(\delta) + \delta H'(\delta)\right]\right) BB' w_h(\delta).
\]

It can be verified that the matrix

\[
\frac{1}{\delta} \left((BB')^{-1} - \left[H(\delta)BB' + \frac{r \Sigma}{\delta}\right]^{-1} \left[H(\delta) + \delta H'(\delta)\right]\right)
\]

is positive definite since \( \delta + \frac{1-G(\delta)}{\delta(\delta)} = \delta H(\delta) \) is decreasing. Therefore

\[
U''(\delta) = D_\delta w_h(\delta) BB' w_h(\delta)
\]

\[
= \frac{1}{\delta} w_h(\delta) BB' \left((BB')^{-1} - \left[H(\delta)BB' + \frac{r \Sigma}{\delta}\right]^{-1} \left[H(\delta) + \delta H'(\delta)\right]\right) BB' w_h(\delta)
\]

\[
\geq 0,
\]

which implies the convexity of \( U(\delta) \).

The following conditions justify the adoption of low-powered incentives in the case with unobservable cost parameter.

**Condition 17.10.6**  \( BB' \) is a diagonal matrix.

**Condition 17.10.7** Matrices \( BB' \) and \( \Sigma \) commute: \( BB' \Sigma = \Sigma BB' \).

**Condition 17.10.8** The following inequality holds:

\[
2 \nu_m^2 + \frac{r}{\delta} \eta > 0.
\]

\[15\] Again, it is true if \( BB' \Sigma \) is symmetric.
There exists a positive number $s$ given any of conditions (17.10.1), (17.10.6), (17.10.7), (17.10.8), there exists $\eta \in \mathbb{R}$ such that

$$\eta = \max \left\{ \min_{i=1,m} \lambda_i \nu_i \left( \frac{\sqrt{k_\lambda + 1}^2 - k_\nu (\sqrt{k_\lambda - 1})^2}{2 \sqrt{k_\lambda}} \right), \min_{i=1,m} \nu_i \lambda_i \left( \frac{\sqrt{k_\nu + 1}^2 - k_\lambda (\sqrt{k_\nu - 1})^2}{2 \sqrt{\nu}} \right) \right\}$$

$$= \begin{cases} \lambda_i \nu_i \left( \frac{\sqrt{k_\lambda + 1}^2 - k_\nu (\sqrt{k_\lambda - 1})^2}{2 \sqrt{k_\lambda}} \right) & \text{if } \sqrt{k_\nu} \leq \sqrt{k_\lambda + 1}, k_\nu \geq k_\lambda \\ \lambda_i \nu_i \left( \frac{\sqrt{k_\lambda + 1}^2 - k_\nu (\sqrt{k_\lambda - 1})^2}{2 \sqrt{k_\lambda}} \right) & \text{if } \sqrt{k_\nu} \leq \sqrt{k_\lambda + 1}, k_\nu < k_\lambda \\ \lambda_i \nu_i \left( \frac{\sqrt{k_\nu + 1}^2 - k_\lambda (\sqrt{k_\nu - 1})^2}{2 \sqrt{k_\lambda}} \right) & \text{if } \sqrt{k_\nu} > \sqrt{k_\lambda + 1}, k_\nu \geq k_\lambda \\ \lambda_i \nu_i \left( \frac{\sqrt{k_\nu + 1}^2 - k_\lambda (\sqrt{k_\nu - 1})^2}{2 \sqrt{k_\lambda}} \right) & \text{if } \sqrt{k_\nu} > \sqrt{k_\lambda + 1}, k_\lambda \geq k_\nu \end{cases}$$

represents the lower bound of eigenvalues of Jordan product $BB'\Sigma + \Sigma BB'$; $\lambda_i, \nu_i$ are the $i$-th eigenvalues of $\Sigma$ and $BB'$ respectively in a descending enumeration. $k_\lambda = \frac{\lambda_i}{\lambda_m}$ and $k_\nu = \frac{\nu_i}{\nu_m}$ denote the spectral condition number of $\Sigma$ and $BB'$ respectively.

**Condition 17.10.9** There exists a positive number $k$ such that $BB' = k\Sigma$.

In the special case where performance measures system is orthogonal, conditions (17.10.1),(17.10.6) and (17.10.7) are satisfied. If $\Sigma$ (resp. $BB'$) is a scalar matrix, then conditions (17.10.1) (resp. (17.10.6)) and (17.10.8) are both satisfied. Besides, condition (17.10.8) could hold even for nondiagonal $BB'$ and $\Sigma$, provided either of them is well-conditioned or $\frac{k}{\bar{k}}$ is sufficiently small.

**Theorem 17.10.2** We have the following conclusions:

1. Given any of conditions (17.10.1), (17.10.6),(17.10.7), (17.10.8), there exists at least one $i \in \{1, \cdots, m\}$, such that $|w_i(\delta)| < |w_i(\bar{\delta})|$ for all $\delta \in [\bar{\delta}, \delta]$;

2. If both conditions (17.10.1) and (17.10.6) are satisfied, namely, the performance measure system is orthogonal, then $|w_i(\delta)| < |w_i(\bar{\delta})|$ for all $\delta \in [\bar{\delta}, \delta]$ and all $i$;

3. Let $\tau_i, i \in \mathcal{L} \equiv \{1,2, \cdots, l\}$ denote $l$ distinct generalized eigenvalues of $BB'$ relative to $\Sigma$, $\mathcal{U}_i \equiv (BB' - \tau_i \Sigma)$ be the eigenspace corresponding to $\tau_i$, $\mathcal{U}_i^\perp$ be its orthogonal complement. Suppose that $B\beta \notin \bigcup_{i \in \mathcal{L}} \mathcal{U}_i^\perp$. Then there exists a positive number $s \in (0, 1)$ such that $w^h = sw^b$ if and only if condition (17.10.9) is met.

**PROOF.** Again, it is omitted here and referred to Meng and Tian (2013). $\Box$

When the agent possesses private information on his own cost, a more efficient type (the agent with higher $\delta$) would acquire information rent by mimicking his less efficient counterpart. To minimize agency costs, optimality requires a downward distortion of the power of inefficient types’
incentive wage. Theorem 17.10.2 gives various conditions ensuring low-powered incentives. If the performance measure sensitivities are orthogonal to each other ($b_i' b_j = 0$ for $i \neq j$), or error terms are uncorrelated ($\sigma_{ij} = 0$ for $i \neq j$), or either $BB'$ or $\Sigma$ is well-conditioned ($k_\nu$ or $k_\lambda$ is close to one), or the agent is nearly risk-neutral ($r$ is very small), or the agent is highly efficient ($\delta$ is very large), then the power of incentives will be lowered for at least one performance measure. For an orthogonal system with all its performance measures congruent ($b_i' \beta \neq 0$ for all $i$), the wage vector in hybrid model is shorter than but points in the same direction as its pure moral hazard counterpart if and only if all the measures share the same signal-to-noise ratio ($b_i' b_i / \sigma_i^2 \equiv k$ for all $i$).

17.11 Biographies

17.11.1 James Mirrlees

James A. Mirrlees (1936-2018) is the founder of the incentive theory and has made a significant contribution to the study of information economics theory as it relates to “moral hazard” and “optimal income taxation”. He received the 1996 Nobel Memorial Prize in Economic Sciences with William Vickrey.

Morris was born in Minigaff, Scotland, which is home to Adam Smith, in 1936. He received a master degree in mathematics from the University of Edinburgh in 1957 and a doctorate in economics from the University of Cambridge in 1963. From 1963 to 1968, Mirrlees was an assistant lecturer, lecturer in economics at Cambridge University and a researcher at Trinity College, Cambridge. He also served as a consultant for the Pakistan Institute of Development Economics, Karachi, during this period. In 1969, he became a professor at University of Oxford, the youngest professor of economics at the university. Professor Morris taught at Oxford from 1969 to 1996, and served as the Professor of Economics and a fellow at Nuffield College.

Mirrlees had been active in the economics profession since the 1960s, and he is known for research on incentive theory. In the 1970s, together with Joseph E. Stiglitz, Alvin E. Roth, Michael Spence, and others, he initiated the study of principal-agent theory. The current method of modelling principal-agent is what Morris pioneered. Mirrlees published three papers in 1974, 1975, and 1976, respectively. The “Notes on Welfare Economics, Information and Uncertainty”, “The Theory of Moral Hazard and Unobservable Behaviour” and “The Optimal Structure of Incentives and Authority within an Organization” laid the basic model framework of principal-agent theory. The framework pioneered by Mirrlees was further developed by Holmstrom et al. and in the literature, and it was called the
Morrlees-Holmstrom Approach.

Morrlees had also made remarkable achievements in economic growth and development. He co-edited the book “Models of Economic Growth” with DAVID I. Stern and co-authored “Project Appraisal and Planning for Developing Countries” with Ian Malcolm David Little. In 1975, he published the paper “A Pure Theory of Underdeveloped Economies using a Relationship between Consumption and Productivity” which conducted a utilitarian analysis of economic policies, especially growth theory, and explored the impact of uncertainty on moderate growth, non-renewable resources theory, inseparable growth theory and non-substitution theorem of durable goods. In the area of development economics, Morrlees proposed a cost-benefit analysis method, established a development model for the low-income economies and studied the effectiveness and consequences of international aid policies.

17.11.2 Joseph E. Stiglitz

Joseph E. Stiglitz (1943—) was born in Gary, Indiana, a small town well known for producing steel and gave birth to two contemporary economists, one is Samuelson and the other is Stiglitz. In 2001, Stiglitz won the Nobel Memorial Prize in Economic Sciences (with George Akerlof and A. Michael Spence) for his fundamental contribution to the creation of information economics.

Stiglitz is the most cited economists on information economics, and the same is true of broader microeconomics and macroeconomics fields. Some of his theories, such as moral hazard, have become standard tools for economists and policymakers. Stiglitz is also one of the most well-known American economics educators. His book “Economics”, which was first published in 1993, was reprinted repeatedly, and translated into many languages, and was recognized as one of the most classic textbooks on economics, being another landmark introduction textbook of economics in the West after Samuelson’s “Economics”.

Stiglitz is more concerned about the situation in developing countries, and often addresses problems from the perspective of developing countries. He advocated highlighting the role of governments in macroeconomic regulation and said that the best way to achieve sustainable growth and long-term efficiency was to find an appropriate balance between government and the market so as to bring the world economy back to a more equitable and stable growth path, which shall benefit everyone. The origins of Stiglitz’s these ideas may have something to do with his growing experience. He came from a hardworking family. His father retired as an insurance agent at the age of 95, and his mother, who retired from the post of primary school teacher at age 67, began teaching people to correct reading until 84. When Stiglitz was in college, he was interested in social activities.
In 1963, the civil rights movement in the United States was in full swing, and Stiglitz took part in a march led by Dr. Martin Luther King in Washington, which culminated in Dr. King’s historic speech “I have a Dream”. These social activities have great influence on his efforts to promote fair and equitable market ideas after becoming famous.

Professor Stiglitz is the author hundreds of academic papers and works, including “Economics of the Public Sector” (Norton Corp), “Readings in the modern theory of economic growth” edited with H. Uzawa (1969), “Lectures on Public Economics” written with Anthony Barnes Atkinson (1980), “Theory of Commodity Price Stabilization: Study in the Economics of Risk” written with D. M. G. Newbery (1981). In 1987, he founded the Journal of Economic Perspectives. Stiglitz’s economics works are extensive, but consistently focus on the role of incomplete information in the process of competition. In several pioneering papers, he showed that: the common assumption that an economic agent has complete information on alternative market opportunities is not as harmless as it seems. These papers are summarized in the paper “Information and Competitive Price Systems”, written with Grossman (1976).

17.12 Exercises

Exercise 17.1 (Debit and Credit under Moral Hazard) A cashless entrepreneur wants to borrow money to run an investment project. With an investment of 1 unit, if he puts in an effort of $\tau > 0$, he will get $q$ units with probability $p$; if he does not put any effort, he will get $q$ with probability $p > 0$ ($\tau > p$). Let $\psi$ denote the entrepreneur’s cost of effort $\tau$. In addition, normalize the current utility of entrepreneur to 0, and assume $pq < r$. For a monopolistic bank, the cost of capital is $r$. When the project is successful, each unit of the loan is repaid $z - x$.

1. Describe the principal-agent issue faced by the bank.

2. Find out the optimal loan contract that the bank maximizes its expected profit under the entrepreneur’s incentive compatibility constraint and participation constraint.

Exercise 17.2 (Risk-averse Principal and Moral Hazard) Suppose that the risk-averse principal delegates the task to a risk-neutral agent. The agent’s effort is denoted as $\epsilon$ and gets $q$ with probability $e$ and $q$ with probability $1 - e$, where $q < q$. The risk-averse principal’s utility function is $v(q - t)$ where $t$ is the transfer to the agent and $v(\cdot)$ is a CARA VNM utility function. The cost of agent’s effort is $\psi(\epsilon)$ with $\psi' > 0$ and $\psi'' > 0$.

1. Suppose that $\epsilon$ is not observable. Find the optimal contract.
2. With a limited liability constraint, find the second-best effort level.

3. Analyze two extreme cases: (a) the principal is infinitely risk-averse; (b) the principal is risk-neutral. Explain your answers.

Exercise 17.3 (More than Two Levels of Effort) Consider the moral hazard model in the chapter and extend the model by allowing more than two levels of effort. Consider the more general case with \( n \) levels of production \( q_1 < q_2 < \cdots < q_n \) and \( K \) levels of effort with \( 0 = e_0 < e_1 < \cdots < e_{K-1} \). We still make the normalization \( \psi_0 = 0 \), and assume that \( \psi_k \) is increasing in \( k \), for \( k = 0, 1, \ldots, K - 1 \). Let \( \pi_{ik} \) denote the probability of producing \( q_i \) when the effort level is \( e_k \). Assume that the agent is risk-averse. Other conditions are the same as the standard moral hazard model.

1. Write down the participation and incentive compatibility constraints for this problem.
2. Solve the optimization problem under complete information.
3. Solve the problem under moral hazard.

Exercise 17.4 (Continuum of Effort Levels) Extend the moral hazard model by allowing for a continuum of effort levels. We re-parameterize the model by assuming that \( \pi(e) = e \), for all \( e \in [0, 1] \). The disutility of effort function \( \psi(e) \) is increasing and convex in \( e \) with \( \psi(0) = 0 \). Moreover, to ensure interior solutions, we assume that the Inada condition holds \( \psi'(0) = +\infty \), and also \( \lim_{e \to 0} \psi'(e)e = 0 \). Other conditions are the same as the moral hazard model in the textbook. Let us finally consider a risk-neutral agent with zero initial wealth who is protected by the limited liability constraint.

1. Write down the participation constraint and incentive compatibility constraint for this problem.
2. Solve the optimization problem under complete information.
3. Solve the problem under moral hazard.

Exercise 17.5 (The First-Order Approach) Consider an economy where the risk-averse agent may exert a continuous level of effort \( e \in [0, \bar{e}] \) and by doing so incurs a disutility \( \psi(e) \) which is increasing, convex and \( \psi(0) = 0 \). To avoid corner solutions, we will also assume that the Inada condition \( \psi'(0) = +\infty \), and also \( \lim_{e \to 0} \psi'(e)e = 0 \). Other conditions are the same as the moral hazard model in the textbook. The agent’s performance is \( q \in [q, \bar{q}] \) with the conditional distribution \( F(q|e) \) and the density function \( f(q|e) \). We assume that \( F(\cdot|e) \) is twice differentiable with respect to \( e \). Principal’s payment to the agent is \( t(q) \).
1. Write down the participation constraint and incentive compatibility constraint for this problem.

2. Prove that when the monotone likelihood ratio property (MLRP) holds, \( t(q) \) is increasing in \( q \).

3. Prove that when the convexity of the distribution function condition (CDFC) holds, which means \( F_{ee}(q|e) > 0 \), the principal’s valuation function \( U(e) \) is concave in \( e \).

4. Prove that when MLRP and CDFC both hold, the solution to the optimal contract can be obtained from the first-order conditions.

5. Verify that the moral hazard model with continuous actions in the textbook satisfies MLRP and CDFC.

6. What are your intuitive explanations for MLRP and CDFC?

**Exercise 17.6** Consider a loan relationship with moral hazard. The risk-neutral borrower wants to borrow \( I \) of funds from the lender to support a risk-free project with a return of \( V \). The project has damage to a third party with probability \( 1 - e \). The borrower’s effort \( e \) for safety care costs \( \psi(e) \) and \( h \) is a compensation to the third party for the damage. A loan contract is \( \{t, \bar{t}\} \), which \( t(\bar{t}) \) is the repayment to the bank when the borrower generates (does not generate) environmental damages.

1. Suppose that \( e \) is observable. Find the first-best level of effort \( e \) for safety care.

2. Suppose that \( e \) is unobservable, the bank is competitive, and the borrower has sufficient repayment ability. Prove that if the bank can compensate the third party \( h \) in the accident, the first-best outcome can still be implementable.

3. Suppose that in the accident, the bank must compensate the third party \( c < h \). Let \( w \) represent the borrower’s initial asset. Prove that when \( w \) gradually gets smaller, the first-best outcome can no longer be implemented.

4. Find the second-best effort level to maximize the borrower’s expected return under the conditions of the bank’s zero-profit constraint, and borrower’s incentive compatibility and limited liability constraints.

5. Prove that raising bank’s repayment obligation \( c \) will reduce the expected welfare level.

6. Prove that when the banking industry is a monopoly industry, this result is no longer valid.
Exercise 17.7 (Risk-averse Agent with Hidden Actions) Consider the principal-agent problem of hidden actions. Suppose that \( h(u) = u + ru^2 \), where \( r > 0, u > -\frac{1}{r} \). Equivalently, \( u(x) = \frac{1+\sqrt{1+2x}}{r} \), where \( x > -\frac{1}{2r} \).

1. Find the second-best transfers required by the principal to induce the agent to exert a high level of effort.

2. Find the second-best cost required by the principal to induce the agent to exert a high level of effort.

3. Find the agent’s optimal utilities, \( u_{SB} \) and \( u_{SB} \), of inducing a high effort for the principal’s incentive-compatible optimization problem.

4. Find the optimal agency cost \( AC \) that is defined as the difference between the principal’s first-best and second-best expected profits.

Exercise 17.8 A risk-averse individual has a utility function of \( u(\cdot) \) and an initial wealth of \( w_0 \). The risk he faces is the potential loss of \( x \) for an accident. In a competitive insurance market, a risk-neutral insurance firm can provide him with a net payout of \( R(x) \) (excluding insurance fees). Assume that the distribution of \( x \) is dependent on the degree of effort \( e \) to prevent accidents and is not continuous at \( x = 0 \) where \( f(0, e) = 1 - p(e) \). When \( x > 0 \), \( f(x, e) = p(e)g(x) \). Assume \( p''(e) > 0 > p'(e) \), and the individual’s effort cost function \( \psi(e) \) is an increasing convex function and is separable from the utility function. Find the first-best and second-best insurance contracts.

Exercise 17.9 (Insurance Contract) Consider the moral hazard problem in an insurance market. The consumer’s VNM utility function is \( u(w) = \sqrt{w} \) and the initial wealth is \( w_0 = 500 \). Assume there are two possible loss levels: \( l = 0 \) and \( l = 200 \). There are also two levels of consumer’s effort: \( e = 0 \) and \( e = 1 \). The cost of effort is \( \psi(e) \) where \( \psi(0) = 0 \) and \( \psi(1) = 1/3 \). The corresponding probability distribution of wealth loss and effort is as follows:

\[
\begin{array}{c|cc}
 e=0 & l=0 & l=200 \\
 \hline
 e=1 & 1/4 & 3/4 \\
 3/4 & 1/4 \\
\end{array}
\]

1. Prove that the above probability distribution satisfies the monotone likelihood ratio property.

2. Suppose that there is only one insurance company, and the consumer can only choose the way of self-insurance. Find the consumer’s level of reservation utility.
3. If there is no insurance company, what is the level of consumer’s effort?

4. Prove that if the information is symmetric, then it is optimal for the insurance company to let the consumer choose a high level of effort.

5. If the information is asymmetric, prove that the result in the previous question will make the consumer choose a low level of effort.

6. Find out the solution to the optimal contract under asymmetric information.

**Exercise 17.10 (Information Learning)** Consider the following principal-agent problem: A risk-neutral principal confronts an unfunded risk-neutral agent protected by limited liability. Assume that there is a risk-free project, and that the principal can get return $0$ with probability of $1$. There is also a risky project, in the absence of information, the principal gets return $\bar{S}$ with probability of $\nu$ and return $\underline{S}$ with probability of $1 - \nu$. Assume that $\nu\bar{S} + (1 - \nu)\underline{S} = 0$. The principal can obtain information on the quality of the risky project and make a decision on whether to invest in the risky project. By paying the cost of $\psi$, the agent can get a signal $\sigma \in \{\sigma, \bar{\sigma}\}$. This signal can provide useful information for future return on the risky project. Assume that $\Pr(\sigma|\bar{S}) = \Pr(\sigma|\underline{S}) = \theta$ where $\theta \in \left[\frac{1}{2}, 1\right]$ is the degree of accuracy of the signal.

1. As the benchmark, suppose that the principal uses information gathering technology by himself. Prove that this project will only be implemented when he observes $\bar{\sigma}$. Write down the optimal information learning conditions.

2. Suppose now that the agent decides whether to implement the risky project, and the principal adopts a contract $\{\bar{t}, \underline{t}, t_0\}$ to motivate the agent. $\bar{t}(\bar{t})$ is the transfer received by the agent when he chooses to implement the risky project with $\bar{S}(\bar{S})$ being realized. $t_0$ is the transfer received when the agent selects the risk-free project. Write down the incentive constraint that guarantees the risky project be implemented only when $\sigma$ is observed.

3. Write down the incentive compatibility constraint that encourages the agent to learn information.

4. Find the contract for the agent and the $\bar{t}$ that prompted the agent to learn information.

5. Find the second-best contract followed by the principal.
17.12. EXERCISES

Exercise 17.11 Consider a principal-agent problem with three exogenous states of nature: \( \theta_1, \theta_2, \theta_3 \), and two effort levels: \( e_H, e_L \). The level of output and the corresponding probability are as follows:

<table>
<thead>
<tr>
<th>nature state</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>output of ( e_H )</td>
<td>20</td>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>output of ( e_L )</td>
<td>20</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The principal is risk-neutral, while the agent has utility function \( \sqrt{w} - \psi(e) \). The cost of effort is normalized to 0 for \( e_L \) and 1 for \( e_H \). The agent’s reservation expected utility is 1.

1. Derive the first-best contract.
2. Derive the second-best contract when only output levels are observable.
3. Suppose that the principal can buy for a price of 1 an information system that allows the parties to verify whether state of nature \( \theta_3 \) happened or not. Will the principal buy this information system?

Exercise 17.12 Consider the following moral hazard model. The principal is risk-neutral, and the agent’s preference is defined in terms of the mean and variance of his income \( w \) and his own effort \( e \). The agent’s expected utility is \( E(w) - \phi \text{Var}(w) - g(e) \) where \( g'(0) = 0, g'(e), g''(e), g'''(e) > 0 \) for all \( e > 0 \). And the profit, denoted by \( \pi \), generated by effort \( e \) is normally distributed with a mean of \( e \) and a variance of \( \sigma^2 \).

1. Consider the linear compensation scheme \( w(\pi) = \alpha + \beta \pi \). Prove that given \( w(\pi), e \) and \( \sigma^2 \), agent’s expected utility is \( \alpha + \beta E - \phi \beta^2 \sigma^2 - g(e) \).
2. Derive the first-best contract when \( e \) is observable.
3. Derive the optimal linear compensation scheme when \( e \) is unobservable and find the effects induced by the changes of \( \beta \) and \( \sigma^2 \), respectively.

Exercise 17.13 A principal needs to hire an agent. The work that the agent receives may be good (\( G \)) or bad (\( B \)). The principal does not know the type of work and only knows that the probability of \( G \) type is \( p \in (0, 1) \). The agent knows the type of work and can choose to exert either high effort \( H \) or low effort \( L \) after being hired. Performance of work \( G \) with high effort is 4 and is 2 with low effort. Performance of work \( B \) with high effort is 2 and is 0 with low effort. The agent’s high effort will cost 1 unit and low effort has no cost. The agent’s reservation utility is 0. The principal observes the performance and then gives the agent a payment.
1. Find the optimal contract.

2. How does the conclusion relate to \( p? \) Please explain.

**Exercise 17.14** Consider the following moral hazard model. The principal is risk-averse. His utility function is \( u(w) = \sqrt{w} \) and the reservation utility is 0. The agent’s effort has three levels: \( E = \{e_1, e_2, e_3\} \). There are two possible profit outcomes: \( \pi_H = 10 \) and \( \pi_L = 0 \). The probability distribution is \( f(\pi_H|e_1) = 2/3 \), \( f(\pi_H|e_2) = 1/2 \) and \( f(\pi_H|e_3) = 1/3 \), respectively. The agent’s effort cost function satisfies \( g(e_1) = 5/3 \), \( g(e_2) = 8/5 \) and \( g(e_3) = 4/3 \).

1. Derive the first-best contract when levels of effort are observable.

2. Prove that if the effort levels are not observable, then \( e_2 \) will not be implementable. Find the value of \( g(e_2) \) such that \( e_2 \) is implementable.

3. When the levels of effort are not observable, find the optimal contract.

**Exercise 17.15** Go on with the previous question. Assume that the relationship between the principal and the agent continues for two periods. The type of work is established before the contract is signed and remains unchanged in both periods. The principal provides the agent a contract for each period. The order is: the agent first knows the type of work, then the principal provides the contract of period 1. The agent chooses the effort level and the performance and payment are realized. The principal provides the contract of period 2, the agent chooses the level of effort and performance and payment are realized. Assume that at the first period, the principal always wants to make the agent put high effort when work is \( B \). And he wants both types of agent to put high effort in period 2.

1. Write down the optimal contract.

2. Suppose that at the beginning of these two periods, the principal can make a commitment. For the two cases above, discuss how this commitment ability affects the principal’s return. Under what conditions will the promise-keeping be optimal?

3. Some organizations regularly perform job rotations for their employees, and this practice is often criticized as sacrificing human capital at specific positions. According to the answers of previous questions, explain why job rotation is a good idea.

**Exercise 17.16** Suppose that a firm hires two types of workers \( \theta_H \) and \( \theta_L \) to produce a product. The proportion of workers of type \( \theta_L \) is \( \lambda \). When a type \( \theta \) worker pays \( T \) dollars to consume \( x \) units of goods, the utility is \( u(x, T) = \theta v(x) - T \) where \( v(x) = \frac{1-(1-x)^2}{2} \). The firm is the only producer of this product, and its production cost per unit is \( C > 0 \).
1. Consider a non-discriminatory monopolist. Derive the monopolist’s pricing strategy. Prove that when \( \theta_L \) or \( \lambda \) is large enough, the monopolist will provide products to these two types of consumers.

2. Now consider the monopolist can distinguish between these two types of consumers, but it can only request a uniform price of \( p_i \) for each type of \( \theta_i \). Describe the optimal pricing of the monopolist.

3. Fine the optimal nonlinear pricing.

**Exercise 17.17 (Debt Financing)** An entrepreneur has two projects, each of which requires an investment of 6 at \( t = 0 \). The cash flow generated by the first project is \( C_1 \in \{10, 80\} \) at \( t = 1 \). The second project generates a cash flow of \( C_2 \in \{0, 90\} \). The probability of obtaining high cash flow in both cases is \( v \), and \( e \) is the entrepreneurial effort level. The entrepreneur’s effort cost is \( 50e^2 \). He can choose three levels of effort: \( e \in \{0, 1, 2\} \). The entrepreneur does not have any assets. All participants are risk-neutral, and there is no discounting.

1. If the entrepreneur has the ability to invest by himself, what level of effort will he choose for each of the two projects?

2. Suppose that the entrepreneur has financing constraints, and all project funds need to be financed through debt. In both projects, how large debt, denoted \( D \), will he choose? If he can get an unconditional loan, which project will he eventually choose?

3. Can the entrepreneur replace debt financing with issuing equities with a share ratio of \( s \) to improve his situation?

**Exercise 17.18 (Insurance Contract)** Consider a risk-averse consumer whose utility function is \( u(\cdot) \). The consumer’s initial wealth is \( W \) and he faces the risk of losing \( L \) with probability \( \theta \) where \( W > L > 0 \). The insurance company’s contract is \( \{c_1, c_2\} \) where \( c_1 \) is the amount of wealth the consumer has in the absence of a loss and \( c_2 \) is the amount of wealth in the event of a loss. In the event that the loss does not occur, he pays the insurance company a premium of \( W - c_1 \) and in the event of a loss, the compensation he receives from the insurance company is \( c_2 - (W - L) \).

1. Suppose that the insurance company is a risk-neutral monopolist. Find the optimal contract when the consumer’s probability of loss \( \theta \) is observable.

2. Suppose that the insurance company cannot observe \( \theta \) that may take values from \( \{\theta_H, \theta_L\} \), and \( p(\theta_L) = \lambda \). Derive the insurance company’s optimal contract. Does the amount of insurance purchase have the rationing property?
Exercise 17.19 (Political Economics of Regulation) Consider an enterprise that implements two projects with values of $S_1$ and $S_2$, respectively. The enterprise can exert effort $e$ to reduce production costs. For project $i$, the enterprise’s production cost is $C_i = \beta_i - e_i$, where $\beta_i \in \{\beta_1, \beta_2\}$, and $\nu = \Pr(\beta = \beta_i)$. Efforts of reducing production cost will bring the enterprise a cost $\psi(e_1, e_2) = \frac{1}{2}(e_1^2 + e_2^2) + \gamma e_1 e_2$ with $\gamma > 0$. Regulator compensates the enterprise $t$ with observable cost $C_1$ and $C_2$. The utility of enterprise is $U = t - \psi(e_1, e_2)$ and the social welfare is $S_1 + S_2 - (1 + \lambda)(t + C_1 + C_2) + U$.

1. What is the optimal mechanism under complete information?
2. What is the optimal regulatory mechanism when $\beta$ is the private information of the enterprise?
3. Suppose that the regulatory mechanism is determined by a majority vote. There are two types of individuals: shareholders or non-shareholders of regulated enterprise. Let $\alpha$ denote the proportion of shareholders. If $\alpha > 1/2$, the goal of regulation is to maximize shareholders’ objective function $\alpha(S_1 + S_2 - (1 + \lambda)(t_1 + C_1 + t_2 + C_2)) + U$. If $\alpha < 1/2$, the goal of regulation is to maximize non-shareholders’ objective function $(1 - \alpha)[S_1 + S_2 - (1 + \lambda)(t_1 + C_1 + t_2 + C_2)]$. Derive the optimal regulatory mechanism under incomplete information in these two cases.

Exercise 17.20 (Bolton and Dewatripont, 2005) Consider the following principal-agent problem. There is a project whose probability of success is $a$ ($a$ is also the effort made by the risk-neutral agent, at cost $a^2$). In case of success the return is $R$, and in case of failure the return is 0. The parameter $R$ can take two values, $X$ with probability $\lambda$ and 1 with probability $1 - \lambda$. To undertake the project, the agent needs to borrow an amount $I$ from the principal. The sequence of events is as follows:

- First, the principal offers the agent a debt contract, with face value $D_0$ (i.e., with a limited liability $D_0$). The agent accepts or rejects this contract.
- Second, nature determines the value $R$ that would occur in case of success. This value is observed by both principal and agent. The principal can then choose to lower the debt from $D_0$ to $D_1$.
- The agent chooses a level of effort $a$. This level is not observed by the principal.
- The project succeeds or not. If the project succeeds, the agent pays the minimum of $R$ and the face value of debt $D_1$.

Answer the following questions:
1. Find the subgame-perfect equilibrium of this game as a function of $I, \lambda$, and $X$.

2. When do we have $D_1 < D_0$? Discuss.

Exercise 17.21 (Regulation of a Risk Averse Enterprise) Consider a regulator who wants to implement a public project that worths $S$. An enterprise can put in a cost of $C = \beta - e$ to implement this project where $\beta \in \{ \beta, \overline{\beta} \}$ is a parameter of efficiency. $e$ is the effort level brings negative effect of $\psi(e)$ ($\psi' > 0$, $\psi'' > 0$, $\psi'''> 0$) to the enterprise. The cost $C$ can be observed by the regulator and the regulator gives the enterprise transfer payment $t$ with the price of $1 + \lambda$. The enterprise is risk-averse and the utility function is $u(t - \psi(\beta))$ ($u' > 0$, $u'' < 0$). The regulator cannot observe $e$ or $\beta$, but $\nu = \Pr(\beta = \beta)$ is common knowledge.

1. Consider the revelation mechanism $\{ t(\overline{t}) = r, C(\overline{\beta}) = \overline{C}; t(\beta) = r, C(\beta) = C \}$. Write down the enterprise’s incentive compatibility and participation constraints.

2. The expected social welfare is defined as:

$$W = S - (1 + \lambda) [\nu (t + C) + (1 - \nu)(\overline{t} + \overline{C})] + u^{-1}(\nu u(\pi) + (1 - \nu) u(\overline{\pi})),$$

where $\pi = t - \psi(\beta - C)$ and $\overline{\pi} = \overline{t} - \psi(\overline{\beta} - \overline{C})$. Explain the social welfare function. Suppose that it is concave for $\pi$ and $\overline{\pi}$. Find out the optimal regulatory mechanism for the principal in the interim stage.

3. Compare the conclusion in the previous question with the case with a risk-neutral enterprise.

4. Consider a special case $v(x) = \frac{1}{\rho}(1 - e^{-\rho x})$. Prove the effort level of type $\overline{\beta}$ increases with $\rho$.

Exercise 17.22 (Information Collection before Signing Contract) Consider a principal-agent problem: An agent produces $q$ units of output of a good at a cost of $\theta q$ where $\theta \in \{ \underline{\theta}, \overline{\theta} \}$ and $\overline{\theta} > \underline{\theta}$. Let $t$ denote the transfer to the agent. The agent’s utility is $U = t - \theta q$. The principal’s utility is $V = S(q) - t$, $S' > 0$, $S'' < 0$.

- On the first day, the principal proposes a list of contracts $(t, q), (\overline{t}, \overline{q})$.
- On the second day, the agent decides whether to learn $\theta$ at a cost of $\psi$. Let $e$ denote this decision: if he learns, then $e = 1$, if not, then $e = 0$. $e$ is a moral risk variable that the principal cannot observe.
- On the third day, the agent decides whether to accept the contract.
• On the fourth day, the agent knows $\theta$ (if he decides not to study on the second day).

• On the fifth day, the contract is implemented.

Consider two contract sets: the $C_1$ type induces the agent to choose $e = 1$ and the $C_2$ type induces the agent to choose $e = 0$.

1. Write down the optimization problem for the principal to maximize his expected utility subject to the agent’s incentive compatibility constraints and budget constraints, regardless of whether it is a contract of $C_1$ or $C_2$ type.

2. Prove that the principal can restrict the contract to $t - \theta q \neq 0$ and $t - \theta q \neq 0$ to achieve a utility lower bound. Prove that a meaningful contract requires $t - \theta q \leq 0$, and that the principal can always imitate a contract in $C_2$ with a contract in $C_1$.

3. Find the optimal contract in $C_1$. (To discuss the scope of $\psi$, distinguish three cases: 1) whether the ex ante participation constraint is binding, 2) the incentive compatibility constraint is binding, and 3) the two constraints are both binding.)

**Exercise 17.23** A risk-neutral principal employs a risk-averse agent. For the principal, the agent’s effort level $e$ is not observable and the principal’s profit level has $n$ possible outcomes which are $q_1 < q_2 < \cdots < q_n$. These $n$ outcomes are observable and verifiable. The probability of realization of profit $q_i$ is $\pi_i(e)$ and the agent’s corresponding income is $I_1, I_2, \cdots, I_n$. The agent’s reservation utility is $\pi$, and the agent’s utility function is $u(I, e) = -e - r(I - e)$.

1. Prove that the first-best effort level is independent of the reservation utility $\pi$.

2. Prove that the second-best effort level is independent of the reservation utility $\pi$.

3. Prove that the optimal income scheme of the agent can be expressed as $(I_1 + k, I_2 + k, \cdots, I_n + k)$ where $k$ is only a function of the reservation utility $\pi$.

4. Suppose that the agent’s utility function is $V(I) - e$ with $V' > 0$ and $V'' < 0$. Are the results of the previous three questions still correct? Explain.

5. Suppose that the agent’s utility function is $V(I) - e$ with $V' > 0$, $V'' < 0$ and $n = 2$. There are only two levels of effort which are work-hard,
and lazy, \( e_L \). In the second-best outcome, the principal wants the agent to take the effort level \( e_H \). Prove that the optimal incentive scheme makes the agent be indifferent between working hard and being lazy.

6. Continue with the previous question and prove that the optimal incentive scheme satisfies \( I_2 > I_1, q_2 - q_1 > I_2 - I_1 \).

**Exercise 17.24** In a two-period problem, \( t = 1, 2 \), the agent chooses the effort level \( e_t = H \) or \( e_t = L \) and the profit is \( q_t = 1 \) or \( q_t = 0 \). The harder the agent works, the greater the probability of achieving a high profit. In particular, \( \Pr(q_t = 1|e_t = H) = p_H, \Pr(q_t = 1|e_t = L) = p_L < p_H \). The agent’s effort cost is \( c(e_t) \) where \( c(H) > c(L) = 0 \). The agent’s utility function is \( u(s, e_1, e_2) = -e^{-(s-c(e_1)-c(e_2))} \) where \( s \) is the wage that the principal pays to the agent. The agent’s reservation utility is a certainty equivalence level of zero and the agent can observe \( q_1 \) before determining the action \( e_2 \). The principal is risk-neutral and his profit is \( q_1 + q_2 - s \). The wage of \( s \) depends on the profit of each period and is paid at the end of the second period.

1. Suppose that the principal wants the agent to work hard in two periods. Prove that the contract that achieves the minimum cost of \((H, H)\) has the following form: \( s(q_1, q_2) = \alpha(q_1 + q_2) + \beta \).

2. If the agent can not observe \( q_1 \) before determining the action \( e_2 \). Is the contract given in question 1 still optimal? Explain.

**Exercise 17.25** A monopolist sells its product at the price of \( p \) and the only production input is labor. In order to produce \( q \) units of product, workers must work hard and the cost of the effort is \( \frac{q^2}{2} \theta \). The number of workers is standardized to 1 and the skill level of the workers is represented by \( \theta \). And the principal cannot observe \( \theta \). Assume that the workers with proportion \( \lambda \) are highly skilled (i.e. \( \theta = \theta_H \)) and the rest are low-skilled (i.e. \( \theta = \theta_L < \theta_H \)). The principal can hire any number of workers. The reservation utility of each worker is 0. The principal’s profit is the product revenue minus the wages paid to the worker.

1. Solve for the optimal contract.

2. Suppose that the output \( q \) is observable and verifiable and that only the wage \( w \) can be written into the contract (transfer for each worker \( i \) is the same, and \( T_i = w_i q_i \)). Solve for the optimal solution of \( w \). Is \( w \) the same for high-skilled and low-skilled workers? Does the optimal \( w \) depend on the proportion of highly skilled workers?
3. Suppose that any contract \((q_i, T_i)\) is possible. Find the optimal contract and compare it with the result of the previous question. You may assume that the proportion of high-skilled workers is quite low and it is not optimal for firms to employ highly skilled workers only.

**Exercise 17.26** Assume that the principal is risk-neutral. The agent chooses the effort levels \(e_H \) and \(e_L \) to achieve three possible outputs of \(q_H > q_M > q_L > 0 \). If the agent chooses \( e_H \), the utility level is \( 2\sqrt{x} - 5 \). If \( e_L \) is selected, the utility level is \( 2\sqrt{x} \). If the agent does not work, the utility level is 3. If the agent chooses to work, his minimum wage is \( w \), \( 0 \leq w \leq 5 \). The principal cannot observe the agent’s effort level, but he can observe the realized output, \( P(q = q_H|e_H) = \frac{1}{2}, P(q = q_M|e_H) = \frac{1}{4}, P(q = q_L|e_H) = 0 \), \( P(q = q_H|e_L) = \frac{1}{4}, P(q = q_M|e_L) = 0, P(q = q_L|e_L) = \frac{3}{4} \). Assume that \( q_H - q_M \) and \( q_M - q_L \) are large enough. Derive the optimal contract for the principal.

**Exercise 17.27** Define monotone likelihood ratio and first-order stochastic dominance.

1. Prove that the above two concepts are equivalent under the case of two realizations.

2. Give an example to show that in the general case, the above two concepts are different. Prove that if the monotone likelihood ratio is satisfied, the first-order stochastic dominance is also satisfied.

**Exercise 17.28** Consider a moral hazard model. The principal is risk-neutral and the agent is risk-averse. The agent can choose two levels of effort, \( e_H \) and \( e_L \). The cost of high effort is \( c \) and the cost of low effort is 0. There are two types of outcomes which are \( x_H \) and \( x_L \), \( p(x_L|a_L) > p(x_L|a_H) \). The agent’s utility function is \( u(w, c_i) = \ln w - c_i \) and the agent’s reservation utility is 0.

1. Describe the principal-agent problem of implementing a high level of effort.

2. Solve for the optimal wage contract.

**Exercise 17.29 (Supervision Cost)** Consider a financial contract with two parties. One is a risk-neutral entrepreneur with wealth constraints, and the other is a wealthy risk-neutral investor. The investment \( I \) is required at \( t = 0 \). The project generates a random gain of \( \pi(\theta, I) = 2 \min\{\theta, I\} \) when \( t = 1 \), where \( \theta \) is a nature state and obeys uniform distribution on \([0, 1]\).

1. Describe the characteristics of the first-best investment level \( I^{FB} \).
2. Suppose that at \( t = 1 \), the realized return can only be observed by the entrepreneur and the investor must pay a cost of \( K > 0 \) to observe \( \pi(\theta, I) \). Consider that the return cannot exceed the difference of return minus the cost of supervision, and the investor’s expected profit is 0. Derive the optimal contract.

3. Prove that the second-best investment level is lower than the first-best investment level \( I_{FB} \).

**Exercise 17.30 (Forged Output)** Consider a risk-averse entrepreneur whose utility function is \( u(\cdot) \) and the entrepreneur’s output \( q \) is a random variable distributed over the interval \([0, \bar{q}]\). The entrepreneur wants to diversify the risk by signing a contract with a risk-neutral financier whose initial wealth is \( w \geq \bar{q} \). The contract provides that the payment to the entrepreneur depends on the level of output. The entrepreneur can observe the output. The financier can also observe output unless the entrepreneur tampers with the accounts. After observing the output \( q \), the entrepreneur may falsify an output report \( R \) which costs \( \psi(q, R) = \frac{1}{4}(q - R) + \frac{1}{2}c(q - R)^2, c > 0 \). Assume that the entrepreneur is protected by limited liability and his reservation utility is higher than \( \bar{q}/2 \).

1. Characterize the optimal contract.

2. If there is no output forgery in the contract, then for all \( q \in [0, \bar{q}] \), we have \( R(q) = q \). Prove that the optimal contract involves falsifying. According to the optimal contract, find the equilibrium solution of the entrepreneur’s falsified output.

3. What is the optimal contract without falsifying? Prove that the falsification-free optimal contract is linear in the output \( q \).

## 17.13 References

**Books and Monographs:**


**Papers:**


Chapter 18
Mechanism Design with Complete Information

18.1 Introduction

In the previous two chapters, we discussed basic models and results on the principal-agent theory with complete contracts. It highlights the various trade-offs between allocative efficiency and information rent extraction under exogenous asymmetric information and trade-offs between incentive intensity and insurance under endogenous asymmetric information. Since they involve only one agent, the design of the principal’s optimal contract can reduce to a constrained optimization problem without having to appeal to sophisticated game theoretic settings.

In this chapter, we will discuss the mechanism design theory that deals with more than one agent under complete information in the sense that all agents know their characteristics each other although the designer does not know characteristics of agents. In such environments, asymmetric information may not only affect the relationship between a principal (often called designer) and each of his agents, but it may also plague the relationships between agents. To describe the strategic interaction between agents and the principal, the game theoretic reasoning is thus used to model social institutions as varied voting systems, auctions, bargaining protocols, and methods for deciding on public projects.

The mechanism design theory has two branches. One is the so-called implementation theory, which studies how to induce individuals, through a rule of game, to do what a designer wants to do, i.e., the designer wishes to find a mechanism (game form) such that individual interests are compatible with socially desirable outcomes. The other is the realization theory, which mainly focuses on informational efficiency of the mechanism. One of the main indicators is that, the smaller the message space is, the smaller the operation cost of the mechanism is. This chapter first discusses the
Incentive problems arise when the social planner cannot distinguish between things that are indeed different. A typical problem is so-called free-ride issue. A free rider can improve his welfare by not telling the truth about his own un-observable characteristic. Like the principal-agent model, a basic insight of the incentive mechanism with more than one agent is that incentive constraints should be considered coequally with resource constraints. One of the fundamental contributions of the mechanism theory has been shown that the free-rider problem may or may not occur, depending on game form (mechanism) that agents play and game theoretical solution concepts. A theme that comes out of the literature is the difficulty of finding mechanisms compatible with individual incentives that simultaneously implements a desired social goal.

Examples of incentive mechanism design that takes strategic interactions among agents exist for a long time. An early example is the Biblical story of the famous judgement of Solomon for determining who is the real mother of a baby. Two women came before the King, disputing who was the mother of a child. The King’s solution used a method of threatening to cut the lively baby in two and give half to each. One women was willing to give up the child, but another women agreed to cut in two. The King then made his judgement and decision: The first woman is the mother, and give the baby to the first woman. Another example of incentive mechanism design is how to cut a pie and divide equally among all participants.

The first major development was in 1970s. When information is private, an appealing solution concept is dominant strategy equilibrium so that truth-telling about their characteristics must be dominant strategy equilibrium. The fundamental conclusion of Gibbard-Hurwicz-Satterthwaite’s impossibility theorem is that we should have a trade-off between the truth-telling and Pareto efficiency. Of course, if one is willing to give up Pareto efficiency, we can have a truth-telling mechanism, such as Vickery-Clark-Groves mechanism. In many situations, one can ignore the first-best or Pareto efficiency, and so one can expect the truth-telling behavior.

On the other hand, we could give up the truth-telling requirement, and want to implement Pareto efficient outcomes. When the information about the characteristics of the agents is shared by individuals but not by the designer, then the relevant equilibrium concept is the Nash equilibrium. In this situation, one can gives up the truth-telling, and uses a general message space. One may design a mechanism that Nash implements Pareto efficient allocations.

We will introduce these results and such trade-offs. In next chapter, we will discuss the case of incomplete information in which agents do not know each other’s characteristics, and we need to consider Bayesian incentive compatible mechanisms.
18.2 Basic Settings

The basic analytical framework that is used to study mechanism design consists of five components: (1) economic environments (fundamentals of economy); (2) social choice goal we want to reach; (3) economic mechanism that specifies the rules of game; (4) description of solution concept on individuals’ self-interested behavior, and (5) implementation of a social choice goal.

18.2.1 Economic Environments

There are two types of individuals: principal (also known as designer or social planner) and agents (also known as participants, individuals, players etc.). The former designs the rules of game, while the latter provides information and participation in games. The designer may be a concrete or an abstract person such as the Congress or may be even a consensus in rules or conventions that are commonly accepted by all the participants in advance. The designer does not observe the real state of the world composed of participants’economic characteristics, and such information is scattered among agents. Let

- \( N = \{1, \cdots, n\} \): The set of agents.
- \( e_i = (Z_i, w_i, \succ_i, Y_i) \): economic characteristic of agent \( i \) which consists of the space of outcomes/alternatives, initial endowment if any, preference relation, and the production set if agent \( i \) is also a producer.
- \( e = (e_1, \ldots, e_n) \): an economy, also known as the economic environment, type or state.
- \( E \): The set of all priori admissible economic environments.
- \( U = U_1 \times \ldots \times U_n \): The set of all admissible utility functions.
- \( \Theta = \Theta_1 \times \Theta_2 \cdots \times \Theta_n \): The set of all admissible parameters \( \theta = (\theta_1, \cdots, \theta_n) \in \Theta \) that determine types of parametric utility functions, \( u_i(\cdot, \theta) \), and so it is called the space of types or the state of the world.

Remark 18.2.1 Thus, the economy \( e \) consists of three components: agents, objects (alternatives or outcomes) and information about the characteristics of economic agents (state of the world). We assume that \( N \) and \( Z = \prod_{i \in N} Z_i \) are state-independent, and thus the only factors affecting the environment involve \( \succ_i (U_i), Y_i \) or \( w_i \). So, \( E \) is a general expression of economic environments, depending on the situations faced by the designer. The set of admissible economic environments under consideration may be just given by \( E = U_i \); \( E = \Theta \), or by the set of all possible initial endowments or production sets.
The designer only knows the allowable range of economic environments, but does not know the specific environment. That is, she does not know the true economic characteristics of individuals, but an individual may know, partially know, or does not know the economic characteristics of other individuals. These three kinds of timing of information structure about economic characteristics are called, respectively, as ex post, interim and ex ante. If each participant knows the economic characteristics of other participants, it is called the complete information environment, otherwise it is called incomplete information environment. In this case, when \( e_i \) can only be observed by individual \( i \), that is, each agent only knows his own economic characteristic and does not know the characteristics of others (but knows the distributions, which may be independent or correlated), the timing is interim.

For simplicity, we assume that preferences are given by parametric utility functions and each agent \( i \) privately observes a type \( \theta_i \in \Theta_i \), which determines his preference over outcomes. The state \( \theta \) is drawn randomly from a prior distribution with density \( \varphi(\cdot) \). Each agent maximizes a von Neumann-Morgenstern expected utility over outcomes, given by (Bernoulli) utility function \( u_i(y, \theta) \). Thus, we assume that \( \varphi(\cdot) \) and \( \{u_i(\cdot, \cdot)\}_{i=1}^{n} \) are common knowledge.

The incomplete information model can be divided into two categories:
1. private value model where the Bernoulli utility function of each individual depends only on his own value and does not depend on the type of other participants, denoted as \( u_i(y, \theta_i) \). At the same time, the private value model can be further subdivided into the independent distribution model of private value types and the correlated distribution model of private value types.
2. interdependent value model where the Bernoulli utility function of each individual depends not only on its own type, but also on the type of other participants, denoted as \( u_i(y, \theta) \). It is important to distinct these differences, because they often lead to very different results.

This chapter discusses the case of complete information, and the next chapter discusses the case of incomplete information.

18.2.2 Social Goal

Another important component of the incentive mechanism design is a goal that the designer wants to reach, simply called social goal or social optimum. Given economic environments, the designer wants to reach some
desired goal that is considered as socially optimal by some criterion. Let

- \( Z = Z_1 \times \ldots \times Z_n \): the outcome space.
- \( A \subseteq Z \): the feasible set.
- \( F : E \rightarrow A \): social rule or called social choice correspondence in which \( F(e) \) is the set of socially desired outcomes under some criteria of social optimality.

Thus, a social choice rule, \( F \), is defined as a mapping from the environment space to the outcome space: If this correspondence is single-valued, it is called a social choice function (SCF). If random choice is allowed, \( Z \) can be substituted for the probability distribution on it, denoted by \( \triangle(Z) \).

Examples of Social Choice Correspondences:

- \( P(e) \): the set of Pareto efficient allocations.
- \( I(e) \): the set of individually rational allocations.
- \( W(e) \): the set of Walrasian allocations.
- \( L(e) \): the set of Lindahl allocations.
- \( FA(e) \): the set of fair allocations.

Those concepts of resource allocations have been discussed in detail in the general equilibrium theory.

Examples of Social Choice Functions:

Solomon’s goal.

Majority voting rule.

18.2.3 Economic Mechanism

If a social planner knows the information on the economic characteristics, she can directly determine the outcomes according to the social choice rule. But since the designer lacks such information, she needs to design an appropriate incentive mechanism (rules of game) to coordinate the personal interests and the collective interests determined by social goal, i.e., under the mechanism, all individuals have incentives to choose actions which result in socially optimal outcomes at equilibrium when they pursue their personal interests. To do so, the designer informs how information she collected from individuals is used to determine outcomes, that is, she first tells the rules of games, then uses the information or actions of agents and the rules of game to determine outcomes of individuals. Thus, a mechanism consists of a message space and an outcome function. Let
\(M_i\): the message space of agent \(i\).

\(M = M_1 \times \ldots \times M_n\): the message space in which communications take place.

\(m_i \in M_i\): a generic message reported by agent \(i\).

\(m = (m_1, \ldots, m_n) \in M\): a profile of messages of all agents.

\(h: M \rightarrow Z\): outcome function that translates messages into outcomes.

\(\Gamma = (M, h)\): a mechanism

Thus, given the rules of the game, each participant is required to send a message \(m_i: E \rightarrow M_i\) based on what he knows about the economic environment, and then the planner determines the outcome of each participant according to the information sent by the participant and outcome function \(h: \prod_{i \in I} M_i \rightarrow Z\). Assume that the state space \(E\) has a Cartesian product structure \(E = \prod_{i \in N} E_i\). If each agent’s message reported is affected only by his own economic characteristic \(E_i\) (or \(U_i\), or “type” \(\theta_i\)), the message he sends is then \(m_i: E_i \rightarrow M_i\). The mechanism is called a **direct mechanism** if the participant’s message space and his type space is the same (i.e., \(E_i = M_i\)).

**Remark 18.2.2** Prior the mechanism design theory, economics was mostly concerned with taking a mechanism as given, and then study under what kind of economic environments, it can reach a given social goal. For instance, in general equilibrium theory, we take the market mechanism as given, and then study under what kind of economic environments a competitive market leads to an efficient allocation of resources. However, for a mechanism designer issue, the question is reversed. Instead of giving an mechanism, given economic environments and the social goals, the designer wants to design an economic mechanism to achieve a social goal so that individual interests are compatible with the social goal he wants to achieve. One of the purposes of mechanism design theory is to study what kind of social goals can be implemented and what kind of social goals can not.

**Remark 18.2.3** A mechanism is often referred to as a game form. The terminology of game form distinguishes it from a game in game theory in a number of ways. (1) In contrast to game theory, which is positive approach, mechanism design is normative approach. Game theory is important because it predicts how a given game will be played by agents. Mechanism design goes one step further: given economic environment and the constraints faced by the designer, what goal can be realized or implemented? What mechanisms are optimal among those that are feasible? (2) The consequence of a profile of message is an outcome in mechanism design rather
than payoffs. Of course, once the preferences of individuals are specified, then a game form or a mechanism induces a game. (3) The preferences of individuals in the mechanism design setting vary, while they are taken as given in game theory. This distinction is critical. Because of this, an equilibrium (such as dominant strategy equilibrium) in mechanism design is much easier to exist than in a given game. (4) In designing mechanisms one must take into account incentive constraints in a way that individual interests are compatible with the social goal.

Remark 18.2.4 The design of incentive mechanisms is different from the information adjustment mechanisms considered in the last section of this chapter. In the design of incentive mechanisms, participants’ behavior is not described by response function or information feedback process, but is determined by participants’ preferences and the way they adopt strategies. However, as shown in the last section, one way to regard an incentive mechanism as an information adjustment mechanism is to regard all points of message space of incentive mechanism as the stationary point of the information adjustment mechanism.

18.2.4 Solution Concepts of Self-Interested Behavior

When sending information to a designer, participants have strategic interactions with each other and the game process happens. The fourth component of mechanism design is the solution concept of equilibrium. When participants play games, different behavioral equilibrium solutions may lead to very different conclusions.

A basic assumption in economics is that individuals are self-interested. Unless they can be better off, they in general do not care about social interests in normal situation. As a result, different economic environments and different rules of game will lead to different reactions of individuals, and thus each individual agent’s strategy (i.e., the message he sends) will depend on his self-interested behavior which in turn depends on the economic environments and the mechanism.

Let \( b(e, \Gamma) \) be the set of equilibrium strategies that describes the self-interested behavior of individuals, and it is a subset of individual strategy space. \( b(e, \Gamma) \) is general notation that can represent different forms of solution concepts of equilibrium, such as dominant strategy equilibrium, Nash equilibrium, subgame perfect equilibrium, Bayesian Nash equilibrium, perfect Bayesian (Nash) equilibrium, etc. The set of resulting equilibrium outcomes is denoted by \( h(b(e, \Gamma)) \).

Thus, given \( E, M, h, \) and \( b \), the resulting equilibrium outcome is a composite function of the rule of game and the equilibrium strategy, i.e., \( h(b(e, \Gamma)) \).
18.2.5 Implementation and Incentive Compatibility

If a designer knew the information about individuals’ economic characteristics, she could directly determine the outcomes according to a social choice rule. However, in most situations, a designer does not know such information. As such, she needs to design appropriate rules of game to reconcile conflicts between individual interests and collective interests so that all individuals have incentives to choose actions which result in socially optimal outcomes even they pursue their individual interests. However, whether it is incentive compatible depends on the design of appropriate incentive mechanisms.

Thus, the fifth component of mechanism design is that the implementation of a social goal so that under the given solution concept, individual interests have no conflicts with the social interest. We will call such problems as implementation of social goal. The purpose of an incentive mechanism design is to implement some desired socially optimal outcomes. Given a mechanism $\Gamma$ and equilibrium behavior assumption $b(e, \Gamma)$, the implementation problem of a social choice rule $F$ studies the relationship of $F(e)$ and $h(b(e, \Gamma))$, which can be illustrated by the following diagram.

Figure 18.1: Diagrammatic Illustration of a Mechanism design Problem.

In Figure 18.1, $E$ is the set of economic environments, $Z$ is the set of outcomes, and the social choice correspondence $F$ specifies the goal that the society wants to achieve. The designer needs to design a mechanism to implement this social goal. Under the mechanism, individuals play a game to reach an equilibrium strategy $b(e, \Gamma)$, and the outcome of the equilibrium solution is given by $h(b(e, \Gamma))$. In general, it does not always lead to socially optimal outcomes. If $h(b(e, \Gamma))$ is in $F(e)$, it forms incentive compatibility, which means that the social goal $F$ is implementable. Formally, we have the following definitions.

**Definition 18.2.1** A mechanism $(M, h)$ is said to
(i) fully implement a social choice correspondence $F$ in equilibrium strategy $b(e, \Gamma)$ on $E$ if for every $e \in E$

(a) $b(e, \Gamma) \neq \emptyset$ (equilibrium solution exists),
(b) $h(b(e, \Gamma)) = F(e)$ (personal interests are fully consistent with social goals);

(ii) implement a social choice correspondence $F$ in equilibrium strategy $b(e, \Gamma)$ on $E$ if for every $e \in E$

(a) $b(e, \Gamma) \neq \emptyset$,
(b) $h(b(e, \Gamma)) \subseteq F(e)$ (individual rationality is compatible with social rationality);

(iii) partially implement a social choice correspondence $F$ in equilibrium strategy $b(e, \Gamma)$ on $E$ if for every $e \in E$

(a) $b(e, \Gamma) \neq \emptyset$,
(b) $h(b(e, \Gamma)) \cap F(e) \neq \emptyset$ (i.e., there is an equilibrium strategy $m^* \in b(e, \Gamma)$ such that $h(m^*) \in F(e)$).

Partial implementation is a weak implementation by which there may exist some socially irrational equilibrium outcomes, that is, they do not belong to $F(e)$.

**Definition 18.2.2** A mechanism $\langle M, h \rangle$ is said to be (fully or partially) incentive-compatible with a social choice correspondence $F$ in $b(e, \Gamma)$-equilibrium if it (fully or partially) implements $F$ in $b(e, \Gamma)$-equilibrium.

In the literature, the terms of above implementation are not unified. The “implementation” defined above is sometime referred to as “strong implementation”, while the “partial implementation” above is called “implementation” or “weak implementation”.

Note that we did not give a specific solution concept yet when we define the implementability and incentive-compatibility. As shown in the following, whether or not a social choice correspondence is implementable will depend on the assumption on the solution concept of self-interested behavior. When information is complete, the solution concept can be dominant equilibrium, Nash equilibrium, strong Nash equilibrium, subgame perfect Nash equilibrium, undominated equilibrium, etc. For incomplete information, equilibrium strategy can be Bayesian Nash equilibrium, undominated Bayesian Nash equilibrium, etc.

In specific issues, we want the social choice rules to be implemented have certain properties. The followings are some of the common properties.
Definition 18.2.3 (Ordinality) If for $\forall (a, b) \in Z^2$ and $\forall (e, e') \in E^2$, $a \succ_i b \iff a' \succ_i b'$, then $F(e) = F(e')$.

This property requires that even when economic environments are changed, but the rank between any two alternatives remains unchanged, then the set of alternatives selected in both states must be exactly the same. In other words, for any two economies $e$ and $e'$, if $F(e) \neq F(e')$, then there must be $(a, b) \in Z^2$ such that $a \succ_i b$ but $b \succ'_i a$.

Definition 18.2.4 Weak Pareto optimality: for $\forall e \in E$ and $a \in Z$, if there does not exist $b \in Z$ such that $b \succ_i a$, $\forall i \in N$, then $a \in F(e)$.

This property suggests that if everyone thinks one alternative is strictly inferior to the other, the former will not be selected.

Definition 18.2.5 Pareto optimality: for $\forall e \in E$ and $a \in Z$, if there is no $b \in Z$ such that $b \succ_i a$, $\forall i \in N$ and $b \succ_j a$ for some $j \in N$, then $a \in F(e)$.

Definition 18.2.6 Pareto indifference: for $\forall (a, e) \in Z \times E$ and $b \in F(e)$, if $a \sim_i b$, $\forall i \in N$, then $a \in F(e)$.

Definition 18.2.7 Dictatorship: There is $i \in I$ such that $F(e) \subseteq C_i(e, Z) \equiv \{a \in Z : a \succ_i b, \forall b \in Z\}, \forall e \in E$.

If there is an individual whose best choice is also socially optimal, then the social choice rule is called dictatorship, with the individual being the dictator.

Definition 18.2.8 Unanimity: for $\forall (a, e) \in Z \times E$, if $a \succ_i b$, $\forall i \in I, \forall b \in Z$, then $a \in F(e)$.

This property shows that if a scheme is considered by everyone to be the best, then it must be selected as the social optimum.

Definition 18.2.9 Strong unanimity: $\forall (a, e) \in Z \times E$, if $a \succ_i b$, $\forall i \in N, \forall b \neq a$, then $F(e) = \{a\}$.

This property shows that if a scheme is strictly preferred to everyone, the scheme must be the only one selected, so that the social choice correspondence becomes a social choice function.

Definition 18.2.10 No veto power: $\forall (a, j, e) \in Z \times N \times E$, if $a \succ_i b$, $\forall i \neq j$, $\forall b \in Z$, then $a \in F(e)$.

The property of no veto power requires that if a scheme is preferred by all but one person, it must be selected. That is to say, no one has the power to veto a scheme unanimously agreed by others.
Definition 18.2.11 Maskin monotonicity: Let $L_i(a,e) = \{ b \in Z : a \succ_i b \}$. If for any two economies $e, e' \in E$ and any $a \in F(e)$, $L_i(a,e) \subseteq L_i(a,e')$ for $\forall i \in N, \forall b \in Z$, then $a \in F(e')$.

This property indicates that if a scheme was originally selected by society, and the change of the environment only led the scheme to be preferred by everyone, then the scheme would be re-selected by society under the new economic environment $e'$. 2 Maskin monotonicity and various variations plays a critical role in mechanism design theory. We will discuss them in detail in this chapter and next chapter.

18.3 Examples

Before we discuss basic results in the mechanism design theory, we first give some examples of economic environments and social goals that are needed to implement by designing certain mechanisms to solve the incentive-compatibility problems.

Example 18.3.1 (Walrasian correspondence) Consider a pure exchange economic environment $e$ as follows:

- Individuals: $N = \{1, 2, \cdots, n\}$ represent the set of consumers.
- Goods: There are $L$ private goods, the endowment of consumer $i$ is $\omega_i \in \mathbb{R}_+^L$, and $X_i$ represents his consumption set, $\mathcal{A} = \{ (x_1, \cdots, x_n) | x_i \in X_i, \sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i, \forall i \in N \}$ represents the set of feasible allocations.
- Type (State of the world): $\theta = (\theta_1, \cdots, \theta_n)$, where $\theta_i$ is consumer $i$’s preference parameter. $\succ_i^{\theta}$ is strictly increasing on its own consumption $x_i$.

The Walrasian social choice rule is defined by

$$W(\theta) = \left\{ x \in \mathcal{A} \mid \exists p \in \mathbb{R}_+^L \text{ such that } px_i \leq p\omega_i \right\}.$$

Example 18.3.2 (Lindahl correspondence) The economic environment $e$ is characterized as follows:

There are $n$ agents, $L$ private goods, and $K$ public goods. $x_i \in \mathbb{R}_+^L$ is a vector of consumer $i$’s private consumption goods. $y \in \mathbb{R}_+^K$ represents the

\[\text{Take an election as an example, if someone, such as Obama, was elected president. Four years later the environment changed, but his position in the minds of every voter rose, and then he would be bound to be re-elected.}\]
 quantity of public goods. Production technology is represented by production set $F$.\footnote{For example, public goods can be produced by private goods with the constant-returns-to-scale, and the initial endowment vector of private goods is $\omega$, then $F = \{ (x, y) \in \mathbb{R}^{l+m}_+ | x + y \leq \sum_{i \in I} \omega_i \}$.}

The utility function of consumer $i$ is $v_i(x_i, y_i, \theta_i)$ (and the corresponding preference is $\succ_i^\theta$), then the Lindahl correspondence is

$$L(\theta) = \left\{ (x, y) \in \mathbb{R}^{L+K}_+ \mid (x, y) \in F, \exists (p, q_1, \cdots, q_n) \in \mathbb{R}^{L+nK}_+ \text{ such that } (x_i, y_i) \text{ maximize } v_i(\cdot, \cdot, \theta_i) \text{ s.t. } px_i + q_i y_i \leq w_i, \text{ where } \sum_{i \in I} w_i = \max_{(x, y) \in F} (px + \sum_i q_i y) \right\}.$$  

**Example 18.3.3 (Solomonic Choice Rule)** The Old Testament of the Bible told the story of two mothers (one of whose child died one night) who brought a baby boy to King Solomon and asked him to decide who was the real mother. When King Solomon suggested splitting the child in half and each mother getting half, one said, "I will give up the child!" and the other one said, "Split him up!" King Solomon declared that the mother who would give up her child was a real mother and gave her the child. We can formulate this problem in the language of mechanism design. Two mothers are denoted by $A$ and $B$, respectively; the state of the world is denoted by $\alpha$ and $\beta$, respectively; $\alpha$ denotes "A is the real mother" and $\beta$ denotes "B is the real mother". There are four alternatives for the king:

- $a$: Anne gets the child;
- $b$: Bets gets the child;
- $c$: Cut the child in half;
- $d$: Death penalty to everyone (two mothers and one baby).

Preferences of the two mothers under the two states are:

- $a \succ^A b \succ^A c \succ^A d$;
- $a \succ^\beta b \succ^\beta c \succ^\beta d$;
- $b \succ^A c \succ^A a \succ^A d$;
- $b \succ^\beta c \succ^\beta a \succ^\beta d$.

The king wants the real mother to get the child, that is, the social choice rule $f(\cdot)$ he wants to implement satisfies: $f(\alpha) = a$, $f(\beta) = b$.

**Example 18.3.4 (Allocating Public Goods)** $n$ households in a community consider whether to build a public project (such as a road or a bridge) or not. Let $y$ denote the size of the project ($y$ can be continuous, such as $y \in Y = [0, \infty]$; also can be discrete, such as $y \in Y = \{ y_1, \cdots, y_k \}$; and
18.3. EXAMPLES

in practice, \( y \in Y = \{0, 1\} \) often denotes “build” if \( y = 1 \) or “not build” if \( y = 0 \). Individual preferences can be represented by a quasi-linear utility function \( v_i(y, \theta) + t_i \) (if the private value model is used, the utility function is \( v_i(y, \theta_i) + t_i \)), where \( y \in Y \) represents the scale of the project, \( \theta_i \) represents the individual’s preference intensity for this public project, \( t_i \) represents the tax or compensation that an individual needs to pay or gain. This environment can be characterized as follows:

- **Individuals**: \( N = \{1, \cdots, n\} \).

- **Goods (Alternatives)**: An outcome consists of the scale of the public project and the transfer payments (tax or subsidy), and the total subsidy should not exceed the total tax. Then the outcome space is \( Z = \mathbb{R}^{n+1} \). The set of feasible outcome is \( A = \{ (y, t_1, \cdots, t_n) : y \in Y, t_i \in \mathbb{R}, \sum_i t_i \leq 0, \forall i \in N \} \).

- **State of the world (Preferences, or types)**: \( \theta = (\theta_1, \cdots, \theta_n) \).

The problem is what mechanisms can be designed to ensure efficient provision of public goods. Therefore, the efficient provision of public projects can be reduced to a typical mechanism design problem.

**Example 18.3.5 (Allocating an Indivisible Private Good)** An indivisible good is to be allocated to a member of a society. For instance, it can be the right to an exclusive license or a firm to be privatized. In this case, the outcome space is \( Z = \{ y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1 \} \), where \( y_i = 1 \) means individual \( i \) obtains the object, and \( y_i = 0 \) means he does not get the object. If individual \( i \) gets the object, his net value is \( v_i \). If he does not get the object, it is 0. Thus, agent \( i \)'s valuation function is

\[
v_i(y) = v_i y_i.
\]

Note that we can regard \( y \) as an \( n \)-dimensional vector of public goods since \( v_i(y) = v_i y_i = v^i y \), where \( v^i \) is a vector with the \( i \)-th component being \( v_i \) and the others being zeros, i.e., \( v^i = (0, \cdots, 0, v_i, 0, \cdots, 0) \). So, the problem of allocating an indivisible private good is also a mechanism design problem.

From these examples, a socially optimal decision clearly depends on the individual’s true valuation function \( v_i(\cdot) \). For the examples of public goods and indivisible goods, let \( h = (y(\cdot), t_1(\cdot), t_2(\cdot), \cdots, t_n(\cdot)) : \Theta \rightarrow Z \) be an outcome function of \( y \) and the transfer \( t_i \), where \((t_1(\cdot), t_2(\cdot), \cdots, t_n(\cdot))\) is called an allocation rule, and \( y(\cdot) \) is a decision rule. We call the decision rule \( y(\cdot) \) **efficient** if and only if it maximizes the social surplus (the sum of individuals’ surpluses), that is,

\[
\sum_{i \in N} v_i(y(\theta), \theta) \geq \sum_{i \in N} v_i(y(\theta'), \theta), \quad \forall \theta' \in \Theta.
\]
For the private value, it becomes
\[ \sum_{i \in N} v_i(y(\theta), \theta_i) \geq \sum_{i \in N} v_i(y(\theta'), \theta_i), \quad \forall \theta' \in \Theta. \]

The social planner wants to truthfully implement an outcome that maximizes the sum of all individuals’ surpluses. A more general utility or valuation function instead of a parameterized utility function can be considered. For instance, let \( V_i \) be the set of all valuation functions \( v_i \) and \( V = \prod_{i \in N} V_i \). Then \( y(\cdot) \) is said to be efficient if and only if
\[ \sum_{i \in N} v_i(y(v)) \geq \sum_{i \in N} v_i(y(v')), \quad \forall v' \in V. \]

### 18.4 Dominant Strategy and Truthful Revelation Mechanisms

This section discusses the incentive mechanism design when dominant strategy equilibrium is used as the solution concept. The reason for this is that we are concerned about whether there is a mechanism such that all participants are willing to truthfully reveal their private information when in equilibrium. As shall be demonstrated, a dominant strategy equilibrium is equivalent to truth-telling. In addition, the strongest solution concept of describing self-interested behavior is dominant strategy. From the point of view of informational efficiency, it needs least information. This is because the strategy chosen by each individual is optimal, regardless of the choices of others. An appealing property of this equilibrium concept is in the weak rationality: an agent need not know what the others are doing. An axiom in game theory is that agents will use a dominant strategy as long as it exists.

For \( e \in E \), a mechanism \( \Gamma = \langle M, h \rangle \) is said to have a dominant strategy equilibrium \( m^* \) if for all \( i \)

\[ h_i(m^*_i, m_{-i}) \succ_i h_i(m_i, m_{-i}), \forall m \in M. \tag{18.4.2} \]

Denote by \( D(e, \Gamma) \) the set of dominant strategy equilibria for \( \Gamma = \langle M, h \rangle \) and \( e \in E \).

Under the existence of a dominant strategy equilibrium, since each agent’s optimal choice does not depend on the choices of others and does not need to know characteristics of the others, the information required is least when an individual makes decisions. Thus, as long as it exists, individuals would adopt it. Also, for a given game, dominant strategies are not likely to exist. However, since we can choose various forms of mechanism, it significantly increases the possibility of the existence of dominant strategies. Another advantage is that bad equilibria are usually not a problem. If an agent has two dominant strategies, then they must be payoff-equivalent, which is in general not true for other equilibrium solution concepts.
Definition 18.4.1 A mechanism $\Gamma = \langle M, h \rangle$ is said to fully implement a social choice correspondence $F$ in dominant strategies $D(e, \Gamma)$ on $E$ if for every $e \in E$, we have

(a) $D(e, \Gamma) \neq \emptyset$;

(b) $h(D(e, \Gamma)) = F(e)$.

Definition 18.4.2 A mechanism $\Gamma = \langle M, h \rangle$ is said to implement a social choice correspondence $F$ in dominant strategies $D(e, \Gamma)$ on $E$ if for every $e \in E$, we have

(a) $D(e, \Gamma) \neq \emptyset$;

(b) $h(D(e, \Gamma)) \subseteq F(e)$.

Definition 18.4.3 A mechanism $\Gamma = \langle M, h \rangle$ is said to partially implement a social choice correspondence $F$ in dominant strategies $D(e, \Gamma)$ on $E$ if for every $e \in E$, we have

(a) $D(e, \Gamma) \neq \emptyset$;

(b) $h(D(e, \Gamma)) \cap F(e) \neq \emptyset$ (i.e., there is a dominant strategy $m^* = d(e) \in D(e, \Gamma)$ such that $h(m^*) \in F(e)$).

In addition to general (indirect) mechanisms, a particular class of mechanisms are revelation mechanisms, in which the message space $M_i$ for each agent $i$ coincides with the set of his economic characteristics $E_i$. In effect, each agent reports an economic characteristic that is not necessarily his true one.

Definition 18.4.4 A mechanism $\Gamma = \langle M, h \rangle$ is said to be a revelation or direct mechanism if $M = E$.

Example 18.4.1 The optimal contracts we discussed in Chapter 16 are revelation mechanisms.

Example 18.4.2 The Vickrey-Clark-Groves mechanism we will discuss below is a revelation mechanism.

The most appealing property of revelation mechanisms is that truth-telling of characteristics always turns out to be an equilibrium. The absence of such a property is called the “free-rider” problem in the theory of public goods.

Definition 18.4.5 (Truthful Implementation) A revelation mechanism $\langle E, h \rangle$ is said to implement a social choice correspondence $F$ truthfully in $b(e, \Gamma)$ on $E$ if for every $e \in E$, 

...
That is, \( F() \) is truthfully implementable in dominant strategies if truth-telling is a dominant strategy equilibrium and the resulting outcome is social optimal.

Note that the truthful implementation in dominant strategies \( D(e, \Gamma) \) is just partial implementation, not implementation, since a misreport, denoted \( e' \), may also be a dominant equilibrium.

Truthful implementation in dominant strategies is also often called dominant strategy incentive compatible, strongly individually incentive-compatible, strategy-proof or strategy straightforward.

Although the message space of a mechanism can be arbitrary, the following Revelation Principle tells us that one only needs to use the revelation mechanism in which the message space consists solely of the set of individuals’ characteristics, and it is unnecessary to seek more complicated mechanisms. It significantly reduces the complexity of constructing a mechanism.

**Theorem 18.4.1 (Revelation Principle)** Suppose that a mechanism \( \langle M, h \rangle \) partially implements a social choice rule \( F \) in dominant strategies. Then there is a revelation mechanism \( \langle E, g \rangle \) which implements \( F \) truthfully in dominant strategies.

**Proof.** Let \( d \) be a selection of dominant strategy correspondence of the mechanism \( \langle M, h \rangle \) that results in a social outcome, i.e., for every \( e \in E \), \( m^* = d(e) \in \mathcal{D}(e, \Gamma) \) such that \( h(d(e)) \in F(e) \). Since \( \Gamma = \langle M, h \rangle \) partially implements social choice rule \( F \), such a selection exists. Also, since the strategy of each agent is independent of the strategies of others, each agent \( i \)’s dominant strategy equilibrium can be expressed as \( m^*_i = d_i(e_i) \).

Define the revelation mechanism \( \langle E, g \rangle \) by \( g(e) \equiv h(d(e)) \) for each \( e \in E \). We first show that truth-telling is always a dominant strategy equilibrium of the revelation mechanism \( \langle E, g \rangle \). Suppose not. Then, there exists a message \( e' \) and an agent \( i \) such that

\[
u_i[g(e'_i, e'_{-i})] > u_i[g(e_i, e'_{-i})].
\]

However, since \( g = h \circ d \), we have

\[
u_i[h(d_i(e'_i), d_{-i}(e'_{-i}))] > u_i[h(d_i(e_i), d_{-i}(e'_{-i}))],
\]

which contradicts the hypotheses that \( m^*_i = d_i(e_i) \) is a dominant strategy equilibrium. This is because, when the true economic environment is \( (e_i, e'_{-i}) \), agent \( i \) has an incentive to not report \( m^*_i = d_i(e_i) \) truthfully, but to report \( m'_i = d_i(e'_i) \), a contradiction.
Finally, since $m^* = d(e) \in D(e, \Gamma)$ and $(M, h)$ partially implements the social choice rule $F$ in dominant strategy, we have $g(e) = h(d(e)) = h(m^*) \in F(e)$. Hence, the revelation mechanism implements $F$ truthfully in dominant strategies.

Thus, by the Revelation Principle, if truthful implementation rather than full implementation or implementation is all that we require, we do not need consider general mechanisms. In particular, when $F$ becomes a single-valued function $f$, $(E, f)$ can also be regarded as a revelation mechanism. Thus, if a mechanism $(M, h)$ implements $f$ in dominant strategies, then the revelation mechanism $(E, f)$ is incentive compatible in dominant strategies.

It should be noted that in the above proof, the difference between the dominant strategy equilibrium and the Nash equilibrium is that the individual choice in the dominant strategy equilibrium is optimal regardless of the choices of other individuals, while the choice of individual strategy in the Nash equilibrium depends on the optimal strategy of other individuals. So we can rewrite the dominant strategy equilibrium as $m^*_i = d_i(e_i)$.

Remark 18.4.1. Notice that the Revelation Principle may be valid only for partial implementation, but not for implementation or full implementation. The Revelation Principle specifies a mapping between a dominant strategy equilibrium of the original mechanism $(M, h)$ and the true profile of characteristics as a dominant strategy equilibrium of the revelation mechanism, and it does not require the revelation mechanism to have a unique dominant equilibrium so that there may also exist non-truthful dominant strategy equilibrium under the revelation mechanism $(E, g)$. Dasgupta, Hammond, and Maskin (1979) provide such an example (see this example in the exercise below). Thus, shifting from a general (indirect) dominant strategy mechanisms to a direct one may introduce undesired dominant strategies which are not truthful. More seriously, these strategies may create a situation where the indirect mechanism is a (full) implantation of a given $F$, while the direct revelation mechanism is not. Thus, even if a mechanism (fully) implements a social choice function, the corresponding revelation mechanism $(E, g)$ may only partially implement, but not implement or fully implement $F$.

However, if agents have strict preferences, i.e., $\forall a, b \in A, a = b$ if and only if $a \sim_i b, \forall i \in N$, any two different outcomes cannot be indifferent. Thus the Revelation Principle is valid for full implementation, but not just for partial implementation.

The following result shows that, although $(\Gamma, \theta)$ may have more than one dominant strategy equilibrium, the resulting equilibrium outcome is unique under strict preferences, and thus the revelation mechanism $(E, g)$ with $g(e) = h(d(e))$ fully implements the social choice correspondence in dominant strategies.
Proposition 18.4.1 Suppose that agents have strict preferences. Then equilibrium outcome of $\Gamma = (M, h)$ that implements a social choice rule $F$ in dominant strategies is unique, and thus the revelation mechanism $\langle E, g \rangle$ with $g(e) \equiv h(d(e))$ implements $F$ in dominant strategies.

**Proof.** Suppose that $\Gamma = (M, h)$ implements $F$ in dominant strategies, but the equilibrium outcome is not unique. Then, there is $m^*_i, m^{*'}_i \in D_i(\Gamma, \theta)$ such that $m^*_i \neq m^{*'}_i$. By dominant strategy, for $\forall m_{-i} \in M_{-i}$, we have

$$h(m^*_i, m_{-i}) \succ^*_i h(m^{*'}_i, m_{-i}),$$

$$h(m^{*'}_i, m_{-i}) \succ^*_i h(m^*_i, m_{-i}).$$

Then, $h(m^*_i, m_{-i}) \sim^*_i h(m^{*'}_i, m_{-i})$. By strict preferences, we have $h(m^*_i, m_{-i}) = h(m^{*'}_i, m_{-i})$. Repeating the process for each $i$, we have $h(m^*_i) = h(m^{*'}_i)$ for all $m^*_i, m^{*'}_i \in D(\Gamma, \theta)$. Therefore, dominant strategy outcome is unique.

Thus, for all $m^* \in D(e, \Gamma)$, $g(e) = h(m^*(e))$ is a single-valued mapping. So the revelation mechanism $\langle E, g \rangle$ completely implements $F$ in dominant strategies.

As a direct corollary of Proposition 18.4.1, any social choice goal that can be fully implemented in dominant strategies must be a single-valued social choice function. Formally, we have

**Corollary 18.4.1** Suppose that agents have strict preferences. Any social choice goal $F$ that can be fully implemented in dominant strategies (i.e., $h(D(e, \Gamma)) = F(e)$) must be a single-valued social choice function.

**Proof.** Since $\Gamma = (M, h)$ fully implements $F$ in dominant strategies, we have $h(D(e, \Gamma)) = F(e)$. Then, by Proposition 18.4.1, we have $h(m^*) = h(m^{*'}_i)$, for all $m^*_i, m^{*'}_i \in D(\Gamma, \theta)$. Thus, $\forall \theta \in \Theta, f(\theta) = h(D(\Gamma, \theta))$ is a single-valued function.

When discussing ex post full implementation in the next chapter, we will give a necessary and sufficient condition for a social choice correspondence under private value to be fully implemented in dominant strategies. If the participant’s utility depends only on his own type, i.e., $u_i(h(m(\theta)), \theta) = u_i(h(m(\theta)), \theta_i), \forall i \in N$, the ex post equilibrium is equivalent to the dominant equilibrium. Thus, the necessary and sufficient conditions for ex post full implementation of the social choice rules are also necessary and sufficient conditions for full implementation in dominant strategies under private value conditions.

However, even if the dominant equilibrium is unique, or different dominant equilibria lead to the same dominant equilibrium outcome, there may exist equilibrium outcomes in other solution concepts. For example, there
may exist a Nash equilibrium that resulting in different equilibrium outcomes. We will give such an example later when we discuss a special form of VCG mechanism, namely the second-price sealed-bid auction mechanism.

### 18.5 Gibbard-Satterthwaite Impossibility Theorem

Social planners want the social choice rules to be implemented to have some desirable properties, such as non-dictatorship; meanwhile, since the information is expected to be as little as possible, the game solution to implement the rule is expected to be as strong as possible, such as dominant equilibrium. The revelation principle says that if one want a social choice goal $f$ to be implementable in dominant strategies, one only needs to show the revelation mechanism $\langle E, f \rangle$ is strongly incentive compatible.

Unfortunately, it is generally impossible to have such a result if economic environments are sufficiently rich. The Gibbard-Satterthwaite impossibility theorem in Chapter 12 tells us that such social goals can only be dictatorial in the sense that an agent’s optimal choice is the social optimum when economic environments are unrestricted. Let us reconsider this theorem from the perspective of mechanism design.

**Theorem 18.5.1 (Gibbard-Satterthwaite Impossibility Theorem)** If the outcome space $A$ contains at least three alternatives, $|A| \geq 3$, the domain of social choice correspondence $F : E \rightarrow A$ contains all strict preferences, which is $E^s \subseteq E$, and $F(\cdot)$ is an onto mapping that can be implemented truthfully in dominant strategies, then $F$ must be dictatorial.

This theorem shows that if individuals’ preferences are unrestricted and each individual is required to tell the truth, then such an incentive mechanism does not exist. Before proving the theorem we give first the following lemmas.

**Lemma 18.5.1** If the domain only contains strict preferences, namely $E = E^s$, the outcome space $A$ contains at least three alternatives, $|A| \geq 3$, and the social choice rule $F : E^S \rightarrow A$ satisfies unanimity and Maskin monotonicity, then it must be dictatorial.

**Proof.**

We prove the lemma with help of Tables 18.1 to 18.6.

- Consider the two alternatives $a, b \in A$. Suppose that $a \succ^e_i x \succ^e_i b, \forall x \in A, \forall i \in I$. According to unanimity, we get $F(e) = a$. In the preference order of agent 1, move $b$ forward one position at a time. According to Maskin monotonicity, as long as $b$ does not exceed $a$, then alternative $a$ is still selected. But when $b$ exceeds $a$, according to
unanimity, the possible outcome is \( a \) or \( b \). If the alternative selected is still \( a \), the same operation is performed on the preference ordering of agent 2, and so on. There must be an agent who changes the social choice from \( a \) to \( b \) when his preference reverses from \( a \) to \( b \), then mark him as \( j \). And also mark the state before the choice between \( a \) and \( b \) reverses as \( e_1 \), and mark the state after that as \( e_2 \). In the states of \( e_1 \) and \( e_2 \), move the preference order of \( a \) to the end for each agent \( i < j \), while move the preference order of \( a \) to the second-to-last position for each agent \( i > j \). The preference of agent \( j \) remains unchanged, and the new states are recorded as \( e_1' \) and \( e_2' \).

- Since \( F(e_2) = b \) and \( L_i(b,e_2) = L_i(b,e_2'), \forall i \in I \), by Maskin monotonicity, we get \( F(e_2') = b \). If we compare \( e_2' \) with \( e_1' \), \( a \) and \( b \) reverse only at the preference order of \( j \), which means \( F(e_1') = a \) or \( b \). If \( F(e_1') = b \), since \( L_i(b,e_1') = L_i(b,e_1), \forall i \in I \), we have \( F(e_1') = b \) by Maskin monotonicity, and come up with a contradiction. So we must have \( F(e_1') = a \).

- Suppose that \( c \neq a,b \) is another alternative, define the state \( e_3 \) according to Table 18.5. Since \( L_i(a,e_3) = L_i(a,e_1') \) for \( \forall i \in I \), we have \( F(e_3) = a \) by Maskin monotonicity.

- At the state \( e_3 \), reverse the preference for \( a \) and \( b \) for each agent \( i > j \), and get state \( e_4 \). By Makin monotonicity again, \( F(e_4) = a \) or \( b \). Obviously, \( c \succ_1 b, \forall i \in I \), and according to unanimity, we must have \( F(e_4) = a \).

- Obviously, for any state \( \alpha \in E^s \), as long as \( a \) is at the top of the list of agent \( j \)'s preference, then \( L_i(a,e_4) \subseteq L_i(a,\alpha) \), \( \forall i \in I \), and we can have \( F(\alpha) = a \) according to Maskin monotonicity. It is clear that individual \( j \) is the dictator of alternative \( a \).

<table>
<thead>
<tr>
<th>individual</th>
<th>preference ( \succ_1^{\theta_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b a ... *</td>
</tr>
<tr>
<td>2</td>
<td>b a ... *</td>
</tr>
<tr>
<td>...</td>
<td>... ... ...</td>
</tr>
<tr>
<td>j-1</td>
<td>b a ... *</td>
</tr>
<tr>
<td>j</td>
<td>a b ... *</td>
</tr>
<tr>
<td>j+1</td>
<td>a ... b</td>
</tr>
<tr>
<td>...</td>
<td>... ... ...</td>
</tr>
<tr>
<td>n</td>
<td>a ... b</td>
</tr>
</tbody>
</table>

Table 18.1: the Preferences under State \( e_1 \)
### 18.5. GIBBARD-SATTERTHWAITE IMPOSSIBILITY THEOREM

<table>
<thead>
<tr>
<th>Individual</th>
<th>Preference $\succ^{\theta_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b * ... * a</td>
</tr>
<tr>
<td>2</td>
<td>b * ... * a</td>
</tr>
<tr>
<td>...</td>
<td>... ... ... ...</td>
</tr>
<tr>
<td>j-1</td>
<td>b * ... * a</td>
</tr>
<tr>
<td>j</td>
<td>a b ... * *</td>
</tr>
<tr>
<td>j+1</td>
<td>* * ... a b</td>
</tr>
<tr>
<td>...</td>
<td>... ... ... ...</td>
</tr>
<tr>
<td>n</td>
<td>* * ... a b</td>
</tr>
</tbody>
</table>

Table 18.2: the Preferences under State $e_1'$

<table>
<thead>
<tr>
<th>Individual</th>
<th>Preference $\succ^{\theta_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b a ... *</td>
</tr>
<tr>
<td>2</td>
<td>b a ... *</td>
</tr>
<tr>
<td>...</td>
<td>... ... ... ...</td>
</tr>
<tr>
<td>j-1</td>
<td>b a ... *</td>
</tr>
<tr>
<td>j</td>
<td>b a ... *</td>
</tr>
<tr>
<td>j+1</td>
<td>a ... b</td>
</tr>
<tr>
<td>...</td>
<td>... ... ... ...</td>
</tr>
<tr>
<td>n</td>
<td>a ... b</td>
</tr>
</tbody>
</table>

Table 18.3: the Preferences under State $e_2$

<table>
<thead>
<tr>
<th>Individual</th>
<th>Preference $\succ^{\theta_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>b * ... * a</td>
</tr>
<tr>
<td>2</td>
<td>b * ... * a</td>
</tr>
<tr>
<td>...</td>
<td>... ... ... ...</td>
</tr>
<tr>
<td>j-1</td>
<td>b * ... * a</td>
</tr>
<tr>
<td>j</td>
<td>b a ... *</td>
</tr>
<tr>
<td>j+1</td>
<td>* ... a b</td>
</tr>
<tr>
<td>...</td>
<td>... ... ... ...</td>
</tr>
<tr>
<td>n</td>
<td>* * ... a b</td>
</tr>
</tbody>
</table>

Table 18.4: the Preferences under State $e_2'$
From the above proof, we can see that under the strict preference, there is a dictator for any alternative \( a \), but there can be no more than one dictator. Suppose that at the state \( e, j \) is the dictator for alternative \( a \), then we have \( F(e) = a \); and suppose that \( j' \) is another dictator for another alternative \( a' \), then we have \( F(e) = a' \), which is a contradiction.

**Lemma 18.5.2** If the social choice rule \( F : E^s \rightarrow A \) is truthfully implemented in dominant strategies and is a surjective mapping, then it must satisfy unanimity and Maskin monotonicity.

**Proof.**

- **Maskin monotonicity:** For states \( e, e' \in E^s \) such that \( F(e) = a \) and \( L_i(a, e) \subseteq L_i(a, e'), \forall i \in I \), if \( F(e'_1, e_{-1}) \neq a \), by truthful implementation in dominant strategies and strict preference, we must have \( a = F(e) \succ_i F(e'_1, e_{-1}) \), that is \( F(e'_1, e_{-1}) \in L_1(a, e) \). Thus \( F(e'_1, e_{-1}) \in L_1(a, e') \), that is, \( a = F(e'_1, e_{-1}) \succ_i F(e'_1, e_{-1}) \). This contradicts the
Unanimity

Suppose Example 18.5.1 (A simple majority voting with only two candidates) least three alternatives”. We start with the counterexample of relaxing the condition of “there are at least three alternatives”.

18.5. GIBBARD-SATTERTHWAITE IMPOSSIBILITY THEOREM

Consider another state $e' \in E^s$ satisfying (i) $a \succ \succ' b \succ c$, $\forall i \neq j, c \neq a, b$; (ii) $b \succ' a \succ' c$, $\forall i \neq j, c \neq a, b$. Then $L_i(a, e) \subseteq L_i(a, e')$, $\forall i \in I$, and $F(e') = b$ by Maskin monotonicity. According to the truthful implementation in dominant strategies $F(e_1, e_{-1}) \succ \succ' F(e_1', e_{-1})$, so $F(e_1, e_{-1}) \succ \succ' F(e_1', e_{-1})$. On the other hand, by the truthful implementation in dominant strategies again, $F(e_1', e_{-1}) \succ \succ' F(e_1, e_{-1})$, so $F(e_1', e_{-1}) = F(e_1, e_{-1}) = a$. By the similar way as in the proof of Lemma 18.5.2., we can get $F(e') = a$. Therefore $F(e') = a \neq M_j(e') = b$, which is a contradiction to the fact that there exists $j \in I$ such that $\forall e \in E^s$, $F(e) = M_j(e)$. So $\forall e \in E \setminus E^s$, we have $F(e) = M_j(e)$.

Gibbard-Satterthwaite Impossibility Theorem is a rather disappointing result, which is essentially equivalent to Arrow Impossibility Theorem. However, for some special classes of economic environments, for example, by relaxing the two conditions “there are at least three alternatives” and the unrestricted domain ($E^s \subseteq E$), its conclusion may not be true. Let us start with the counterexample of relaxing the condition of “there are at least three alternatives”.

Example 18.5.1 (A simple majority voting with only two candidates) Suppose that there are $N$ voters casting their votes on two candidates for president election, $a$ and $b$. Each voter reports his preference by naming the candidate
he supports, and they follow the simple majority rule:

\[ F(e) = \begin{cases} 
  a & \text{if } \# \{ i \in I | a \succ_i b \} > \# \{ i \in I | b \succ_i a \}, \\
  b & \text{if } \# \{ i \in I | a \succ_i b \} \leq \# \{ i \in I | b \succ_i a \}. 
\end{cases} \]

It can be seen that every voter will truthfully report his preference in this process, regardless whether others are telling the truth or not. Therefore, \( F(\cdot) \) can be truthfully implemented in dominant strategies.

Another example is the VCG mechanism to be discussed below, which is defined on the domain of quasi-linear utility functions and leads to the social surplus maximization.

The following Hurwicz impossibility theorem gives an even stronger impossibility result: even if further restricted to the neo-classical economic environment, Pareto efficiency and truth-telling are fundamentally incompatible.

### 18.6 Hurwicz Impossibility Theorem

Before Hurwicz impossibility theorem, when economists think of the differences between public goods and private goods, it is generally believed that a competitive market mechanism can handle private goods well. It was thought that, for economies with only private goods, the efficient allocations of resources were consistent with individual’s self-interested behavior.

For economic environments with public goods, it is relatively easy to understand that the efficient allocation of resources is incompatible with the self-interest behavior of individuals, because individuals have incentives to “free ride” and benefit from the contribution of others. So they are unwilling to report their true preferences for public goods so as to contribute less. If the tax paid by an individual is determined according to the degree of personal consumption reported by himself, he may underreport his wants. But, once a public good is produced, he can benefit from the consumption of public goods in the same way. This is very different from the intuition of private goods, in which an individual spends money on items that benefit himself only.

This strategic phenomenon of hiding one’s true preference was originally proposed by Samuelson (1954, 1955) in the criticism of the Lindahl equilibrium solution for public goods. He further speculated that for an economic environment with public goods, there was no decentralized economic mechanism that could lead to Pareto efficient allocations with participants reporting their true preferences. According to direct revelation mechanism and the incentive-compatibility, Samuelson’s assertion means
that participants have no incentives to report their true economic characteristics. Surprisingly, this assertion holds not only for the environment of public goods, but also for that of private goods.

In 1972, Hurwicz proved that for the neoclassical economic environments with a finite number of participants, there was no economic mechanism that could lead to individually rational and Pareto optimal allocations, and simultaneously enable everyone to report truthfully their economic characteristics.

**Theorem 18.6.1 (Hurwicz Impossibility Theorem, 1972)** For the neoclassical private goods economies, there is no mechanism \( \langle M, h \rangle \) that implements Pareto efficient and individually rational allocations in dominant strategies. Consequently, any revelation mechanism \( \langle M, h \rangle \) that yields Pareto efficient and individually rational allocations is not strongly individually incentive compatible (i.e., Truth-telling about their preferences is not a dominant strategy Equilibrium).

**Proof.** By the Revelation Principle, we only need to show that we can find a specific pure exchange economy such that any revelation mechanism does not implement truthfully Pareto efficient and individually rational allocations in a dominant strategy equilibrium. Then it is sufficient to show that truth-telling is not even a Nash equilibrium for any revelation mechanism that yields Pareto efficient and individually rational allocations of the pure exchange economy.

Consider a private goods economy with two agents \((n = 2)\) and two goods \((L = 2)\):

\[
\begin{align*}
    w_1 &= (0, 2), w_2 = (2, 0), \\
    \hat{u}_i(x, y) &= \begin{cases} 3x_i + y_i & \text{if } x_i \leq y_i, \\ x_i + 3y_i & \text{if } x_i > y_i. \end{cases}
\end{align*}
\]

The set of feasible allocations are then given by

\[
A = \{ [(x_1, y_1), (x_2, y_2)] \in R_+^4 : x_1 + x_2 = 2, y_1 + y_2 = 2 \}.
\]

Let \(U_i\) be the set of all neoclassical utility functions, i.e., they are monotone, continuous and quasi-concave functions that each agent \(i\) can report to the designer. Thus, his true utility function, \(\hat{u}_i\) is included in \(U_i\). Then,

\[
U = U_1 \times U_2
\]

and

\[
h : U \rightarrow A
\]
constitute a revelation mechanism. Note that if the true utility function profile \( \hat{u}_i \) were a Nash equilibrium, it would satisfy
\[
\hat{u}_i(h_i(\hat{u}_1, \hat{u}_{-i})) \geq \hat{u}_i(h_i(u_i, \hat{u}_{-i})).
\]  

(18.6.3)

We want to show that \( \hat{u}_i \) is not a Nash equilibrium. Note from Figure 18.2 that,
1. \( P(\hat{u}) = O_1O_2 \) (contract curve);
2. \( IR(\hat{u}) \cap P(\hat{u}) = \overline{ab} \);
3. \( h(\hat{u}_1, \hat{u}_2) = d \in \overline{ab} \).

Now, suppose that agent 2 misreports his utility function as:
\[
u_2(x_2, y_2) = 2x + y.
\]  

(18.6.4)

Then, with \( u_2 \), the new set of individually rational (IR) and Pareto efficient allocations is given by
\[
IR(\hat{u}_1, u_2) \cap P(\hat{u}_1, u_2) = \overline{ae}.
\]  

(18.6.5)

Note that \( d \) is strictly inferior to any point between \( a \) and \( e \) for agent 2, thus an allocation determined by any mechanism which is IR and Pareto efficient under \((\hat{u}_1, u_2)\) is some point, say, point \( c \) in the figure, on the line determined by endpoints \( a \) and \( e \). Hence, we have
\[
\hat{u}_2(h_2(\hat{u}_1, u_2)) > \hat{u}_2(h_2(\hat{u}_1, \hat{u}_2)).
\]  

(18.6.6)
since $h_2(u_1, u_2) = c \in \pi c$. Similarly, if $d$ is on $\pi c$, then agent 1 has incentive to deviate. Thus, no mechanism that yields Pareto efficient and individually rational allocations is incentive compatible.

The Hurwicz impossibility theorem tells us that, even for the neoclassical economic environments with private goods only, as long as the number of participants is finite, there is no information-decentralized economic mechanism such that Pareto efficient allocations can be implemented in dominant strategies. This is because the finite number of participants is essentially inconsistent with perfect competition. This makes one realize that the general equilibrium theory is based on the assumption that the number of participants is infinitely large, which is of course unrealistic. However, when the number of economic agents is continuum, “truthful revelation of preferences” is possible, which is also unrealistic.

Thus, if one wants a mechanism that implements Pareto optimal allocations, we must give up the requirement of truth-telling. For economic environments with public goods, no matter whether the number of individuals is finite or not, Ledyard and Roberts (1974) proved a similar impossibility result. In this sense, there is no essential difference between the two economic environments, namely economic environments with public goods and the economic environments without public goods. Of course, as shown in the following section, a mechanism that guarantees truth-telling and the social surplus maximization (not Pareto optimality since feasibility in general is not satisfied) is possible.

In conclusion, the Hurwicz’s impossibility theorem implies that Pareto efficiency and the truthful revelation of individuals’ characteristics are fundamentally incompatible. If we desire a positive result, we must give up either the requirement of Pareto efficiency or the requirement of truthful revelation. For example, if one is willing to give up Pareto efficiency, say, one only requires the efficient provision of public goods, is it possible to find an incentive compatible mechanism that truthfully reveals individual’s characteristics? The answer is positive. For the class of quasi-linear utility functions, the so-called Vickrey-Clark-Groves mechanism can be such a mechanism.

18.7 Vickrey-Clark-Groves (VCG) Mechanisms

From Chapter 15 on public goods, we know that public goods economies may have an inefficiency issue with a decentralized resource allocation mechanism because of the free-rider problem. Private provision of a public good generally results in less than an efficient amount of the public good. Voting may result in too much or too little of a public good. Are there any mechanisms that result in the “right” amount of the public good? This is a question of the incentive compatible mechanism design.
The Vickrey-Clark-Groves (in short, VCG) mechanism can do so for private-value economic environments. Vickrey (William Vickrey, 1914-1996, his biography can be seen in section 21.8.1) was the first to propose such a mechanism in the format of the second-price auction in the early 1960s, and Clark then gave the pivotal mechanism, though they did not consider the efficient provision of indivisible private goods and public goods from the perspective of mechanism design. Groves provided a general form of the VCG mechanism, which covers Vickrey’s second-price auction mechanism and Clark’s pivotal mechanism as special cases of Groves’ general mechanism.

In the following, we will show that for economic environments with quasi-linear utility functions under private value, the VCG mechanism generates an efficient provision of public goods, and is the only incentive-compatible mechanism when the economic environment is sufficiently “thick”.

We begin the case of indivisible public goods.

### 18.7.1 VCG Mechanisms for Discrete Public Goods

Consider the provision of a discrete public good. Suppose that the economy has $n$ agents. Let

- $c$: the cost of producing the public project.
- $r_i$: the maximum willingness to pay of $i$.
- $s_i$: the share of the cost by $i$.
- $v_i = r_i - s_i c \equiv r_i - g_i$: the net value of $i$.

The public project is determined according to

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} v_i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

From the discussion in Chapter 15, it is efficient to produce the public good, $y = 1$, if and only if

$$\sum_{i=1}^{n} v_i \geq 0.$$

Since the maximum willingness to pay for each agent, $r_i$, is private information and so is the net value $v_i$, the designer needs to induce the information about $v_i$ from individuals to determine if the project is built. What mechanism one should use? One mechanism we might use is simply to ask each agent to report his or her net value and provide the public good if the sum of the reported values is positive. The problem with such a scheme is that it does not provide right incentives for individuals to reveal their true
willingness-to-pay. Individuals may have incentives to underreport their willingness-to-pay.

Thus, a question is how we can induce agents to truthfully reveal their values for the public good. The so-called Vickrey-Clark-Groves (VCG) mechanism gives such a mechanism.

Suppose that the utility functions are quasi-linear in the net demand of private goods, \( x_i - w_i \):

\[
\bar{u}_i(x_i - w_i, y) = x_i - w_i + r_i y \\
s.t. \quad x_i + g_i y = w_i + t_i,
\]

where \( t_i \) is the transfer to agent \( i \). Then, we have

\[
u_i(t_i, y) = t_i + r_i y - g_i y \\
= t_i + (r_i - g_i) y \\
= t_i + v_i y.
\]

A Simple Form of VCG Mechanism:

In a VCG mechanism, agents are required to report their net values. Thus the message space of each agent \( i \) is \( M_i = \mathbb{R} \). The VCG mechanism is defined as follows:

\[
\Gamma = \langle M_1, \ldots, M_n, t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot), y(\cdot) \rangle \equiv \langle M, t(\cdot), y(\cdot) \rangle,
\]

where

1. \( b_i \in M_i = \mathbb{R} \): each agent \( i \) reports a “bid” for the public good, i.e., report the net value of agent \( i \) which may or may not be his true net value \( v_i \).

2. The level of the public good is determined by

\[
y(b) = \begin{cases} 
1 & \text{if } \sum_{i=1}^n b_i \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

3. Each agent \( i \) receives a transfer payment

\[
t_i(b) = \begin{cases} 
\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geq 0, \\
0 & \text{otherwise}. 
\end{cases} \quad (18.7.7)
\]

If \( t_i < 0 \), it is a lump-sum tax; if \( t_i > 0 \), it is a subsidy. Then, the payoff of agent \( i \) is given by

\[
\phi_i(b) = \begin{cases} 
v_i + t_i(b) = v_i + \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geq 0, \\
0 & \text{otherwise}. 
\end{cases} \quad (18.7.8)
\]
We want to show that it is optimal for each individual to report the true net value, namely $b_i = v_i$, regardless of what the other agents report. That is, truth-telling is a dominant strategy equilibrium.

There are two cases to be considered.

Case 1: $v_i + \sum_{j \neq i} b_j > 0$. Then agent $i$ desires the public good to be provided and can ensure this by reporting $b_i = v_i$. Indeed, if $b_i = v_i$, then $\sum_{j \neq i} b_j + v_i = \sum_{i=1}^n b_j > 0$ and thus $y = 1$. In this case, $\phi_i(v_i, b_{-i}) = v_i + \sum_{j \neq i} b_j > 0$. Also, the pursuit of individual utility maximization and the social goal of providing this public good are incentive compatible.

Case 2: $v_i + \sum_{j \neq i} b_j \leq 0$. Agent $i$ does not desire the public good and can ensure it not to be provided by reporting $b_i = v_i$, so that $\sum_{i=1}^n b_i \leq 0$. In this case, $\phi_i(v_i, b_{-i}) = 0 \geq v_i + \sum_{j \neq i} b_j$. As such, the pursuit of individual utility maximization and the social goal of not providing this public good are incentive compatible.

Thus, for either cases, agent $i$ has incentives to tell the true value of $v_i$ regardless the choices of other individuals. Hence, it is a dominant strategy for agent $i$ to tell the truth. There is no incentive for agent $i$ to misreport his true net value regardless of what other agents do.

The above VCG mechanism has a major weakness: the total transfer payment may be very large, and so it is costly to induce the agents to tell the truth. As such, the implementing (information) cost of this mechanism is high.

Ideally, we desire to have a mechanism where the sum of individual transfer payments is equal to zero so that the feasibility condition holds, and consequently it results in Pareto efficient allocations; but, it is in general impossible by Proposition 18.7.4 below. Nevertheless, we can reduce the information cost by modifying the above mechanism and asking each agent to pay a tax, but not receive a subsidy. As this tax may incur a deadweight loss, the allocation of public goods is still not Pareto efficient.

The basic idea of paying a tax is to add an extra amount to agent $i$’s transfer payment, $d_i(b_{-i})$ that depends only on what the other agents do.

**General VCG Mechanism:**

Let each individual pay an additional tax amount $d_i(b_{-i})$ that is independent of $b_i$.

In this case, the transfer payment is given by

$$ t_i(b) = \begin{cases} \sum_{j \neq i} b_j + d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i \geq 0, \\ d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i < 0. \end{cases} $$

The payoff to agent $i$ now takes the form:

$$ \phi_i(b) = \begin{cases} v_i + t_i(b) = v_i + \sum_{j \neq i} b_j + d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i \geq 0, \\ d_i(b_{-i}) & \text{otherwise.} \end{cases} \quad (18.7.9) $$
For exactly the same reason as for the mechanism above, one can prove that it is a dominant strategy for each agent \( i \) to report his true net value. We then have the following proposition.

**Proposition 18.7.1** For discrete private-value public good economies, truth-telling is a dominant strategy for all participants under the VCG mechanism that truthfully implements the efficient decision rule (the efficient provision of public goods) in dominant strategy.

**Remark 18.7.1** This conclusion relies on the private value assumption. If an individual’s valuation function also depends on the type of other people, the transfer payment function defined above also depends on his own type, and then the VCG mechanism may not lead to the truthful implementation of social decision rule in dominant strategy. However, the generalized VCG mechanisms to be discussed in Chapter 21 about auction theory implements truthfully the efficient decision rule.

**Remark 18.7.2** The VCG mechanism plays an important role in general mechanism design, auction theory and dynamic mechanism design. We will give various modifications and extensions of the VCG mechanism.

If the function \( d_i(b_{-i}) \) is properly chosen, the size of the transfer payment can be significantly reduced. A typical choice was given by Clark (1971). The Clark mechanism is also called Pivotal mechanism.

The Pivotal Mechanism is a special case of the VCG Mechanism in which

\[
 d_i(b_{-i}) = \begin{cases} 
 -\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j \geq 0, \\
 0 & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j < 0, \\
 \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j < 0, \\
 0 & \text{otherwise}.
\end{cases}
\]

(18.7.10)

In this case, it gives

\[
 t_i(b) = \begin{cases} 
 0 & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j \geq 0, \\
 -\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j < 0, \\
 -\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^{n} b_i < 0 \text{ and } \sum_{j \neq i} b_j \geq 0, \\
 0 & \text{if } \sum_{i=1}^{n} b_i < 0 \text{ and } \sum_{j \neq i} b_j < 0,
\end{cases}
\]

(18.7.11)

Therefore, the payoff of agent \( i \) is

\[
 \phi_i(b) = \begin{cases} 
 v_i & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j \geq 0, \\
 v_i + \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^{n} b_i \geq 0 \text{ and } \sum_{j \neq i} b_j < 0, \\
 -\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^{n} b_i < 0 \text{ and } \sum_{j \neq i} b_j \geq 0, \\
 0 & \text{if } \sum_{i=1}^{n} b_i < 0 \text{ and } \sum_{j \neq i} b_j < 0.
\end{cases}
\]

(18.7.12)
Thus, from the transfers given in (18.7.11), one can see that, if the bidding strategy of agent $i$ changes the social decision, he is a pivotal agent. The amount of the tax agent $i$ must pay is the amount by which agent $i$’s bid damages the other agents. The price that agent $i$ must pay to change the amount of public good is equal to the total loss that he imposes on the other agents.

### 18.7.2 VCG Mechanism with Continuous Public Goods

Now we discuss the provision of continuous public goods. Consider a public goods economy with $n$ agents, one private good, and $K$ public goods. Denote

- $x_i$: the consumption of the private good by $i$;
- $y$: the consumption of the public goods by all individuals;
- $t_i$: transfer of agent $i$;
- $g_i(y)$: the contribution made by $i$;
- $c(y)$: the cost function of producing public goods $y$ that satisfies the condition:

$$\sum g_i(y) = c(y).$$

Then, agent $i$’s budget constraint should satisfy

$$x_i + g_i(y) = w_i + t_i, \quad (18.7.13)$$

and his utility function is given by

$$\bar{u}_i(x_i - w_i, y) = x_i - w_i + u_i(y). \quad (18.7.14)$$

Combining the above two equations gives

$$u_i(t_i, y) = t_i + (u_i(y) - g_i(y))$$

$$\equiv t_i + v_i(y),$$

where $v_i(y)$ is called the valuation function of agent $i$. From the budget constraint,

$$\sum_{i=1}^{n} (x_i + g_i(y)) = \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} t_i, \quad (18.7.15)$$

we have

$$\sum_{i=1}^{n} x_i + c(y) = \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} t_i. \quad (18.7.16)$$
From the above equation, the balancedness condition is equivalent to
\[ \sum_{i=1}^{n} t_i = 0. \] (18.7.17)

Recall that Pareto efficient allocations are completely characterized by
\[
\max \sum_{i=1}^{n} a_i \bar{u}_i(x_i, y) \\
\text{s.t.} \quad \sum_{i=1}^{n} x_i + c(y) = \sum_{i=1}^{n} w_i.
\]

For quasi-linear utility functions, it is easily seen that the weights \( a_i \) must be the same for all agents since \( a_i = \lambda \) for interior Pareto efficient allocations (no income effect) where \( \lambda \) is the Lagrangian multiplier, and thus \( y \) is uniquely determined by \( u_i(t_i, y) = t_i + v_i(y) \). Then, the above characterization problem becomes
\[
\max_{t_i, y} \left[ \sum_{i=1}^{n} (t_i + v_i(y)) \right], \quad (18.7.18)
\]
or equivalently

(1) maximization of social surplus:
\[
\max_{y} \sum_{i=1}^{n} v_i(y); \quad (18.7.19)
\]

(2) balancedness condition:
\[ \sum_{i=1}^{n} t_i = 0. \] (18.7.20)

Then, the Lindahl-Samuelson condition is given by:
\[ \sum_{i=1}^{n} \frac{\partial v_i(y)}{\partial y_k} = 0, \]
that is,
\[ \sum_{i=1}^{n} \frac{\partial u_i(y)}{\partial y_k} = \frac{\partial c(y)}{\partial y_k}. \]

Thus, Pareto efficient allocations for quasi-linear utility functions are completely characterized by the Lindahl-Samuelson condition \( \sum_{i=1}^{n} v_i'(y) = 0 \) and the balancedness condition \( \sum_{i=1}^{n} t_i = 0 \).
In general, the mechanism designer does not know the true valuation function \( v_i(\cdot, \cdot) \), and then needs to design an incentive mechanism that results in the efficient provision of public goods, and each participant has incentives to truthfully report his valuation function.

In term of direct revelation mechanism \( \langle V, h \rangle \), the designer requires each participant to report his valuation function, which may be a truthful reporting or not, and then determine the outcome function \( h(\cdot) = (y(\cdot), t_1(\cdot), \cdots, t_n(\cdot)) : V \rightarrow \mathbb{R}_+ \times \mathbb{R}^n \) so that such a truthful revelation mechanism leads to the efficient provision of public goods, \( y^* \).

In a simple form of VCG mechanism, it is assumed that each agent is asked to report the valuation function \( v_i(y) \). Denote the valuation function reported by \( b_i(y) \).

To get the efficient level of public goods, the designer may announce that it will provide a level of public goods \( y^* \) that maximizes

\[
\max_y \sum_{i=1}^n b_i(y). \tag{18.7.21}
\]

The VCG mechanism has the form:

\[
\Gamma = \langle V, h \rangle, \tag{18.7.22}
\]

where \( V = V_1 \times \cdots \times V_n \) is the message space that consists of the set of all possible valuation functions with element \( b_i(y) \in V_i \), and \( h(b) = (y(b), t_1(b), t_2(b), \cdots, t_n(b)) \) are outcome functions. It is determined by:

1. Ask each agent \( i \) to report his/her valuation function \( b_i(y) \) which may or may not be the true valuation \( v_i(y) \);
2. Determine \( y^* \): the level of the public goods, \( y^* = y(b) \), is determined by

\[
\max_y \sum_{i=1}^n b_i(y); \tag{18.7.23}
\]

3. Determine \( t \): the transfer payment of agent \( i \), \( t_i \), is determined by

\[
t_i(b) = \sum_{j \neq i} b_j(y^*). \tag{18.7.24}
\]

The payoff of agent \( i \) is then given by

\[
\phi_i(b(y^*)) = v_i(y^*) + t_i(b) = v_i(y^*) + \sum_{j \neq i} b_j(y^*). \tag{18.7.25}
\]

Under this mechanism, each participant \( i \) maximizes his utility:

\[
\max_y [v_i(y) + \sum_{j \neq i} b_j(y)]. \tag{18.7.26}
\]
When each individual $i$ truthfully reports his true value function, i.e.,
$b_i(y) = v_i(y)$, his maximization problem (18.7.26) and the social surplus
maximization problem (18.7.21) are exactly the same. By reporting $b_i(y) = v_i(y)$,
agent $i$ ensures that the designer will choose $y^*$ which also maxi-
mimizes his payoff while the designer maximizes the social welfare. That is,
individuals’ interests are compatible with the social interest that is deter-
mined by the Lindahl-Samuelson condition. Thus, truth-telling, $b_i(y) = v_i(y)$,
is a dominant strategy equilibrium.

In general, $\sum_{i=1}^n t_i(b(y)) \neq 0$, which means that a VCG mechanism
does not result in Pareto efficient allocations even if it results in an efficient pro-
vision of public goods.

As in the discrete case, the total transfer payment might be very large.
The VCG mechanism can then be modified to
\[
t_i(b) = \sum_{j \neq i} b_j(y) + d_i(b_{-i}),
\]
where $d_i(b_{-i})$ is an arbitrary function that is independent of $b_i$.

The general form of the VCG mechanism (the VCG mechanism) is then
\[
\Gamma = \langle V, t, y(b) \rangle
\]
such that
\begin{align*}
(1) & \quad \sum_{i=1}^n b_i(y) = \sum_{i=1}^n b_i(y), \quad \forall y \in Y; \\
(2) & \quad t_i(b) = \sum_{j \neq i} b_j(y) + d_i(b_{-i}).
\end{align*}

We then have the following proposition.

**Proposition 18.7.2** For continuous public goods economies under private values,
the truth-telling is a dominant strategy under the VCG mechanism, and hence it
implements truthfully the efficient decision rule $y^*(\cdot)$ (the efficient provision
of public goods) in dominant strategies.

A special case of the VCG mechanism is independently described by
Clark and is called the Clark mechanism (also called the pivotal mechanis-
m) in which $d_i(b_{-i}(y))$ is given by
\[
d_i(b_{-i}) = -\max_y \sum_{j \neq i} b_j(y).
\]
That is, the pivotal mechanism, $\Gamma = \langle V, t, y(b) \rangle$, is to choose $(y^*, t^*_i)$ such
that
\begin{align*}
(1) & \quad \sum_{i=1}^n b_i(y^*) \geq \sum_{i=1}^n b_i(y), \quad \forall y \in Y; \\
(2) & \quad t_i^*(b) = \sum_{j \neq i} b_j(y^*) - \max_y \sum_{j \neq i} b_j(y).
\end{align*}

It is interesting to point out that the Clark mechanism contains the well-
known second-price auction mechanism as a special case. Under the Vick-
ery mechanism, the highest-bid participant obtains the object, and he pays
the second highest bid price. To see this, let us explore this relationship in the case of a single good auction (Example 18.3.5 in the beginning of this chapter). In this case, the outcome space is $Z = \{y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1\}$ where $y_i = 1$ implies that agent $i$ gets the object, and $y_i = 0$ means he does not get the object. Agent $i$’s valuation function is then given by

$$v_i(y) = v_i y_i.$$

Since we can regard $y$ as an $n$-dimensional vector of public goods, by the Clark mechanism above, we know that $y^* = g(b) = \{y \in Z : \max_i \sum_{i=1}^n b_i y_i\} = \{y \in Z : \max_{i \in N} b_i\}$, and truth-telling is a dominant strategy. Thus, if $g_i(v) = 1$, then $t_i(v) = \sum_{j \neq i} v_j y^*_j - \max_y \sum_{j \neq i} v_j y_j = -\max_{j \neq i} v_j$. If $g_i(b) = 0$, then $t_i(v) = 0$. Thus the object is allocated to the individual with the highest valuation and he pays an amount equal to the second highest valuation. This is exactly the outcome predicted by the Vickrey auction format.

As we mentioned earlier, even if the dominant equilibrium is unique, or different dominant equilibria lead to the same dominant equilibrium outcome, there may be other equilibrium outcomes in other solution concepts. Such an example is given below.

**Example 18.7.1** In the second-price auction mechanism, besides the truth-telling is the dominant strategy equilibrium that results in the efficient outcome of the object, there is also a Nash equilibrium that results in an inefficient outcome, that is, the agent with the highest valuation does not win the object.

To see this, without loss of generality, assume that the valuations of the object satisfy $v_1 > v_2 > \cdots > v_n > 0$. Consider the following strategy $b^*$:

- $b^*_j > v_1$;
- $b^*_1 < v_j$;
- $b^*_i = 0$ for all $i \notin \{1, j\}$.

Obviously, $b^*$ is a Nash equilibrium. It is easy to verify that there is no agent $k$ who has incentive to deviate the strategy $b^*_k$.

It is worth noting that although the second-price auction mechanism truthfully implements the efficient allocation in dominant strategies it is generally not the full implementation in dominant strategies, or even not the full implementation in Nash equilibrium, nor the full ex post implementation either. Cason, Saijo, Sjostrom and Yamato (2006) demonstrated that for private-values models, the second-price auction mechanism does
18.7. VICKREY-CLARK-GROVES (VCG) MECHANISMS

not satisfy Maskin monotonicity and ex post monotonicity, which are the necessary conditions for full implementation in Nash equilibrium and in ex post equilibrium, respectively.

However, for the interdependent-value model, although the second-price auction mechanism does not satisfy Maskin’s monotonicity, it satisfies ex post monotonicity and incentive compatibility so that it is fully implementable in ex post equilibrium (see Bergemann and Morris, 2008).

18.7.3 Uniqueness of VCG Mechanism

Do there exist other mechanisms truthfully implementing the efficient decision rule \( y^* \)? The answer is no if valuation functions \( v(y, \cdot) \) are sufficiently “rich” in some sense. To see this, consider the class of parametric valuation functions \( v_i(y, \cdot) \) with a continuum type space \( \Theta_i = [\hat{\theta}, \bar{\theta}] \).

Even if \( \{ v_i(\cdot, \theta_i) : \theta_i \in \Theta_i \} = V \) where \( V \) is a collection of all possible utility functions is not required, we can get similar conclusion (Laffont and Maskin (1980)), whose proof is relatively simple.

Proposition 18.7.3 (Laffont and Maskin (1980)) Suppose that \( Y = \mathbb{R}, \Theta = [\hat{\theta}, \bar{\theta}], \) and \( v_i : Y \times \Theta_i \to \mathbb{R}, \forall i \in N \) is differentiable. Then any mechanism that implements an efficient decision rule \( y^*(\cdot) \) in dominant strategies is a VCG mechanism. That is, if a social choice rule \( f(\theta) = \{ y(\theta), t_1(\theta), \ldots, t_n(\theta) \} \) is implemented truthfully in dominant strategies, and

\[
y(\theta) = \arg \max_{y \in Y} \sum_{i=1}^{n} v_i(y, \theta_i),
\]

then we must have \( t_i(\theta) = \sum_{j \neq i} v_j(y(\theta), \theta_j) + d_i(\theta_{-i}) \), where \( d_i(\theta_{-i}) \) is some function independent of \( \theta_i \).

**Proof.** Since

\[
y(\theta) = \arg \max_{y \in Y} \sum_{i=1}^{n} v_i(y, \theta_i),
\]

we have

\[
\sum_{i=1}^{n} \frac{\partial v_i}{\partial y} (y(\theta), \theta_i) = 0. \tag{18.7.28}
\]

By the requirement of truthful implementation in dominant strategies,

\[
v_i(y(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(y(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i}), \forall \theta_i, \hat{\theta}_i, \theta_{-i}.
\]

Then

\[
\frac{\partial v_i}{\partial y} (y(\theta), \theta_i) \frac{\partial y}{\partial \theta_i} (\theta) + \frac{\partial t_i}{\partial \theta_i} (\theta, \theta_{-i}) = 0, \forall (\theta_i, \theta_{-i}) \in \Theta. \tag{18.7.29}
\]
Let \( d_i(\theta) = t_i(\theta) - \sum_{j \neq i} v_j(y(\theta), \theta_j) \). We need to show \( d_i(\theta_i, \theta_{-i}) \) is independent of \( \theta_i \). Indeed, since
\[
\frac{\partial d_i}{\partial \theta_i} = \frac{\partial t_i}{\partial \theta_i}(\theta_i, \theta_{-i}) - \sum_{j \neq i} \frac{\partial v_j}{\partial y}(y(\theta), \theta_j) \frac{\partial y}{\partial \theta_i}(\theta_i, \theta_{-i})
\]
\[
= \frac{\partial t_i}{\partial \theta_i}(\theta_i, \theta_{-i}) + \frac{\partial v_i}{\partial y}(y(\theta), \theta_i) \frac{\partial y}{\partial \theta_i}(\theta_i, \theta_{-i}) - \sum_{j=1}^{n-1} \frac{\partial v_j}{\partial y}(y(\theta), \theta_j) \frac{\partial y}{\partial \theta_i}(\theta_i, \theta_{-i})
\]
\[
= 0
\]
by using (18.7.28) and (18.7.29), we have \( d_i(\theta) = d_i(\theta_{-i}) \).

The intuition behind the proof is that each agent \( i \) has the incentive to tell the truth only when he is given the residual claimant of total surplus, i.e., \( U_i(\theta) = \sum_{i=1}^{n} v_i(y(\theta), \theta_i) + d_i(\theta_{-i}) \), which is the feature of the VCG mechanism.

This result was also obtained by Green and Laffont (1979), but under less restrictive assumptions, namely they allow agents to have all possible valuations over a finite decision set \( X \), which can be described by the Euclidean type space \( \Theta_i = \mathbb{R}^{K} \) (the valuation of agent \( i \) for decision \( k \) is represented by \( \theta_{ik} \)). The proof can be seen in Mas-Colell, Whinston and Green (1995) without imposing differentiability on \( v_i \).

### 18.7.4 Balanced VCG Mechanisms

Implementation of a social surplus-maximizing decision rule \( y^*(\cdot) \) is a necessary condition for (ex-post) Pareto efficiency, but it is not sufficient. To have Pareto efficiency, we must also ensure that there be an ex post balanced budget (no waste of numeraire goods). That is, for all \( b \in V \),
\[
\sum_{i=1}^{n} t_i(b) = 0. \tag{18.7.31}
\]

Substituting \( t_i \) in the VCG mechanism into (18.7.31) leads to Pareto-efficient allocations if and only if
\[
(n - 1) \sum_{i=1}^{n} v_i(y(\nu)) + \sum_{i=1}^{n} d_i(b_{-i}) \equiv 0. \tag{18.7.32}
\]

A balanced budget implies that there is no waste of resources. For example, in order to build a public project, the government levies taxes on some residents and subsidizes some residents who are adversely affected. If the total tax revenue is equal to the total subsidies, the government’s use of funds is efficient; otherwise, if \( \sum_{i=1}^{n} t_i(\theta) < 0 \), then there is budget surplus and the implementation of decision rule \( y(\cdot) \) is not Pareto efficient.
Pareto optimal allocation must satisfy both the social surplus maximization (18.7.19) and the balanced budget condition (18.7.20). An interesting question is thus: can we find a balanced VCG mechanism? Green and Laffont (1979), Holmstrom (1979), and Laffont and Maskin (1980) all gave a negative answer.

**Proposition 18.7.4 (Holmstrom (1979) and Green and Laffont (1979))** Suppose that \( V(\theta) \triangleq \max_{y \in \mathcal{R}} \sum_{i=1}^{n} v_i(y, \theta_i) \). Then the VCG mechanism is budget balanced (so resulting in a Pareto efficient allocation) if and only if

\[
\frac{\partial^n V(\theta)}{\partial \theta_1 \cdots \partial \theta_n} = 0, \quad \forall \theta \in \Theta,
\]

where \( \frac{\partial^{n-1}}{\partial \theta_{i-1} \cdots \partial \theta_n} = \frac{\partial^{n-1}}{\partial \theta_1 \cdots \partial \theta_{i-1} \partial \theta_i \cdots \partial \theta_n} \).

**Proof.** **Necessity.** If \( \sum_{i=1}^{n} t_i(\theta) = 0 \), by the definition of VCG mechanism, we have

\[
\sum_{i=1}^{n} \sum_{j \neq i} v_j(y(\theta), \theta_j) + \sum_{i=1}^{n} d_i(\theta_{-i}) = (n - 1)V(\theta) + \sum_{i=1}^{n} d_i(\theta_{-i}) = 0.
\]

It is easy to verify that \( \frac{\partial^n V(\theta)}{\partial \theta_1 \cdots \partial \theta_n} \sum_{i=1}^{n} d_i(\theta_{-i}) = 0 \), so \( \frac{\partial^n V(\theta)}{\partial \theta_1 \cdots \partial \theta_n} = 0 \).

**Sufficiency.** If \( \frac{\partial^n V(\theta)}{\partial \theta_1 \cdots \partial \theta_n} = 0 \), there must be a series of functions \( d_i(\theta_{-i}), i = 1, \ldots, n \) such that \( V(\theta) = \sum_{i=1}^{n} d_i(\theta_{-i}) \). Let

\[
t_i(\theta) = \sum_{j \neq i} v_j(y(\theta), \theta_j) - (n - 1)d_i(\theta_{-i}).
\]

Then

\[
\sum_{i=1}^{n} t_i(\theta) = (n - 1)V(\theta) - (n - 1) \sum_{i=1}^{n} d_i(\theta_{-i}) = 0.
\]

A trivial case for which we can achieve ex post efficiency is given in the following example.

**Example 18.7.2** If there exists some agent \( i \) such that \( \Theta_i = \{ \theta_i \} \), a singleton, then there is no incentive issue for agent \( i \); and we can set

\[
t_i(\theta) = -\sum_{j \neq i} t_j(\theta), \quad \forall \theta \in \Theta.
\]

This trivially guarantees the balanced budget.

However, in general, there is no such a positive result. When the set of \( v(\cdot, \cdot) \) is sufficiently “dense”, then there may be no social choice rule \( f(\cdot) = (y^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot)) \), where \( y^*(\cdot) \) is ex post Pareto optimal, which is truthfully implementable in dominant strategies and also satisfies (18.7.31). So it does not result in Pareto efficient allocations. We let \( V \) represent a collection of all possible utility functions \( u : \mathcal{D} \rightarrow \mathcal{R} \).
Proposition 18.7.5 Suppose that \( V = \{ v_i(\cdot, \theta_i) : \theta_i \in \Theta_i \} \) is the set of all admissible utility functions that are twice continuously differentiable with \( \frac{\partial^2 v_i}{\partial y^2} < 0 \) and \( \frac{\partial^2 v_i}{\partial y \partial \theta_i} \neq 0 \) at all \( (y, \theta_i) \in \mathbb{R}_+ \times [\underline{\theta}, \bar{\theta}] \). Then there is no such balanced VCG mechanism.

**Proof.** Obviously, the necessary and sufficient conditions for a balanced budget (18.7.33) cannot be satisfied in general. For instance, if \( n = 2 \), by the Envelope Theorem, we have
\[
\frac{\partial V(\theta)}{\partial \theta_1} = \frac{\partial v_1(y(\theta_1, \theta_2), \theta_1)}{\partial \theta_1}.
\]
Then,
\[
\frac{\partial^2 V(\theta)}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 v_1(y(\theta_1, \theta_2), \theta_1)}{\partial y \partial \theta_1} \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_2} + \frac{\partial^2 v_1(y(\theta_1, \theta_2), \theta_1)}{\partial \theta_1 \partial \theta_2} \neq 0 \tag{18.7.34}
\]
as long as \( \frac{\partial^2 v_i}{\partial y \partial \theta_i} \neq 0 \) and \( \frac{\partial^2 v_i}{\partial y^2} \neq 0 \).

However, when the economic environment is particularly small (nowhere dense in the space of utility functions), we may have some positive results, namely there is a balanced VCG mechanism. Groves and Loeb (1975) showed this for a quadratic utility function form
\[
v_i(y, \theta_i) = \theta_i y - \frac{y^2}{2}
\]
when \( n > 2 \).

Then,
\[
V(\theta) = \max_{y \in \mathbb{R}} \sum_{i=1}^{n} v_i(y, \theta_i) = \frac{1}{2n} (\sum_{i=1}^{n} \theta_i)^2.
\]
So,
\[
\frac{\partial^n V(\theta)}{\partial \theta_1 \cdots \partial \theta_n} = 0, \forall \theta \in \Theta, \forall n \geq 3.
\]

Tian (1996a), and Liu and Tian (1999) obtain similar results for more general valuation functions
\[
\{ v_i(y, \theta_i) = \psi_i(\theta_i) \phi(y) - (b \phi(y) + c)^{\alpha} \}_{i=1}^{n}
\]
and
\[
v_i(y, \theta_i) = \psi_i(\theta_i) \phi(y) - G_i(y) + \varphi_i(\theta_i).
\]

In addition, as we will see next chapter, given the probability distribution \( \varphi \) on the state \( \theta \), if the transfer for every agent \( i \), when others reporting true types, is given by
\[
t_i(\hat{\theta}) = E_{\theta_{-i}}[\sum_{j \neq i} v_j(y(\hat{\theta}_i, \theta_{-i}), \theta_j)] + d_i(\hat{\theta}_{-i}),
\]
we can achieve interim Pareto efficiency in this case.
Remark 18.7.3 Although the VCG mechanism provides goods efficiently, it generally does not meet the balanced budget requirement and is therefore not Pareto efficient. Shao and Zhou (2016a, 2016b) proved that when adopting certain weighted average criterion (taking expectations of social welfare function), VCG mechanism was not the most efficient because participants need to pay additional funds to ensure the efficient provision of the goods. If the cost of efficiently providing the goods was also taken into account, Shao and Zhou (2016a) proved that the most efficient mechanism was certain voting mechanism. And for the indivisible goods environment with two participants, Shao and Zhou (2016b) proved that the most efficient mechanism was a fixed price mechanism or an option mechanism, both of which satisfy the balanced budget requirement and incentive compatibility in dominant strategies.

18.8 Nash Implementation

18.8.1 Nash Equilibrium and Nash Implementation

From Hurwicz’s impossibility theorem, we know that, if one wants to have a mechanism that implements Pareto efficient and individually rational allocations, one must give up the dominant strategy implementation, and then we must look at more general mechanisms, \( \langle M, h \rangle \), instead of using a revelation mechanism.

The dominant strategy equilibrium is a strong solution concept. An individual does not need any information about other individuals’ characteristics. Although the VCG mechanism can make agents report true preferences so as to provide public goods efficiently, it in general does not result in Pareto optimal allocations. Thus, if we want to find a mechanism to implement Pareto efficient and individually rational allocations, we must give up dominant strategy implementation and adopt other solution concept to describe individuals’ self-interested behavior.

Now, if we adopt the Nash equilibrium as a solution concept, can we design an incentive compatible mechanism which implements Pareto efficient and individually rational allocations? The answer is yes, and this is the problem we consider in this section and the next section.

Definition 18.8.1 For \( e \in E \), a mechanism \( \langle M, h \rangle \) is said to have a Nash equilibrium \( m^* \in M \) if

\[
h_i(m^*) \succ_i h_i(m_i, m^*_{-i})
\]

for all \( m_i \in M_i \) and all \( i \).

Denote by \( \mathcal{N}(e, \Gamma) \) the set of all Nash equilibria of the mechanism \( \Gamma \) for \( e \in E \). It is obvious that every dominant strategy equilibrium is a Nash equilibrium, but the converse may not be true.
Definition 18.8.2 A mechanism $\Gamma = \langle M, h \rangle$ is said to fully Nash-implement a social choice correspondence $F$ on $E$ if for every $e \in E$,

(a) $N(e, \Gamma) \neq \emptyset$;
(b) $h(N(e, \Gamma)) = F(e)$.

We say it Nash implements a social choice correspondence $F$ on $E$ if for every $e \in E$,

(a) $N(e, \Gamma) \neq \emptyset$;
(b) $h(N(e, \Gamma)) \subseteq F(e)$.

The following proposition shows that if truth-telling about their characteristics is a Nash equilibrium of the revelation mechanism, it must be a dominant strategy equilibrium of the mechanism. Thus, for the revelation mechanism, the dominant equilibrium and the Nash equilibrium are equivalent.

Proposition 18.8.1 For a revelation mechanism $\Gamma = \langle E, h \rangle$, a truth-telling strategy $e$ is a Nash equilibrium if and only if it is a dominant strategy equilibrium.

PROOF. For every $e \in E$ and $i$, by Nash equilibrium, we have

$$h(e_i, e_{-i}) \succ_i h(e'_i, e_{-i})$$ for all $e'_i \in E_i$.

Since $(e'_i, e_{-i})$ is arbitrary, it is a dominant strategy equilibrium. $\square$

Thus, we cannot get any new results if insisting on the revelation mechanisms. To obtain more satisfactory results, one have to give up the revelation mechanism and look for a more general mechanism with general message spaces. When we adopt the general message space and Nash equilibrium as a solution concept, it is possible to reach both incentive compatibility and Pareto efficiency. Even if everyone pursue their personal interests, by adopting suitable rules of game, Pareto efficient allocations of resources or other social goals are implementable.

Notice that, when one adopts the Nash equilibrium as a solution concept, partial implementation, rather than (full) implementation in Nash solution may be a too weak requirement, which may result in too many Nash equilibria. To see this, consider any social choice correspondence $F$ and the following mechanism: each individual’s message space consists of the set of economic environments, i.e., it is given by $M_i = E \times Z$ with a generic element $m = (e, a)$. The outcome function is defined as $h(m) = a$ when all agents report the same message $m_i = (e, a)$ with $a \in A \subseteq Z$, and otherwise each agent is seriously punished by offering a worse outcome. Then, it is clear that the profile that all agents report their true characteristics $e$ and an outcome $a \in A$ is a Nash equilibrium. However, it has many, in fact
infinitely many other Nash equilibria. For instance, if they all report a false economic environment $e'$ and an outcome $a' \in A$ so that $m_i = (e', a')$, then this also forms a Nash equilibrium. So, when we use Nash equilibrium as a solution concept, we need a social choice rule to be implemented or fully implemented in Nash equilibrium. Thus, all of our discussions below are about Nash implementation or full Nash implementation.

### 18.8.2 Characterization of Nash Implementation

Now we discuss what kind of social choice rules can be (fully) implementable in Nash equilibrium. Maskin in 1977 gave necessary and sufficient conditions for a social choice rule to be fully Nash implementable (This paper was not published till 1999 due to some incompleteness in the proof). Maskin’s result is fundamental since it not only helps us understand what kind of social choice correspondence can be fully Nash implementable, but also provides basic techniques and methods in characterizing implementability of a social choice rule in many other solution concepts. Maskin used the proof of Repello (1987) when his paper was officially published in 1999. Because Maskin’s work on characterization of full Nash implementation of any social rule had a profound influence on the development of mechanism design theory, Maskin won the Nobel Prize in Economics in 2007.

The necessary condition for full Nash implementation, called Maskin monotonicity, is defined at the beginning of this chapter. In order to facilitate the comparison of the two different expressions, we provide them below. It is not difficult to see that they are equivalent. The monotonic illustration is shown in Figure 18.3.

![Figure 18.3: An illustration of Maskin’s monotonicity.](image-url)
Definition 18.8.3 (Maskin Monotonicity) A social choice correspondence $F : E \rightarrow A$ is said to be Maskin monotonic if for any $e, \bar{e} \in E$ and $x \in F(e)$ such that for all $i$ and all $y \in A$, $x \succ_i y$ implies that $x \succ_i \bar{y}$, then $x \in F(\bar{e})$.

In words, Maskin monotonicity requires that if an outcome $x$ is socially optimal with respect to economy $e$, changing economy from $e$ to $\bar{e}$ makes all individuals even more like $x$, then $x$ remains socially optimal with respect to $\bar{e}$. This is equivalent to the following expression.

Definition 18.8.4 (A Variation of Maskin Monotonicity) An equivalent condition for a social choice correspondence $F : E \rightarrow A$ to be Maskin monotonic is that, if for any two economic environments $e, \bar{e} \in E$, $x \in F(e)$ such that $x \not\in F(\bar{e})$, there is an agent $i$ and another $y \in A$ such that $x \prec_i y$ and $y \succ_i \bar{x}$.

It means that if the result $x$ is socially optimal choice in economy $e$, but when the environment changes from $e$ to $\bar{e}$, $x$ is no longer the social optimal choice, then $x$ is not the best alternative at least for some participant.

Maskin monotonicity is a weak condition, and many well known social choice rules satisfy Maskin monotonicity.

Example 18.8.1 (Weak Pareto Efficiency) The weak Pareto optimal correspondence $P_w : E \rightarrow A$ is Maskin monotonic.

**Proof.** If $x \in P_w(e)$, then for all $y \in A$, there exists $i \in N$ such that $x \succ_i y$. Now if for any $j \in N$, $x \succ_j y$ implies $x \succ_j y$ for all $y \in A$. Then for particular $i$, we have $x \succ_i y$ for all $y \in A$. Thus we must have $x \in P_w(\bar{e})$. □

Example 18.8.2 (Majority Rule) The majority rule or called the Condorcet correspondence $CON : E \rightarrow A$ for strict preference profile, which is defined by

$$CON(e) = \{x \in A : \# \{i | x \succ_i y\} \geq \# \{i | y \succ_i x\} \text{ for all } y \in A\},$$

is Maskin monotonic. 4

**Proof.** If $x \in CON(e)$, then for all $y \in A$,

$$\# \{i | x \succ_i y\} \geq \# \{i | y \succ_i x\}.$$  (18.8.36)

But if $\bar{e}$ is an economy such that, for all $i$, $x \succ_i y$ implies $x \succ_i \bar{y}$, then the left-hand side of (18.8.36) cannot fall when we replace $e$ by $\bar{e}$. Furthermore,

4It was proposed by the 18th century French mathematician and philosopher Marquis de Condorcet. The basic procedure for an election is: Each voter must make a rank for all the candidates, in the order in which the most favourite one is ranked first. For example, if there are three candidates, denoted A, B, and C, you must mark 1, 2, and 3 in their name, not just a circle or a fork. After all the voters have been marked, the two candidates are compared in this order to see who is ranked first, and the person who wins the most votes is elected. Of course, as seen in Chapter 12, there may be a Condorcet paradox.
if the right-hand side rises, then we must have $x \succ_i y$ and $y \succ_i x$ for some $i$, a contradiction of the relation between $e$ and $\bar{e}$, given the strictness of preferences. So $x$ is still a majority winner with respect to $\bar{e}$, i.e., $x \in CON(\bar{e})$.

In addition to the above two examples, Walrasian correspondence and Lindahl correspondence with interior allocations are Maskin monotonic. The class of preferences that satisfy “single-crossing” property and individuals’ preferences over lotteries satisfying the von Neumann-Morgenstern axioms also automatically satisfy Maskin monotonicity.

The following theorem shows that Maskin monotonicity is a necessary condition for any social rule to be fully Nash-implementable.

**Theorem 18.8.1** For a social choice correspondence $F : E \rightarrow A$, if it is fully Nash implementable, then it must be Maskin monotonic.

**Proof.** Suppose that a social choice correspondence is fully Nash implementable. Then there exists an implementable mechanism denoted $\Gamma = \langle M, h \rangle$ such that $h[N(e, \Gamma)] = F(e)$ for all $e \in E$. For any two economies $e, \bar{e} \in E$, $x \in F(e)$, then by full Nash implementability of $F$, there is $m \in M$ such that $m$ is a Nash equilibrium and $x = h(m)$. This means that $h(m) \succ_i h(m_i', m_{-i})$ for all $i$ and $m_i' \in M_i$. Given $x \succ_i y$ implies $x \succ_i y$, $h(m) \succ_i h(m_i', m_{-i})$, which means that $m$ is a Nash equilibrium in $\bar{e}$. Thus, by full Nash implementability again, we have $x \in F(\bar{e})$.

We can use another equivalent definition of Maskin monotonicity to prove the necessity of full Nash implementation. Indeed, consider the economic environment $e$ and allocation outcome $x \in F(e)$. Since $F$ can be fully Nash implemented, there exists a Nash equilibrium $m \in N(e, \Gamma)$ making $h(m) = x$. If there exists an another economic environment $\bar{e}$ making $x \not\in F(\bar{e})$, then the fact that $\Gamma = \langle M, h \rangle$ fully Nash complements $F$ implies that $m$ is not a Nash equilibrium in $\bar{e}$. Then there must be an $i$ and $\bar{m}_i$ making $h(\bar{m}_i, m_{-i}) \succ_i h(m_i, m_{-i})$. Since $m$ is a Nash equilibrium in $e$, we have $h(m) \succ_i h(\bar{m}_i, m_{-i})$. Let $y = h(\bar{m}_i, m_{-i})$. Thus, we have $x \succ_i y$, but $x \succ_i y$.

It is worth noting that despite Maskin monotonicity that is a necessary condition for any social goal to be fully Nash implementable is a very weak condition, certain restrictions are imposed on social choice rules. For example, Lemma 18.5.1 shows that, with the unrestricted domain, any social choice rule that satisfies Maskin monotonicity and unanimity must be dictatorial. Hurwicz and Schmeidler (1978) proved that if the domain was not restricted, the social choice rules that satisfy Maskin monotonicity were not necessarily Pareto-efficient.\(^5\) Saijo (1987) proved that on a non-

\(^5\)This conclusion is proved as follows. Because the domain is unrestricted, there is $e \in E$ making $\bigcap_{i=1}^n M_i(e) = \emptyset$. Suppose that $f(e) = a$. Then there must be some individuals
restricted domain, any single-valued social choice rule must be constant, i.e., \( f(e) = a, \forall e \in \Theta \).

Moreover, Maskin monotonicity per se cannot guarantee that a social choice correspondence be fully Nash implementable. However, under the so-called no-veto power, it becomes sufficient.

Recall that a social choice correspondence \( F : E \rightarrow A \) satisfies no-veto power (NVP) if whenever for any \( i \) and \( e \) such that \( x \succ_j y \) for all \( y \in A \) and all \( j \neq i \), \( x \in F(e) \). That is, if \( n - 1 \) agents regard an outcome as their best choice, it is socially optimal.

The no-veto power property is a very weak condition. It is satisfied by many “standard” social choice rules, including weak Pareto efficient and Condorcet correspondences. It is also satisfied by any social choice rules when individual preferences are restricted. For example, for private goods economies with at least three agents, if each agent’s utility function is strongly monotone, then there is no other allocation such that it is an optimal one for more than one agent, so the no-veto power condition holds.

The following theorem is given by Maskin (1977, 1999), but the proof of the theorem was due to Repullo (1987).

**Theorem 18.8.2** Under the no-veto power property, if Maskin monotonicity condition is satisfied, then \( F \) is fully Nash implementable.

**Proof.** The proof is by construction. For each agent \( i \), his message space is defined by

\[
M_i = E \times A \times N,
\]

where \( N = \{1, 2, \ldots\} \). Its elements are denoted by \( m_i = (e_i, a_i, v_i) \), which means agent \( i \) announces a profile of economic characteristics of all agents including himself, an outcome, and an integer. Notice that we have used \( e_i \) and \( a_i \) to denote all individuals’ economic characteristics and outcomes, but not just agent \( i \)’s economic characteristic and outcome.

The outcome function is constructed in three cases:

Case (1). If \( m_1 = m_2 = \ldots = m_n = (e, a, v) \) and \( a \in F(e) \), the outcome function is defined by

\[
h(m) = a.
\]

In words, if players are unanimous in their strategy, and the alternative \( a \) proposed is \( F \)-optimal given their economic characteristics \( e \), then outcome is \( a \).

\^6Suppose that there exist \( e, e' \in E \) making \( f(e) = a \neq a' = f(e') \), then we can build a new environment \( e'' \in E \) satisfying: \( a \sim_{i''} a' \succ_{e''} b, \forall b \notin \{a, a'\} \), and according to monotonicity \( \{a, a'\} \subseteq f(e'') \). This is obviously contradictory to the monotonicity.
Case (2). For all \( j \neq i, m_j = (e, a, v), m_i = (e', a', v') \neq (e, a, v) \), and \( a \in F(e) \), define:

\[
h(m) = \begin{cases} 
a' & \text{if } a' \in L(a, e_i), 
a & \text{if } a' \notin L(a, e_i),
\end{cases}
\]

where \( L(a, e_i) = \{ b \in A : a \ R_i b \} \) is the lower contour set of \( R_i \) at \( a \). That is, suppose that all players but one play the same strategy and, given their characteristics, their proposed alternative \( a \) is \( F \)-optimal. Then, the outcome is selected from the worse one among \( a \) and \( a_i \) according to agent \( i \)'s preference \( \succ_i \). As such, no one can gain by deviating from the unanimous strategy.

Case (3). If neither Case (1) nor Case (2) applies, then define

\[
h(m) = a^{i*},
\]

where \( i^* = \max\{ i \in \mathcal{N} : v^i = \max_j v^j \} \). In other words, when neither Case (1) nor Case (2) applies, the outcome is the alternative proposed by the player with the highest index among those whose proposed number is maximum.

Now we show that the mechanism \( (M, h) \) defined above fully Nash implements social choice correspondence \( F \), i.e., \( h(N(e)) = F(e) \) for all \( e \in E \). We first show that \( F(e) \subseteq h(N(e)) \) for all \( e \in E \), i.e., we need to show that for all \( e \in E \) and \( a \in F(e) \), there exists a \( m \in M \) such that \( a = h(m) \) is a Nash equilibrium outcome. To do so, we only need to show that any \( m \) which is given by Case (1) is a Nash equilibrium. Note that \( h(m) = a \) and for any given \( m'_i = (e^i, a^i, v^i) \neq m_i \) by Case (2), we have

\[
h(m'_i, m_{-i}) = \begin{cases} 
a' & \text{if } a' \in L(a, e_i), 
a & \text{if } a' \notin L(a, e_i),
\end{cases}
\]

and thus

\[
h(m) R_i h(m'_i, m_{-i}) \quad \forall m'_i \in M_i.
\]

Hence, \( m \) is a Nash equilibrium.

We now show that for each economic environment \( e \in E \), if \( m \) is a Nash equilibrium at \( e \), then \( h(m) \in F(e) \). First, consider the Nash equilibrium \( m \) given by Case (1) so that \( a \in F(e) \), but the true economic environment is \( e' \), i.e., \( m \in N(e', \Gamma) \) (if it is a Nash equilibrium at \( e \), then proof is immediate). We need to show \( h(m) = a \in F(e') \). Let \( b \in L(a, e_i) \) (so that \( a R_i b \)), and let the new message for \( i \) be \( m'_i = (e, b, v') \). Then \( h(m'_i, m_{-i}) = b \) by Case (2). Now, since \( a \) is a Nash equilibrium outcome with respect to \( e' \), we have \( a = h(m) R_i h(m'_i, m_{-i}) = b \). Thus, we have shown that for all \( i \in N \) and \( b \in A \), \( a R_i b \) implies \( a R'_i b \). Thus, by Maskin monotonicity condition, we have \( a \in F(e') \).

Next, suppose Nash equilibrium \( m \) for \( e' \) is in the Case (2), i.e., for all \( j \neq i, m_j = (e, a, v), a \in F(e) \), and \( m_i \neq (e, a, v) \). Let \( a' = h(m) \). By Case
(3), each \( j \neq i \) can induce any outcome \( b \in A \) by choosing \((R^j, b, v^j)\) with a sufficiently large \( v^j \) (which is greater than \( \max_{k \neq j} v_k \)), as the outcome at \((m'_j, m_{-j})\), i.e., \( b = h(m'_j, m_{-j}) \). Hence, \( m \) is a Nash equilibrium with respect to \( e' \), which implies that for all \( j \neq i \), we have

\[
a' R'_j b.
\]

Thus, by no-veto power assumption, we have \( a' \in F(e') \).

Using the same argument, we can show that if \( m \) is a Nash equilibrium under environment \( e' \) given by Case (3), then we have \( a' \in F(e') \).

Although Maskin monotonicity is very weak and satisfied by many social choice rules, it is violated by some social choice rules so they are not fully Nash implementable. It has already been mentioned that the second-price auction mechanism is not Maskin monotonic, and hence is not fully Nash implementable. The Walrasian allocation, Lindahl allocation, and Pareto-efficient allocation that contain boundary solutions do not satisfy Maskin monotonicity, and thus are not fully Nash implementable (if they just contain interior allocations, then they are fully Nash implementable). In addition, Solomon’s mechanism violates Maskin monotonicity. Since each woman knows who is the real mother, namely it is a situation of complete information, so that Solomon’s solution can be considered with Nash implementation. However, his solution to threaten two women to cut the child into two halves does not satisfy Maskin monotonicity condition, and thus is not fully Nash implementable. What would he judge if the fake mother behaved in the same way as the real mother? In the beginning of this chapter, we formally formed Solomon’s problem by the language of mechanism design.

Two women: Anne and Bets,

Two economies (states): \( E = \{ \alpha, \beta \} \), where

\( \alpha \): Anne is the real mother,

\( \beta \): Bets is the real mother.

Solomon has four alternatives so that the feasible set is given by \( A = \{ a, b, c, d \} \), where

\( a \): Anne gets the baby,

\( b \): Bets gets the baby,

\( c \): Cut the baby in half,

\( d \): Death penalty.

Solomon’s desirability (social goal) is to give the baby to the real mother,
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\[ f(\alpha) = a \text{ if } \alpha \text{ happens,} \]
\[ f(\beta) = b \text{ if } \beta \text{ happens.} \]

Preferences of Anne and Bets:

For Anne,

\[ \begin{align*}
& \text{at state } \alpha, \ a \succ^{\alpha}_A b \succ^{\alpha}_A c \succ^{\alpha}_A d, \\
& \text{at state } \beta, \ a \succ^{\beta}_A c \succ^{\beta}_A b \succ^{\beta}_A d.
\end{align*} \]

For Bets,

\[ \begin{align*}
& \text{at state } \alpha, \ b \succ^{\alpha}_B c \succ^{\alpha}_B a \succ^{\alpha}_B d, \\
& \text{at state } \beta, \ b \succ^{\beta}_B a \succ^{\beta}_B c \succ^{\beta}_B d.
\end{align*} \]

To see why Solomon’s solution does not work, we only need to show his social choice goal is not Maskin monotonic. Notice that for Anne, since

\[ a \succ^{\alpha}_A b, c, d; \]
\[ a \succ^{\beta}_A b, c, d, \]
and \( f(\alpha) = a \), by Maskin’s monotonicity, we should have \( f(\beta) = a \), but we actually have \( f(\beta) = b \). So Solomon’s social choice goal is not fully Nash implementable. As such, one needs to adopt a solution concept that is a refinement of Nash equilibria such as sub-game perfect Nash equilibrium so that the set of equilibria becomes smaller.

It is worth noting that the sufficiency conditions of the above theorems apply only to the case with \( n \geq 3 \). If \( n = 2 \), Maskin monotonicity and NVP may not ensure full Nash implementation. Below we give a counterexample. In the remainder of this section, let \( L_i(\mathbf{a}, \theta) = \{ \mathbf{b} \in A : a \sim_\theta^\alpha \mathbf{b} \} \) be the lower contour set for \( \sim^\theta_\alpha \) at \( a \).

**Example 18.8.3** Suppose that \( A = \{a, b\}, N = \{1, 2\}, E = \{\theta, \phi, \xi\} \), and individual preferences and the social choice rule \( F(\cdot) \) are given below (see in Figure 18.4):

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \phi )</th>
<th>( \xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Individual 1</td>
<td>( b )</td>
<td>( a )</td>
</tr>
<tr>
<td>Individual 2</td>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>( F(\cdot) )</td>
<td>( F(\theta) = {a, b} )</td>
<td>( F(\phi) = {a, b} )</td>
</tr>
</tbody>
</table>

Figure 18.4: A Counterexample
It is easy to verify that $F(\cdot)$ satisfies monotonicity and NVP. Suppose that there exists a mechanism $\Gamma = (M_1 \times M_2, h)$ that can fully Nash complements $f(\cdot)$, then there is $(m_1, m_2) \in \mathcal{N}(\Gamma, \theta)$, $(m'_1, m'_2) \in \mathcal{N}(\Gamma, \phi)$, satisfying $h(m_1, m_2) = h(m'_1, m'_2) = a$. So,

$$(m_1, m_2) \in \mathcal{N}(\Gamma, \theta) \Rightarrow h(M_1, m_2) \subseteq L_1(a, \theta) = L_1(a, \xi) = \{a\},$$
$$(m'_1, m'_2) \in \mathcal{N}(\Gamma, \phi) \Rightarrow h(m'_1, M_2) \subseteq L_2(a, \phi) = L_2(a, \xi) = \{a\}.$$

Therefore, $h(m'_1, m_2) = a$ and $(m'_1, m_2) \in \mathcal{N}(\xi)$. We have $a \in F(\xi)$, a contradiction.

From the above analysis, we see that monotonicity is a necessary but not sufficient condition for full Nash implementation. In cases with three and more participants, Maskin monotonicity and NVP are sufficient but not necessary conditions for full Nash implementation. So, is it possible to find a necessary and sufficient condition for full Nash implementation? Moore and Repullo (Econometrica, 1990) gave such conditions.

**Definition 18.8.5 (conditions $\mu$)** We call condition $\mu$ is satisfied if there is a set $B \subseteq A$ such that for any $i \in N, e \in E, a \in F(e)$, there is a set $C_i = C_i(a, e) \subseteq B$ such that $a \in C_i \subseteq L_i(a, e) \cap B$ and satisfying the following conditions:

(i) if $C_i \subseteq L_i(a, e^*)$, $\forall i \in N$, then $a \in F(e^*)$;

(ii) for some $i$, if $c \in C_i \subseteq L_i(c, e^*)$ and $B \subseteq L_j(c, e^*), \forall j \neq i$, then $c \in F(e^*)$;

(iii) if $c \in B \subseteq L_i(c, e^*), \forall i$, then $c \in F(e^*)$.

**Proposition 18.8.2** Suppose that $n \geq 3$. Then, the necessary and sufficient condition for a social choice rule $F(\cdot)$ to be fully Nash implementable is that it satisfies the condition $\mu$.

**Proof.**

- **Necessity:** Notice that if $F(\cdot)$ can be fully Nash implemented by mechanism $\Gamma = \{\Pi_{i=1}^n M_i, h\}$, then $\forall e \in E, \forall a \in F(e)$, there exists $m(a, e) \in \mathcal{N}(\Gamma, e)$, such that $g(m(a, e)) = a$. Let $B = \{a = g(m) \in A \mid \exists m \in M\}$, and $C_i = C_i(a, e) \equiv g(M_i, m_{-i}(a, e)) \equiv \{a = g(\hat{m}_i, m_{-i}(a, e)) \mid \exists \hat{m}_i \in M_i\}$. Obviously, $a \in C_i \subseteq L_i(a, e) \cap B$.

  a. If $C_i \subseteq L_i(a, e^*) \forall i \in N$, then $m(a, e) \in \mathcal{N}(\Gamma, e^*)$. Since $F(\cdot)$ can be fully Nash implemented, $a = g(m(a, e)) \in g(\mathcal{N}(\Gamma, e^*)) \equiv F(e^*)$. The condition (i) is satisfied.
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b. Suppose that there exists \( c \) satisfying \( c \in C_i(a, e) \subseteq L_i(c, e^*) \) and \( B \subseteq L_j(c, e^*), \forall j \neq i \). Let \( c_i = g(\hat{m}_i, m_{-i}(a, e)) \). Obviously, \( (\hat{m}_i, m_{-i}(a, e)) \in \mathcal{N}(\Gamma, e^*) \), so \( c = g(\hat{m}_i, m_{-i}(a, e)) \in g(\mathcal{N}(\Gamma, e^*)) = F(e^*) \). Then the condition (ii) is satisfied.

c. If there exists \( c \in B \) satisfying \( B \subseteq L_j(c, e^*), \forall i \in N \), then
\[
\begin{align*}
g^{-1}(c) &= \{ m \in M | g(m) = c \} \in \mathcal{N}(\Gamma, e^*) \quad \text{and} \quad c = g(\mathcal{N}(\Gamma, e^*)) = F(e^*) \text{, and then the condition (iii) is satisfied.} 
\end{align*}
\]

- **Sufficiency:** We need to construct a mechanism \( \Gamma = \langle \Pi_{i=1}^{\infty} M_i, g \rangle \) so that \( F(\cdot) \) can be fully Nash implemented by \( \Gamma \), i.e., \( F(e) = g(\mathcal{N}(\Gamma, e)), \forall e \in E \). Let
\[
M_i = \{(e_i, a_i, b_i, v_i) \in E \times A \times B \times N | a_i \in F(e_i) \},
\]

and the outcome function \( g(\cdot) \) is defined as follows:

**Case 1:** If \( m_i \equiv (e, a, b, v), \forall i \in N \), then \( g(m) = a \);

**Case 2:** If there exists \( i \in N \) such that \( m_j \equiv (e, a, b, v), \forall j \neq i \) and \( m_i \neq (e, a, b, v) \), then
\[
g(m) = \begin{cases} 
b_i, & \text{if } b_i \in C_i(a, e) \\
a, & \text{others.} 
\end{cases}
\]

**Case 3:** Other cases, \( g(m) = b_i \) with \( i = \min \{ i \in N : v_i = \max_{j \in N} v_j \} \).

We first show that \( F(e) \subseteq g(\mathcal{N}(\Gamma, e)), \forall e \in E \). For any \( e \in E \), let \( a \in F(e) \) and \( m_i \equiv (e, a, 0), \forall i \in N \). Then \( m \in \mathcal{N}(\Gamma, e) \), \( g(m) = a \), and thus \( F(e) \subseteq g(\mathcal{N}(\Gamma, e)), \forall e \). Hence we have \( g(\mathcal{N}(\Gamma, e^*)) \subseteq F(e^*), \forall e^* \).

If \( m \in \mathcal{N}(\Gamma, e^*) \), there must be one of the following three cases:

a. \( m_i \equiv (e^*, a, b, v), \forall i \in N \). Then \( g(m) = a \in F(e^*) \).

b. \( m_i \neq m_j \equiv (e^*, a, b, v), \forall j \neq i \). Then we must have \( g(m) \in C_i \subseteq L_i(g(m), e^*), B \subseteq L_j(g(m), e^*), j \neq i \). According to property (ii), \( g(m) = F(e^*) \).

c. For \( m_i, i \in N \), at least two agents have different strategies. Then anyone can get the most desirable alternative among \( B \) by reporting the largest possible \( v_i \). So \( B \subseteq L_i(g(m), e^*), \forall i \in N \). According to property (iii), \( g(m) \in F(e^*) \), so \( g(\mathcal{N}(\Gamma, e^*)) \subseteq F(e^*) \).

From all above, we have \( g(\mathcal{N}(\Gamma, e^*)) = F(e^*) \).

Moore and Repullo (Econometrica, 1990) also provided necessary and sufficient conditions for full Nash implementation of any social choice correspondence for any economies with only two agents.
Definition 18.8.6 (Condition $\mu_2$) We say condition $\mu_2$ is satisfied, if condition $\mu$ above is satisfied, and in addition, the following condition holds:

- For any $(a, e', b, \hat{e}) \in (A \times E \times A \times E)$ with $a = F(e'), b = F(\hat{e})$, there is $c = c(a, e', b, \hat{e}) \in C_1(a, e') \cap C_2(b, \hat{e})$ such that for any $e^* \in E$, whenever $c \in C_1(a, e') \subseteq L_1(c, e^*)$ and $c \in C_2(b, \hat{e}) \subseteq L_2(c, e^*)$, we have $c \in F(e^*)$.

Proposition 18.8.3 Suppose that $n = 2$. Then a social choice rule $F(\cdot)$ is fully Nash implementable if and only if $\mu_2$ is satisfied.

18.9 Well-Behaved Mechanism Design

Maskin’s Theorem and Moore-Repullo Theorem give necessary and sufficient conditions for a social choice correspondence to be fully Nash implementable. However, due to the general nature of the social choice rules under consideration, the implementing mechanisms in proving characterization theorems turn out to be quite complex and infeasible. Characterization results show what is possible for the (full) implementation of a social choice rule, but not what is realistic.

Thus, like most characterization results in the literature, the mechanisms given by Maskin and Moore-Repullo are not natural in the sense that it is not continuous; small variations in an agent’s strategy choice may lead to large jumps in the resulting allocations, and further it has a message space of infinite dimension since it includes preference profiles as a component. So unless a very special or small set of preferences is considered, the information processing cost could be very high. As such, although it is possible to characterize such social choice rules that can be fully Nash implementable, the mechanisms tend to be very unrealistic and cannot work in practice. A natural question is that, is there a mechanism that performs well and also is easy to operate in practice? In this section, we give several mechanisms that have some desired properties.

18.9.1 Groves-Ledyard Mechanism

Groves and Ledyard (1977, *Econometrica*) were the first to propose a specific mechanism that has some desired properties and Nash implements Pareto efficient allocations for public goods economies.

To show the basic structure of the Groves-Ledyard mechanism, consider a simplified version here. Public goods economies under consideration have one private good $x_i$, one public good $y$, and three agents ($n = 3$). The production function is given by $y = f(v) = v$.

The mechanism is defined as follows:
• $M_i = R_i$, $i = 1, 2, 3$. Its elements, $m_{ij}$, can be interpreted as the proposed contribution (or tax) that agent $i$ is willing to make.

• $t_i(m) = m_i^2 + 2m_jm_k$: the actual contribution $t_i$ determined by the mechanism with the reported $m_i$.

• $y(m) = (m_1 + m_2 + m_3)^2$: the level of public good $y$.

• $x_i(m) = w_i - t_i(m)$: the consumption of the private good.

Then the mechanism is balanced since

$$
\sum_{i=1}^{3} x_i + \sum_{i=1}^{3} t_i(m) = \sum_{i=1}^{3} x_i + (m_1 + m_2 + m_3)^2 = \sum_{i=3}^{n} x_i + y = \sum_{i=1}^{3} w_i.
$$

The payoff function is given by

$$
v_i(m) = u_i(x_i(m), y(m)) = u_i(w_i - t_i(m), y(m)).
$$

To find a Nash equilibrium, we set

$$
\frac{\partial v_i(m)}{\partial m_i} = 0. \quad (18.9.37)
$$

Then,

$$
\frac{\partial v_i(m)}{\partial m_i} = \frac{\partial u_i}{\partial x_i}(-2m_i) + \frac{\partial u_i}{\partial y}2(m_1 + m_2 + m_3) = 0, \quad (18.9.38)
$$

and thus

$$
\frac{\partial u_i}{\partial x_i} = \frac{m_i}{m_1 + m_2 + m_3}. \quad (18.9.39)
$$

When $u_i$ is quasi-concave, the first order condition turns out to be a sufficient condition for Nash equilibrium.

Making summation, we have at Nash equilibrium that

$$
\sum_{i=1}^{3} \frac{\partial u_i}{\partial y}m_i = \sum_{i=1}^{3} \frac{m_i}{m_1 + m_2 + m_3} = 1 = \frac{1}{f'(v)}. \quad (18.9.40)
$$

that is,

$$
\sum_{i=1}^{3} MRS_{yx_i} = MRTS_{yv}.
$$
Thus, the Lindahl-Samuelson and balanced conditions hold, which means every Nash equilibrium allocation is Pareto efficient.

They claimed that they have solved the free-rider problem in the presence of public goods. However, economic issues are usually complicated. Some agreed that they indeed solved the problem, others did not. The Groves-Ledyard Mechanism has two weaknesses: (1) it is not individually rational: the payoff at a Nash equilibrium may be lower than at the initial endowment, and they are reluctant to participate in this mechanism for resource reallocation; (2) it is not individually feasible: for some non-equilibrium strategies, $x_i(m) = w_i - t_i(m)$ may be negative, and the resources allocated through the mechanism are not within the individual’s consumption space.

Then one may ask: how can we design the incentive mechanism to pursue Pareto efficient and individually rational allocations?

### 18.9.2 Walker’s Mechanism

We know that Lindahl and Walrasian allocations are Pareto efficient and individually rational, so as long as we design mechanisms which fully Nash implements these allocations, they will lead to both Pareto efficiency and individual rationality. Hurwicz (1979a, 1979b) gave such mechanisms under the economic environment of public goods and private goods. Walker (1981) gave a similar mechanism. We will introduce it below.

Again, consider public goods economies with $n$ agents, one private good, and one public good, and the production function is given by $y = f(v) = v$. We assume $n \geq 3$. Suppose that participant $i$’s utility function $u_i(x_i, y)$ is continuously differentiable, quasi-concave and monotonic, and satisfies Inada condition $\frac{\partial u}{\partial x_i}(0) = +\infty$, as well as $\lim_{x_i \to 0} \frac{\partial u}{\partial x_i} x_i = 0$, such that interior solutions exist.

Let us first recall the definition of a Lindahl equilibrium allocation: Allocation $z = (x, y) = (x_1, x_2, \ldots, x_n, y) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ is called a Lindahl equilibrium allocation if it is feasible and there exists a personalized price vector $(q_1, \ldots, q_n) \in \mathbb{R}^n_+$ such that

1. $x_i + q_i y = w_i, \quad i = 1, \ldots, n$;
2. for all $i = 1, \ldots, n$, $u_i(x'_i, y') > u_i(x_i, y)$ implies $x'_i + q_i y' > w_i$;
3. $\sum_{i=1}^n q_i = 1$.

We know it is both Pareto efficient and individually rational. The Walker’s mechanism is defined by:

- $M_i = \mathbb{R}$: $m_i$ is the contribution which participant $i$ is willing to pay;
- $y(m) = \sum_{i=1}^n m_i$: the level of public good.
• \( q_i(m) = \frac{1}{n} + m_{i+1} - m_{i+2} \): personalized price of public good.
• \( t_i(m) = q_i(m)y(m) \): the contribution (tax) made by agent \( i \).
• \( x_i(m) = w_i - t_i(m) = w_i - q_i(m)y(m) \): the private good consumption.

Then, the budget constraint holds:
\[
x_i(m) + q_i(m)y(m) = w_i, \quad \forall m_i \in M_i.
\] (18.9.41)

Making summation, we have
\[
\sum_{i=1}^{n} x_i(m) + \sum_{i=1}^{n} q_i(m)y(m) = \sum_{i=1}^{n} w_i,
\]
and thus
\[
\sum_{i=1}^{n} x_i(m) + y(m) = \sum_{i=1}^{n} w_i,
\]
which means the mechanism is balanced.

The payoff function is
\[
v_i(m) = u_i(x_i, y) = u_i(w_i - q_i(m)y(m), y(m)).
\]

The first order condition for interior allocations is given by
\[
\frac{\partial v_i}{\partial m_i} = - \frac{\partial u_i}{\partial x_i} \left[ \frac{\partial q_i}{\partial m_i} y(m) + q_i(m) \frac{\partial y(m)}{\partial m_i} \right] + \frac{\partial u_i}{\partial y} \frac{\partial y}{\partial m_i} = 0
\]
\[
\Rightarrow \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial y} = 0
\]
\[
\Rightarrow \frac{\partial q_i}{\partial m_i} = q_i(m) \quad \text{(FOC for the Lindahl Allocation)}
\]
\[
\Rightarrow N(e) \subseteq L(e).
\]

Since \( u_i \) is quasi-concave, it is also a sufficient condition for the existence of Lindahl equilibrium. We can also show every Lindahl allocation is a Nash equilibrium allocation, i.e.,

\[
L(e) \subseteq N(e).
\] (18.9.42)

Indeed, suppose that \([x^*, y^*], q_1^*, \ldots, q_n^*\) is a Lindahl equilibrium. Let \( m^* \) be the solution of the following equation:
\[
q_i^* = \frac{1}{n} + m_{i+1} - m_{i+2},
\]
\[
y^* = \sum_{i=1}^{n} m_i.
\]
Then we have \( x_i(m^*) = x_i^* \), \( y(m^*) = y^* \) and \( q_i(m^*) = q_i^* \) for all \( i \in N \).

Thus from \((x(m_i^*, m_{-i}), y(m_i^*, m_{-i})) \in R_+^2 \) and \( x_i(m_i^*, m_{-i}) + q_i(m^*) y(m_i^*, m_{-i}) = w_i \) for all \( i \in N \) and \( m_i \in M_i \), we have \((x_i(m_i^*), y(m_i^*)) R_i (x_i(m_i^*, m_{-i}), y(m_i^*, m_{-i})) \), which means that \( m^* \) is a Nash equilibrium.

Thus, Walker’s mechanism fully implements Lindahl allocations which are Pareto efficient and individually rational.

Walker’s mechanism also has the same shortcoming that it is not feasible either although it does solve the individual rationality problem. The mechanisms of Hurwicz and Walker are still not very satisfactory, because these mechanisms cannot guarantee individual feasibility requirements, i.e., they are not feasible. If a person claims large amounts of \( t_i \), consumption of private good may be negative, i.e., \( x_i = w_i - t_i < 0 \). And mechanism in Huricz (1979a) makes use of a larger dimension of message space. Hurwicz (1979b) give an allocation mechanism that guarantees individual feasibility and results in efficient resource allocation. However, it is not continuous, even small information processing errors can lead to large variations in resource allocation; in addition, it is not budget balanced, that is, resources allocated through the mechanism are more than the total resources of society.

Then we can ask a question whether one can design a mechanism that is both individually feasible and budget balanced. Tian (1991) answers this question by proposing a mechanism that overcomes the defects of Walker mechanism. Tian’s mechanism is individually feasible, balanced, continuous, and further it has a minimal dimension of message space. We will discuss it below.

### 18.9.3 Tian’s Mechanism

In Tian (1991)’s mechanism, everything is the same as all other parts of the Walker mechanism, except that the outcome function for public goods \( y(m) \) is given by (see Figure 18.5):

\[
y(m) = \begin{cases} 
a(m) & \text{if } \sum_{i=1}^n m_i > a(m); \\
\sum_{i=1}^n m_i & \text{if } 0 \leq \sum_{i=1}^n m_i \leq a(m); \\
0 & \text{if } \sum_{i=1}^n m_i < 0,
\end{cases}
\]

where \( a(m) = \min_{i \in N'} \frac{w_i}{q_i(m)} \) can be regarded as the feasible upper bound for having a feasible allocation. Here \( N'(m) = \{ i \in N : q_i(m) > 0 \} \).

An interpretation of this formulation is that if the total contribution that the agents are willing to pay were between zero and the feasible upper bound, the level of public good to be produced would be exactly the total contribution proposed by agents; if the total contribution is less than zero, no public good would be produced; if the total contribution exceeds the feasible upper bound, the level of the public good would be equal to the
feasible upper bound. Thus, one can easily see Tian’s mechanism has all the desired properties. Namely, it is individually feasible, balanced, continuous, and further it has a minimal dimension of message space.

To show that this mechanism fully Nash implements Lindal allocations, we need to assume that preferences are strongly monotonically increasing and convex, and further assume that every interior allocation is preferred to any boundary allocations. That is, for all \( i \in N \), we have \((x_i, y) \succ_P (x'_i, y')\), for all \((x, y) \in R^2_+ \) and \((x', y') \in \partial R^2_+\), where \( \partial R^2_+ \) is the boundary of \( R^2_+ \). This condition guarantees that all Lindahl equilibrium allocation be interior points; otherwise the Lindahl correspondence with the boundary point cannot guarantee that the Maskin monotonicity condition be satisfied, and thus it is impossible to be fully implemented by the feasible mechanism.

To show the equivalence between Nash allocations and Lindahl allocations. We first prove the following lemmas.

**Lemma 18.9.1** If \((x_1(m^*), y_1(m^*)) \in \mathcal{N}(\Gamma , e)\), then \((x_i(m^*), y(m^*)) \in R^2_+\) for all \( i \in N \).

**Proof.** We prove by way of contradiction. Suppose that \((x_i(m^*), y_1(m^*)) \in \partial R^2_+\) for some \( i \in I \). Then \( x_i(m^*) = 0 \) or \( y(m^*) = 0 \). Consider the quadratic equation

\[
y = \frac{w^*}{2(y + c)},
\]

where \( w^* = \min_{i \in N} w_i; c = b + n \sum_{i=1}^{n} |m^*_i| \), where \( b = 1/n \). The larger root of the quadratic equation (18.9.44) is positive and denoted by \( \tilde{y} \). Suppose that player \( i \) chooses his/her message \( m_i = \tilde{y} - \sum_{j \neq i} m^*_j \). Then \( \tilde{y} = \)

![Figure 18.5: The feasible public good outcome function \( y(m) \).](image)
If the mechanism has a Nash equilibrium, then we have
\[ q(x, \mathbf{m}) = \sum_{s=1}^{n} |m_s^*| \tilde{y} \]
\[ = w_j - (\tilde{y} + b + n \sum_{s=1}^{n} |m_s^*|) \tilde{y} \]
\[ = w_j - w^*/2 \geq w_j/2 > 0 \quad (18.9.45) \]

for all \( j \in N \). Thus, \( y(m^*_i, m_{-i}) = \tilde{y} > 0 \) and \( x_j(m^*_i, m_{-i}) = w_j - q_j(m^*_i, m_{-i}) y(m^*, m_{-i}) = w_j - q_j(m^*_i, m_{-i}) \tilde{y} > 0 \) for all \( j \in N \). Thus \((x(m^*_i, m_{-i}), y(m^*_i, m_{-i})) \in \mathcal{N}(\Gamma, e)\) by that every interior allocation is preferred to any boundary allocations, which contradicts the hypothesis \((x^*(m^*), y(m^*)) \in \mathcal{N}(\Gamma, e)\). \( \square \)

**Lemma 18.9.2** If \((x(m^*), y(m^*)) \in \mathcal{N}(\Gamma, e)\), then \(y(m^*)\) is an interior point of \([0, a(m)]\) and thus \(y(m^*) = \sum_{i=1}^{n} m_i^*\).

**Proof.** By Lemma 18.9.1, \(y(m^*) > 0\). So we only need to show \(y(m^*) < a(m^*)\). Suppose, by way of contradiction, that \(y(m^*) = a(m^*)\). Then \(x_j(m^*) = w_j - q_j(m^*) y(m^*) = w_j - q_j(m^*) a(m^*) = w_j - w_j = 0\) for at least some \( j \in N \), which is a contradiction to \(x(m^*) > 0\) by Lemma 18.9.1. \( \square \)

**Proposition 18.9.1** If the mechanism has a Nash equilibrium \(m^*\), then \((x(m^*), y(m^*))\) is a Lindahl allocation with \((q_1(m^*), \ldots, q_n(m^*))\) as the Lindahl price vector, i.e., \(\mathcal{N}(\Gamma, e) \subseteq L(e)\).

**Proof.** Let \(m^*\) be a Nash equilibrium. Now we prove that \((x(m^*), y(m^*))\) is a Lindahl allocation with \((q_1(m^*), \ldots, q_n(m^*))\) as the Lindahl price vector. Since the mechanism is feasible and \(\sum_{i=1}^{n} q_i(m^*) = 1\) as well as \(x_i(m^*) + q_i(m^*) y(m^*) = w_i\) for all \( i \in N \), we only need to show that each individual is maximizing his/her preference. Suppose, by way of contradiction, that there is some \((x_i, y_i) \in \mathbb{R}^2_+\) such that \((x_i, y_i) P_i (x_i(m^*), y(m^*))\) and \(x_i + q_i(m^*) y_i \leq w_i\). Because of monotonicity of preferences, it will be enough to confine ourselves to the case of \(x_i + q_i(m^*) y = w_i\). Let \((x_i, y_i) = (\lambda x_i + (1 - \lambda) x_i(m^*), \lambda y + (1 - \lambda) y(m^*))\). Then by convexity of preferences we have \((x_i, y_i) P_i (x_i(m^*), y(m^*))\) for any \(0 < \lambda < 1\). Also \((x_i, y_i) \in \mathbb{R}^2_+\) and \(x_i + q_i(m^*) y_i = w_i\).

Suppose that player \(i\) chooses his/her message \(m_i = y_i - \sum_{j \neq i}^n m_j^*\). Since \(y(m^*) = \sum_{i=1}^{n} m_i^*\) by Lemma 18.9.2, \(m_i = y_i - y(m^*) + m_i^*\). Thus as \(\lambda \to 0\), \(y_i \to y(m^*)\), and then \(m_i \to m_i^*\). Since \(x_j(m^*) = w_j - q_j(m^*) y(m^*) > 0\) for all \( j \in N \) by Lemma 18.9.1, we have \(w_j - q_j(m_i^* m_{-i}) y_i \to 0\) for all \( j \in N \) when \(\lambda\) is sufficiently small. Thus, \(y(m_i^* m_{-i}) = y_i \), and \(x_i(m_i^* m_{-i}) = w_i - q_i(m_i^* m_{-i}) y_i = x_i\). Since \((x_i, y_i) P_i (x_i(m^*), y(m^*))\), we have \((x_i(m_i^* m_{-i}), y(m_i^* m_{-i})) P_i (x_i(m^*), y(m^*))\). This contradicts to \((x(m^*), y(m^*)) \in \mathcal{N}(\Gamma, e)\). \( \square \)
Proposition 18.9.2 If \((x^*, y^*)\) is a Lindahl allocation with the Lindahl price vector \(q^* = (q_1^*, \ldots, q_n^*)\), then there is a Nash equilibrium \(m^*\) of the mechanism such that \(x_i(m^*) = x_i^*, q_i(m^*) = q_i^*,\) for all \(i \in N\), \(y(m^*) = y^*\), i.e., \(L(e) \subseteq N(\Gamma, e)\).

**Proof.** We need to show that there is a message \(m^*\) such that \((x^*, y^*)\) is a Nash allocation. Let \(m^*\) be the solution of the following linear equations system:

\[
q_i^* = \frac{1}{n} + m_{i+1} - m_{i+2}, \quad y^* = \sum_{i=1}^n m_i
\]

Then we have \(x_i(m^*) = x_i^*, y(m^*) = y^*\) and \(q_i(m^*) = q_i^*\) for all \(i \in N\). From \((x(m_i^*, m_{-i}), y(m_i^*, m_{-i})) \in \mathbb{R}_2^n\) and \(x_i(m_i^*, m_{-i}) + q_i(m^*)y(m_i^*, m_{-i}) = w_i\) for all \(i \in N\) and \(m_i \in M_i\), we have \((x_i(m^*), y(m^*)) R_i (x_i(m_i^*, m_{-i}), y(m_i^*, m_{-i}))\).

Thus, Tian’s mechanism fully Nash implements Lindahl allocations.

It should be pointed out that many studies in mechanism design theory considered the issue of incentive compatibility and the issue of information efficiency separately. The incentive compatibility theory only considers the conditions under which a given goal can be implemented under a solution concept of self-interested behavior, without considering the information requirements of the mechanism. The realization theory on information efficiency to be introduced in the last section of this chapter only considers the amount of information required to achieve a social goal (i.e., the dimension of the message space), while ignoring the incentive problem of the mechanism.

Reichelstein-Reiter (1988) considered both issues simultaneously. They demonstrated that under the condition of Nash incentive compatibility, the amount of information needed to implement a social goal is at least as the amount of information needed to realize the same goal without considering the incentive problem. As for the environment of public goods with only one public good, the minimum dimension of implementing the Lindahl mechanism can be the same as the minimum dimension of realizing the Lindahl allocation. Walker (1981) and Tian (1990, 1991) gave specific incentive mechanisms. However, for the environment of private goods, Reichelstein-Reiter (1988) proved that the minimum dimension of the full implementation of the Walrasian mechanism is greater than the minimum dimension of realizing the Walrasian allocation.

18.10 Other Implementations of Social Choice Rules

Although the above sections give some examples of Nash implementable social goals, and the necessary and sufficient conditions for Nash imple-
mentation, there are many social choice rules that do not satisfy Maskin monotonicity (such as the King Solomon’s problem) and hence are not implementable in Nash equilibrium. There are two ways that can greatly expand the scope of implementable social goals. One is to refine Nash equilibrium solutions. Using Nash equilibrium as a solution concept may lead to multiple Nash equilibrium solutions. The concept of refining Nash equilibrium gives a way to eliminate those undesirable Nash equilibria, which makes the set of Nash equilibrium outcomes smaller and makes it implementable under the solution concept of refining Nash equilibrium. The other is so-called virtual implementation of a social choice rule, which only requires the equilibrium outcome to be arbitrarily close to the social choice correspondence. In addition, the method of the Nash and strong Nash double implementation can make the set of Nash equilibrium outcomes smaller.

18.10.1 Implementation in Refining Nash Equilibrium

What kind of social goals are implementable in refining Nash equilibrium? When it is impossible to make the set $N(\Gamma, e)$ of Nash equilibrium outcomes of any mechanisms be a subset of the set of social goals $F(e)$, the social goal is not Nash implementable. However, since the set of refining Nash equilibrium outcomes can be much smaller than the set of Nash equilibrium outcomes, the former set may be a subset of the set of social goals. Thus, it may be implementable in a refining Nash equilibrium solution. This is indeed the case for several concepts of refinements of Nash equilibria.

A concept of refining Nash equilibrium solution is strong Nash equilibrium. When a strategy is a strong equilibrium, then for any alliance formed by a group of individuals and given the other participant’s strategy, no one in the alliance can benefit more from the coalition. Obviously, the strong Nash equilibrium strategy is a stronger than Nash equilibrium strategy, and every strong Nash equilibrium is Nash equilibrium, whereas the converse may not be true. Therefore, the set of strong Nash equilibrium is a subset of the set of Nash equilibrium. And since the set of strong Nash equilibria is smaller than the set of Nash equilibria, the set of social choice rules implementable by strong Nash equilibrium may be larger. Maskin (1979) proved that: any social choice rule that leads to Pareto optimal allocation and rational allocation for appropriate preferences is implementable in strong Nash equilibrium.

The strategic hypothesis of individuals’ self-interested behaviors can also be subgame perfect Nash equilibrium introduced by Selton or other refinements of Nash equilibrium. In addition, there are many kinds of strategic equilibrium solutions to characterize self-interested behaviors, such as the undominated Nash equilibrium or other refinements of Nash equilibri-
um solutions. Moore-Repullo (1988), Abreu-Sen (1990) and among others proved that almost all social goals are implementable in subgame Nash equilibrium. Palfrey- Srivastava (1991) also proved the same result under the assumption of undominated Nash equilibrium.

18.10.2 Virtual Nash Implementation

We can also expand the scope of implementable social goals by means of virtual Nash implementation. Although the set of Nash equilibrium outcomes $N(\Gamma, e)$ cannot be a subset of social goals $F(e)$, as long as each Nash equilibrium allocation can be arbitrarily close to some allocation in $F(e)$, we say that this social goal is virtually implementable. Matsushima (1988), Abreu-Sen (1991), Tian (1997) and among others proved that almost all social goals are virtually implementable.

18.10.3 Double Implementation in Nash and Strong Nash Equilibrium

Nash equilibrium strategy is a non-cooperative concept of strategic equilibrium, which excludes the possibility of cooperation. Although Nash equilibrium is relatively easy to achieve, it may be unstable: participants tend to respond to the designers through some forms of cooperation that may yield greater benefits. The self-interested behavior assumption in the sense of Nash equilibrium may be unrealistic, and thus it may be more reasonable to adopt the solution concept of strong Nash equilibrium. Since the set of strong Nash equilibrium is obviously a subset of Nash equilibrium, it is more likely to implement a social goal through strong Nash equilibrium.

Although strong Nash equilibrium is more reasonable, it may not exist or be difficult to solve for. In order to be relatively easy to achieve an equilibrium and simultaneously guarantee equilibrium stability, one naturally requires that a social choice be doubly implemented in Nash and strong Nash equilibrium. Suh (1997) gave the necessary and sufficient conditions for a social choice rule to be doubly implementable by Nash and strong Nash equilibrium. Since the characterization results do not consider the complexity of the mechanism, Peleg (1996a, 1996b) and Tian (1999a, 2000a, 2000b, 2000c, 2000d) presented the mechanisms that have some desired properties and doubly implement both Walrasian and Lindahl allocations and some other social choice rules that lead to Pareto optimal and rational allocations.

18.11 Informational Efficiency in Mechanism Design

There are three basic requirements in mechanism design: allocative efficiency, incentive compatibility, and informational efficiency. They are high-
ly desired properties for an economic mechanism to have. Pareto optimality requires that resources be allocated efficiently. Incentive compatibility requires the consistence of individuals’ interests and social goals. Informational efficiency requires that an economic system have the least informational cost of operation. These properties are of fundamental importance in evaluating and choosing economic institutions.

Studies on the incentive compatibility and informational requirements of an economic system resulted in implementation theory and the realization theory, respectively, which together make up the theory of mechanism design originated by Hurwicz (1960, 1972, 1973, 1979a).

Till now, we only considered the design of incentive mechanisms, and we now focus on the informational efficiency of economic mechanisms while ignoring the incentive-compatibility issues. In this section, we discuss the minimal informational size that realize Pareto optimal allocations in terms of the message space size and determines which economic institution is informationally the most efficient.

### 18.11.1 Framework of Information Mechanism Design

From informational point of view, an economic mechanism can be viewed as a process of informational exchange and adjustment; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function that translates messages into outcomes (allocations of resources). Attention may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment.

Decentralized decision-making is essentially of a feature of incomplete information—information is dispersed across decision makers of production and consumption activities. Individuals make decisions about production and consumption through the exchange and transmission of information on economic activities such as demand and supply. The decentralization of information is closely related to the most desirable feature of the competitive market mechanism argued by Smith-Hayek-Friedman.⁷ So, what is the information? What is the formal definition of information

---

⁷There are two most important theorems in microeconomics, namely the first fundamental theorem and the second fundamental theorem of welfare economics discussed in the general equilibrium theory in Part IV which give the relationship between the market mechanism and the Pareto efficiency of resource allocation. The first fundamental theorem of welfare economics states that a perfectly competitive market mechanism leads to a Pareto efficient allocation. It assumes that there are no externalities and that individual preferences satisfy local non-satiation (self-interestedness). The second fundamental theorem of welfare economics states that any Pareto efficient allocation can be achieved by a perfectly competitive market mechanism through redistribution of endowments. But there are also other important assumptions, such as the convexity of individual preferences and the absence of increasing returns to scale in production.
What should it include? In what sense the information cost is viewed as large or small? When discussing these issues, we need a unified model to study what is an economic mechanism. This model is better to include the informational decentralization process, informational centralization process, market-economy mechanism, planned-economy mechanism, as well as possible mixes of these mechanisms. We describe the general model of information adjustment processes below, by which we can study the informational size of an economic mechanism.

Suppose that there are \( n \) participants in an economic society. Each participant can be both a producer and a consumer, can be either a producer or a consumer, and can be a household or a government agency. The set of participants is denoted by \( N \). The production possibility set of a producer is denoted by \( Y_i \). Each consumer has a consumption set, denoted by \( X_i \), and a preference ordering or a utility function (if exists), denoted by \( R_i \) (or \( u_i \)). Each participant \( i \) has an initial endowment vector, denoted by \( w_i \). The economic characteristic of such a participant is denoted by \( e_i = (X_i, w_i, u_i, Y_i) \). As such, the economic society that consists of the characteristics of all participants is called an economy or an economic environment, denoted by \( e = (e_1, e_2, \ldots, e_n) \). The set of all possible economic environments is denoted by \( E \). The set of all possible resource allocations is called the resource allocation space, denoted by \( Z \).

From the perspective of information transmission, an economic mechanism transmits messages from one economic unit to another economic unit; from the perspective of informational physical forms, a message can be a letter, an email, a phone call or an image; from the perspective of quantitative representation of a message, it can be a group of numbers, a vector or a matrix. Mechanism design focuses on reducing the complexity of informational transmission, that is, to make a mechanism operate appropriately while paying the least amount of information cost. The message reported by agent \( i \) is denoted by \( m_i \), all of which consist of agent \( i \)'s message space, denoted by \( M_i \). The message space of a mechanism is thus written as \( M = \prod_{i \in N} M_i \).

To describe the dynamic adjustment process, we denote by \( m(t) = (m_1(t), \ldots, m_n(t)) \) the vector of messages of \( n \) agents at time \( t \). Since each agent constantly adjusts and reports his own message according to the reported messages of the other participants, agent \( i \)'s response at time \( t + 1 \) to the message of time \( t \) can be expressed as the following first-order difference equation:

\[
m_i(t + 1) = \varphi_i(m(t), e), \quad i \in N,
\]

in which \( \varphi_i : E \rightarrow M \) is called the response function that determines the adjustment process of informational responses. Agents shall not adjust their reported messages as long as the process arrives at the stationary state, that
is, \( m \) is the fixed point of the response function such that \( m_i = \varphi_i(m, e) \) for \( i \in N \). When arriving at the terminal time \( T \), the allocation outcome is determined by the outcome function \( h(\cdot) : M \to Z \), that is, \( z = h(m) \).

The informational adjustment process and the resource allocation process determine an economic mechanism, which consists of a message space, a response function and an outcome function, denoted by \( \langle M, \varphi, h \rangle \).

Different from the previous model studying incentive-compatible mechanisms, here we have an additional informational adjustment process, represented by the response function \( \varphi \). The message space defines the possible message that can be reported by each participant according to his characteristic; the response function defines the message of next period of time, which reflects agents’ responses to the messages received at this moment, depends on economic environment \( e \), and also determines the stationary message state; the allocation rule \( h \) determines resource allocation based on the messages reported by all participants.

Any mechanism operates under certain constraints, which are determined by all participants from the government, the legal system and the economic system. Each participant chooses under such constraints messages that benefit him most. The message set may consist of his demand or supply of certain commodity, his preference ordering over alternative commodities, or his description of production cost. The allocation rule determines resource allocation, that is, it translates the process of informational transmission into the allocation process of physical resources. As such, it defines a correspondence from the message space to the resource allocation space, and specifically, it determines social output and individual consumption based on the information collected from individuals, firms and other economic sectors.

The set of stationary points of the response function given in (18.11.46) in fact defines a correspondence from economic environment space \( E \) to message space \( M \), denoted by \( \mu_i : E \to M \), that is \( \mu_i(e) = \{ m \in M : m = m_T \text{ or } m_i = \varphi_i(m, e), i \in N \} \). Thus, the (joint) equilibrium message correspondence \( \mu : E \to M \) is defined by

\[
\mu(e) = \bigcap_{i=1}^{N} \mu_i(e)
\]

which assigns to every economy \( e \) the set of stationary (equilibrium) messages determined by equation (18.11.46). As such, the informational adjustment mechanism can be equivalently defined as \( \langle M, \mu, h \rangle \).

The adjustment process determined by equation (18.11.46) is an informationally-centralized adjustment process as the message reported by agent \( i \) at next period depends on not only his economic characteristic \( e_i \), but also the characteristics of all other participants, \( e_{-i} \). If in an economic mechanism the message reported by each agent just depends on his own economic charac-
18.11. INFORMATIONAL EFFICIENCY IN MECHANISM DESIGN

Teristic, the mechanism is called an *informationally decentralized mechanism*. It is a special case of equation (18.11.46), that is,

\[ m_i(t + 1) = \varphi_i(m(t), e_i), \; i \in N, \]  

(18.11.47)

which defines a *privacy-preserving mechanism* with the (joint) equilibrium message correspondence being \( \mu(e) = \cap_{i=1}^N \mu_i(e_i) \), in which \( \mu_i(e_i) = \{ m \in M : \varphi_i(m, e_i) \} \).

Readers may find such economic mechanisms too abstract. The following discussion about the informational efficiency and uniqueness of competitive market mechanisms may help readers understand the components of the economic mechanism. We shall discuss how to define market mechanism as an informationally decentralized mechanism. The message space of market mechanism consists of prices and exchanges, and it realizes the competitive market equilibrium.

In practice, the message content of communications is usually a vector. So, as long as the adjustment process and informational decentralization are defined, we can evaluate a mechanism in terms of the dimension of the corresponding message space. When considering real-world mechanisms, we find that some mechanisms need to transmit more information than do others. From the viewpoint of informational efficiency, we always hope to realize a social goal using the mechanism that entails the smallest possible operation cost.

If the message space is a general topological space, the notion of informational size can be considered as a concept that characterizes the relative sizes of topological spaces that are used to convey information in the resource allocation process. It would be natural to consider that a space, say \( S \), has more information than the other space \( T \) whenever \( S \) is topologically "larger" than \( T \). This suggests the following definition, which was introduced by Walker (1977).

**Definition 18.11.1** Let \( S \) and \( T \) be two topological spaces. The space \( S \) is said to have as much information as the space \( T \) by the Fréchet ordering, denoted by \( S \preceq_F T \), if \( T \) can be embedded homeomorphically in \( S \), i.e., if there is a subspace of \( S' \) of \( S \) that is homeomorphic to \( T \).

**Definition 18.11.2** Let \( \mathcal{P}(e) \) be a subset of Pareto efficient allocations for \( e \in E \). An allocation mechanism \( \langle M, \mu, h \rangle \) is said to be *non-wasteful* on \( E \) with respect to \( \mathcal{P} \) if for all \( e \in E \), \( \mu(e) \neq \emptyset \) and \( h(m) \in \mathcal{P}(e) \) for all \( m \in \mu(e) \).

**Definition 18.11.3** An informationally decentralized non-wasteful mechanism \( \langle M, \mu, h \rangle \) is said to be *informationally efficient* on \( E \) if the size of its message space \( M \) is the smallest one among all other informationally decentralized non-wasteful mechanisms.
Definition 18.11.4 Let \( \langle M, \mu, h \rangle \) be the message mechanism from the economic environment space \( E \) to the outcome space \( Z \). The performance correspondence of the mechanism, denoted by \( G : E \rightarrow Z \), is defined as:

\[
G(e) = \{ z \in Z : z = h(m), \text{ for some } m \in \mu(e) \}.
\]

Definition 18.11.5 Given a social choice correspondence \( F : E \rightarrow Z \) and a message mechanism \( \langle M, \mu, h \rangle \), the mechanism is said to realize the social choice goal \( F \) if \( G(e) \neq \emptyset \) and \( G(e) \subseteq F(e) \) for any economic environment \( e \in E \). The message mechanism is said to fully realize the social choice goal \( F \) if \( G(e) \neq \emptyset \) and \( G(e) = F(e) \) for all \( e \in E \).

The efficient allocation of resources is a widely accepted social goal, and we know that competitive market mechanism results in Pareto efficient resource allocation when individuals’ preferences are locally non-satiated. We may ask: for neoclassical economic environments (i.e., commodities are divisible, preferences are continuous, monotonic and weakly convex, and production set is closed without exhibiting increasing returns to scale), are there some other informationally decentralized mechanisms (such as the socialist market economic mechanism) that also realize Pareto efficient allocations but are informationally more efficient than the competitive market mechanism? Hurwicz et al. proved in 1970s that the answer to the question is no for pure-exchange neoclassical economic environments. Jordan (1982) further proved that for pure-exchange neoclassical economic environments, competitive market mechanism is the unique mechanism that realizes Pareto efficient allocations and simultaneously uses the least amount of information.

Since pure-exchange economies exclude the possibility of production, these theoretical results are not sufficiently appreciated from the political economic perspective. Whether competitive mechanism is of the unique informational efficiency in economies with production had been an unresolved problem for quite a while. Tian (2006) then proved that for private economies with production, competitive market mechanism is still the unique mechanism that realizes Pareto efficient allocations and simultaneously uses the least amount of information. Since economies with production contain pure-exchange economies as special cases, we in what follows just introduce the general production economies.

18.11.2 Informational Efficiency and Uniqueness of Competitive Market Mechanism

Consider production economies with \( L \) private goods, \( I \) consumers and \( J \) firms so that total number of agents is \( n := I + J \). Consumer \( i \)'s characteristic is given by \( e_i = (X_i, w_i, R_i) \), where \( X_i \subseteq \mathbb{R}^L \), \( w_i \in \mathbb{R}^L^+ \), and \( R_i \) is convex, continuous on \( X_i \), and strictly monotonic on the set of interior
points of $X_i$. Producer $j$’s characteristic is given by $e_j = (Y_j)$. We assume that, for $j = I + 1, \ldots, n$, $Y_j$ is nonempty, closed, convex, contains 0 (possibility of inaction), and $Y_j - R^L_+ \subseteq Y_j$ (free-disposal). We also assume that the economies under consideration have no externalities or public goods.

An economy is the full vector $e = (e_1, \ldots, e_I, e_{I+1}, \ldots, e_N)$ and the set of all such production economies is denoted by $E$ and is called neoclassical production economies. $E$ is assumed to be endowed with the product topology.

Let $x_i$ denote the net increment in commodity holdings (net trade) by consumer $i$ and $y_j$ producer $j$’s (net) output vector, by which we have $x = (x_1, \ldots, x_I)$ and $y = (y_{I+1}, \ldots, y_N)$.

An allocation of the economy $e$ is a vector $z := (x, y) \in R^{NL}$. An allocation $z = (x, y)$ is said to be individually feasible if $x_i + w_i \in X_i$ for $i = 1, \ldots, I$, and $y_j \in Y_j$ for $j = I + 1, \ldots, n$. An allocation $z = (x, y) \in R^{nl}$ is said to be balanced if $\sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j$. An allocation $z = (x, y)$ is said to be feasible if it is balanced and individually feasible for every individual.

An allocation $z = (x, y)$ is said to be Pareto efficient if it is feasible and there does not exist another feasible allocation $z' = (x', y')$ such that $(x'_i + w_i) P_i(x_i + w_i)$ for all $i = 1, \ldots, I$ $(x'_j + w_j) P_j(x_j + w_j)$ for some $i = 1, \ldots, I$. Denote by $P(e)$ the set of all such allocations.

An important characterization of a Pareto optimal allocation is associated with the following concept. Let $\Delta^{L-1} = \{p \in R^L_+ : \sum_{i=1}^L p^i = 1\}$ be the $L - 1$ dimensional unit simplex.

A nonzero vector $p \in \Delta^{L-1}$ is called a vector of efficiency prices for a Pareto optimal allocation $(x, y)$ if

(a) $p \cdot x_i \leq p \cdot x'_i$ for all $i = 1, \ldots, I$ and all $x'_i$ such that $x'_i + w_i \in X_i$ and $(x'_i + w_i) P_i(x_i + w_i)$;

(b) $p \cdot y_j \geq p \cdot y'_j$ for all $y'_j \in Y_j$, $j = I + 1, \ldots, N$.

In the terminology of Debreu (1959, pp. 93), $(x, y)$ is an equilibrium relative to the price system $p$. It is well known that under certain regularity conditions such as convexity, local non-satiation, etc., every Pareto optimal allocation $(x, y)$ has an efficiency price associated with it as shown in Second Theorem of Welfare Economics in Chapter ??.

We also want a mechanism to be individually rational. However, as Hurwicz (1979b) pointed out, it is not quite obvious what the appropriate generalization of the individual rationality concept should be for an economy with production. The following definition of individual rationality of an allocation for an economy with production was introduced by Hurwicz (1979b).

An allocation $z = (x, y)$ is said to be individually rational with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$ if $(x_i + w_i) R_i(\gamma_i(e) + w_i)$ for all
\[ i = 1, \ldots, I. \] Here, \( \gamma_i(e; \theta) \) is given by
\[
\gamma_i(e; \theta) = \frac{p \cdot \sum_{j=I+1}^{N} \theta_{ij}y_j}{p \cdot w_i} \quad i = 1, \ldots, I, \tag{18.11.48}
\]
where \( p \) is an efficiency price vector for \( e \) and the \( \theta_{ij} \) are non-negative fractions such that \( \sum_{j=1}^{I} \theta_{ij} = 1 \) for \( j = I + 1, \ldots, N \), which can be interpreted as the profit shares of consumer \( i \) from producer \( j \). Note that this definition on the individual rationality contains pure exchange as well as constant returns as special cases. Denote by \( I_{\theta}(e) \) the set of all such allocations.

Now we define the competitive equilibrium of a private ownership economy in which the \( i \)-th consumer owns the share \( \theta_{ij} \) of the \( j \)-th producer, and is, consequently, entitled to the corresponding fraction of its profits. Thus, the ownership structure can be denoted by the matrix \( \theta = (\theta_{ij}) \). Denoted by \( \Theta \) the set of all such ownership structures.

An allocation \( z = (x, y) = (x_1, x_2, \ldots, x_I, y_{I+1}, y_{I+2}, \ldots, y_N) \in \mathcal{R}^{IL} \times \mathcal{Y} \) is a \( \theta \)-Walrasian allocation for an economy \( e \) if it is feasible and there is a price vector \( p \in \Delta^{L-1} \) such that
\begin{enumerate}
  \item \( p \cdot x_i = \sum_{j=I+1}^{N} \theta_{ij}p \cdot y_j \) for all \( i = 1, \ldots, I \);
  \item for all \( i = 1, \ldots, I \), \( (x'_i + w_i) P_i (x_i + w_i) \) implies \( p \cdot x'_i > \sum_{j=I+1}^{N} \theta_{ij}p \cdot y_j \); and
  \item \( p \cdot y_j \geq p \cdot y'_j \) for all \( y'_j \in \mathcal{Y}_j \) and \( j = I + 1, \ldots, N \).
\end{enumerate}

Denote by \( W_{\theta}(e) \) the set of all such allocations, and by \( \mathcal{W}_{\theta}(e) \) the set of all such price-allocation pairs \((p, z)\).

Let \( E^c \subseteq E \) be the subset of production economies on which \( W(e) \neq \emptyset \) for all \( e \in E^c \) and call such a subset as the Walrasian production economies.

It may be remarked that, every \( \theta \)-Walrasian allocation is clearly individually rational with respect to the \( \gamma_i(e; \theta) \), and also, by the local non-satiation of preferences, it is Pareto efficient. Thus we have \( W_{\theta}(e) \subseteq I_{\theta}(e) \cap P(e) \) for all \( e \in E^c \).

We now define the Walrasian process that is a privacy-preserving process and realizes the Walrasian correspondence \( W_{\theta} \), and in which messages consist of prices and trades of all agents. In defining the Walrasian process, it is assumed that the private ownership structure matrix \( \theta \) is common knowledge for all the agents.

Define the excess demand correspondence of consumer \( i \), where \( i = 1, \ldots, I \), as \( D_i : \Delta^{L-1} \times \Theta \times \mathcal{R}^I \times E_i \rightarrow \mathcal{R}^L \) by
\[
D_i(p, \theta, \pi_{I+1}, \ldots, \pi_N, e_i) = \{ x_i : x_i + w_i \in X_i, p \cdot x_i = \sum_{j=I+1}^{N} \theta_{ij} \pi_j (x'_i + w_i) P_i (x_i + w_i) \implies p \cdot x'_i > \sum_{j=I+1}^{N} \theta_{ij} \pi_j \}, \tag{18.11.49}
\]
where \( \pi_j \) is the profit of firm \( j \), for each \( j = I + 1, \ldots, N \).
For a given private ownership structure matrix \( S \), contained within a Euclidean space of dimension \( L \), these equations are not independent by Walras’ Law, any Walrasian message is bounded by the allocation \( y \) and \( x \), we consider a special class of economies, denoted by \( \mathcal{E} \).

Remark 18.11.1 For a given private ownership structure matrix \( \theta \), since an element, \( m = (p, x_i, x_{i+1}, y_j, y_{j+1}, y_{j+2}) \in R^L \times R^L, \) of the Walrasian message space \( M \), satisfies the conditions \( \sum_{i=1}^{L} p_i = 1, \sum_{i=1}^{L} x_i = \sum_{j=I+1}^{N} y_j, \) and \( p \cdot x_i = \sum_{j=I+1}^{N} \theta_{ij} p \cdot y_j \) for each \( i = 1, \ldots, I \), and one of these equations is not independent by Walras’ Law, any Walrasian message is bounded by the allocation \( y \) and \( x \), we consider a special class of economies, denoted by \( \mathcal{E} \).

To establish the informational efficiency of the competitive mechanism, we consider a special class of economies, denoted by \( \mathcal{E} \).
where preference orderings are characterized by Cobb-Douglas utility functions, and efficient production technology are characterized by quadratic functions.

For \( i = 1, \ldots, I \), consumer \( i \)'s admissible economic characteristics in \( E^c_i \) are given by the set of all \( e_i = (X_i, w_i, R_i) \) such that \( X_i = R_i^L, w_i > 0 \), and \( R_i \) is represented by a Cobb-Douglas utility function \( u(\cdot, a_i) \) with \( a_i \in \Delta^{L-1} \) such that \( u(x_i + w_i, a_i) = \prod_{l=1}^{L} (x_i^l + w_i^l)^{a_l} \).

For \( i = I + 1, \ldots, N \), producer \( i \)'s admissible economic characteristics are given by the set of all \( e_i = Y_i = Y_i(b_i) \) such that

\[
Y_i(b_i) = \{ y_i \in \mathbb{R}^L : b_i^I y_i^I + \sum_{l=2}^{L} (y_i^l + b_l^I/2)(y_i^l)^2 \leq 0, \ -1/b_i^I \leq y_i^I \leq 0 \text{ for all } l \neq 1 \},
\]

where \( b_i = (b_1^I, \ldots, b_L^I) \) with \( b_l^I > \frac{1}{y_i^I} \). It is clear that any economy in \( E^c \) is fully specified by the parameters \( a = (a_1, \ldots, a_I) \) and \( b = (b_{I+1}, \ldots, b_N) \). Furthermore, production sets are nonempty, closed, and convex by noting that \( 0 \in Y(b_j) \) and their efficient points are represented by quadratic production functions in which \((y_2^I, \ldots, y_L^I)\) are inputs and \( y_1^I \) is possibly an output.

Given an initial endowment \( \bar{w} \in \mathbb{R}^{I+L}_+ \), define a subset \( E^{eq} \) of \( E^c \) by

\[
\bar{E}^{eq} = \{ e \in E^c : w_i = \bar{w}, \forall i = 1, \ldots, I \}. \]

That is, endowments are constant over \( \bar{E}^{eq} \). Let \( E^e \) be the set of all production economies in which Walrasian equilibrium exists.

We then have the following theorem that shows that the Walrasian mechanism is informationally efficient among all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the set \( E^e \).

**Theorem 18.11.1 (Informational Efficiency Theorem, Tian, 2006)** Suppose that \( (M, \mu, h) \) is an allocation mechanism defined on the class of production economies \( E^e \) such that:

(i) are informationally decentralized;
(ii) are non-wasteful with respect to \( P \);
(iii) have Hausdorff topological message spaces;
(iv) satisfy the local threadedness property at some point \( e \in \bar{E}^{eq} \).

\(^8\)A stationary message correspondence \( \mu \) is said to be locally threaded at e if there is a neighborhood \( N(e) \subseteq E \) and a continuous function \( f : N(e) \to M \) such that \( f(e') \in \mu(e') \) for all \( e' \in N(e) \). The stationary message correspondence \( \mu \) is said to be locally threaded on \( E \) if it is locally threaded at every \( e \in E \).
Then the Walrasian allocation mechanism \( \langle M_\text{c}, \mu_\text{c}, h_\text{c} \rangle \) is informationally efficient among all allocation mechanisms \( \langle M, \mu, h \rangle \) on \( E^c \). That is, \( M_\text{c} = F \mathcal{R}^{(L-1)I + LJ} \leq_F M \).

This theorem establishes the informational efficiency of the competitive mechanism within the class of all smooth resource allocation mechanisms which are informationally decentralized and non-wasteful over the class of Walrasian production economies \( E^c \).

The following theorem further shows that the competitive allocation process is the unique most informationally efficient decentralized mechanism for production economies among mechanisms that achieve Pareto optimal and individually rational allocations.

**Theorem 18.11.2 (The Uniqueness Theorem, Tian, 2006)** Suppose that \( \langle M, \mu, h \rangle \) is an allocation mechanism on the class of production economies \( E^{cq} \) such that:

1. it is informationally decentralized;
2. it is non-wasteful with respect to \( \mathcal{P} \);
3. it is individually rational with respect to the fixed share guarantee structure \( \gamma_i(e; \theta) \);
4. \( M \) is a \((L - 1)I + LJ\) dimensional manifold;
5. \( \mu \) is a continuous function on \( E^{cq} \).

Then, there is a homeomorphism \( \phi \) on \( \mu(E^{cq}) \) to \( M_\text{c} \) such that

(a) \( \mu_\text{c} = \phi \cdot \mu \);

(b) \( h_\text{c} \cdot \phi = h \).

The conclusion of the theorem is summarized in the following commutative homeomorphism diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\mu_\text{c}} & M_\text{c} \\
\downarrow{\mu} & \nearrow{\phi} & \downarrow{h_\text{c}} \\
\mu(E) & \xrightarrow{h} & Z
\end{array}
\]

Thus, the above theorem shows that the competitive mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies \( E^c \). For a mechanism \( \langle M, \mu, h \rangle \), when there exists no homeomorphism \( \phi \) on \( \mu(E^{cq}) \) to \( M_\text{c} \) such that (1) \( \mu_\text{c} = \phi \cdot \mu \) and (2) \( h_\text{c} \cdot \phi = h \), we may call such a mechanism \( \langle M, \mu, h \rangle \) a non-Walrasian allocation mechanism. The Uniqueness Theorem then implies that any non-Walrasian allocation mechanism defined on \( E^{cq} \) must use a larger message space.
For public goods economies, Tian (2000e) obtained the similar results, and showed that Lindahl mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies that guarantees the existence of Lindahl equilibrium. For more discussions about mechanism design of informational efficiency in public goods economies, readers may refer to Tian (1994a).

We thus have the following important conclusion: regardless of the planned-economy mechanism, state-owned economy, collective economy, mixed economy, or any other non-market economic institutions, in order to realize efficient resource allocations, they must use more information than does the competitive mechanism, and hence they are not informationally efficient. The implication is thus that as long as competitive mechanism can realize Pareto efficient allocations, allocations should be done by market. Only when competitive market fails will some other mechanisms be designed to complement the market. This result lays out an important theoretical foundation that helps explain why a transition economy must let market play the fundamental and decisive role, and let private economy rather than the state-owned economy play the dominant role in economic resource allocation. By defining competitive mechanism as an informationally decentralized economic mechanism, the message space of market mechanism consists of two vectors: one is price vector and the other is resource allocation vector.

When adopting the command planned-economy mechanism, firms must report and transmit various information, including their production functions (technology conditions and production capabilities), to the central planners. For example, if production functions are given by polynomial functions, they may be of an arbitrarily high degree, and hence the dimension of the message space reported could be arbitrarily large. The central planners also need relevant information from the demand side of consumers. As heterogeneous consumers may have different preferences and hence different utility functions, this also makes the dimension of message space facing the planners be arbitrarily large and be even infinity. As a result, in order to make production and consumption arrangements, the central planners must deal with a huge amount of information, resulting in quite high operation cost.

On the other hand, for an economic mechanism facing a message space of a small dimension, the allocation rule might become highly complicated. It may turn out that such a mechanism entails a total operation cost even higher than those mechanisms with message spaces of much higher dimensions. Nevertheless, studies on searching for the smallest message space of a mechanism are worthwhile because they enable us to identify the associated informational requirements or operation cost. In addition, they are useful for exploring other aspects of mechanisms, such as the complexity
of mechanisms.

Moreover, for general non-neoclassical economic environments (such as indivisible commodities, and non-convex preference ordering or production set), do there exist informationally decentralized mechanisms that realize efficient resource allocations? If yes, what is the relationship between the mechanisms and the corresponding informational size? Hurwicz and among others proved the existence of such mechanisms for quite general economic environments. However, these mechanisms entail quite high informational cost of operation. Calsamiglia (1977) proved for a class of non-classical economic environments, especially non-convex economic environments, that a message space of an infinite dimension is required for realizing Pareto optimal allocations.

18.12 Biographies

18.12.1 Leonid Hurwicz

Leonid Hurwicz (1917-2008), the father of mechanism design theory, was awarded the Nobel Memorial Prize in Economic Sciences in 2007 for “having laid the foundations of mechanism design theory.” Despite his pioneering achievements in many areas of economics, Hurwicz did not earn any degree in economics.

Hurwicz was born in a Jew family in Poland in 1917 and came to the United States during the World War II. The highest degree he received is a law degree equivalent to a master’s degree in Poland. In the outbreak of World War II, Hurwicz was in Switzerland. He did not return to Poland but went to the United States. He said, “If I stay in Poland, I could very well be a victim of the Auschwitz concentration camp.” After coming to the United States, he did not go to get a doctoral degree, but instead became an assistant for Paul Samuelson, and then he was promoted directly to a full professor from an assistant professor.

As early as in 1960s of the 20th century, Hurwicz had begun to think about research projects derived from general equilibrium theory, social choice theory and game theory. In the early 1960s, his article entitled “Optimality and Informational Efficiency in Resource Allocation Processes” kicked off the prelude to mechanism design theory. Hurwicz’s mechanism design theory is closely linked with major economic issues such as market regulation, market screening and public goods provision. Subsequently, many economists have involved in this research. Both Myerson and Maskin, who were young at the time, were active members.

Later, Hurwicz wrote the well-known papers such as “Revealed Preference without Demand Continuity Assumptions” and “On the Concept and Possibility of Informational Decentralization”, and gradually improved the
CHAPTER 18. MECHANISM DESIGN WITH COMPLETE INFORMATION

Theoretical basis. In 1973, Hurwicz published the paper “The Design of Mechanisms for Resource Allocation” in American Economic Review, which proposed the core issue in the theoretical framework of mechanism design - incentive compatibility, and laid the framework of the mechanism design theory.

The theory of mechanism design can be regarded as the most important development in economics science in the past half century. It aims to systematically analyze resource allocation institution and process, reveal the important role of information, communication, incentives and economic agents’ processing capabilities in decentralized resource allocation, and enables us to identify sources of market failure. In fact, many field of modern economics are influenced by the mechanism design theory pioneered by Hurwicz. His “incentive compatibility” has now become a core concept in economics. The two most objective realities in the real world are the individual’s self-interested behavior and asymmetric nature of individual private information. In simple terms, incentive compatibility means that through the design of rules of game, let other individuals do what a designer wants to do. Information asymmetry can lead to market failure, so some incentive mechanism should be designed to induce individuals’ right incentives.

Hurwicz had close connection with Lange, a representative of the Lausanne school, and Hayek, a representative of the Austrian school. His work on mechanism design theory was also influenced by these two aspects of economic theory and methodology, and rigorously unifies these two seemingly very different schools of thought into mechanism design theory. Hurwicz also interacted academically with representatives of the new institutional economics group, such as Douglass C. North (1920-2015, whose biography was seen in Section 15.7.2), trying to establish a dialogue channel and a connection path between the two theories.

Hurwicz also did a lot of other pioneering work: in the late 1940s, he laid the groundwork for the identification problem of dynamic econometric models; as early as 1947, he first proposed and defined the concept of rational expectation that is the most critical solution concept in neoclassical macroeconomics. The school of rational expectations has become the mainstream of today’s macroeconomics; Hurwicz also made important work on integrability, i.e., deriving the utility functions from the demand function, which is an important result from the perspective of political economy. Traditional political economics believes that utility is an idealistic concept. It does not exist. He and Kenneth J. Arrow also made pioneering work on the stability of the general equilibrium of competitive markets.

In addition, Hurwicz attaches great importance on the preciseness of expressing economic problems. According to Hurwicz, one of the biggest problems in many traditional economic theories is that it is too arbitrary to express concepts. The greatest significance of the axiomatic approach
lies in the clarity of the expression of theory, which makes the discussion and criticism have a universal research paradigm and analysis framework. This is also a strong feeling he got from the "socialist controversy" — the debate between Mises-Hayek and Lange-Lerner in twenties and thirties of the last century, because he felt that the two sides of the discussion are incommensurable in theoretical expressions. According to Hurwicz, the study of the economic system can adopt either an empirical scientific approach or a normative scientific approach. Whatever the method is, as long as it is analytical, a crucial first step is to carry out an standardized concept of economy. Thus, it also prompted him to create theory of mechanism design that allows different mechanisms to be compared under a unified framework.

On April 14, 2007, the University of Minnesota hosted the 90th Birthday Party for Professor Leonid Hurwicz. This became a gathering of masters of economics, including Eric Maskin and Roger Myerson who won Nobel Memorial Prize in Economic Sciences with Hurwicz in 2007, and Arrow who won the 1972 Nobel Memorial Prize in Economic Sciences, and his student McFadden who won the 2000 Nobel Memorial Prize in Economic Sciences and many other economists. They come together to celebrate the great birthday. Six months later, Hurwicz and Eric Maskin and Roger Myerson jointly won the Nobel Memorial Prize in Economic Sciences for their contribution to the mechanism design theory. At the same time, it also created a record that Hurwicz became the oldest Nobel Prize winner in history. After learning that he had won the Nobel Memorial Prize in Economic Sciences, with a sense of humor for his later life, he said, “I thought my time had passed. For the Nobel Prize, I am really too old. But this bonus is indeed good for a retired old man.”

18.12.2 Eric S. Maskin

Eric S. Maskin (1950—), a professor of economics at Harvard University, has made outstanding contributions to many areas in modern economics, especially in social choice theory, game theory, incentives and information theory, and mechanism design. With his profound theoretical contributions, rigorous academic attitude and outstanding contributions to the training of advanced research talents in economics, Maskin has been regarded as the most prestigious economist in the current economics profession. His research has profound influence on economics, political science and law. He was awarded the Nobel Memorial Prize in Economic Sciences in 2007 for having laid the foundations of mechanism design theory.

When Maskin was a graduate student at Harvard University, he became interested in information economics after taking the course taught by Kenneth Arrow, and later he completed his doctoral dissertation under Arrow’s guidance. Maskin’s early research focused on the implementation theory.
This theory emphasizes how to design an incentive mechanism such that individual rationality is compatible with collective rationality.

The most outstanding contribution of Maskin is to introduce game theory into mechanism design. Before him, the most important scholar in mechanism design was Leonid Hurwicz. Researchers previously focused on the mechanism design of central planners from the perspective of information efficiency. Maskin developed an incentive mechanism design for any social goal, that is, even if individuals aim to pursue their own benefit, it also will achieve the social goal under this mechanism. In 1977, Maskin completed the paper *Nash Equilibrium and Welfare Optimality*, but this article was not officially published until 1999. This paper is a milestone in mechanism design, in which Maskin proposed and proved the sufficient and necessary conditions of implementation in Nash Equilibrium, and one of the constructed necessary conditions is called Maskin Monotonicity, which was widely circulated. The contribution of Maskin to the field of industrial organization is as follows: he and Tirole jointly initiated the game theory analytical framework of industrial organization, thus fully demonstrating the strong power of game theory.

His researches on implementation theory has become one of the important cornerstones of mechanism design. Also, Maskin has also made important contributions to repeated games, political economy and comparing different voting systems. He brought soft budget constraint, a concept widely used in the analysis of socialist economies, into the study of market economies, which has a great influence on the study of comparing different economic systems. Maskin has published more than 100 articles in international economics journals, nearly 40 of which are published in the top economics journals.

### 18.13 Exercises

**Exercise 18.1** Suppose that $X$ is a set of alternatives, and $R = R_1 \times \cdots \times R_n$ is the set of preference profiles defined on $X$. Answer questions in response to the following statement:

A social choice function $f : R \rightarrow X$ is partially implementable in dominant strategies if and only if the social choice function is truthfully implementable by a revelation mechanism in dominant strategies.

1. Give the formal definitions of the following terms: (1) dominant strategy mechanism; (2) the revelation mechanism; (3) dominant-strategy incentive compatible revelation mechanism; (4) truthful implementation in dominant strategies.

2. State whether the above statement is correct or not. If correct, prove it; otherwise, give a counter-example.
Exercise 18.2 With a restricted domain, the Gibbard-Satterthwaite Impossibility Theorem may not be true. Justify your answer.

Exercise 18.3 (Majority Voting with Two Candidates) Suppose that $N$ voters cast their ballots on two candidates $a$ and $b$, and each voter stated his or her preferences on the candidates for whom he supports. The winner will be determined by the following simple majority rule:

$$F(e) = \begin{cases} 
    a, & \text{if } \#\{i \in I | a \succ_i b\} > \#\{i \in I | b \succ_i a\}, \\
    b, & \text{if } \#\{i \in I | a \succ_i b\} \leq \#\{i \in I | b \succ_i a\}.
\end{cases}$$

1. Prove that the simple majority rule is strategy-proof, that is, every voter will truthfully report his preference, regardless of whether others are telling the truth or not, so that $F(\cdot)$ is truthfully implementable in dominant strategies.

2. Will this conclusion be true for more than two candidates?

Exercise 18.4 Consider pure exchange economic environments with two individuals and two goods. The consumption spaces are $X_1 = X_2 = \mathbb{R}^2_+$. For every $i = 1, 2$ and $e^i \in E^i = \{u^i_1, u^i_2\}$, the utility functions are given by

$$u^i_{ci}(x^1_{i}, x^2_{i}) = x^1_{i} x^2_{i},$$
$$u^i_{li}(x^1_{i}, x^2_{i}) = x^1_{i} + 2x^2_{i}.$$  

Then, for such economic environments $E = E^1 \times E^2$, there are four possible economies. For every economy, the endowment is fixed, namely, $\omega^1 = (1, 0), \omega^2 = (1, 2)$.

1. Find the set of all Pareto efficient allocations, and show the sets of individual rational and Pareto efficient allocations as well as their intersection in graph.

2. Prove that for this economic environment, there is no incentive-compatible direct mechanism that truthfully implements individually rational and Pareto efficient allocations.

Exercise 18.5 Suppose that the Clark (pivotal) mechanism is used to determine whether a public good is provided. It is known that individuals $a, b, c$ report the net values of the public good as 3, 5, $-7$, respectively.

1. Should the public good be provided?

2. Who is the pivotal agent under this mechanism?

3. What should be the transfer to each individual?
4. If the true value of the public good to consumer $b$ is 3, will he still keep reporting 5? Why?

5. Is the pivotal mechanism balanced?

Exercise 18.6 Prove that for continuous public goods economies with private value, under the Vickrey-Clark-Groves mechanism, truth-telling
\[(b_1(y), b_2(y), \ldots, b_n(y)) = (v_1(y), v_2(y), \ldots, v_n(y))\]
is a dominant strategy. Thus, this mechanism truthfully implements the efficient social rule in the dominant equilibrium $y^*(\cdot)$.

Exercise 18.7 Suppose that there are two production technologies that can abate pollution, each owned by firm $i$, $i \in \{1, 2\}$. The production technology set is convex, and relies on private information $\theta_i$. For simplicity, we assume that the cost function is $C_i(q_i) = \frac{1}{2} \theta_i q_i^2$, in which $q_i$ is the amount of pollution that is abated by firm $i$. At the same time, we assume that marginal utility is of a linear form, $MU(q) = 1 - 2q$. The total abatement of pollution is $q = q_1 + q_2$.

1. Find the efficient amount of pollution abated by the two firms, and it is a function of $(\theta_i, \theta_j)$.

2. Find the efficient level of transfers in the pivotal mechanism.

Exercise 18.8 Consider the provision of public goods. The two towns, on the banks of a river, decide whether to build a bridge between the two towns at cost $c$ with $1 > c > 0$. Let $\theta_i$ be the number of residents in town $i$ who intend to use bridges. Suppose that the prior distribution of $\theta_i$ is a uniform distribution on $[0, 1]$, and $\theta_1$ is independent of $\theta_2$. After crossing the bridge to the town on the other side of the river, the residents of town $i$ will have an effect $\gamma \theta_i$ on the residents of town $j$, which can be either a positive or a negative effect. Therefore, for town $i$, the total value of building bridges is $\theta_i + \gamma \theta_j$.

1. For this problem, what is the VCG mechanism? Find the ex post equilibrium associated with it.

2. What is the social choice rule that is implementable in dominant strategies?

Exercise 18.9 Consider a public good economy with individuals $a, b, c$ and the public good $y$. The utility function of individual $i$ is given by $u_i(t_i, y) = t_i + v_i(y) = \bar{\theta}_i \ln y - y$, where $t_i$ is the transfer to individual $i$, and the true $\bar{\theta}_i$ is unknown to the designer.
1. Find the VCG mechanism $t_a(\theta), t_b(\theta), t_c(\theta), y(\theta)$ where $\theta_i$ is the reporting of the true value $\bar{\theta}_i$ for individual $i$.

2. Find the Clark mechanism (the pivotal mechanism).

Exercise 18.10 Suppose that $v(\cdot, \theta_i)$ is twice continuously differentiable, and for $\theta_i \in [\bar{\theta}_i, \underline{\theta}_i]$, $\partial^2 v_i(y, \theta_i)/\partial y^2 < 0$, and $\partial^2 v_i(y, \theta_i)/\partial y \partial \theta_i > 0$. Prove that a continuously differentiable social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if $y(\theta)$ is the nondecreasing function of $\theta_i$ and

$$t_i(\theta_i, \theta_{-i}) = t_i(\bar{\theta}_i, \theta_{-i}) - \int_{\bar{\theta}_i}^{\theta_i} \frac{\partial v_i(y(s, \theta_{-i}), s)}{\partial y} \frac{\partial y(s, \theta_{-i})}{\partial s} ds.$$ 

Exercise 18.11 Prove that, if the domain of agents’ preferences for public goods is unrestricted (that is, any utility function $v_i : D \to \mathbb{R}$ is possible), then the VCG mechanism is the only one that can truthfully implement the efficient provision of public goods.

Exercise 18.12 Individuals 1 and 2 will decide on the ownership of an object. Consider the following mechanism: individuals 1 and 2 report prices, $p_1$ and $p_2$, separately. If $p_1 > p_2$, then individual 1 will get the object and pay $p_2$ to individual 2; if $p_1 < p_2$, then individual 2 will get the object and pay $p_1$ to individual 1; if $p_1 = p_2$, then individual 1 and 2 have half the chance to get the object and pay $p_1$ to the other. Answer the following questions:

1. Is this mechanism incentive compatible in dominant strategies?

2. Is the mechanism individually rational?

3. Is the mechanism budget balanced?

Exercise 18.13 Consider the quasi-linear economic environments with $n$ participants and a public good $y$. Let $y^*(\cdot)$ be an ex post Pareto efficient allocation mechanism, and define $V^*(\theta) = \sum_i v_i(y^*(\theta), \theta_i)$.

1. Prove that the necessary and sufficient condition for the existence of ex post Pareto efficient social choice function that can be truthfully implemented in dominant strategies is that $V^*(\cdot) = \sum_i V_i(\theta_{-i})$, where $V_i(\cdot)$ is a function of $\theta_{-i}$ only for all $i$.

2. Using the above conclusion, prove that when $n = 3$, $Y = \mathbb{R}, \Theta_i = \mathbb{R}_+$, and $v_i(y, \theta) = \theta_i y - \frac{1}{2} y^2$ for all $i$, there is a Pareto efficient social choice function that is truthfully implementable in dominant strategies.
3. Now suppose that $v_i(y, \theta_i)$ are $n$-th continuously differentiable, $\frac{\partial^2 v_i}{\partial y^2} < 0$, and $\frac{\partial^2 v_i}{\partial y \partial \theta_i} \neq 0$ for all $(y, \theta_i) \in \mathcal{R}_+ \times \mathcal{R}_+$. Prove that a necessary and sufficient condition for the existence of an ex post Pareto efficient social choice function is: for any $\theta$,$$
abla^n V^*(\theta) = 0.$$4. Using the conclusion in question (3) to verify that when $n = 2$, there is no ex post Pareto efficient social choice function that is truthfully implementable in dominant strategies.

Exercise 18.14 Consider the economy consisting of $n \geq 3$ individuals and public good $y$. The utility function of individual $i$ is $u_i(t_i, y) = t_i + v_i(y)$ with $v_i(y) = -1/2y^2 + \bar{\theta}_i y$.

1. Find the VCG mechanism.
2. Prove that if $d_i(\theta_{-i}) = \frac{1}{2n} \sum_{j \neq i} \theta_j^2 + \frac{n-1}{2n(n-2)} \sum_{j \neq i} \sum_{k \neq j, i} \theta_j \theta_k$, then the VCG mechanism is balanced.
3. Is the VCG mechanism Pareto efficient?

Exercise 18.15 (Jackson, 2003) Consider a public good provision problem with two participants. Let $I = \{1, 2\}$, and the type of each participant be distributed as $\theta_i \in \{0, 1\}, \forall i = 1, 2$. The decision of public good provision is discrete $Y = \{0, 1\}$. Assume that the cost for providing the public good $y = 1$ is $c = \frac{3}{2}$. The utility functions of two participants are quasi-linear, namely $u_i(y, t_i, \theta_i) = y \theta_i - t_i$, in which $y \in \{0, 1\}$ is the provision decision and $t_i$ is the transfer for participant $i$. $(y(\theta), t_i(\theta), i \in I)$ is budget balanced if satisfying:$$\sum_{i \in I} t_i(\theta) = 0, \forall \theta.$$$(y(\theta), t_i(\theta), i \in I)$ is individually rational if satisfying:$$u_i(y(\theta), t_i(\theta)) \geq 0, \forall \theta, \forall i \in I.$$

1. Prove that the VCG mechanism is not budget balanced for this economy.
2. Prove that the VCG mechanism does not satisfy individual rationality.
Exercise 18.16 (Non-Implementability by Induced Revelation Mechanism)
The textbook points out: although the original general mechanism (fully) implements a social choice correspondence \( F \), the revelation mechanism induced, \( \langle E, g \rangle \), may only partially implement, but not fully implement nor implement \( F \). Now consider the example given by Dasgupta, Hammond, and Maskin (1979). Suppose that the set of alternatives is \( A = \{a, b, c, d, e, p, q, r\} \), the sets of preferences of individuals 1 and 2 are denoted by \( R_1 = \{R_1, R'_1\} \) and \( R_2 = \{R_2, R'_2\} \), respectively, which are described as follows:

\[
R_1 = \begin{bmatrix}
q \\
a - c - e \\
d - b - p \\
r
\end{bmatrix}, \quad R'_1 = \begin{bmatrix}
c - b - p \\
a - d - e \\
q - r \\
r
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
r \\
d - a - e \\
b - c - p \\
q
\end{bmatrix}, \quad R'_2 = \begin{bmatrix}
d \\
b - c \\
e - p - q - r \\
a
\end{bmatrix}.
\]

Consider a social choice rule defined as follows:

\[
F(R_1, R_2) = \{a, e\}; \\
F(R'_1, R_2) = \{c, p, b\}; \\
F(R_1, R'_2) = \{d\}; \\
F(R'_1, R'_2) = \{b\}.
\]

1. Prove that social choice rule \( F \) satisfies Maskin monotonicity and Pareto efficiency.
2. Prove that \( F \) can be implemented in dominant strategies by \( g_1 \) that is defined by the following table, in which the rows are the strategic choices of participant 1 and the columns are the strategic choices of participant 2.

\[
\begin{array}{ccc}
a & d & e \\
c & b & p \\
a & b & e \\
\end{array}
\]

3. Prove that \( f \) cannot be implemented in dominant strategies by the following direct mechanism \( g_2 \) that is defined by the following table.

\[
\begin{array}{ccc}
R_1 & R_2 & R'_2 \\
\hline
R_1 & a & d \\
R'_1 & c & b \\
\end{array}
\]

4. Prove that the equilibrium outcome that cannot be implemented by a direct mechanism is not Pareto efficient.

Exercise 18.17 Prove or disprove whether the following social choice correspondences satisfy Maskin monotonicity.
1. Weakly Pareto efficient correspondence.

2. Condorcet correspondence.

3. Individually rational allocation correspondence.

4. Walrasian correspondence.

5. Lindahl correspondence.


7. Constrained Lindahl correspondence.

8. For the above social choice correspondences that do not satisfy Maskin monotonicity, what kind of restriction on the domain can make it satisfy Maskin monotonicity? Prove your claim.

Exercise 18.18 Prove or disprove whether the following social choice correspondences satisfy the no-veto power property.

1. Weakly Pareto efficient correspondence.

2. Condorcet correspondence.

3. Individually rational allocation correspondence.

4. Walrasian correspondence.

5. Lindahl correspondence.


7. Constrained Lindahl correspondence.

Exercise 18.19 In the King Solomon example, as the social goal does not satisfy Maskin monotonicity, it is not implementable in Nash equilibrium. Can King Solomon’s goal be implementable in subgame perfect equilibrium or virtually Nash implementable? Give your ideas.

Exercise 18.20 If the no-veto power property is not satisfied, Maskin monotonicity does not guarantee that the social choice rule be fully Nash implemented. Illustrate this conclusion with an example.

Exercise 18.21 Is the non-veto power condition a necessary condition for a social choice rule to be fully Nash implementable? If so, give the proof; if not, give a counterexample.
Exercise 18.22 (Borda rule) There are \( N \) individuals in an economy. Suppose that \( X \) contains a finite number of alternatives. For any alternative \( x \), \( B_i(x) \) denotes the number of alternatives that is not as good as \( x \) in \( X \) for individual \( i \). Borda rule is defined as:

\[
x R y \iff \sum_{i=1}^{N} B_i(x) \geq \sum_{i=1}^{N} B_i(y).
\]

1. Prove that this correspondence satisfies Maskin monotonicity.

2. Does the correspondence satisfy the no veto power condition?

Exercise 18.23 Let \( N = \{1, 2, 3\} \), \( E = \{e, \bar{e}\} \) and \( A = \{a, b, c\} \). The preferences of individuals are given as follows:

- Under state \( e \), 1 : \( b \succ a \succ c \); 2 : \( a \succ c \succ b \); 3 : \( a \succ c \succ b \).
- Under state \( \bar{e} \), 1 : \( b \succ c \succ a \); 2 : \( c \succ a \succ b \); 3 : \( c \succ a \succ b \).

For any \( \theta \in E \), define the social choice rule \( F \) as follows:

1. \( a \in F(\theta) \) if and only if \( a \) is better than \( b \) by the majority rule;
2. \( b \in F(\theta) \) if and only if \( b \) is better than \( a \) by the majority rule;
3. \( c \in F(\theta) \) if and only if \( \{x \in A | x \preceq_\theta c\} = A \) for any \( i \in N \).

Answer the following questions:

1. Does \( F \) satisfy Maskin monotonicity? Justify your answer.
2. Does \( F \) satisfy no veto power? Justify your answer.

Exercise 18.24 Let \( N = \{1, 2, 3, 4\} \), and there be four indivisible objects, \( a_1, a_2, a_3, a_4 \). For any \( i \in N \), the initial endowment is \( w_i = a_i \), \( E = \{e_1, e_2\} \). The preferences of individuals are given as follows:

- Under state \( e_1 \), 1 : \( a_1 \succ a_2 \succ a_3 \succ a_4 \); 2 : \( a_2 \succ a_1 \succ a_3 \succ a_4 \);
  \hspace{1cm} 3 : \( a_3 \sim a_4 \succ a_1 \succ a_2 ; 4 : a_3 \sim a_4 \succ a_2 \succ a_1 \).
- Under state \( e_2 \), 1 : \( a_1 \succ a_2 \succ a_3 \succ a_4 \); 2 : \( a_2 \succ a_1 \succ a_3 \succ a_4 \);
  \hspace{1cm} 3 : \( a_3 \succ a_4 \succ a_1 \succ a_2 ; 4 : a_3 \succ a_4 \succ a_2 \succ a_1 \).

The social goal is to allocate the four objects to the individuals, namely each one gets one object. Specifically, the social goal is a correspondence \( \sigma : N \rightarrow A \). Let \( Z \) represent the set of all possible allocations. The allocation rule \( F \) satisfies individual rationality, namely: for any \( e_i \in E \), we have \( F(e_i) = \{\sigma \in Z | \sigma_j \succeq e_i \ w_j, \text{ for any } e_i \in E, j \in N\} \). Answer the following questions.
1. Does $F$ satisfy Maskin monotonicity? Justify your answer.

2. Does $F$ satisfy no veto power? Justify your answer.

Exercise 18.25 Let $N = \{1, 2, 3\}$, $E = \{e, \hat{e}\}$ and $A = \{a, b, c\}$. The specific preferences are given as follows:

Under state $e$,

- $1 : a \succ b \succ c$;
- $2 : c \succ a \succ b$;
- $3 : b \succ c \succ a$.

Under state $\hat{e}$,

- $1 : a \succ b \succ c$;
- $2 : c \succ a \succ b$;
- $3 : c \succ b \succ a$.

The social choice rule $F$ is given by the majority voting rule, with $F(e) = \{a, b, c\}$ and $F(\hat{e}) = \{c\}$. Is $F$ Nash implementable? Justify your answer.

Exercise 18.26 (Moore and Repullo, 1990) Consider the Nash implementation problem with two participants. Let $N = \{1, 2\}$. The following condition is called $\mu_2$.

$\mu_2$: Condition $\mu$ given in Definition 18.8.6 holds. In addition, for each 4-tuple $(a, e', b, \hat{e}, \bar{e}) \in (A \times E \times A \times E)$ with $a = F(e')$ and $b = F(\hat{e})$, there exists $c = c(a, e', b, \hat{e}) \in C_1(a, e') \cap C_2(b, \hat{e})$ such that for all $e^* \in E$, the following condition is satisfied:

If $c \in C_1(a, e') \subseteq L_1(c, e^*)$ and $c \in C_2(b, \hat{e}) \subseteq L_2(c, e^*)$, then $c \in F(e)$.

Prove: in a two-agent environment, $F$ is Nash implementable if and only if Condition $\mu_2$ is satisfied.

Exercise 18.27 (Moore and Repullo, 1990) Consider the problem of Nash implementation in two-agent environments. A social choice rule $F$ satisfies the restricted veto power property, i.e., if for all $i \in I, e \in E, a \in A$, there exist $b \in \text{range}(F) = \{a \in A : a = F(e) \text{ for some } e \in E\}$ such that:

$$\text{if } A \subseteq \bigcap_{j \neq i} L_j(a, e) \text{ and } a \succeq_i b, \text{ then } a \in F(e).$$

We call outcome $z$ a bad outcome, if for all $e \in E$ and all $a \in F(e)$, we have $a \succeq_i (e)z$, in which $\succeq_i (e)$ is the preference of participant $i$ under the environment $e$.

Prove that if a two-agent choice rule $F$ satisfies monotonicity and restricted veto power, and if there is a bad outcome, then $F$ can be Nash implemented.

Exercise 18.28 (Withholding Mechanism, Tian, 1993) Consider the set of economic environments $E$ with one private good $x_i$, $K$ public goods $y$, and $n \geq 3$ individuals. The production function is $y = f(v) = v$. Suppose that the preference relations $\succeq_i$ of participant $i$ is convex and strongly monotone in private goods, and for all $i \in N, (x, y) \in R_i(x^i, y^i)$, with $x_i \in R_{++}$, $x^i \in \partial B_{++}$, and $y, y^i \in R^K_{+}$. The $\partial R^m_{+}$ is the boundary of $R^m_{+}$.
For each $i \in N$, define the message space as

$$M_i = (0, 1] \times (0, \hat{w}_i] \times \mathbb{R}^K \times \mathbb{R}^K,$$

whose general elements can be represented as $(\delta_i, w_i, \phi_i, y_i)$, and let $M = \prod_{i=1}^n M_i$. The personalized price is defined as

$$q_i(m) = \frac{1}{n} + m_{i+1} - m_{i+2}.$$

Define the correspondence $B: M \to 2^{\mathbb{R}_+^K}$,

$$B(m) = \{ y \in \mathbb{R}_+^K : (1 - \delta(m))w_i - q_i(m) \cdot y \geq 0 \ \forall \ i \in N \},$$

in which $\delta(m) = \min\{\delta_1, \cdots, \delta_n\}$. Define the outcome function of public goods as $Y: M \to B$,

$$Y(m) = \{ y : \min_{y \in B(m)} \| y - \tilde{y} \| \},$$

and $\tilde{y} = \sum_{i=1}^n y_i$. For each $i$, define the tax function $T_i: M \to \mathbb{R}$,

$$T_i(m) = q_i(m) \cdot Y(m).$$

It can be seen that

$$\sum_{i=1}^n T_i(m) = q(m) \cdot Y(m).$$

The outcome function of private goods $X(m): M \to \mathbb{R}_+$ is defined by

$$X_i(m) = w_i - q_i(m) \cdot Y(m).$$

1. Prove that the correspondence $B: M \to 2^{\mathbb{R}_+^K}$ is a nonempty, compact and convex continuous correspondence.

2. Prove that $Y(m)$ is a single-valued continuous function on $M$.

3. Prove that the above mechanism is a continuous and feasible mechanism.

4. Prove that the above mechanism fully Nash implements the Lindahl correspondence on $E$.

**Exercise 18.29 (Fully Implementing Walrasian allocations, Tian, 1992)** Consider a pure exchange economy $e = (\{X_i, w_i, \succ_i\})$, and let $E$ be the set of all economies in which Walrasian equilibrium exists. Define the mechanism that is individually feasible, balanced, and continuous as follows:

For each $i \in N$, define the message space as

$$M_i = \mathbb{R}_+^L \times \mathbb{R}_n^L,$$
whose generic elements can be represented as \( m_i = (p_i, x_{i1}, \cdots, x_{in}) \), and let \( M = \prod_{i=1}^{n} M_i \). Define the price vector function as \( p: M \rightarrow R_{++}^L \),

\[
p(m) = \begin{cases} 
\sum_{i=1}^{n} \frac{a_i}{p_i}, & \text{if } a > 0, \\
\sum_{i=1}^{n} \frac{1}{p_i}, & \text{if } a = 0,
\end{cases}
\]

in which \( a_i = \sum_{j,k \neq i} \|p_j - p_k\|, a = \sum_{i=1}^{n} a_i \) and \( \| \cdot \| \) represents the Euclidean norm.

Define the correspondence that is individually feasible and budget balanced as \( B: M \rightarrow 2^{R_{++}^L} \),

\[
B(m) = \{ x \in R_{++}^n : \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i \& p(m) \cdot x_i = p(m) \cdot w_i, \forall i \in N \}.
\]

Let \( \tilde{x}_j = \sum_{i=1}^{n} x_{ij}, \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \). The outcome function of private goods, \( X: M \rightarrow R_{++}^n \), is given as follows:

\[
X(m) = \{ y \in R_{++}^n : \min_{y \in B(m)} \| y - \tilde{x} \| \}.
\]

1. Prove the price vector function \( p: M \rightarrow R_{++}^L \) is continuous.

2. Prove the correspondence \( B: M \rightarrow 2^{R_{++}^L} \) is a continuous correspondence with nonempty, compact, convex values.

3. Prove that \( X(m) \) is single-valued continuous function on \( M \).

4. Prove: for all \( e \in E \), every Nash equilibrium outcome of this mechanism is a Walrasian equilibrium allocation.

5. Prove: for all \( e \in E \), every Walrasian equilibrium allocation is a Nash equilibrium outcome of the mechanism. Thus the mechanism fully Nash implements the Walrasian correspondence on \( E \).

**Exercise 18.30 (Moore and Repullo, 1988)** Consider an economy with two participants. Suppose that the economy has two states, \( E = \{C, L\} \): when \( e = C \), the preferences of two individuals are Cobb-Douglas utility function, and the allocation is \( f(C) \in A \); when \( e = L \), both individual preferences are perfectly complementary utility functions (i.e., in the form of Leontief function), and the allocation is \( f(L) \in A \). Suppose that \( A = \{ f(C), f(L), x, y \} \). The states and allocations can be seen in Figure 18.6. Consider the following dynamic game \( g \).

First stage: participant 1 provides report \( r_1 \). If \( r_1 = L \), the game is over and the outcome is \( f(L) \); if \( r_1 = C \), the game goes into the second stage.

Second stage: participant 2 provides report \( r_2 = C \), the game is over and the outcome is \( f(C) \); otherwise the game goes into the third stage.
Third stage: participant 1 chooses an outcome from \( \{x, y\} \), and the game is over.

Prove: the above social choice rule \( f \) can be implemented in subgame perfect equilibrium by the dynamic game \( g \).

**Exercise 18.31 (Jackson, 1992)** For environments \((N, A, E)\), in which \( N = \{1, \ldots, n\} \) is the set of participants. A mechanism \( \Gamma = (M, g) \) consists of the message space of participants \( M = M_1 \times \cdots \times M_n \) and the outcome function \( g : M \to A \). We say that \( m_i \) is a **undominated message**, if there does not exist message \( m_i' \) that weakly dominates \( m_i \). Let the set of all undominated messages of mechanism \( \Gamma \) under the environment \( e \) be \( U(\Gamma, e) \), all the Nash Equilibria be \( N(\Gamma, e) \), and the set of all the undominated Nash equilibria be \( UN(\Gamma, e) = U(\Gamma, e) \cap N(\Gamma, e) \). The social choice rule \( F : E \to A \) is said to be **implementable in undominated Nash equilibrium**, if there exists a mechanism \((M, g)\) such that \( F(e) = g(UN(\Gamma, e)), \forall e \in E \).

Suppose that \( N = \{1, 2\}, A = \{a, b\}, E = \{(R^1_1, R^2_2), (R^1_1, R^2_2)\} \), and \( aP^1_1 b, bP^1_1 aP^2_2 b \). The social choice rule satisfies \( F(R^1_1, R^2_2) = b \) and \( F(R^1_1, R^2_2) = a \). The following mechanism \((M, g)\) is shown in Figure 18.7.

1. Prove that \( F \) does not satisfy Pareto efficiency and the Maskin monotonicity (so it is not Nash implementable).

2. Prove that \( F \) can be implemented in undominated Nash equilibrium by mechanism \((M, g)\).
Exercise 18.32 (Matsushima, 1988; Abreu and Sen, 1991; Diamantaras, 2009)

For environments \((N, X', E)\), let \(A' = \{a_1, a_2, \ldots, a_K\}\) be a set of socially feasible outcomes. \(A\) is the set of all lotteries defined on \(A'\), that is, \(A = \{x \in [0, 1]^K : \sum_{k=1}^K x_k = 1\}\).

We call the environment no-total indifference, if for every \(i\) and \(e\), there are two outcomes \(a, a' \in A'\) such that \(u_i(a, e) > u_i(a', e)\), where \(u^i\) is the utility function of participant \(i\).

We call the two social choice rules \(F\) and \(H\) are \(\epsilon\)-close, if for any \(e \in E\), there is a one-to-one mapping \(\tau : F(E) \rightarrow H(E)\) such that 
\[
|\sqrt{\sum_{k=1}^K (x_k - \tau(x)_k)^2} < \epsilon \text{ for any } x \in F(e).
\]

A social choice rule \(F\) is called virtually Nash implementable if there is a social rule \(G\) that Nash-implementable as well as \(G\) and \(F\) are \(\epsilon\)-close.

\(F\) is called ordinal, if for \(F(e) \neq F(e')\), there is a participant \(i\) and \(x, x' \in A\) such that \(u^i(x, e) \geq u^i(x', e), u^i(x', e') > u_i(x, e')\).

Prove that if \(|N| \geq 3\), and the environment is no-total indifference, then any ordinal social choice rule can be virtually Nash implementable.

18.14 References

Books and Monographs


Papers


18.14. REFERENCES


Chapter 19

Mechanism Design with Incomplete Information

19.1 Introduction

This chapter is intended to analyze the implementation problem with incomplete information. The so-called incomplete information refers to that agents do not know each other’s economic characteristics such as preferences, technology, endowments and other information about agents’ economic characteristics.

So far, we only considered the implementation of social choice rules under two extreme solution concepts in terms of information requirement. One is dominant equilibrium solution, and the other is Nash equilibrium solution and its refinements. Dominant equilibrium is the strongest solution concept of the self-interest behavior of individuals since it requires the least amount of information for both designers and agents, who do not need to know any other participants’ information. Although the mechanism with dominant equilibrium as the solution concept is able to induce participants to report truthfully their preferences, and the VCG mechanism can even have the efficient provision of public goods or indivisible goods, the truthful revelation mechanism cannot implement Pareto optimal allocations since the balancedness condition in general does not hold.

The other extreme solution concept is Nash equilibrium and its refinements which are very weak solution concepts since it assumes that each participant needs know not only his/her own economic characteristics but also those of all other participants. Although Pareto optimal allocations can be implemented in Nash equilibrium as the solution concept, it is necessary to abandon the requirement that participants report truthfully.

Under these two solution concepts, Hurwicz impossibility theorem, VCG mechanism, and Nash implementable mechanisms all show that Pareto efficiency and truth-telling are generally impossible to be achieved simulta-
neously. Then, is there a solution concept between them that can achieve both properties simultaneously? The answer is yes. As long as the assumption of Bayesian incentive-compatibility is satisfied (every one has incentive to tell the truth provided all the others do the same), there exist mechanisms such that these two desirable features are achieved at the same time. As such, Bayesian incentive-compatibility is a requirement of conditional truth-telling in the sense that every has an incentive to telling truth only under the condition that all the others tell the truth.

As such, this chapter uses Bayesian-Nash equilibrium as solution concept to study implementation of social choice rules. Bayesian-Nash equilibrium assumes that although each agent does not know economic characteristics of the others, he/she knows the probability distributions of others’ economic characteristics. The corresponding implementation is Bayesian-Nash implementation. One can design various Bayesian incentive compatible mechanisms and characterize the full implementability of general social choice rules.

In the following, we will first set up the basic model with incomplete information, and the notations and definitions of Bayesian implementation in various senses. We then consider Bayesian incentive compatibility and Bayesian implementation problems of social choice functions (rather than the set of social choice functions). In the last two sections of this chapter, we will consider full Bayesian implementation and ex post implementation of a social choice set.

19.2 Basic Analytical Framework

19.2.1 Model

Throughout this chapter we follow the notation introduced in the previous chapter. Let $Z$ denote the set of outcomes, $A \subseteq Z$ be the set of feasible outcomes, and $\Theta_i$ be agent $i$’s space of types. For simplicity, till the last section, we assume each individual’s preference is given by a parametric utility function with private values, denoted by $u_i(x, \theta_i)$, where $x \in Z$ and $\theta_i \in \Theta_i$. Assume that all agents and the designer know $\theta = (\theta_1, \ldots, \theta_n)$ is distributed according to the probability density $\varphi(\theta)$ on $\Theta = \prod_{i \in N} \Theta_i$.

Each agent knows his own type $\theta_i$, and computes the conditional distribution of the other agents’ types:

$$\varphi(\theta_{-i}|\theta_i) = \frac{\varphi(\theta_i, \theta_{-i})}{\int_{\Theta_{-i}} \varphi(\theta_i, \theta_{-i}) d\theta_{-i}}.$$

\footnote{The notion of Bayesian-Nash equilibrium was introduced by Harsanyi. Nash, Selten and Harsanyi received the 1994 Nobel Prize in economics for their fundamental contributions on Nash equilibrium, sub-game perfect Nash equilibrium and Bayesian-Nash equilibrium, respectively.}
Let $M = M_1 \times \cdots \times M_n$ be the message space and $h : M \to Z$ be the outcome function. As usual, a mechanism is a pair, $\Gamma = \langle M, h \rangle$. Given $m$ with $m_i : \Theta_i \to M_i$, agent $i$’s expected utility at $\theta_i$ is given by

$$
\Pi^i_{\langle M, h \rangle}(m(\theta); \theta_i) = E_{\theta_{\neg i}}[u_i(h(m_i(\theta_i), m_{\neg i}(\theta_{\neg i})), \theta_i) | \theta_i] = \int_{\Theta_{\neg i}} u_i(h(m(\theta)), \theta_i) \phi(\theta_{\neg i}|\theta_i) d\theta_{\neg i}. \tag{19.2.1}
$$

**Definition 19.2.1** A strategy $m^*(\cdot)$ is a Bayesian-Nash equilibrium (BNE) of $\langle M, h \rangle$ if for all $\theta_i \in \Theta_i$,

$$
\Pi^i(m^*(\theta_i); \theta_i) \geq \Pi^i(m_i, m^*_{\neg i}(\theta_{\neg i}); \theta_i) \quad \forall m_i \in M_i.
$$

That is, if player $i$ believes that other players are playing strategies $m^*_{\neg i}(\cdot)$ then he maximizes his expected utility by playing strategy $m^*_i(\cdot)$. Denote by $\mathcal{B}(\Gamma)$ the set of all Bayesian-Nash equilibria of the mechanism.

**Remark 19.2.1** In the present private value incomplete information setting, a message $m_i : \Theta_i \to M_i$ is a dominant strategy for agent $i$ in mechanism $\Gamma = \langle M, h \rangle$ if for all $\theta_i \in \Theta_i$ and all possible strategies $m_{\neg i}(\theta_{\neg i})$,

$$
E_{\theta_{\neg i}}[u_i(h(m_i(\theta_i), m_{\neg i}(\theta_{\neg i})), \theta_i) | \theta_i] \geq E_{\theta_{\neg i}}[u_i(h(m^*_i, m_{\neg i}(\theta_{\neg i})), \theta_i) | \theta_i] \tag{19.2.2}
$$

for all $m_i' \in M_i$.

Since condition (19.2.2) holds for all $\theta_i$ and all $m_{\neg i}(\theta_{\neg i})$, it is equivalent to the condition that, for all $\theta_i \in \Theta_i$,

$$
u_i(h(m_i(\theta_i), m_{\neg i}), \theta_i) \geq u_i(h(m_i', m_{\neg i}), \theta_i) \tag{19.2.3}
$$

for all $m_i' \in M_i$ and all $m_{\neg i} \in M_{\neg i}$. This leads to the following equivalent definition on dominant equilibrium.

**Definition 19.2.2** For the private value model, a message $m^*$ is a dominant strategy equilibrium of mechanism $\Gamma = \langle M, h \rangle$ if for all $i$ and all $\theta_i \in \Theta_i$,

$$u_i(h(m^*_i(\theta_i), m_{\neg i}), \theta_i) \geq u_i(h(m_i', m_{\neg i}), \theta_i)$$

for all $m_i' \in M_i$ and all $m_{\neg i} \in M_{\neg i}$, which is the same as that in the case of complete information.

**Remark 19.2.2** It is clear that every dominant strategy equilibrium is a Bayesian-Nash equilibrium, but the converse may not be true. Bayesian-Nash equilibrium requires more sophisticated belief structures from the agents than dominant strategy equilibrium does. Each agent, in order to find his optimal strategy, must have a correct prior, $\phi(\cdot)$, over type profiles, and must correctly predict the equilibrium strategies used by other agents.

---

\(^2(19.2.2)\) implies (19.2.3) by choosing $m_{\neg i}(\theta_{\neg i}) = m_{\neg i}$ for all $\theta_{\neg i} \in \Theta_{\neg i}$; on the other hand, (19.2.3) implies $u_i(h(m_i(\theta_i), m_{\neg i}(\theta_{\neg i})), \theta_i) \geq u_i(h(m_i', m_{\neg i}(\theta_{\neg i})), \theta_i)$ for each $\theta_{\neg i} \in \Theta_{\neg i}$, which further implies (19.2.2) by taking expectation with respect to $\theta_{\neg i}$.
19.2.2 Bayesian Incentive-Compatibility and Bayesian Implementation

To see specifically what is implementable in Bayesian-Nash equilibrium (BNE), till the last two sections, the social choice goal is given by a single-valued social choice function rather than the set of social choice functions. We focuses on the issues of Bayesian incentive-compatibility and Bayesian implementation of a social choice function. Furthermore, by Revelation Principle, without loss of generality, we can focus on direct revelation mechanisms.

Let $f : \Theta \rightarrow A$ be a social choice function. Since at Bayesian-Nash equilibrium, every agent reaches his interim utility maximization, Bayesian implementation sometimes is also called interim implementation. Like implementation in dominant strategies and Nash strategy, we similarly have (fully) Bayesian implementation and partial Bayesian implementation.

**Definition 19.2.3** A mechanism $\Gamma = (M, h)$ is said to **Bayesian implement** a social choice function $f$ on $\Theta$ if for all Bayesian-Nash equilibrium $m^*$, we have $h(m^*(\theta)) = f(\theta)$, $\forall \theta \in \Theta$. If such a mechanism exists, we call $f$ is Bayesian implementable.

When there are multiple Bayesian-Nash equilibria, it may result in some undesirable equilibrium outcome, i.e., it is not equal to $f$ (we will provide such an example). Then, we have the following weaker concept on Bayesian implementability.

**Definition 19.2.4** A mechanism $\Gamma = (M, h)$ is said to **partially Bayesian implement** a social choice function $f$ on $\Theta$ if there exists a Bayesian-Nash equilibrium $m^*$, such that $h(m^*(\theta)) = f(\theta)$, $\forall \theta \in \Theta$. If such a mechanism exists, we call $f$ is partially Bayesian implementable.

Like dominant implementation, we will see that a social choice function is partially Bayesian implementable if and only if it is truthfully Bayesian implementable.

**Definition 19.2.5** We call a social choice function $f$ is **truthfully Bayesian implementable** or Bayesian incentive compatible, if truth-telling, namely $m^*(\theta) = \theta$, $\forall \theta \in \Theta$, is a Bayesian-Nash equilibrium of revelation mechanism $\Gamma = (\Theta, f)$, i.e.,

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i})), \theta_i] \geq E_{\theta_{-i}}[u_i(f(\theta'_i, \theta_{-i})), \theta_i], \quad \forall i, \forall \theta_i, \theta'_i \in \Theta_i.$$ 

Bayesian incentive-compatibility means that for social choice rule $f$, every agent will report his type truthfully provided all the other agents report their truthful strategies and thus every truthful strategy profile is a

---

3Since the social choice is a function, Bayesian-Nash implementation is the same as full Bayesian-Nash implementation.
Bayesian-Nash equilibrium of the direct mechanism \((\theta, f)\). Notice that Bayesian incentive compatibility is only the conditional truth-telling. It does not say what is the best response of an agent when other agents are not playing truthful strategies. When a mechanism has multiple equilibria, non-truth-telling may be a Bayesian-Nash equilibrium resulting in an undesirable outcome that is not equal to \(f\). When Bayesian incentive compatibility is just a basic requirement, it in general only ensures a social choice rule to be truthfully Bayesian implementable but not (fully) Bayesian implementable, i.e., it may also have some undesirable equilibrium outcomes (they are not social optimal) when a mechanism has multiple equilibria.

Similarly, we have the following revelation principle.

**Proposition 19.2.1 (Revelation Principle)** A social choice rule \(f(\cdot)\) is partially implementable in Bayesian-Nash equilibrium (in short, BNE) if and only if it is truthfully implementable in BNE.

**Proof.** The proof is the same as before: Suppose that there exists a mechanism \(\Gamma = (M_1, \ldots, M_n, g(\cdot))\) and an equilibrium strategy profile \(m^*(\cdot) = (m^*_1(\cdot), \ldots, m^*_n(\cdot))\) such that \(g(m^*(\cdot)) = f(\cdot)\) and \(\forall i \in \Theta_i\). We then have

\[
E_{\theta_i}[u_i(g(m^*_i(\theta_i), m^*_{-i}(\theta_{-i})), \theta_i) | \theta_i] \geq E_{\theta_i}[u_i(g(m'_i, m^*_{-i}(\theta_{-i})), \theta_i) | \theta_i], \forall m'_i \in M_i.
\]

One way to deviate for agent \(i\) is by pretending that his type is \(\hat{\theta}_i\) rather than \(\theta_i\), i.e., sending message \(m'_i = m^*_i(\hat{\theta}_i)\). This gives

\[
E_{\theta_i}[u_i(g(m'_i, m^*_{-i}(\theta_{-i})), \theta_i) | \theta_i] \geq E_{\theta_i}[u_i(g(m^*_i(\hat{\theta}_i), m^*_{-i}(\theta_{-i})), \theta_i) | \theta_i], \forall \theta_i \in \Theta_i.
\]

But since \(g(m^*(\theta)) = f(\theta), \forall \theta \in \Theta\), we must have \(\forall i, \forall \theta_i \in \Theta_i\) that

\[
E_{\theta_i}[u_i(f(\theta_i, \theta_{-i}), \theta_i) | \theta_i] \geq E_{\theta_i}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i) | \theta_i], \forall \hat{\theta}_i \in \Theta_i.
\]

Although Bayesian incentive-compatibility is a necessary and sufficient for \(f\) to be truthfully Bayesian implementable, it is not a sufficient condition for \(f\) to be Bayesian implementable. Bayesian implementability requires all BNE outcomes be equal to \(f(\theta)\). The goal for designing a mechanism is to reach a desirable equilibrium outcome, but it may also result in an undesirable equilibrium outcome. Thus, while the incentive compatibility requirement is central, it may not be sufficient for a mechanism to give all of desirable outcomes. The severity of this multiple equilibrium problem has been exemplified by Demski and Sappington (1984), Postlewaite and Schmeidler (1986), Repullo (1988), and among others. The implementation problem involves designing mechanisms to ensure that all equilibria result in desirable outcomes.

Under what conditions, does there exist a mechanism that has a unique Bayesian incentive-compatible equilibrium outcome? If such a mechanism
exists, by definition, we know a social choice function $f$ is Bayesian implementable. One such condition is the existence of undominated Bayesian equilibrium.

**Definition 19.2.6** A message $m \in M$ is weakly dominated, if there is an agent $i$, $\theta_i$, and another Bayesian message $m'_i \neq m_i$, such that for all $m_{-i} \in M_{-i}$,

$$\Pi^i(m_i(\theta_i), m_{-i}; \theta_i) \leq \Pi^i(m'_i(\theta_i), m_{-i}; \theta_i)$$

with strict inequality for some $m_{-i} \in M_{-i}$.

**Definition 19.2.7** $m \in M$ is an undominated Bayesian equilibrium of $\Gamma = \langle M, h \rangle$, if it is a Bayesian-Nash equilibrium that is not weakly dominated.

Palfrey and Srivastava (1989, JPE) proved the following proposition.

**Proposition 19.2.2** For the class of private value economic environments, if there exists a mechanism such that all agents do not have weakly dominated strategies, then any Bayesian incentive-compatible social choice function is Bayesian implementable.

In the remainder of the chapter, we mainly consider truthful Bayesian implementability of a social choice rule. The last two sections of the chapter will discuss the necessary and sufficient conditions for full Bayesian implementability of a general social choice rule—the set of social choice functions under more general interdependent value model.

### 19.3 Truthful Implementation of Pareto Efficient Outcomes

Once again, let us return to the quasilinear setting. From the discussion on dominant strategy implementation in the last chapter, we know that there is, in general, no ex-post Pareto efficient implementation in dominant strategies. However, in quasilinear environments, changing the equilibrium concept from DE to BNE allows us to truthfully Bayesian implement ex post Pareto efficient choice rule $f(\cdot) = (y(\cdot), t_1(\cdot), \cdots, t_i(\cdot))$, where $\forall \theta \in \Theta$,

$$y(\theta) \in \arg\max_{y \in Y} \sum_{i=1}^{n} v_i(y, \theta_i), \quad (19.3.4)$$

and there is a balanced budget:

$$\sum_{i=1}^{n} t_i(\theta) = 0. \quad (19.3.5)$$
19.3. TRUTHFUL IMPLEMENTATION OF PARETO EFFICIENT OUTCOMES

The expected externality mechanism described below was suggested independently by D’ Aspremont and Gerard-Varet (1979) and Arrow (1979) so as called the AGV mechanism in the literature. This mechanism enables us to have ex-post Pareto efficient implementation under the following additional assumption.

**Assumption 19.3.1** Types are distributed independently: \( \varphi(\theta) = \prod_i \varphi_i(\theta_i), \forall \theta \in \Theta \).

To see this, consider the VCG transfer for agent \( i \). Instead of using other agents’ types announced, take the expectation over their possible types and let

\[
t_i(\hat{\theta}) = E_{\theta_{-i}} \left[ \sum_{j \neq i} v_j(y(\hat{\theta}_i, \theta_{-i}), \theta_j) + d_i(\hat{\theta}_{-i}) \right].
\]

By Assumption 19.3.1, the expectation over \( \theta_{-i} \) does not need to be taken conditionally on \( \theta_i \). Note that unlike VCG mechanisms, the first term only depends on agent \( i \)'s announcement \( \hat{\theta}_i \), and not on other agents' announcements. This is because it sums the expected utilities of agents \( j \neq i \) assuming that they tell the truth and given that \( i \) announced \( \hat{\theta}_i \), and does not depend on the actual announcements of agents \( j \neq i \). This means that \( t_i(\cdot) \) is less “variable”, but on average it will cause \( i \)'s incentives to be lined up with the social welfare.

To see that agent \( i \)'s incentive compatibility is satisfied given that agents \( j \neq i \) announce truthfully, observe that agent \( i \) solves

\[
\max_{\hat{\theta}_i \in \Theta_i} E_{\theta_{-i}} \left[ v_i(y(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i}) \right] = \max_{\hat{\theta}_i \in \Theta_i} E_{\theta_{-i}} \left[ \sum_{j=1}^{n} v_j(y(\hat{\theta}_i, \theta_{-i}), \theta_j) \right] + E_{\theta_{-i}} d_i(\theta_{-i}). \tag{19.3.6}
\]

Again, agent \( i \)'s announcement only matters through the decision \( y(\hat{\theta}_i, \theta_{-i}) \). Moreover, by the definition of the efficient decision rule \( y(\cdot) \), the designer always chooses \( y(\cdot) \) to max social surplus (19.3.4) for each realization of \( \theta_{-i} \in \Theta_{-i} \). Then, when agent \( i \) announces truthfully, namely \( \hat{\theta}_i = \theta_i \), maximization of (19.3.6) is consistent with the social goal (19.3.4). As such, when the social surplus (19.3.4) is maximized, agent \( i \)'s expected utility (19.3.6) is maximized by choosing the decision \( y(\theta_i, \theta_{-i}) \), which can be achieved by announcing truthfully, i.e., \( \hat{\theta}_i = \theta_i \). Therefore, truthful announcement maximizes the agent \( i \)'s expected utility as well. Thus, BIC is satisfied.

**Remark 19.3.1** The argument relies on the assumption that other agents announce truthfully. Thus, it is in general not a dominant strategy for agent \( i \) to announce the truth. Indeed, if agent \( i \) expects the other agents to cheat, i.e., they announce \( \hat{\theta}_{-i}(\theta_{-i}) \neq \theta_{-i} \), then agent \( i \)'s expected utility is

\[
E_{\theta_{-i}} [v_i(y(\hat{\theta}_i, \hat{\theta}_{-i}(\theta_{-i})), \theta_i) + \sum_{j \neq i} v_j(y(\hat{\theta}_i, \theta_{-i}), \theta_j)] + E_{\theta_{-i}} d_i(\theta_{-i}),
\]
which may not be maximized by truthful announcement.

Furthermore, to achieve ex post Pareto efficiency, we can now choose functions $d_i(\cdot)$ so that the budget is balanced. To see this, let

$$
\xi_i(\theta_i) = E_{\theta_{-i}}[\sum_{j \neq i} v_j(y(\theta_i, \theta_{-i}), \theta_j)],
$$

so that the transfers in the expected externality mechanism are $t_i(\theta) = \xi_i(\theta_i) + d_i(\theta_{-i})$. We will show that we can use the $d(\cdot)$ functions to "compensate" the $\xi(\cdot)$ functions in the following way.

Let

$$
d_j(\theta_{-j}) = -\sum_{i \neq j} \frac{1}{n-1} \xi_i(\theta_i).
$$

Then

$$
\sum_{j=1}^{n} d_j(\theta_{-j}) = -\frac{1}{n-1} \sum_{j=1}^{n} \sum_{i \neq j} \xi_i(\theta_i) = -\frac{1}{n-1} \sum_{j \neq i}^{n} \sum_{i=1}^{n} \xi_i(\theta_i)
$$

$$
= -\frac{1}{n-1} \sum_{i=1}^{n} (n-1) \xi_i(\theta_i) = -\sum_{i=1}^{n} \xi_i(\theta_i),
$$

and therefore

$$
\sum_{i=1}^{n} t_i(\theta) = \sum_{i=1}^{n} \xi_i(\theta_i) + \sum_{i=1}^{n} d_i(\theta_{-i}) = 0.
$$

Thus, we have shown that when agents’ Bernoulli utility functions are quasilinear and agents’ types are statistically independent, ex-post Pareto efficient social choice function is truthfully Bayesian implementable.

### 19.4 Characterization of DIC and BIC for Linear Preferences

In this section, we shall give a full characterization of dominant incentive compatibility (DIC) and Bayesian incentive compatibility (BIC) when participants’ utility functions are of linear forms. The following BIC characterization theorem shall be very useful. As a corollary, this theorem implies that for any two Bayesian incentive compatible mechanisms that truthfully implement the same decision rule $y(\cdot)$, their interim expected utilities $E_{\theta_{-i}} U_i(\theta)$ and transfer payments $E_{\theta_{-i}} t_i(\theta)$ coincide up to a constant. We shall apply this theorem to prove the well-known revenue equivalence theorem in auction theory in Chapter 21.
19.4. CHARACTERIZATION OF DIC AND BIC FOR LINEAR PREFERENCES

Consider the quasi-linear environment with the decision set $Y \subseteq \mathbb{R}^n$, and the type spaces $\Theta_i = [\underline{\theta}_i; \overline{\theta}_i] \subseteq \mathbb{R}$ for all $i$. Each agent $i$’s utility takes the form

$$\theta_i y_i + t_i.$$ 

Note that these payoffs satisfy the single-crossing property (SCP).

The decision set $Y$ may have different forms, depending on the application under consideration.

Two examples:

1. Allocating an indivisible private good: $Z = \{y \in \{0, 1\}^n : \sum_i y_i = 1\}$.
2. Provision of a public good: $Y = \{(q, \ldots, q) \in \mathbb{R}^n : q \in \{0, 1\}\}$.

We can fully characterize both dominant incentive-compatible and Bayesian incentive-compatible social choice rules in this environment.

Proposition 19.4.1 (DIC Characterization Theorem) In the linear model, a social choice rule $(y(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is dominant incentive compatible if and only if for all $i \in N$,

1. (dominant monotonicity (DM)): $y_i(\theta_i, \theta_{-i})$ is nondecreasing in $\theta_i$ for all $\theta_{-i} \in \Theta_{-i}$;
2. (dominant incentive-compatibility first-order condition (DIC-FOC)): $U_i(\theta_i, \theta_{-i}) = U_i(\theta_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} y_i(\tau, \theta_{-i}) d\tau, \forall \theta \in \Theta$.

Proof. Necessity: Dominant incentive compatibility implies that for all $\theta'_{i} > \theta_i$, we have

$$U_i(\theta_i, \theta_{-i}) \geq U_i(\theta'_{i}, \theta_{-i}) + t_i(\theta', \theta_{-i}) = U_i(\theta'_{i}, \theta_{-i}) + (\theta'_{i} - \theta_i)y_i(\theta'_{i}, \theta_{-i})$$

and

$$U_i(\theta'_{i}, \theta_{-i}) \geq U_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i}) = U_i(\theta_i, \theta_{-i}) + (\theta'_{i} - \theta_i)y_i(\theta_i, \theta_{-i}).$$

Thus

$$y_i(\theta', \theta_{-i}) \geq \frac{U_i(\theta'_{i}, \theta_{-i}) - U_i(\theta_i, \theta_{-i})}{\theta'_{i} - \theta_i} \geq y_i(\theta_i, \theta_{-i}). \quad (19.4.7)$$

Expression (19.4.7) implies that $y_i(\theta_i, \theta_{-i})$ must be nondecreasing in $\theta_i$ by noting that $\theta'_{i} > \theta_i$. In addition, letting $\theta'_{i} \rightarrow \theta_i$ in (19.4.7) implies that for all $\theta_i$, we have

$$\frac{\partial U_i(\theta)}{\partial \theta_i} = y_i(\theta),$$
and thus
\[ U_i(\bar{\theta}_i, \theta_{-i}) = U_i(\bar{\theta}_i, \theta_{-i}) + \int_{\bar{\theta}_i}^{\theta_i} y_i(\tau, \theta_{-i}) d\tau, \forall \theta \in \Theta. \]

** Sufficiency:** Consider any two types \( \theta_i' \) and \( \theta_i \). Without loss of generality, suppose \( \theta_i > \theta_i' \). If DM and DICFOC hold, then for all \( \theta_{-i} \in \Theta_{-i} \):
\[
U_i(\theta_i, \theta_{-i}) - U_i(\theta_i', \theta_{-i}) = \int_{\theta_i'}^{\theta_i} y_i(\theta, \theta_{-i}) d\tau \geq \int_{\theta_i'}^{\theta_i} y_i(\theta, \theta_{-i}) d\tau = (\theta_i - \theta_i') y_i(\theta, \theta_{-i}).
\]

Thus, we have
\[
U_i(\theta_i, \theta_{-i}) \geq U_i(\theta_i', \theta_{-i}) + (\theta_i - \theta_i') y_i(\theta, \theta_{-i}) = \theta_i y_i(\theta_i', \theta_{-i}) + t_i(\theta_i', \theta_{-i}).
\]

Similarly, we have
\[
U_i(\theta_i', \theta_{-i}) \geq U_i(\theta_i, \theta_{-i}) + (\theta_i - \theta_i') y_i(\theta, \theta_{-i}) = \theta_i' y_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i}).
\]

Hence, the decision rule \((y(\cdot), t_1(\cdot), \ldots, t_n(\cdot))\) is dominant-strategy incentive compatible.

Similarly, we can provide the necessary and sufficient conditions for a social choice rule to be Bayesian incentive compatible. When agent \( i \) announces \( \hat{\theta}_i \) and the others announce \( \theta_{-i} \) truthfully, we have the interim expected consumption \( \bar{y}_i(\hat{\theta}_i) = E_{\theta_{-i}} y_i(\hat{\theta}_i, \theta_{-i}) \) and transfer \( t_i(\hat{\theta}_i) = E_{\theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i}) \). Then, agent \( i' \)'s interim expected utility is given by \( \bar{U}_i(\theta_i, \hat{\theta}_i) = \theta_i y_i(\hat{\theta}_i) + t_i(\hat{\theta}_i) \).

Define
\[
\bar{U}_i(\theta_i) \equiv \bar{U}_i(\theta_i, \hat{\theta}_i) |_{\hat{\theta}_i = \theta_i} = E_{\theta_{-i}} U_i(\theta_i) \equiv \theta_i y_i(\theta_i) + t_i(\theta_i),
\]
which is agent \( i \)'s interim expected utility when he announces \( \theta_i \) truthfully. Then, BIC means that truth-telling is optimal at the interim stage. By the same way used in the proof of the above proposition for \( U(\theta_i) \), we can prove the following result.

**Proposition 19.4.2 (BIC Characterization Theorem)** In the linear model, a social choice rule \((y(\cdot), t_1(\cdot), \ldots, t_n(\cdot))\) is Bayesian incentive compatible if and only if for all \( i \in N \),

1. **(Bayesian monotonicity (BM)):** \( E_{\theta_{-i}} y_i(\theta_i, \theta_{-i}) \) is nondecreasing in \( \theta_i \) for all \( \theta_{-i} \in \Theta_{-i} \);
2. **(Bayesian incentive-compatibility first-order condition (BIC-FOC)):** \( E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) = E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) + \int_{\theta_i}^{\theta_i} E_{\theta_{-i}} y_i(\tau, \theta_{-i}) d\tau, \forall \theta \in \Theta. \)
19.5. BIC MECHANISM AND OPTIMAL CONTRACT DESIGN

The above proposition implies that for any two BIC mechanisms that implement the same decision rule \( y(\cdot) \) truthfully in Bayesian-Nash equilibrium, the interim expected utilities \( E_{\theta_i} U_i(\theta) \) and thus the transfers \( E_{\theta_i} t_i(\theta) \) coincide up to a constant. We then have the following important corollary.

**Corollary 19.4.1** In the linear model, for any BIC mechanism that implements the ex post Pareto efficient decision rule \( y(\cdot) \) truthfully in Bayesian-Nash equilibrium, there exists a VCG mechanism with the same interim expected transfers and utilities.

**Proof.** If \( (y(\cdot), t(\cdot)) \) is a BIC mechanism and \( (y(\cdot), \tilde{t}(\cdot)) \) is a VCG mechanism, then by BIC Characterization Theorem (Proposition 19.4.2), \( E_{\theta_i} t_i(\theta) = E_{\theta_i} \tilde{t}_i(\theta) + c_i, \forall \theta_i \in \Theta_i \). Letting \( t_i(\theta) = \tilde{t}_i(\theta) + c_i \) then \( (y(\cdot), \tilde{t}(\cdot)) \) is also a VCG mechanism, and \( E_{\theta_i} \tilde{t}_i(\theta) = E_{\theta_i} t_i(\theta) \). \( \square \)

For example, the expected externality mechanism is interim-equivalent to a VCG mechanism. Its only advantage is that it allows to balance the budget ex post for each state of the world. More generally, if a decision rule \( y(\cdot) \) is truthfully implementable in dominant strategies, the only reason to truthfully implement it in a BIC mechanism is that we care about ex post transfers/utilities rather than just their interim or ex ante expectations.

### 19.5 BIC Mechanism and Optimal Contract Design

In general mechanism design considered in this chapter so far, we have not yet been thinking of the mechanism as a contract or a principal-agent problem. Since if a mechanism is a contract, it must be voluntary, i.e., it should satisfy participation constraints. Thus, if we would put the principal-agent theory into the framework of general mechanism design, a mechanism should also be individually rational. As such, when agents’ types are not observed, a social choice function that can be successfully implemented should satisfy not only the incentive compatibility in dominant strategies or Bayesian Nash dominant strategies, depending on the equilibrium concept used, but also the participation constraints that are relevant in the environment under study.

When using dominant strategy as a solution concept, Hurwicz impossibility theorem and especially VCG mechanism tell us that, there is no truth-telling mechanism that results in Pareto efficient allocations even for quasi-linear economic environments. As for Bayesian incentive compatibility, such a mechanism exists and is called the expected externality mechanism. Then, a question is, if imposing the individual rationality constraint, so that we can think of a mechanism as a contract, does such a mechanism still exist? The answer tends out, unfortunately, to be negative.
19.5.1 Participation Constraints

If thinking of the mechanism as a contract, the following issues are raised:

- Will the agents accept it voluntarily, i.e., are their participation constraints satisfied?

- If one of the agents designs the contract, he will try to maximize his own payoff subject to the other agents’ participation constraints. What will the optimal contract look like?

To answer these questions, we need to imposes additional restrictions on the social choice rule in the form of participation constraints. These constraints depend on when agents can withdraw from the mechanism, and what they get when they do so. Let \( \hat{u}_i(\theta_i) \) be the reservation utility of agent \( i \). According to the timing of information revelation, we have three kinds of participation constraints.

**Definition 19.5.1** The social choice rule \( f(\cdot) \) is

1. **ex post individually rational** if for all \( i \),
   \[
   U_i(\theta) \equiv u_i(f(\theta), \theta) \geq \hat{u}_i(\theta_i), \forall \theta \in \Theta;
   \]
2. **interim individually rational** if for all \( i \),
   \[
   E_{\theta_{-i}}[U_i(\theta_i, \theta_{-i})] \geq \hat{u}_i(\theta_i), \forall \theta_i \in \Theta_i;
   \]
3. **ex ante individually rational** if for all \( i \),
   \[
   E_{\theta}[U_i(\theta)] \geq E_{\theta_i}[\hat{u}_i(\theta_i)].
   \]

Note that ex post IR imply interim IR, which in turn imply ex ante IR, but the reverse may not be true. Then, the constraints imposed by voluntary participation are most sever when agents can withdraw at the ex post stage.

Ex post IR arises when the agent can withdraw in any state after the announcement of the outcome so that there is no regret when the ex post IR is satisfied. For example, they are satisfied by any decentralized bargaining procedure. They are the hardest constraints to satisfy.

The interim IR arises when the agent can withdraw after learning his type \( \theta_i \); but before learning anything about other agents’ types. Once the agent decides to participate, the outcome can be imposed on him. These constraints are easier to satisfy than ex post IR.

With the ex-ante participation constraint the agent can commit to participating even before his type is realized. These are the easiest constraints
to satisfy. For example, in a quasi-linear environment, whenever the mechanism generates a positive expected total surplus, i.e.,

$$E_\theta[\sum_i U_i(\theta)] \geq E_\theta[\sum_i \hat{u}_i(\theta_i)],$$

all agents’ ex ante IR can be satisfied by reallocating expected total surplus among agents through lump-sum transfers, which will not disturb agents’ incentive constraints or budget balance.

For this reason, we will focus mainly on interim IR. In the following, we will illustrate further the limitations on the set of implementable social choice functions that may be caused by participation constraints by the important theorem given through Myerson-Satterthwaite (1983).

### 19.5.2 Myerson-Satterthwaite Impossibility Theorem

Even though Pareto efficient mechanisms do exist (e.g., the expected externality mechanism), it remains unclear whether such a mechanism may result from private contracting among the parties. We have already seen that private contracting need not yield Pareto efficiency. In the principal-agent model, the principal offers an optimal contract to extract the agent’s information rent.

However, this raises a question whether there is some contracting or bargaining procedure that would yield Pareto efficiency. For instance, in the principal-agent model, if the agent makes an offer to the principal who has no private information on his own, the agent would extract all the surplus and truthfully implement efficient allocation. Thus, we focus on a bilateral situation in which both parties have private information. In this situation, it turns out that generally there does not exist an efficient mechanism that satisfies both agents’ participation and balanced constraints.

Consider the setting of allocating an indivisible object with two agents - a seller and a buyer: $I = \{S, B\}$. Each agent’s type is $\theta_i \in \Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subseteq \mathbb{R}$, where $\theta_i \sim \varphi_i(\cdot)$ are independent, and $\varphi_i(\cdot) > 0$ for all $\theta_i \in \Theta_i$. Let $y \in \{0, 1\}$ indicate whether $B$ receives the good. A social choice rule is then $f(\theta) = (y(\theta), t_1(\theta), t_2(\theta))$. The agents’ utilities can thus be written as

$$u_B(y, \theta_B) = \theta_B y + t_B,$$
$$u_S(y, \theta_S) = -\theta_S y + t_S.$$

It is easy to see that an efficient decision rule $y(\theta)$ must have

$$y(\theta_B, \theta_S) = \begin{cases} 1 & \text{if } \theta_B > \theta_S, \\ 0 & \text{if } \theta_B < \theta_S. \end{cases}$$

---

*Given our full support assumption on the distributions, ex ante efficiency dictates that the decision rule coincide with $y(\cdot)$ almost everywhere.*
We could use an expected externality mechanism to truthfully implement an efficient decision rule in BNE with being ex post budget balanced. However suppose that we have to satisfy interim IR:

\[
E_{\theta_S}[\theta_B y(\theta_B, \theta_S) + t_B(\theta_B, \theta_S)] \geq 0,
\]

\[
E_{\theta_B}[-\theta_S y(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] \geq 0.
\]

As for budget balance, let us relax this requirement by requiring only ex ante budget balance:

\[
E_\theta[t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] \leq 0.
\]

Unlike ex post budget balance considered before, ex ante budget balance allows us to borrow money as long as we break even on average. It also allows us to have a surplus of funds. The only constraint is that we cannot have an expected shortage of funds.

We then have the following impossibility theorem:

**Theorem 19.5.1 (Myerson-Satterthwaite Theorem, 1983)** In the two-party trade setting above, suppose that each agent’s type is \( \theta_i \in \Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subseteq \mathcal{R} \), where \( \theta_i \sim \varphi_i(\cdot) \) are independent and \( (\theta_B, \theta_B) \cap (\theta_S, \theta_S) \neq \emptyset \) (gains from trade are possible but types are not certain). Then there is no Bayesian incentive-compatible social choice rule that has the efficient decision rule and satisfies ex ante budget balance and interim IR.

**Proof.** Consider first the case where \([\underline{\theta}_B, \overline{\theta}_B] = [\underline{\theta}_S, \overline{\theta}_S] = \{0, \overline{\theta}\} \]

By Corollary 19.4.1 above, we know that for any BIC mechanism in the linear environment that implements the ex post Pareto efficient decision rule \( y(\cdot) \), there exists a VCG mechanism with the same interim expected transfers and utilities. As such, we can restrict attention to VCG mechanisms while preserving ex ante budget balance and interim IR. Such mechanisms take the form

\[
t_B(\theta_B, \theta_S) = -\theta_S y(\theta_B, \theta_S) + d_B(\theta_S),
\]

\[
t_S(\theta_B, \theta_S) = \theta_B y(\theta_B, \theta_S) + d_S(\theta_B).
\]

By interim IR of \( B' \)’s type \( \overline{\theta} \), \( E_{\theta_S}[\theta_B y(\theta_B, \theta_S) + t_B(\theta_B, \theta_S)] \geq 0 \) using the fact that \( y(\overline{\theta}, \overline{\theta}_S) = 0 \) with probability 1, we must have \( E_{\theta_S} d_B(\theta_S) \geq 0 \). Similarly, by interim IR of \( S' \)’s type \( \overline{\theta} \), \( E_{\theta_B}[-\theta_S y(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] \geq 0 \) using the fact that \( y(\theta_B, \overline{\theta}) = 0 \) with probability 1, we must have \( E_{\theta_B} d_S(\theta_B) \geq 0 \). Thus, adding the transfers, we have

\[
E_\theta[t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] = E_\theta[(\theta_B - \theta_S) y(\theta_B, \theta_S)] + E_{\theta_S}[d_B(\theta_S)] + E_{\theta_B}[d_S(\theta_B)]
\]

since \( \Pr(\theta_B > \theta_S) > 0 \). Therefore, ex ante budget balance cannot be satisfied.
Now, in the general case, let \((\theta, \hat{\theta}) = (\theta_B, \hat{\theta}_B) \cap (\theta_S, \hat{\theta}_S)\), and observe that any type of either agent above \(\hat{\theta}\) (or below \(\theta\)) has the same decision rule, and therefore must have the same transfer, as this agent’s type \(\hat{\theta}\) (resp. \(\hat{\theta}\)). Therefore, the payments are the same as if both agents had valuations distributed on \([\theta, \hat{\theta}]\); with possible atoms on \(\theta\) and \(\hat{\theta}\). The argument thus still applies.

The intuition for the proof is simple: In a VCG mechanism, in order to induce truthful revelation, each agent must become the residual claimant for the total surplus (since it is given by the sum of individuals’ values plus \(d_i(\hat{\theta}_{-i})\)). This means that in case trade is implemented, the buyer pays the seller’s cost for the object, and the seller receives the buyer’s valuation for the object. Any additional payments to the agents must be nonnegative, in order to satisfy interim IR of the lowest-valuation buyer and the highest-cost seller. Thus, each agent’s utility must be at least equal to the total surplus. In BNE implementation, agents receive the same expected utilities as in the VCG mechanism, thus again each agent’s expected utility must equal at least to the total expected surplus. This cannot be done without having an expected infusion of funds equal to the expected surplus, which breaks the budget balance.

Myerson-Satterthwaite Impossibility Theorem is based on the assumption that agents’ types are independent. If agents’ types are correlated, Cremer-McLean Full Surplus Extraction Theorem to be discussed next subsection tells us that the above conclusion may not be true.

When agents face ex ante rather than interim IR, we can easily have BIC Pareto efficient optimal contract. For instance, in a quasi-linear environment, whenever the mechanism such as AVG mechanism generates a positive expected total surplus \(E_\theta[\sum_i U_i(\theta)] \geq E_\theta[\sum_i \hat{u}_i(\theta_i)]\), all agents’ ex ante IR can be satisfied by reallocating expected surplus among agents through lump-sum transfers, which will not disturb agents’ incentive constraints or budget balance.

The above impossibility theorem shows that when an agent at the interim stage, he would be interested not only in efficiency but also in maximizing his own payoff. As such, it cannot form incentive compatibility. A better interpretation of the result is then as an upper bound on the efficiency of decentralized bargaining procedures. Indeed, any such procedure can be thought of as a mechanism, and must satisfy interim IR (even ex post IR), and ex ante budget balance (indeed, even ex post budget balance). The theorem says that decentralized bargaining in this case cannot be efficient. In the terminology of the Coase Theorem, private information creates a “transaction cost”.

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19.6 Cremer-McLean Full Surplus Extraction Theorem

A central theme of mechanism design under incomplete information is about surplus extraction ability of agents. Whatever the realistic observation or economic intuition tells us that the existence of private information makes the principal has to give up some positive information rent. However, the Myerson-Satterthwaite Theorem is based not only on the assumption of private value but also on the assumption that agents’ types are independent. If agents’ types are correlated, can the Myerson-Satterthwaite Impossibility Theorem that there is no BIC social choice rule that has the efficient decision rule and satisfies ex ante budget balance and interim IR be still true?

The answer is that Myerson-Satterthwaite Impossibility Theorem is no longer holds, which is the basic conclusion of the full surplus extraction theorem given by Cremer and McLean (1988). This theorem proves that, when private types are correlated, the principal of the mechanism can extract all information rents of agents through verifying information reported, and thus obtains first-best outcomes. Using the ideas of Cremer and McLean (1988), we can see that with correlated types, “almost anything is implementable”. Such a conclusion is called Cremer-McLean Theorem, which has a wide application. In auction theory, seller can design a Pareto efficient design mechanism. Cremer-McLean’s full surplus extraction theorem holds for any kind of correlated types.

Consider an economy with $n$ agents. Each agent’s type is discrete, $\theta_i \in \Theta_i \equiv \{\theta_{i1}, \theta_{i2}, \ldots, \theta_{ik_i}\}, \forall i \in N = \{1, \ldots, n\}$, and his utility function is quasi-linear with private values, i.e., $U_i(y, t, \theta_i) = v_i(y, \theta_i) + t_i$. Assume that agents’ types are correlated and have density function $\pi(\theta)$. When agent $i$’s type is $\theta_i$, his beliefs about other agents types are given by

$$\pi_i(\theta_{-i}|\theta_i) = \frac{\pi(\theta)}{\sum_{\theta_{-i} \in \Theta_{-i}} \pi(\theta_{-i}|\theta_i)}.$$  

For each agent $i$, his beliefs form a matrix $\Pi_i$ with element $\pi_i(\theta_{-i}|\theta_i)$. So matrix $\Pi_i$ has $k_i$ rows and $\prod_{j \neq i} k_j$ columns. Each row represents agent $i$’s beliefs distribution about other agents’ types when his type is $\theta_i$. If agents’ types are independent, all rows are the same, and thus the rank of the matrix $\Pi_i$ is 1. If types are correlated, then different rows represent different beliefs. We assume that $\Pi_i$ is row full rank that is $k_i$, which implies an agent’s beliefs distribution about others’ beliefs is different.

The following well-known Cremer-McLean Full Surplus Extraction Theorem shows that, even under the requirement of dominant incentive-compatibility, the principal can extract all surplus. The idea of the proof is pretty simpler. We first take any efficient and dominant incentive-compatible mechanism such as VCG mechanism, and then form a corresponding efficient
and dominant incentive-compatible mechanism with biding individually-rational constraint so that information rent is zero. As a result, we obtain a dominant incentive-compatible mechanism that implements first-best outcomes. Formally, we have the following Cremer-McLean Full Surplus Extraction Theorem.

**Theorem 19.6.1 (Cremer-McLean Full Surplus Extraction Theorem)** Suppose that agents’ types under private value model are correlated, and the information matrix $\Pi_i$ has row full rank. Then, for any efficient and dominant incentive-compatible social choice rule $(y(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$, there exists another dominant incentive-compatible and efficient social choice rule $(\tilde{y}(\cdot), \tilde{t}_1(\cdot), \ldots, \tilde{t}_n(\cdot))$ with the same decision rule $y(\cdot)$ in which interim individual rationality constraints are binding, and thus the first-best is implementable.

**Proof.** For any dominant incentive-compatible social choice rule $(y(\cdot), t_1(\cdot), t_2(\cdot))$, since agents have private-value quasi-linear utility functions, agent $i$’s interim expected utility with $\theta_i$ at equilibrium can be written as

$$U_i(\theta_i) = \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i)[u_i(y(\theta), \theta_i)) + t_i(\theta)].$$

Let $\tilde{u}_i^\star = [\tilde{U}(\theta_1^i), \tilde{U}(\theta_2^i), \ldots, \tilde{U}(\theta_{k_i}^i)]'$. Since $\Pi_i$ is row full rank, there is a vector $c_i = (c_i(\theta_{-i}))_{\theta_{-i} \in \theta_{-i}}$ with $\prod_{j \neq i} k_j$ columns, such that

$$\Pi_i c_i' = -\tilde{u}_i^\star;$$

that is, for $\forall \theta_i$, we have

$$\tilde{U}_i(\theta_i) + \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i)c_i(\theta_{-i}) = 0.$$

To ensure $i$’s interim individual rationality be binding, let

$$\tilde{t}_i(\theta) = t_i(\theta) + c_i(\theta_{-i}).$$

Construct a new mechanism, called Cremer-McLean mechanism:

$$(y(\theta), \tilde{t}_i(\cdot)) = (y(\theta), t_i(\theta) + c_i(\theta_{-i}), \forall i \in N).$$

Since $c_i(\theta_{-i})$ is independent of $\theta_i$, Cremer-McLean mechanism is still dominant incentive-compatible and efficient. But, under Cremer-McLean mechanism, agent $i$’s interim expected utility at equilibrium becomes

$$\bar{V}_i(\theta_i) = \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i)[u_i(y(\theta), \theta_i)) + \tilde{t}_i(\theta)]$$

$$= \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i)[u_i(y(\theta), \theta_i)) + t_i(\theta) + c_i(\theta_{-i})]$$

$$= \bar{U}_i(\theta_i) + \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i)c_i(\theta_{-i})$$

$$= 0.$$
Thus, all agents’ interim individual-rationality constraints are binding, the designer extracts all the surplus, and in particular, the designer can extract agents’ surplus through VCG mechanism to reach the first-best outcome. 

The above Cremer-McLean Theorem is based on the assumption of private values. In fact, Cremer-McLean Theorem still holds for interdependent environments. Although VCG mechanism with interdependent values is not dominant incentive-compatible, the generalized VCG mechanism (by modifying VCG mechanism) is dominant incentive-compatible so that Cremer-McLean Theorem still holds for interdependent model. Such a mechanism will be discussed in auction theory with interdependent values in Chapter 21.

It may be pointed out that Cremer-McLean Full Surplus Extraction Theorem relies on the assumptions of quasi-linear utility function and unlimited liability, and it also relies on an implicit assumption that there is no collusion; otherwise, it may not be true. A major criticism towards CM’s result comes from its vulnerability to collusion among agents. In the FSE mechanism, payments to and from agents depend on the reports of other agents. Therefore, it is highly susceptible to collusion among the agents, especially in nearly independent environments where these payments are very large.

The pioneering work that studies collusion in principal-multiagent setting is due to Laffont and Martimort (1997, 2000). In procurement/public good settings with two agents, they show that if the types are correlated, preventing collusion entails strict cost to the principal (LM, 2000). Che and Kim (2006) showed that the results obtained by Laffont and Mortimort (1997, 2000) depends the two-agent assumption. For \( n > 2 \), they show that agents’ collusion, including both reporting manipulation and arbitrage, is harmless to the principal in a broad class of circumstances.

For two-agent nonlinear pricing private goods environment, when types are correlated and arbitrage is allowed, Meng and Tian (2017) showed that it makes a big difference if types are positively or negatively correlated. Under negative correlation, the principal can exploit the conflict of interest inside the coalition to prevent collusion at no cost, i.e., CM’s result is still true. However, under positive correlation, however, the threat collusion forces the principal to distort the allocation away from the first-best level obtained without collusion.

19.7 Characterization of Bayesian-Nash Implementation

This section discusses the necessary and sufficient conditions for full Bayesian implementability of any set of social choice functions in general economic
19.7. CHARACTERIZATION OF BAYESIAN-NASH IMPLEMENTATION

environments (say, allowing for interdependent types).

Assume that agent $i$’s parametric utility function depends on type $\theta$, denoted by $u_i(x; \theta)$, where $x \in Z$ and $\theta \in \Theta$. The designer and agents know $\theta = (\theta_1, \cdots, \theta_n)$ has density function $\varphi(\theta)$ on $\Theta = \prod_{i \in N} \Theta_i$.

Let $X = \{x : \Theta \rightarrow A\}$ be the set of all feasible outcomes, and $\hat{F} \subseteq X$ a social choice rule that is the set of social choice functions $\hat{F} = \{f_1, f_2, \cdots\}$. When a social choice rule contains only one social choice function $\hat{F} = \{f\}$, it is called a social choice function, denoted by $f$. We will see below that a set of social choice functions $\hat{F}$ in general differs from social choice correspondence, unless every status $\theta \in \Theta$ is common knowledge event and the set satisfies closure that will be defined.

Given a mechanism $\langle M, h \rangle$, like Nash implementation, Bayesian implementation also involves the relationship between $\hat{F}$ and $B(\Gamma)$.

**Definition 19.7.1** A mechanism $\Gamma = \langle M, h \rangle$ is said to fully Bayesian implement $\hat{F}$, if

(i) for every $f \in \hat{F}$, there is a Bayesian-Nash equilibrium $m^*$ such that $h(m^*) = f(\cdot)$;

(ii) If $m^*$ is a Bayesian-Nash equilibrium, then $h(m^*) \in \hat{F}$.

If such a mechanism exists, we call $\hat{F}$ is fully Bayesian implementable.

**Definition 19.7.2** A mechanism $\Gamma = \langle M, h \rangle$ is said to Bayesian implement $\hat{F}$, if whenever $m^*$ is a Bayesian-Nash equilibrium, then $h(m^*) \in \hat{F}$, i.e., condition (ii) above holds.

If such a mechanism exists, we call $\hat{F}$ is Bayesian implementable. When $\hat{F}$ contains only one social choice function, Bayesian implementation and full Bayesian implementation are the same.

**Definition 19.7.3** A mechanism $\Gamma = \langle M, h \rangle$ partially Bayesian implements social choice set $\hat{F}$, if for every $f \in \hat{F}$, there exists a Bayesian-Nash equilibrium $m^*$, such that $h(m^*) = f(\cdot)$, i.e., condition (i) above holds.

If such a mechanism exists, we call that $\hat{F}$ is partially Bayesian Implementable. The reason we call partial Bayesian implementation is that there may exist another Bayesian-Nash equilibrium at which the resulting outcome is not in the social choice set.

Like implementation of a social goal under other solution concepts, full Bayesian implementability needs to deal with two issues: Condition (i) requires to seek a mechanism such that any outcome determined by the social choice rule is a Bayesian-Nash equilibrium outcome of the mechanism, while Condition (ii) requires all Bayesian-Nash equilibrium outcomes of the mechanism are the outcomes under the social choice rule. As for
Bayesian implementation, while the incentive compatibility requirement is central, it may not be sufficient for a mechanism to give all of desirable outcomes due to the existence of multiple equilibria. The severity of this multiple-equilibrium problem is a critical issue to be solved in (full) Bayesian implementation. Then, we need find conditions such that there exists a mechanism in which all Bayesian-Nash equilibrium outcomes are the outcomes under the social choice rule.

We first consider an example given by Palfrey and Srivastava (1989), which shows that even for a social choice function, how the issue of multiple Bayesian-Nash equilibria affects full Bayesian implementability.

**Example 19.7.1 (Palfrey and Srivastava, 1989)** Consider an exchange economy with two agents and two goods. Agent 1 has two preferences (types) $\theta_1$ and $\theta'_1$, and agent 2 only has one preference $\theta_2$. Agent 1’s preference is private information. Under preference profile $\theta = (\theta_1, \theta_2)$, Pareto efficient allocation is $x(\theta)$; and under $\theta' = (\theta'_1, \theta_2)$, Pareto efficient allocation is $x(\theta')$, see Figure 19.1. In the above two allocations, $x(\theta)$ and $x(\theta')$, agent 1 with type $\theta_1$ likes $x(\theta)$ more, while agent 1 with $\theta'_1$ feels two allocations are indifferent, but $\theta_2$ likes $x(\theta')$ more. Consider the social choice rule satisfying $f(\theta) = x(\theta)$ and $f(\theta') = x(\theta')$, which is Pareto efficient.

In the above exchange economy, consider the following mechanism $\Gamma = (\Theta, h(\cdot), m(\cdot))$: Each agent (mainly agent 1) reports his types $M_1 = \Theta_1 = \{\theta_1, \theta'_1\}$, $M_2 = \{\theta_2\}$. If the message reported is $m = (\theta_1, \theta_2)$, then $h(\theta_1, \theta_2) = x(\theta)$; if the message reported is $m = (\theta'_1, \theta_2)$, then $h(\theta'_1, \theta_2) = x(\theta')$. This mechanism has two Bayesian-Nash equilibria: One is $m_1(\theta_1) = \theta_1, \forall \theta_1 \in \Theta_1; m_2(\theta_2) = \theta_2$, and the other is $m_1(\theta_1) = \theta_1, \forall \theta_1 \in \Theta_1; m_2(\theta_2) = \theta_2$. However, these two Bayesian-Nash equilibria are equally desirable from Pareto optimality criterion. When the true type profile is $\theta' = (\theta'_1, \theta_2)$, the second Bayesian-Nash equilibrium outcome is not Pareto efficient. This example shows that we have a multiple-equilibria problem in Bayesian implementation.

To eliminate undesirable Bayesian-Nash equilibria, we consider the following mechanism, see Table 19.1.

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>$\theta_1$</th>
<th>$\theta'_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 2</td>
<td>$\theta_2$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$x(\theta)$</td>
<td>$x(\theta')$</td>
<td>$x(\theta)$</td>
</tr>
</tbody>
</table>

Table 19.1: Fully Bayesian implementable social choice rule

In this mechanism, agent 2 has two different reports $\theta_2$ and $\rho$. If agent 2 reports $\rho$, then $h(\theta_1, \rho) = x(\theta')$ and $h(\theta'_1, \rho) = x(\theta)$. Under this mechanism, $m_1(\theta_1) = \theta_1$ is not agent 1’s Bayesian-Nash equilibrium, otherwise agent 2
would report $\rho$, resulting in outcome $x(\theta')$. This is because $x(\theta')$ is strictly preferred to $x(\theta)$ for agent 2. When agent 2 chooses $\rho$, the optimal report of agent 1 with $\theta_1$ will be $m_1(\theta_1) = \theta'_1$.

One can easily see the mechanism has two Bayesian-Nash equilibria:

Equilibrium 1: $m(\theta_1) = \theta_1, m(\theta'_1) = \theta'_1, m_2(\theta_2) = \theta_2$, and its equilibrium outcome is $h(\theta_1, \theta_2) = x(\theta)$ and $h(\theta'_1, \theta_2) = x(\theta')$;

Equilibrium 2: $m(\theta_1) = \theta'_1, m(\theta'_1) = \theta_1, m_2(\theta_2) = \rho$, resulting in equilibrium outcome $h(\theta_1, \rho) = x(\theta)$ and $h(\theta'_1, \rho) = x(\theta')$.

Thus, although this mechanism has two Bayesian-Nash equilibria, the outcome at these two equilibria is the same as the outcome under the social choice rule, i.e., this mechanism fully Bayesian implements $f$.

For a general social choice set $\hat{F}$, Pastlewaite–Schmeidler (1986), Palfrey-Srivastava (1987, 1989, 1989), Mookherjee-Reichelstein (1990), Jackson (1991), Dutta-Sen (1994), Tian (1996, 1999) provided necessary and sufficient conditions for a social choice set $\hat{F}$ to be fully Bayesian implementable. Under some technical conditions, they showed that a social choice set $\hat{F}$ is fully Bayesian implementable if and only if $\hat{F}$ is Bayesian incentive compatible (it deals with the first problem that any outcome determined by a social choice rule is a Bayesian-Nash equilibrium outcome of the mechanism), and it is Bayesian monotonic (it solves the second problem: all Bayesian-Nash equilibrium outcomes of the mechanism are the outcomes under the social choice rules).

We first discuss these conditions. It will be seen that, Bayesian monotonicity is similar to Maskin monotonicity, using expected utility function replaces Bernoulli utility function.

The following important concept of Bayesian incentive-compatibility
A social choice set

**Definition 19.7.4** A social choice set \( \hat{F} \) is said to be Bayesian incentive-compatible or truthfully Bayesian implementable, if for every \( f \in \hat{F} \), truth-telling, i.e., \( m^*(\theta) = \theta \ \forall \theta \in \Theta \), is a Bayesian-Nash equilibrium of the revelation mechanism \( \Gamma = (\Theta, f) \), i.e., for any \( f \in \hat{F} \), we have

\[
E_{\theta_i}[u_i(f(\theta_i, \theta_{-i})), \theta) | \theta_i] \geq E_{\theta_i}[u_i(f(\theta'_i, \theta_{-i})), \theta) | \theta_i], \quad \forall i, \forall \theta_i, \theta'_i \in \Theta_i.
\]

We then have the first necessary condition for full Bayesian implementability of a social choice set \( \hat{F} \).

**Proposition 19.7.1** If social choice set \( \hat{F} \) is partially Bayesian implementable, then it satisfies Bayesian incentive-compatibility.

**Proof.** Suppose that mechanism \( \Gamma = (M, h) \) partially Bayesian implements social choice set \( \hat{F} \). If some social choice function \( f \in \hat{F} \) is not Bayesian incentive-compatible, then there are \( i \) and \( \theta_i, \theta'_i \in \Theta \) such that

\[
E_{\theta_i}[u_i(h(m_i(\theta_i)), m_{-i}(\theta_{-i})), \theta) | \theta_i] < E_{\theta_i}[u_i(h(m_i(\theta'_i)), m_{-i}(\theta_{-i})), \theta) | \theta_i]. \tag{19.7.8}
\]

Let \( m \in B(\Gamma) \) such that \( h \circ m = f \). When agent \( i \)'s type is \( \theta_i \), if he chooses \( m_i(\theta_i) \), then his expected utility is given by

\[
E_{\theta_i}[u_i(h(m_i(\theta_i)), m_{-i}(\theta_{-i})), \theta) | \theta_i] = E_{\theta_i}[u_i(f(\theta_i, \theta_{-i})), \theta) | \theta_i].
\]

If he chooses a message \( m'_i = m_i(\theta'_i) \), his expected utility is

\[
E_{\theta_i}[u_i(h(m_i(\theta'_i)), m_{-i}(\theta_{-i})), \theta) | \theta_i] = E_{\theta_i}[u_i(f(\theta'_i, \theta_{-i})), \theta) | \theta_i].
\]

By (19.7.8), we know the agent has incentive to deviate from \( m_i(\theta_i) \), contradicting to \( m \in B(\Gamma) \). \( \square \)

We now discuss the second necessary condition for a social choice set to be fully Bayesian implementable, i.e., Bayesian monotonicity condition. Consider a revelation mechanism, an agent \( i \), and a strategy \( \alpha_i : \Theta_i \rightarrow \Theta_i \). If agent \( i \) tells the truth, it implies \( \alpha_i(\theta_i) = \theta_i, \forall \theta_i \in \Theta_i \); otherwise \( \alpha_i \) is called a deception strategy of agent \( i \). We call \( \alpha(\theta) = (\alpha_1(\theta_1), \cdots, \alpha_n(\theta_n)) \) is a deception, if at least one agent has a deception strategy.

Let

\[
\alpha_{-i}(\theta_{-i}) = (\alpha_1(\theta_1), \cdots, \alpha_{i-1}(\theta_{i-1}), \alpha_{i+1}(\theta_{i+1}), \cdots, \alpha_n(\theta_n)).
\]

For a social choice function \( f \) and deception \( \alpha \), \( f \circ \alpha \) denotes the social choice under deception. When social status is \( \theta \), social choice outcome is \( f \circ \alpha(\theta) = f(\alpha(\theta)) \). For every \( \theta' \in \Theta \), define \( f_{\alpha_i(\theta_i)}(\theta') \equiv f(\alpha_i(\theta_i), \theta'_{-i}) \), i.e., it is the outcome when only agent \( i \) chooses the deception strategy.

Similar to Maskin monotonicity, we have the following Bayesian monotonicity condition.
19.7. CHARACTERIZATION OF BAYESIAN-NASH IMPLEMENTATION

**Definition 19.7.5 (Bayesian monotonicity)** A social choice set $\hat{F}$ is said to satisfy Bayesian monotonicity, if for any $f \in \hat{F}$ and deception $\alpha$ that results in $f \circ \alpha \notin \hat{F}$, there exists an agent $i$ and a function $y : \Theta_i \to A$ such that

$$E[u_i(f(\theta_i, \theta_{-i}), \theta)|\theta_i] \geq E[u_i(y(\theta_{-i}), \theta)|\theta_i]$$

(19.7.9)

for all $\theta_i \in \Theta_i$ and for some $\theta'_i$, we have

$$E[u_i(f(\theta'_i, \theta_{-i}), \theta)|\theta'_i] < E[u_i(y(\theta_{-i}), \theta')|\theta'_i].$$

(19.7.10)

Bayesian monotonicity is a variation of Maskin monotonicity under incomplete information. Its role is to avoid an undesirable outcome to be Bayesian-Nash equilibria. Consider a mechanism $\Gamma = \langle M, h \rangle$ that fully Bayesian implements a social choice set $\hat{F}$, and a social choice function $f \in \hat{F}$ that can be Bayesian implemented by Bayesian-Nash equilibrium $m^*$, i.e., for any $\theta \in \Theta$, $h(m(\theta)) = f(\theta)$. Suppose that agent $i$ adopts a deception strategy $\alpha$. Then strategy $m \circ \alpha$ results in an outcome given by $f \circ \alpha$. If $f \circ \alpha \notin \hat{F}$, then $m \circ \alpha$ is not a Bayesian-Nash equilibrium. The inequality in Bayesian monotonicity condition (19.7.10) avoids the possibility that $m \circ \alpha$ becomes a Bayesian-Nash equilibrium, while another inequality (19.7.9) ensures no one have incentive to cheat.

We then have another necessary condition for full Bayesian implementability, i.e., Bayesian monotonicity.

**Proposition 19.7.2** If a social choice set $\hat{F}$ is fully Bayesian implementable, then $\hat{F}$ must satisfy Bayesian monotonicity.

**Proof.** Suppose that $\Gamma = \langle M, h \rangle$ fully Bayesian implements a social choice set $\hat{F}$. Then, for any social choice function $f \in \hat{F}$, there is a Bayesian-Nash equilibrium $m^*$ such that $f(\theta) = h(m^*(\theta)), \forall \theta \in \Theta$. Let $\alpha$ be a deception such that $f \circ \alpha \notin \hat{F}$. Consider strategy $m^* \circ \alpha$. Under this strategy, for any $\theta \in \Theta$, agents’ strategy profile is $m^* \circ \alpha(\theta) = m^*(\alpha(\theta))$. Since $f \circ \alpha \notin \hat{F}$, full Bayesian implementability implies that $m^* \circ \alpha$ is not a Bayesian-Nash equilibrium, which implies that there is $\theta'_i \in \Theta_i$ such that the agent has incentive to choose some $m'_i \neq m^*_i(\alpha_i(\theta'_i))$ such that

$$E[u_i(h(m^* \circ \alpha(\theta'_i, \theta_{-i})), \theta'_i)|\theta_i] < E[u_i(h(m'_i, m^*_i(\alpha_{-i}(\theta_{-i}))), (\theta'_i, \theta_{-i}))|\theta_i].$$

(19.7.11)

Define $y : \Theta_i \to A$: $y(\theta_{-i}) = h(m'_i, m^*_i(\alpha_{-i}(\theta_{-i})))$. We have

$$y(\alpha_{-i}(\theta_{-i})) = h(m^*_i(m'_i, \alpha_{-i}(\theta_{-i}))).$$

Thus, from the above inequality (19.7.11), we can obtain (19.7.10), while (19.7.9) comes from the fact that $f$ can be Bayesian implemented by $h \circ m^*$. In this case, for any $\theta_i$, all agents have incentive to tell the truth. \qed
In addition to Bayesian incentive-compatibility and Bayesian monotonicity, to enable them also to be sufficient conditions, some technical conditions are needed. For fully implementable social choice set \( \hat{F} \), first they need satisfy the closure condition. We call a subset of a type space \( \Theta' \subseteq \Theta \) a common knowledge event, if for any \( \theta' = (\theta'_i, \theta'_{-i}) \in \Theta', \theta = (\theta_i, \theta_{-i}) \notin \Theta' \), we have \( \varphi(\theta'_{-i}|\theta_i) = 0, \forall i \).

If an agent does not know the true status, for all possible statuses, the agent need predict what messages the other agents will report. All such possible states consist of common knowledge, which are the bases for reporting messages.

**Definition 19.7.6 (Closure of Social Choice Set)** Let \( \Theta_1 \) and \( \Theta_2 \) be a partition of \( \Theta \), i.e., \( \Theta_1 \cap \Theta_2 = \emptyset \) and \( \Theta_1 \cup \Theta_2 = \Theta \). A social choice set \( \hat{F} \) is said to satisfy closure, if for any \( f_1, f_2 \in \hat{F} \), there is \( f \in \hat{F} \) such that \( f(\theta) = f_1(\theta), \forall \theta \in \Theta_1; f(\theta) = f_2(\theta), \forall \theta \in \Theta_2 \).

If every state \( \theta \in \Theta \) is common knowledge, it reduces to an economic environment with complete information discussed in the previous chapter, and thus a social choice set that satisfies closure becomes a social choice correspondence. If a social choice set \( \hat{F} \) does not satisfy closure, say, \( \Theta = \{\theta, \theta'\} \), and every state is common knowledge, \( \hat{F} = \{f_1, f_2\} \) satisfies \( f_1(\theta) = f_2(\theta') = a; f_1(\theta') = f_2(\theta) = b, a \neq b \), then the social choice set \( \hat{F} \) is not fully Bayesian implementable. This is because, if \( \hat{F} \) is fully Bayesian implementable, we need them to be Bayesian-Nash equilibria under two states \( a \) and \( b \), and then by the definitions of \( f_1 \) and \( f_2 \), there is no way that ensures implementable outcomes be different under these two different states. We should notice that in this case, social choice set is not the same as social choice correspondence, i.e., \( \hat{F} \neq F \), where \( F(\theta) = F(\theta') = \{a, b\} \).

Jackson (1991) characterizes the sufficient condition for full Bayesian implementation of a social choice set in incomplete information environments. We first give the definition of economic environment in incomplete information.

**Definition 19.7.7** An environment is called an economic environment, if for any state \( \theta \in \Theta \) and any outcome \( y \in Y \), there are two agents \( i \) and \( j \), and two outcomes \( y_i \) and \( y_j \) such that

\[
u_i(y_i, \theta) > u_i(y, \theta)\]

and

\[
u_j(y_j, \theta) > u_j(y, \theta).\]

Intuitively, for incomplete information economic environments, for any social choice function and state, there are at least two agents who hope to change social choice outcome under the state, which implies that a social choice outcome cannot make all agents reach their satiated points. If every
utility functions are monotonic in private goods economies, this condition is satisfied. Such a condition is also satisfied for public good economies and economies with externalities.

We now state the following theorem given by Jackson (1991) without proof.

**Proposition 19.7.3 (Necessity and Sufficiency for Full Bayesian Implementability)**

For economic environments with \( N \geq 3 \) agents, suppose that a social choice set \( \hat{F} \) satisfies the closure condition. Then, \( \hat{F} \) is fully Bayesian implementable if and only if it satisfies Bayesian incentive-compatibility and Bayesian monotonicity.

Under more general environments, Jackson (1991) strengthens Bayesian monotonicity by introducing monotonicity-no-veto-power condition and showed that in economic environments with \( N \geq 3 \) agents, if a social choice function satisfies Bayesian incentive-compatibility and monotonicity-no-veto-power condition, it is fully Bayesian implementable. For environments with two agents, Dutta and Sen (1994) provide sufficient conditions for a social choice function to be fully Bayesian implementable.

One can similarly investigate implementation of a social choice set in refinements of Bayesian-Nash equilibria and virtual Bayesian implementation. Palfrey-Srivastava (1989) showed that for economic environments with \( N \geq 3 \) agents, a necessary and sufficient condition for a social choice set \( F \) to be fully Bayesian implementable in undominated Bayesian equilibria is that it satisfies Bayesian incentive-compatibility. Abreu-Matsushima (1990), Matsushima (1993), Duggan (1993) and Tian (1997) show under various technical conditions that a social choice set is virtual Bayesian implementable if and only if the social goal is Bayesian incentive compatible.

### 19.8 Characterization of Ex-post Implementability

So far, we only discussed the Bayesian implementable mechanism design problem with Bayesian equilibrium as the solution. One of the basic assumptions discussed in this issue is that agents in the game have common knowledge about the distribution of participants’ types. However, assuming a common knowledge of the type distribution is a serious limitation on applications. This is due to the fact that, the common knowledge is very difficult to achieve under the incomplete information, including the mechanism designer, if not completely unachievable. Wilson (1987) sums up the validity and limitations of game theory on the analysis of real problems. There was a popular quote: ‘Game Theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one player’s probability assessment about another’s preferences or information. I foresee the progress of game theory as depending on successive
reductions in the base of common knowledge required to conduct useful analysis of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality.”

In the development of mechanism design theory, especially in the practice of auctions design, many researchers have focused on a more robust equilibrium concept, ex post equilibrium. Roughly speaking, the so-called ex post equilibrium means that the participant’s interim equilibrium strategy will not have the incentive to change or repent afterwards (that is, after knowing the true type of other people). The dominant equilibrium under private value we discussed earlier is a typical ex-post equilibrium. Because under the dominant strategy, participants do not have to consider the probability distribution of other participant types, i.e., strategy is independent of belief. If a participant’s utility function depends on the types of other participants, then the concept of ex-post equilibrium needs to be introduced. In this section, we mainly discuss the issue of the social choice set (function or correspondence) that can be ex post implemented.

Similar to the Nash implementation and Bayesian implementation discussed earlier, ex post implementation also has its own relevant incentive compatibility conditions and monotonicity conditions. This section discusses the necessary and sufficient conditions for ex post implementation. The discussion is mainly based on Bergemann and Morris (2005, 2008).

Assume that the set of participants is \( N = \{1, \ldots, n\} \), a type of participant \( i \) is \( \theta_i \in \Theta_i \), and a profile of types for all participants is \( \theta \in \Theta \). Assume that the set of feasible outcomes is \( A \), and the utility function of participant \( i \) is \( u_i : A \times \Theta \rightarrow R \). The value of participants depends on the type of other people, that is, it is an interdependent value model. A social choice function is denoted as \( f \). The set of all social choice functions is denoted as \( F \), and a subset of social choices is denoted as \( \hat{F} \subseteq F \). \( \Gamma = \langle M_1, \ldots, M_n, h(.) \rangle \) is a mechanism, a participant’s message pure strategy is \( m_i : \Theta_i \rightarrow M_i \), after which the ex post equilibrium definition is introduced.

**Definition 19.8.1 (Ex post equilibrium)** A message profile \( m^* = (m^*_1, \ldots, m^*_n) \) is an ex post equilibrium of mechanism \( \Gamma = \langle M_1, \ldots, M_n, h(.) \rangle \) if for any \( i \in N, \theta \in \Theta, m_i \in M_i \), we have

\[
u_i(h(m^*_i, m^-_i), \theta) \geq u_i(h(m_i, m^*_i), \theta).
\]

Obviously, if the utilities of the participants depend only on their own types, that is \( u_i(h(m(\theta), \theta)) = u_i(h(m(\theta)), \theta), \forall i \in N \), ex post equilibrium is equivalent to dominant equilibrium. As a corollary, here the necessary and sufficient conditions for full ex post implementation of the set of social choice functions are also necessary and sufficient conditions for a social choice correspondence to be fully implementable in dominant strategy equilibrium under private values. In games of incomplete information, ex post equilibrium has the characteristic of ex post no regret, that is, even if
the participant knows the types of other participants ex post, he will not change his strategy.

**Definition 19.8.2 (Full Ex post Implementation)** A social choice set \( \hat{F} \) is (pure-strategy) fully ex post implementable or fully implementable in ex post strategies if there is a mechanism \( \Gamma = (M, h(\cdot)) \) such that:

(i) For any \( f \in \hat{F} \), there is an ex post equilibrium, \( m^* \), satisfying:
\[
h(m^*(\theta)) = f(\theta), \quad \forall \theta \in \Theta;
\]

(ii) For any ex post equilibrium, \( m^* \), there is a social choice function, \( f \in \hat{F} \), satisfying:
\[
h(m^*(\theta)) = f(\theta), \quad \forall \theta \in \Theta.
\]

The above implementation requires that the equilibrium of the mechanism be exactly the same as the outcome of the social choice set, that is, to achieve full implementation. Similarly, we can define ex post implementation and partial ex post implementation.

Like full Nash implementation with complete information and Bayesian implementation with incomplete information, a social choice set usually needs to satisfy the conditions of incentive compatibility and monotonicity to be ex post implementable. Here we introduce ex post incentive compatibility and ex post monotonicity.

**Definition 19.8.3 (Ex Post Incentive Compatibility)** A social choice set \( \hat{F} \) is ex post incentive compatible if for any \( f \in \hat{F}, i \in N, \theta \in \Theta \) and \( \theta' \in \Theta_i \), we have
\[
u_i(f(\theta), \theta) \geq u_i(f(\theta', \theta_{-i}), \theta).
\]

Social choice set \( \hat{F} \) is strictly ex post incentive compatible if the above inequality strictly holds, that is, for all \( i \in N, \theta \in \Theta, \theta' \neq \theta_i \), we have
\[
u_i(f(\theta), \theta) > u_i(f(\theta', \theta_{-i}), \theta).
\]

In a direct mechanism, consider participant \( i \)'s manipulation of information disclosure. Let \( \alpha_i : \Theta_i \to \Theta_i \) be a deception of participant \( i \), denoted as \( \alpha_i(\theta_i) \neq \theta_i \), and \( \alpha(\theta) \) be a deception combination of participants. Under the social choice function \( f \), this deception will result in an outcome with \( f(\alpha(\theta)) \neq f(\theta) \). The ex post monotonicity condition guarantees that for each deception, there must exist an alert provided by one of the participants, and the provision of the alert satisfies incentive compatibility.

**Definition 19.8.4 (Ex Post Monotonicity)** Social choice set \( \hat{F} \) satisfies ex post monotonicity if for any \( f \in \hat{F} \) and a deception \( \alpha \) with \( f \circ \alpha \notin \hat{F} \), there is a participant \( i \in N \) and an outcome \( y \) such that:
\[
u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta),
\] (19.8.12)
When comparing ex post monotonicity to Maskin monotonicity. However, for environments that have the single-crossing property, if a social choice set \( F \) is fully ex post implementable, then it must satisfy ex post incentive compatibility and ex post monotonicity.

\[ u_i(f(\theta_i, \alpha_{-i}(\theta_{-i})), (\theta_i', \alpha_{-i}(\theta_{-i}))) \geq u_i(y, (\theta_i', \alpha_{-i}(\theta_{-i}))), \quad \forall \theta_i' \in \Theta_i. \]

(19.8.13) means that when a deception \( \alpha \) appears, the participant \( i \) has incentive to provide the alert \( y \); but the inequality (19.8.13) means that the alert will not appear in the absence of deception. For convenience of discussion, we define a set that given other participants’ types are \( \theta_{-i} \), participant \( i \) does not have incentives to provide an alert for all \( \theta_i \in \Theta_i \):

\[ Y_i^f(\theta_{-i}) = \{ y | u_i(y, (\theta_i', \theta_{-i})) \leq u_i(f(\theta_i', \theta_{-i}), (\theta_i', \theta_{-i})), \forall \theta_i' \in \Theta_i \}. \]

(19.8.14) implies that \( y \in Y_i^f(\theta_{-i}) \). \( Y_i^f(\theta_{-i}) \) is called a reward set, that is, participant \( i \) who reports information truthfully gets this reward, which depends on the social choice function \( f \). In the meantime, we define a successful reward set: \( Y_i^{f*}(\theta_{-i}) = \{ y : u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta) \} \cap Y_i^f(\theta_{-i}) \). In this set, the participant \( i \) only sends out an alert on deception, thus avoiding some bad choices. Here we discuss the necessary and sufficient conditions for ex post full implementation.

**Remark 19.8.1** When comparing ex post monotonicity to Maskin monotonicity, one may feel that the former is stronger than the latter. In the definition of ex post monotonicity, the truth-telling constraint is satisfied at \((\theta_i', \alpha_{-i}(\theta_{-i}))\) for all \( \theta_i' \in \Theta_i \), while Maskin monotonicity only requires it to hold on \( \alpha(\theta) \). However, this difference has no effect on ex post implementation and Nash implementation. Indeed, as proved by Bergemann and Morris (2008), in general, ex post monotonicity does not lead to Maskin monotonicity, nor does Maskin monotonicity. For example, the weakly Pareto efficient allocation satisfies Maskin monotonicity, but does not satisfy ex post monotonicity. The single unit auction with interdependent valuations satisfies ex post monotonicity, but does not satisfies Maskin monotonicity. However, for environments that have the single-crossing property, it can be shown that these two are equivalent.

**Theorem 19.8.1 (Necessary conditions for full ex post implementation)** If a social choice set \( \hat{F} \) is fully ex post implementable, then it must satisfy ex post incentive compatibility and ex post monotonicity.

**Proof.** Let \( \Gamma = (M_1, \cdots, M_n; h(\cdot)) \) be a mechanism that fully ex post implements social choice set \( \hat{F} \). For any given \( f \in \hat{F} \), by full ex post implementability, there is an ex post equilibrium \( m^* \) such that \( f = h \circ m^* \). Since \( m^* \) is an ex post equilibrium, for any \( i \in N, \theta_i' \in \Theta_i, \theta \in \Theta \), we have:

\[ u_i(h(m^*(\theta)), \theta) \geq u_i(h(m^*_i(\theta'_i), m^*_{-i}(\theta_{-i})), \theta). \]
As \( f(\theta) = h(m^*(\theta)) \), \( f(\theta', \theta_{-i}) = h(m^*_{\theta'}(\theta_{-i}), m^*_{-\theta'}(\theta_{-i})) \), the social choice set \( \hat{F} \) satisfies ex post incentive compatibility.

Consider any deception \( \alpha \) that satisfies \( f \circ \alpha \notin \hat{F} \), then \( m^* \circ \alpha \) must not be an equilibrium under some \( \theta \). Then, there is \( i \in N \) and \( m_i \in M_i \) such that:
\[
u_i(h(m_i, \alpha_{-i}(\theta_{-i})), \theta) > u_i(h(m^*(\alpha(\theta))), \theta).
\]
Letting \( y \equiv h(m_i, \alpha_{-i}(\theta_{-i})) \), we have
\[
u_i(y, \theta) > u_i(h(m^*(\alpha(\theta))), \theta).
\]

Since \( m^* \) is an ex post equilibrium and \( f = h \circ m^* \), we have:
\[
u_i(f(\theta', \alpha_{-i}(\theta_{-i})), (\theta', \alpha_{-i}(\theta_{-i}))) = u_i(h(m^*(\theta', \alpha_{-i}(\theta_{-i}))), (\theta', \alpha_{-i}(\theta_{-i})))
\geq u_i(h(m_i, m^*_{\alpha_{-i}(\theta_{-i})}), (\theta', \alpha_{-i}(\theta_{-i})))
= u_i(y, (\theta', \alpha_{-i}(\theta_{-i}))).
\]

Therefore, the alert \( y \) satisfies the incentive compatibility of participant \( i \) or \( y \in Y_i(\alpha_{-i}(\theta_{-i})) \). We thus have shown the ex post monotonicity of \( \hat{F} \). \( \square \)

For the economic environments specified in Definition 19.7.7, Bergemann and Morris (2008) also obtain sufficient conditions similar to those for Bayesian implementation.

**Theorem 19.8.2 (Sufficient conditions of ex post implementation)** In economic environments with more than three participants, if a social choice set \( \hat{F} \) satisfies ex post incentive compatibility and ex post monotonicity, then it is ex post implementable.

**Proof.** For the social choice set \( \hat{F} \), Bergemann and Morris (2008) constructed the following mechanism.

First of all, the message space of each participant contains message \( m_i = (\theta_i, f_i, z_i, y_i) \), in which \( \theta_i \) is the type of participant \( i \) himself, \( f_i \) is social choice function suggested by participant \( i \), \( z_i \) is similar to the positive integer in the Maskin implementation mechanism, but the scope of \( z_i \) here is \( N = \{1, \cdots, n\} \), and \( y_i \) is the reward (outcome) that participant \( i \) proposed. The message space of \( i \) is \( M_i = \Theta_i \times \hat{F} \times N \times Y_i \), in which \( Y_i \) is the set of all possible outcomes. The mechanism is restricted by the following rules:

**Rule 1.** If \( \forall i, f_i = f \), then \( h(m) = f(\theta) \).

**Rule 2.** If there is a participant \( j \) and a social choice rule \( f \in \hat{F} \) such that for \( \forall i \neq j \) we have \( f_i = f \) and \( f_j \neq f \), then we choose \( y_j \) when \( y_j \in Y_j(f(\theta_{-i})) \); otherwise, we choose \( f(\theta) \).

**Rule 3.** In addition to the two cases above, if \( j(z) = \sum_{i=1}^{n} z_i (mod n) \), then we choose outcome \( y_j(z) \).

Now we prove the sufficiency of the theorem.
First, we rewrite the message of participant $i$ as:

$$m_i(\theta_i) = (m_i^1(\theta_i), m_i^2(\theta_i), m_i^3(\theta_i), m_i^4(\theta_i)) \in \Theta_i \times \hat{F} \times N \times Y.$$ 

We shall complete the proof in three steps.

Step one: For a given $f \in \hat{F}$, there is an ex post equilibrium $m^*$ such that $h(m^*(\theta)) = f(\theta), \forall \theta \in \Theta$.

Consider the message profile $(m_i^*(\theta_i) = (\theta_i, f, \ldots), i \in N)$. By rule 1, $h(m^*(\theta)) = f(\theta)$. We prove that this message profile is an ex post equilibrium. Given other participants' messages $(m_j^*(\theta_j), j \neq i)$, consider that participant $i$ deviates from message $m_i^*(\theta_i)$, choosing to send a message $m_i(\theta_i) = (\theta_i', f_i, \ldots)$. When $y_i \not\in Y_i^f(\theta_{-i})$, then by rule 2, the society will choose $f(\theta_i', \theta_{-i})$, then for participant $i$, the payoff of deviation is:

$$u_i(f(\theta_i', \theta_{-i}), (\theta_i, \theta_{-i})) - u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) \leq 0.$$ 

The above inequality is derived by the ex post incentive compatibility, and the participants do not have incentives to deviate. When $f_i \neq f, y_i \in Y_i^f(\theta_{-i})$, by rule 2, the society will choose $y_i$, for participant $i$, the payoff of deviation is:

$$u_i(y_i, (\theta_i, \theta_{-i})) - u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) \leq 0.$$ 

The inequality above is derived by the definition (19.8.14) of $Y_i^f(\theta_{-i})$ or the ex post monotonicity condition (19.8.13). When $f_j = f$, according to rule 1, the other message choices of the participant will not change the choice of whole society.

Step two: For any ex post equilibrium $m^*$, there is a social choice function $f \in \hat{F}$ such that $m_i^2(\theta_i) = f, \forall i \in N, \theta \in \Theta$. Then by rule 1, $h \circ m^* = f$.

We prove this by way of contradiction. Suppose that, for ex post equilibrium $m^*$, and for any $f \in \hat{F}$, there is an $i$ and a $\theta_i$ such that $m_i^2(\theta_i) \neq f$. Then, there is a state $\theta$ such that rule 1 is not applicable.

We first assume that rule 2 can be applied to state $\theta$, which means that there is a $j$ and an $f$ such that $f_j = f, i \neq j$. Now, for any participant $i \neq j$ whose state is $\theta_i$, he sends a message $m_i(\cdot, f_i, z_i, y_i)$ with taking the other participants' states as $\theta_{-i}$, in which $f_i \neq f, i = \sum_{k=1}^{n} z_k (mod \ n)$. Therefore, the social rule will choose $y_i$ under rule 3 and the utility is $u_i(y_i, \theta)$. Hence, if $m^*$ is an ex post equilibrium, for any $y \in Y, i \neq j$, it must satisfy $u_i(h(m^*(\theta)), \theta) \geq u_i(y, \theta)$, which contradicts with the condition of the economic environment.

Now we assume rule 3 can be applied in state $\theta$, which means for any participant with state $\theta_i$, he will send the message $m_i(\cdot, f_i, z_i, y_i)$ when taking the other participants' states as $\theta_{-i}$, in which $i = \sum_{k=1}^{n} z_k (mod \ n)$. Then by rule 3, the social rule will choose $y_i$ and the utility level is $u_i(y_i, \theta)$. Therefore, if $m^*$ is an ex post equilibrium, for any $y \in Y, i \neq j$, there must
be \( u_i(h(m^*(\theta)), \theta) \geq u_i(y, \theta) \). It contradicts with the condition of the economic environment.

Step three: For any social choice function \( f \in \hat{F} \) and any ex post equilibrium \( m^* \) that satisfies \( m^*\theta_i = f, \forall i, \theta_i, \) we must have \( f \circ m^*1 \in \hat{F} \).

Suppose by way of contradiction that \( f \circ m^*1 \notin \hat{F} \). According to ex post monotonicity, there is an \( i, \theta, y \in Y_i(f(m^*1(\theta_{-i}))) \) satisfying:

\[
u_i(y, \theta) > u_i(f(m^*1(\theta)), \theta).
\]

Consider a participant \( i \) with state \( \theta_i \), believing other participants’ states to be \( \theta_{-i} \), he will send the message \( m_i = (.., f_i, .., y) \) that satisfies \( f_i \neq f \). Given other participants choose their equilibrium strategies, by rule 2, the social choice outcome is \( h(m_i, m^*\theta_{-i}) = y \). The participant \( i \)'s expected utility under \( m_i \) is:

\[
u_i(h(m_i, m^*(\theta_{-i})), \theta) = u_i(y, \theta) > u_i(f(m^*1(\theta)), \theta) = u_i(h(m^*(\theta)), \theta),
\]

which contradicts the fact that \( m^* \) is an ex post equilibrium.

From the above three steps, we have proved that under the economic environments, the ex post incentive compatibility and ex post monotonicity are sufficient conditions for ex post implementation.

For the non-economic environment, Bergemann and Morris (2008) show that with an additional condition of “no veto power”, ex post incentive compatibility and ex post monotonicity are sufficient conditions for full ex post implementation. For the economic environment that satisfies the single-crossing property, they also prove that any social choice set with strict ex post incentive compatibility and consisted of interior points satisfies ex post monotonicity condition and is therefore fully ex post implementable. Besides, when the number of participants is greater than 2, Chapter 21 will introduce the generalized direct VCG mechanism under the interdependent valuation environments, and it also satisfies ex post monotonicity and has a unique ex post equilibrium.

In addition, Ohashi (2012) gave similar sufficient and necessary conditions for full ex post implementation in the two-participant environments. Also, using the concept of ex post equilibrium, Bergemann and Morris (2005) established an analytical framework for robust mechanism design.

19.9 Biographies

19.9.1 Michael Spence

A. Michael Spence (1943—) was born in New Jersey of the United States of America, and he received his PhD degree from Harvard University in 1972. He is a professor at New York University and a senior fellow at Stanford.
University's Hoover Institution. In 2001, he won the Nobel Memorial Prize in Economic Sciences, along with George Akerlof and Joseph Stiglitz, for their pioneering research on the dynamics of information flows and market development.

Spence’s most important research achievement is how the individuals with informational advantages transmit the information “signal” reliably to the individuals with informational disadvantages, in order to avoid problems related to adverse selection in the market. Signals require individuals to take observational and costly measures to convince other individuals of their abilities or, more generally, of the value or quality of their products. Spence’s contribution lies in formalizing this idea, and analyzing its influence at the same time. Spence used education as a signal of productivity in the labor market in his ground breaking study (based on his doctoral thesis) in 1973. The basic point of view was that, the signal would not have a successful effect unless the signal cost was significantly different among job seekers. Employers could not distinguish highly competent job seekers from those with low abilities, unless when the latter chose a lower level of education the former found that their investment in education paid off. If the employer could not distinguish between high and low labor abilities, it would lead the labor market to employ the low ability with low wages, forming the phenomenon of “Bad money drives out good” in the labor market. Spence also pointed out the possibility of different “expected” equilibria between education and wages, where men and whites earn more than women and blacks when productivity is equal.

Spence’s subsequent research included extensive applied research that expanded this theory, confirmed the importance of different market signaling, and analyzed a large number of economic phenomena, such as expensive advertising and full warrant as productivity signals; the active price reduction as a signal of market power; the strategy of delaying wage quotation as a signal of bargaining power; debt financing as a sign of profitability rather than the financing method of issuing new shares; monetary policy at the expense of recession as a signal of strong commitment to lower high inflations.

19.9.2 Roger B. Myerson

Roger Bruce Myerson (1951—) is an economics professor at the University of Chicago, and was awarded Nobel Prize in economics with Leonid Hurwicz and Eric Maskin in 2007 for his contribution to the creation and development of mechanism design theory and auction theory, including the optimal mechanism design, revenue-equivalence theorem and etc.

Myerson was born in Boston on March 29, 1951. In 1976, he received his Ph.D. in applied mathematics from Harvard University. His doctoral thesis is “A Theory of Cooperative Games”. He has been at the University of
Chicago since 2007 and his research expertise includes mechanism design theory, the game theory, and the voting system in political science. Myerson’s “Optimal Auction Design” published in 1981 was the cornerstone of the optimal mechanism design. He also published books “Game Theory: Analysis of Conflict” and “Probability Models for Economic Decisions”.

After Vickrey, economists began to work on auction theory. Among them, Myerson deserves special mention. He used the newly developed mechanism design theory to revisit the auction theory and promoted Vickrey’s theory on this basis. Myerson came to the conclusion through rigorous mathematical analysis: In a series of assumptions (such as symmetric independent private values etc.), all standard auction mechanisms will bring the same expected revenue to the auctioneer. Obviously, this conclusion surpassed previous research ideas of Vickrey and other scholars, who just focused on specific auction forms, and greatly advanced the auction theory.

Myerson is an economist who can make abstract economic theory practical. He had solved economic problems for California’s power crisis, and is well known in the economics community. In the 1980s, electricity market reform in California was to break the drawbacks of power monopoly, but it was impossible for the power industry to implement perfect competition. The best way is to use oligopoly. Myerson applied the mechanism design theory to design a plan for California’s electricity market reform. In addition, he also solved the problem of recruiting students at the medical school in the United States. Myerson’s contribution to the real economy has impressed economics profession: It is therefore possible to create an “Economic Engineering” and to make economics as practical as engineering, and to design economic phenomena as well.

19.10 Exercises

Exercise 19.1 Consider economic environments with one seller and one buyer. The value of buyer is $v_b$ and the value of seller is $v_s$. Both $v_b$ and $v_s$ follow the uniform distribution on $[0, 1]$, and are independent. The trading mechanism is given as follows: the buyer and the seller simultaneously make offers at the purchasing price $p_b$ and the selling price $p_s$, respectively. If $p_b \geq p_s$, the transaction is conducted with the price $p = (p_b + p_s) / 2$, and the profits of buyers and sellers are, respectively, given by:

$$v_b - p,$$
$$p - v_s.$$

If $p_b < p_s$, there will be no transaction and the profits of buyers and sellers are 0. It is known that under this mechanism, there are multiple-equilibrium
bidding strategies. For example, there is a linear equilibrium bidding strategy: \( p_i(\upsilon_i) = a_i + c_i \upsilon_i, i \in b, s \). Find the linear equilibrium strategy for buyers and sellers and prove that whether it can truthfully implement ex post Pareto efficient outcomes.

**Exercise 19.2 (Arrow, 1979; d’Aspremont and Gerard-Varet, 1979)** Suppose that the set of participants is \( N \), with \( |N| = n \), and the set of types for participant \( i \) is \( \Theta_i \). The utility functions are \( u_i(d, \theta_i, t_i) = v_i(d, \theta_i) + t_i \). AVG mechanism \( (d(\theta), t_i(\theta)) \) satisfies:

\[
d(\theta) \in \arg\max_d \sum_{i \in N} v_i(d, \theta_i);
\]

\[
t_i(\theta) = E_{\Theta_i}[\sum_{j \neq i} v_j(d(\theta), \theta_j)|\theta_i] - \frac{1}{n} \sum_{k \neq i} E_{\Theta_i}[\sum_{j \neq k} v_j(d(\theta), \theta_j)|\theta_k].
\]

Prove that if participants’ type distributions are independent, the above AVG mechanism satisfies Bayesian incentive compatibility and is interim budget balanced.

**Exercise 19.3 (Diamantaras, 2009)** Consider economic environments with two participants and one public good. The set of types for each participant is \( \Theta_i = \{0, 1\}, i = 1, 2 \). The two types have equal probability, and the types of two participants are independently distributed. Participant \( i \)'s evaluation of the public good is \( \theta_i \). Assume the cost of public good is \( c = \frac{3}{4} \).

Prove that in such environments there is no mechanism that simultaneously satisfies the following features: (i) interim participation constraints; (ii) Bayesian incentive compatibility; (iii) the decision-making mechanism of public good provision is efficient; (iv) mechanism is budget balanced, i.e., it does not require extra resources from outside of the economy under consideration.

**Exercise 19.4 (Palfrey and Srivastava, 1989)** Consider economies in which the set of participants is \( I = \{1, 2, 3\} \), the set of alternatives is \( A = \{a, b\} \) and the set of each participant is \( T^i = \{t_a, t_b\} \). Suppose that the types of participants are all independently and identically distributed, namely \( \text{Prob}(t^i = t_b) = q > 0.5^{1/2} \), and each participant’s utility function is the same and satisfies:

\[
u^i(a, t_a) = 1 > 0 = u^i(b, t_a);
\]

\[
u^i(b, t_b) = 1 > 0 = u^i(a, t_b).
\]

Suppose that the social choice function \( f \) is given as follows:

\[
f(t^1, t^2, t^3) = \begin{cases} 
  t_a, & \text{if } t^1 = t^2 = t_a, \text{ or } t^1 = t^3 = t_a, \text{ or } t^2 = t^3 = t_a; \\
  t_b, & \text{if } t^1 = t^2 = t_b, \text{ or } t^1 = t^3 = t_b, \text{ or } t^2 = t^3 = t_b.
\end{cases}
\]
1. Prove that $f(\cdot)$ is the only allocation rule that satisfies the following five properties: (i) It is Bayesian incentive compatible; (ii) It is ex ante, interim, and ex post efficient; (iii) $f(t)$ is the majority voting decision rule; (iv) It maximizes the Arrow social welfare function; (v) It can be implemented in dominant strategies by a direct revelation mechanism.

2. Prove that $f(\cdot)$ is not fully Bayesian implementable (hint: prove that it does not satisfy the Bayesian monotonicity).

**Exercise 19.5 (Palfrey and Srivastava, 1989)** Suppose that the set of participants is $I = \{1, 2, 3\}$ and the set of alternatives is $A = \{a, b\}$. Participant's type spaces are $T_i = \{t^a, t^b\}$, which is independently and identically distributed, with $\text{prob}(t_i = t^a) = q > 0.5^{1/2}$, and their utility functions are the same and depends on the type of other participants and satisfies:

$$ u_i(a, t) = \begin{cases} 1, & \text{if at least two participants are of the type } t^a, \\ 0, & \text{otherwise}. \end{cases} $$

$$ u_i(b, t) = \begin{cases} 1, & \text{if at least two participants are of the type } t^b, \\ 0, & \text{otherwise}. \end{cases} $$

Suppose that the social choice function $f$ is given as follows:

$$ f(t_1, t_2, t_3) = \begin{cases} t^a, & \text{if } t_1 = t_2 = t^a, \text{or } t_1 = t_3 = t^a, \text{or } t_2 = t_3 = t^a, \\ t^b, & \text{if } t_1 = t_2 = t^b, \text{or } t_1 = t_3 = t^b, \text{or } t_2 = t_3 = t^b. \end{cases} $$

Prove that this social choice function cannot be implemented in undominated Bayesian equilibrium. The so-called undominance means that for each participant, there is no other strategy that weakly dominates this strategy.

**Exercise 19.6 (Characterization of Bayesian Incentive Compatibility)** Consider the linear model of independent distribution of private values. Prove that social choice rule $(y(\cdot), t_1(\cdot), \cdots, t_n(\cdot))$ is Bayesian incentive compatible if and only if for all $i \in N$,

1. $E_{\theta \rightarrow} y_i(\theta_i, \theta_{-i})$ is non-decreasing in $\theta_i$;

2. $E_{\theta \rightarrow} U_i(\theta_i) = E_{\theta \rightarrow} [U_i(\theta_i, \theta_{-i})] + \int_{\theta_i}^{\theta^b} E_{\theta \rightarrow} y_i(\tau, \theta_{-i}) \, d\tau, \forall \theta_i \in \Theta_i$.

**Exercise 19.7** A seller and a buyer are bargaining over an indivisible item. The valuation of the buyer is $\theta_b = 10$. There are two possible valuations for the seller, i.e., $\theta_s \in \{0, 9\}$. Let $t$ be the period when the transaction occurs ($t = 1, 2, \cdots$), and $P$ represent the price agreed-upon. The discount factor for both the buyer and the seller is $\delta$. 
1. In the current economic environments, what is the set of feasible alternatives?

2. Imagine that under a Bayesian Nash equilibrium of the bargaining process, when the valuation of the seller is $0$, the transaction occurs immediately; and when the valuation of the seller is $\theta_s$, the transaction price agreed-upon is $(10 + \theta_s)/2$. When the valuation of the seller is $9$, what is the earliest possible time for the transaction to take place?

Exercise 19.8 Consider Myerson-Satterthwaite bilateral trade model under discrete conditions. Given the valuation of buyer is $\nu$ and the cost of seller is $c$ that is uniformly distributed over $1, 2, 3, 4$.

1. Prove that the efficient trade is Bayesian incentive-compatible.

2. What does this example tell us about the Myerson-Satterthwaite impossibility theorem?

Exercise 19.9 Consider a direct incentive-compatible mechanism $(q, t)$, in which $t$ is ex ante budget balanced, that is $E_\theta \sum_{i=1}^N t_i(\theta) = 0$. Suppose that the type is independently distributed.

1. Prove that there is another Bayesian incentive-incapable mechanism $(q, t')$ such that $t'$ is ex post budget balanced, namely $\sum_{i=1}^N t'_i(\theta) = 0$.

2. Prove that $E_\theta - \sum_{i=1}^N t_i(\theta) = E_\theta - \sum_{i=1}^N t'_i(\theta)$.

3. Prove that if there is an individually rational VCG mechanism $(q^*, t)$ such that $E_\theta \sum_{i=1}^N t_i(\theta) \geq 0$, then there is a Bayesian incentive-compatible, budget balanced, individually rational and efficient mechanism denoted $(q^*, t')$.

Exercise 19.10 Consider a bilateral trading economy where two participants initially own a unit of commodity. Each participant’s valuation for every unit of the product consumed is $\theta_i (i = 1, 2)$. Suppose that $\theta_i$ follows a uniform distribution over $[0, 1]$ and is independent of each other.

1. Characterize a trading rule in an ex post efficient social choice function.

2. Consider the following mechanism: Each participant submits a bid, and the highest bidder obtains the unit of good from the opponent participant and pays the offer. Give a symmetric Bayesian-Nash equilibrium for this mechanism.

3. What is the social choice function that can be implemented by this mechanism? Verify that it is incentive compatible. Is it efficient? Does it satisfy individual rationality? Intuitively, why does it differ from the conclusion of Myerson-Satterthwaite theorem?
Exercise 19.11 Consider the problem of bilateral trading between a seller and a consumer. The seller’s production cost is determined by its cost type, and the cost type is the seller’s private information. When the seller’s type is \( \theta \), if the consumer’s total payment is \( x \), then the former’s profit is \( x - \theta q \). At the same time, for \( q \) units of goods, the value of the consumer depends on the seller’s type \( \theta \), which is expressed as \( \pi(q|\theta) = (4 + 2\theta)q^{1/2} \). Therefore, the consumer obtains \( q \) unit goods by paying \( x \) from the seller whose type is \( \theta \), and the net surplus is \( (4 + 2\theta)\sqrt{q} - x \). Any participant can refuse to participate in the transaction, and then \( q = 0 \) and \( x = 0 \).

1. When the seller’s type \( \theta \) is common knowledge, find the expression of the quantity of goods, denoted by \( q^*(\theta) \), that maximizes the social surplus.

2. Suppose that the seller’s type is either \( \theta_L = 2 \) or \( \theta_H = 3 \). The probability of being a low-cost \( \theta_L \)-type is \( p_L \), and the probability of being a high-cost \( \theta_H \)-type is \( p_H = 1 - p_L \). For the consumer’s expected net return maximization problem that takes into account incentive constraints and participation constraints, find a trading plan and give the constrained optimization problem. (Hint: \( \pi(q|\theta_L) = 8\sqrt{q} \) and \( \pi(q|\theta_H) = 10\sqrt{q} \).)

3. Write down the problem for finding the consumer’s optimal Bayesian incentive compatible trading plan.

4. Consider the following case: the consumer believes that the seller’s cost type be uniformly distributed over \([2, 3]\). Write down the problem for finding the consumer’s optimal Bayesian incentive compatible trading plan, which maximizes the consumer’s expected surplus subject to the participation constraint and incentive-compatibility constraint.

5. Consider another case: the seller’s cost type is either \( \theta_L = 2 \) or \( \theta_H = 3 \). In all incentive-compatible plans where the consumer’s surplus is non-negative for any type of seller, find the optimal trading plan for both types of sellers.

6. For the solution to the problem (5), prove that if \( p_L \) is close enough to 0 and \( p_H \) is close enough to 1, then there is a pooling strategy that can give the consumer a non-negative expected surplus, which is better than the trading plan in question (5).

Exercise 19.12 There are two construction firms that bid for the construction of a government building. Each firm is one of two types: an \( H \)-type firm can use the cost \( C_H \) to construct a high-quality project that values \( v_H \); and an \( L \)-type firm can use the cost \( C_L \) to construct a low-quality
project that values \( v_L \). Here, \( C_H > C_L \) and \( v_H > v_L \). Suppose that \( v_H - C_H > v_L - C_L > 0 \). The types are independent of each other and are the private information of firms. Each firm has a probability of \( v \) to be a high-quality firm. Here we consider the direct revelation mechanism. When one firm declares its type as \( \theta \) and the other firm declares its type as \( \theta' \), we use \( q(\theta, \theta') \), \( \theta, \theta' \in \{H, L\} \) to represent the probability that the previous firm wins the bid; similarly, we use \( T(\theta, \theta') \) to represent the government’s payment. Given that both the government and the two firms are risk-neutral, the government’s goal is to maximize its expected surplus, denoted \( E(v - \omega) \).

1. Write down the government’s optimal bidding design problem that satisfies participation constraints and incentive-compatibility constraints.

2. Derive the government’s optimal bidding mechanism.

**Exercise 19.13 (Dana and Spier, 1994)** Suppose that there are two firms, \( j = 1, 2 \), who shall obtain the production rights in a given market through competition. A social planner designs an optimal production-rights auction to maximize the expectation of a social surplus function. The social surplus function is defined by

\[
W = \sum_j \pi_j + S + (\lambda - 1) \sum_j t_j,
\]

in which \( \pi_j \) is the firm \( j \)'s total (pre-transfer) profit, \( S \) is the consumer’s surplus, \( \lambda > 1 \) is the shadow cost of public funds, and \( t_j \) denotes the transfer from firm \( j \) to the planner. The auction determines each firm’s transfer and the structure of a market, that is, either two firms do not obtain production rights, or one firm obtains production rights, or both have the right to production.

Every firm \( j \) privately observes its fixed production cost, denoted by \( \theta_j \). The fixed costs \( \theta_1 \) and \( \theta_2 \) follow an independent and identical distribution over \( [\theta^1, \theta^2] \). The density function \( \phi(\cdot) \) and the distribution function \( \Phi(\cdot) \) are all continuously differentiable. Suppose that \( \frac{\Phi'(\cdot)}{\phi(\cdot)} \) is increasing in \( \theta \). The firms have a common marginal cost \( c < 1 \) and produce homogeneous products. The inverse market demand is \( p(x) = 1 - x \). If both firms obtain production rights, they become Cournot competitors.

1. Write the problem of the optimal auction mechanism for production rights.

2. Characterize the optimal auction mechanism.

**Exercise 19.14** There is a seller \( (i = 0) \) who owns two identical indivisible goods. At the same time, there are two buyers \( (i = 1, 2) \). For a unit of each good, buyers and sellers each have a willingness to pay, denoted by \( \theta_i \),
which are private information. The three participants’ willingness to pay follows a uniform distribution over $[0, 1]$ and is independent of each other. This is a common knowledge. For the portion of the product that exceeds one unit, everyone’s willingness to pay is 0.

1. Describe the efficient allocation function $f(\theta_0, \theta_1, \theta_2)$ for these two units of goods.

2. For an incentive-compatible mechanism that truthfully Bayesian implements the efficient allocation of these two units of goods, what is the interim expected utility of each participant?

3. What is the interim participation constraint for each participant?

4. When there is no external transfer, is there a mechanism that satisfies the participation constraint in question (3), as well as incentive-compatibility constraints, and can efficiently allocate these two units of goods?

Exercise 19.15 There are two types of students, denoted by $L$ and $H$, searching for jobs in the market. There are at least two potential employers. The profit of hiring type-$i$ students is $\pi_i$, and $\pi_H > \pi_L$. Recruitment is conducted as follows: (1) student observes his own type (not observed by the employer); (2) student chooses an unproductive education level $e$, which is costly and can be observed by the employer; (3) the employer offers a salary $w$ to future workers (current students) based on the observed level of education (Bertrand competition); (4) student chooses an employer and becomes a worker. The utility function for $i$-type worker who accepts $w$ is $U(w, e) = w - c_i(e)$. The corresponding employer’s profit is $\pi_i - w$. Suppose $C'_H(e) > C'_L(e)$ for all $e > 0$. Only consider pure strategy equilibrium.

1. Does this model satisfy Spence-Mirrlees single-crossing property? Explain your answer.

2. What is the level of education obtained by a student of type $L$ in the separating equilibrium? Derive the expressions of the maximum and minimum education levels of workers of type $H$ in the separating equilibrium.

3. Derive the expressions of the maximum and minimum education levels of all the students in the pooling equilibrium.

4. Which equilibrium outcome is more dominant? Explain your answer.

5. Suppose that the government taxes $t$ for education level (that is, a person who receives $e$ years of education will pay a tax of $te$), and the government holds all the tax revenue. How does this tax change the
separating and pooling equilibrium conditions you derived in questions (2) and (3)?

6. Only consider efficient separating equilibrium. Based on question (5), derive the partial derivatives of $H$- and $L$-type workers’ utility functions with respect to tax $t$. Give the sign of the partial derivatives. How does this tax affect the welfare of workers?

**Exercise 19.16** Answer the following questions:

1. Under conditions of complete information, can single-crossing property imply Maskin monotonicity? If yes, prove it; if no, give a counterexample.

2. Under conditions of incomplete information, can single-crossing property imply Bayesian monotonicity? If yes, prove it; if no, give a counterexample.

**Exercise 19.17** Give an example to illustrate the independence between Maskin monotonicity and ex post monotonicity.

**Exercise 19.18** (Bergemann and Morris, 2008) Consider a modified VCG mechanism under interdependent values. We call $(\Theta, y^*, t^*)$ a generalized VCG mechanism if the highest bidder gets the good and the payment price is $\max_{j \neq i} u_j(\varphi_i(\theta_{-i}), \theta_{-i})$ (not directly dependent on $\theta_i$).

1. Prove the generalized VCG mechanism satisfies ex post monotonicity.

2. Prove the generalized VCG mechanism satisfies ex post incentive compatibility.

3. Prove the generalized VCG mechanism does not satisfy Maskin monotonicity, so that it is not Nash implementable.

**Exercise 19.19** Prove that for economic environments with 2 participants, if the social choice function $f$ satisfies ex post incentive compatibility and ex post monotonicity, then it is ex post implementable.

**Exercise 19.20** For economic environments with 2 participants, suppose that the social choice set $F$ contains more than one element. Explain whether ex post incentive compatibility and ex post monotonicity are the sufficient conditions for $F$ to be ex post implementable.

**Exercise 19.21** (Bergemann and Morris, 2008) They obtained the following conclusion: for a general case with the number of participants being $n \geq 3$, if social choice function $f$ satisfies ex post incentive compatibility, ex post monotonicity and no-veto-power condition, then it is ex post implementable. Explain whether this conclusion can be generalized to the case with $n = 2$. 

Exercise 19.22 (Bergemann and Morris, 2005) Consider two participants 1 and 2. The sets of their types are $\Theta_1 = \{\theta_1, \theta_1'\}$ and $\Theta_2 = \{\theta_2, \theta_2'\}$, respectively. The set of socially feasible outcomes is $A = \{a, b, c\}$. With different outcomes and types of participants, their utilities can be described by the following table: (the two numbers in each box represent respectively the utilities of participant 1 and 2):

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A social planner considers the following social choice rule, $F$, to maximize the sum of the utilities of these two participants, and it is described by the following table:

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Prove that social choice rule $F$ can not be ex post implemented.

Exercise 19.23 (Bergemann and Morris, 2005) Consider two participants 1 and 2. Their type sets are $\Theta_1 = \{\theta_1, \theta_1', \theta_1''\}$ and $\Theta_2 = \{\theta_2, \theta_2'\}$, respectively. Suppose that the socially feasible outcome set is $A = \{a, b, c, d\}$. With different outcomes and types of participants, their utilities can be described by the following table: (the two numbers in each box represent respectively the utilities of participants 1 and 2):

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<td>(0,0)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$\theta_1'$</td>
<td>(0,0)</td>
<td>(0,2)</td>
<td>(1,0)</td>
</tr>
<tr>
<td>$\theta_1''$</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td>(0,2)</td>
</tr>
</tbody>
</table>
A social planner considers the following social choice rule, $F$, to maximize the sum of the utilities of these two participants, and it is described by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta'_1$</th>
<th>$\theta'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>a</td>
<td>a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta'_1$</td>
<td>b</td>
<td>c</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta''_1$</td>
<td>d</td>
<td>d</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Suppose that the social planner can use transfers among the participants. Each participant has a quasi-linear utility function. Prove that when the social choice rule $F$ is ex post budget balanced, it cannot be ex post implemented.

### 19.11 References

**Books and Monographs**


**Papers**


Part VII

Market Design
The previous part discusses the principal-agent theory with one agent and the theory of mechanism design with more than one agent. The last part of this book will focus on two forefront subfields of modern microeconomics: auction theory and matching theory, which can be regarded as the extensions of the theory of mechanism design. Auction design and matching design are the core issues of market design where traditional nature market fails. As such, the market design is regarded as microeconomic engineering and has wide applications in reality. In recent years, the research frontiers have promoted the development of auction design from single-unit auctions to multiple objects auctions, and matching format design has been applied to school choice, organ matching and many other aspects.

Traditional economics such as neoclassical general equilibrium theory discusses the interaction between market and participants, as well as the equilibrium outcome and welfare properties of the interaction in a given market. Since a market usually fails in cases such as the allocation of indivisible goods and the provision of public goods, it is often necessary to amend the market and hence mechanism design arises.

Similarly, market design does not take market as given, but applies the empirical or experimental results of economics and game theory to the design of market rules. With the rapid development, auction theory mainly considers auction design where the price mechanism that are characterized by monetary transfers play an essential role—agents who are willing to pay the highest to win an object in auction issues, while the matching theory mainly considers mechanism design where the price mechanism however usually does not play a dominant role in matching issues, and especially, even no monetary transfers are made in many cases. Auction theory and matching theory have established their own analytical frameworks, which have increasingly become independent subfields of modern microeconomic theory.

Chapter 21 discusses basic contents of auction theory, including the most fundamental auction formats. Auction mechanism design is an important part of market design. Auction theory mainly uses price mechanism characterized by transfer payments to design the allocation of resources in the market. Meanwhile, many results on auction theory can be regarded as the extensions of the theory of mechanism design. We will see that many basic conclusions in auction theory can be easily obtained by applying the results in the mechanism design theory introduced in the previous part.

Chapter ??, as the last chapter of the book, discusses another important category in market design: design of allocation of indivisible objects generally without transfers or without playing a dominant role in matching issues. In most cases (such as human organ transplantation, school choice, and office allocation), transfer payment is not allowed for the allocation of such indivisible objects, so that it is a non-price mechanism. We will first discuss matchings in two-sided market and one-sided market, respectively,
and then analyze the applications of matching theory: matching mechanism in school choice and organ transplantation.
Chapter 21

Auction Theory

21.1 Introduction

Auction theory is in the frontier of modern economics and has wide applications. It has developed systematic theoretical results, which can be used to analyze and understand the superiority of auction and tender mechanisms, and also provided many useful insights for practical operation. Since the auction theory can be considered as a branch or an extension of the theory of mechanism design, by applying the results discussed in the previous part, it is easy to get some basic results in the auction theory.

As an effective trading mechanism of objects, auction is applied more and more widely and has penetrated into all aspects of daily economic activities. In reality, many major objects or projects are auctioned or tendered. Objects that are often auctioned include tangible assets such as antiques, paintings, jewelery, used cars, buildings and agricultural products, as well as intangible assets such as land use rights, oilfield exploitation rights, and even some special telephone numbers and license plate numbers. The central banks of many countries often sell government bonds by auction, and the Ministry of the Interior conducts regular auctions of oil exploration rights. In recent years, the U.S. Department of Communications adopted auctions designed by economists to allocate licenses to provide personal communications services and has gained unprecedented benefits. It has also brought tremendous revenue to the U.S. Treasury, and similar auctions have been adopted in European countries.

In fact, the history of auction can be traced back to at least the ancient Babylonian period of 700 BC. Some auction events even influenced the whole course of history. In AD 193, Roman emperor Pertinax was killed by his guards because he wanted to rectify military discipline. Subsequently, the controlling Praetorian Guard auctioned the throne in the barracks. Didius Julianus won the auction by paying almost two kilos of gold to each soldier (the actual payment was 25,000 sesterces), thus getting the support
of the Guard and the emperor’s throne. However, it did not last long. Two months later, the insurgent army stormed into Rome, and this politician throned by auction eventually ended up with death.

Although the practice of auction has lasted for thousands of years, some of which even imposed a huge historical impact, the scientific research on auction based on rigorous economic theory did not begin until the 1960s. The first work was Vickrey’s less than 30-page paper titled “Counterspeculation, Auctions, and Competitive Sealed Tenders” in the 1961 Journal of Finance (for William Vickrey’s biography, see Section 21.7.1). In this classic paper, he discussed the four most widely used types of auction for single item, which this chapter will first focus on. He provided a landmark conclusion on the development of the auction theory, that is, the revenue equivalence theorem, and proved that all the bidders will bid truthfully in the second price sealed-bid auction. This chapter mainly refers to the classic book on auction theory by Krishna (2010).

**Four Basic Auctions**

There are many kinds of auction formats, among which four auction formats are most widely used and studied while others are variations of those four auction formats.

The first is *ascending-price auction* (or *English auction*). Under this format, the price of the auctioneer is *publicly* called, and it will increase until there is only one bidder that bids. Then this bidder wins the object and pays the bid. One variation of this auction is that all the bidders bid and the price increases until there is only one bidder bidding and pays his/her bid. This format is often used in live auction, especially on the cultural relics, calligraphy and painting, second-hand goods and commercial land (such as in China).

The second is *first-price (sealed-bid) auction*. Under this rule, each bidder submits his bid document to the auctioneer in sealed form in the specified time independently and indicates the price he wishes to pay. Thus the bid of other bidders cannot be seen. The auctioneer then invites all bidders to open their bids on-site at a specified time. Then the bidder with the highest bid wins the object by his bid. Thus the first-price auction is also called *hig
price auction, bidding auction or mail auction.

The fourth is the second-price (sealed-bid) auction, which is also called Vickrey auction. Under this format, the bid is also sealed, but after opening the bid, the bidder whose bid is the highest wins the object and pays the second highest bid. This form is rarely used in practice, but it has a good theoretical nature. This mechanism has been discussed in Chapter 18 as a special case of VCG mechanism. It was first proposed by Vickrey in 1961, and economists began to make an in-depth study of the auction since then.

In each type of auction, if several bidders bid the same with the highest price, then the auctioneer will randomly select among them as a winner. The above introduction is only for single-unit item auction, and we will also discuss auctions of multi-unit items. Their auction formats are different, but are basically the variants of the above four mechanisms.

It can be seen from the above introduction of the four basic auctions that auction mainly adopts two major formats: the first two are open outcry methods, such as antique, calligraphy and painting, second-hand goods (such as used cars) and so on, which require that the bidders collect in the same place, while the latter two are sealed auction, which may be submitted by mail, so a bidder may observe the behavior of other bidders in one format and not in the other. In many auctions or bidding activities, the confidentiality of participants’ business secrets should be ensured, so the auction or bidding methods adopted should avoid commercial leaks. Compared with the open outcry methods, sealed price methods have the advantage in this respect, so the use of sealed price bidding is necessary.

However, for rational decision makers, some of these differences are superficial. The Dutch open descending price auction is strategically equivalent to the first-price sealed-bid auction in the sense that they have the same equilibrium strategies. When values are private, the English open ascending auction is also outcome equivalent to the second-price sealed-bid auction in equilibrium outcome in a weak sense that they have the same equilibrium outcomes.

In addition, sellers often impose two restrictions for each auction format. One is to set the base price, which is called the reserve price; the other is the charge for bidding. Under the first-price and second-price sealed-bid auction formats, the bidder must bid higher than or equal to the reserve price, otherwise no transaction will be made. As for the second-price auction, if there is only one bidder bidding and bids above the reserve price, he will get the item and pays the reserve price. The reserve price has a similar effect in both the ascending-price and descending-price auctions.

There are two basic issues to be discussed in the auction theory: (1) Can the resources be efficiently allocated using these auctions? (2) Which auction format can make the highest revenue for the auctioneer? To answer these two questions, we need to set up the basic analytical framework of
the auction, including a description of bidders’ preferences and the information structure – private value, interdependent value, or common value (a special case of interdependent value). Then we will analyze their bidding strategies to see if they lead to efficient outcomes and compare the auctioneer’s expected revenue.

In the following, we will first examine auctions in the case of private value and focus mainly on the symmetric, risk-neutral, unlimited liability private value economic environment with single object. We then discuss auctions under the context of interdependent value for single object and auctions for multi-unit objects. As an auction is a special case of incomplete information mechanism design, the solution concept used is mainly the Bayesian Nash equilibrium. In order to make it easier to understand, we try to use the same terms and notations as in Chapters 18-20.

21.2 Single Object Auctions with Private Values

21.2.1 Basic Analytical Framework

We consider the private value benchmark model of auction theory for single object, which usually includes the following six hypotheses:

1. **Private Value**: The value of the item by bidders is private information, depending only on their own type and having nothing to do with the type of others;

2. **Independence**: The types of bidders are independent;

3. **Symmetry**: The types of bidders have the same probability distribution;

4. **Risk Neutrality**: The expected utility functions of bidders are risk-neutral;

5. **Unlimited liability**: Bidders have no budget constraints and have the ability to pay the bidding price;

6. **Non-collusion**: All buyers decide on their own bidding strategy independently and there is no binding cooperative agreement.

The above description of such economic environments is called **Symmetric Independent Private Value (in short, SIPV) model**. We mainly consider solving symmetric Bayesian-Nash equilibrium under this model and discuss their properties, although there may be asymmetric solutions for a general utility function.

Consider an auction with an indivisible object owned by the auctioneer and \( n \) risk-neutral bidders. Every bidder \( i \) has a private value of the object, denoted by \( \theta_i \), the type of the bidder. As such, the auction model
21.2. SINGLE OBJECT AUCTIONS WITH PRIVATE VALUES

can be considered as a special case of indivisible object model we studied in Chapters 18-19: \( Y = \{ y \in \{0,1\}^n : \sum_i y_i = 1 \} \), and the payoff of buyer \( i \) can be written as

\[
\theta_i y_i + t_i,
\]

where \( t_i \) is the corresponding transfer payment. The value \( \theta_i \) for buyer \( i \) is a random variable distributed on \( [\tilde{\theta}_i, \bar{\theta}_i] \) with independent density \( \varphi_i(\cdot) > 0 \), where \( \tilde{\theta}_i < \theta_i \), and the cumulative distribution function is denoted by \( \Phi_i(\cdot) \).

We first examine equilibrium outcomes of these common mechanisms under symmetric scenarios, and discuss the forms and properties of the Bayesian-Nash equilibrium, such as validity. Then we compare the seller’s expected revenue under these auctions.

In private value auctions, the first-price (sealed-bid) auction and the Dutch auction have the highest price called or accepted as the auction price, and so these two auctions are strategically equivalent. In an English auction, a bidder will obtain a positive income from participating if and only if she can win the auction with a bid that is less than her valuation of the object. Therefore, the bidder with the highest valuation wins the auction and pays an amount slightly higher than the value of the second highest bidder. This is why the English auction is sometimes referred to as an open second-price auction. Thus, the second-price (sealed-bid) auction and the English auction where the second highest price called or accepted is the auction price, so the two auctions are outcome-equivalent. However, this equivalence applies only for private values, or if there are just two bidders.

Therefore, without loss of generality, we only need to discuss the first-price (sealed-bid) auction and second-price (sealed-bid) auction for the private value environment.

Below we focus on the case of symmetry. Assuming that all bidders’ private values are symmetric, that is, for any \( i \), there is \( \theta_i \sim \varphi(\cdot) \) distributed on \( [0, w] \) with distribution function \( \Phi(\cdot) \). At the same time we consider the symmetric Bayesian-Nash equilibrium, in which we focus on examining bidder 1’s strategic choices.

21.2.2 First-Price Sealed-Bid Auctions

In a first-price (sealed-bid) auction, given bidder \( i \)’s value \( \theta_i \), if his bid is \( b_i \) and the other bidder’s bid is \( b_j, j \neq i \), then bidder \( i \)’s utility is \( U_i = \theta_i y_i(b_i, b_{-i}) + t_i(b_i, b_{-i}) \), where the allocation rule is given by

\[
y_i(b_i, b_{-i}) = \begin{cases} 
1 & \text{if } b_i > \max_{j \neq i} b_j \\
0 & \text{if } b_i < \max_{j \neq i} b_j,
\end{cases}
\]

and transfer payment is given by

\[
t_i(b_i, b_{-i}) = \begin{cases} 
-b_i & \text{if } b_i > \max_{j \neq i} b_j \\
0 & \text{if } b_i < \max_{j \neq i} b_j.
\end{cases}
\]
If \( b_i = \max_{j \neq i} b_j \), the object is randomly assigned by the same probability. Consider a symmetric equilibrium denoted by \( \beta^I(\cdot) \), which is a mapping \( \beta^I : [0, \omega] \rightarrow [0, \infty) \). Without loss of generality, suppose \( i = 1 \). Let \( \vartheta_1, \ldots, \vartheta_{n-1} \) be the order from the largest to smallest of \( \theta_2, \ldots, \theta_n \), or be called the 1st, \( \ldots, n-1 \)th sequential order statistics of \( \theta_2, \ldots, \theta_n \). Suppose that \( \beta^I(\cdot) \) is an increasing function, as we will show later.

When other bidders choose strategy \( \beta^I(\cdot) \), then only if \( b_1 > \beta^I(\vartheta_1) \), bidder 1 can get the object, otherwise he would not. Since \( \vartheta_1 \) is a continuous random variable, the probability of \( b_1 = \beta^I(\vartheta_1) \) is zero, so we can omit the case. As \( \theta_j, j \neq 1 \), are all distributed according to the density function \( \varphi(\cdot) \) (the corresponding distribution function is \( \Phi(\cdot) \)), it is easy to obtain that \( \vartheta_1 \)'s distribution function is \( \Psi(\cdot) = \Phi^{n-1}(\cdot) \) (the corresponding density function is \( \psi(\cdot) \)).

Given that other bidders choose strategies \( \beta(\cdot) = \beta^I(\cdot) \), the (interim) expected utility of bidder 1 with value \( \theta \) and bid \( b \) then is:

\[
E_{\theta_1} U_1 = \Psi(\beta^{-1}(b))(\theta - b),
\]

where \( \beta^{-1}(\cdot) \) is the inverse function of \( \beta(\cdot) \).

The first order condition for \( b \) is:

\[
\frac{\psi(\beta^{-1}(b))(\theta - b)}{\beta'(\beta^{-1}(b))} - \Psi(\beta^{-1}(b)) = 0.
\]

In equilibrium, \( b = \beta(\theta) \), then we have:

\[
\beta'(\theta) = \frac{\psi(\theta)}{\Psi(\theta)}(\theta - \beta(\theta)), \tag{21.2.3}
\]

or equivalently,

\[
\frac{d}{d\theta}(\Psi(\theta)\beta(\theta)) = \theta \psi(\theta). \tag{21.2.4}
\]

Since \( \beta(0) = 0 \), the solution of first-order differential equation (21.2.3) is:

\[
\beta(\theta) = \frac{1}{\Psi(\theta)} \int_0^\theta \vartheta_1 \psi(\vartheta_1)d\vartheta_1 = E[\vartheta_1|\vartheta_1 < \theta].
\]

Obviously \( \beta(\theta) < \theta \). From the differential equation (21.2.3), we get \( \beta'(\cdot) > 0 \), i.e., \( \beta(\cdot) \) is an increasing function. We then have the following proposition.

**Proposition 21.2.1** With symmetric independent private value, the symmetric Bayesian-Nash equilibrium of first-price auction is

\[
\beta^I(\theta) = \frac{1}{\Psi(\theta)} \int_0^\theta \vartheta_1 \psi(\vartheta_1)d\vartheta_1 = E[\vartheta_1|\vartheta_1 < \theta].
\]
21.2. SINGLE OBJECT AUCTIONS WITH PRIVATE VALUES

\[\text{PROOF.}\] We obtain a (first-order) necessary condition for the strategy to be a Bayesian-Nash equilibrium from the above argument, and in the following we only need to show the sufficiency of the proposition.

If other bidders all choose \(\beta = \beta^i\), under \(\theta\), bidder \(i\) chooses bid \(\beta(\tilde{\theta})\), and then his expected utility is:

\[
\bar{U}_i(\theta, \tilde{\theta}) = E_{\theta-1} [\theta y_i(\beta(\tilde{\theta}), \beta_{-i}(\theta_{-i})) + t_i(\beta(\tilde{\theta}), \beta_{-i}(\theta_{-i}))]
\]

\[
= \Psi(\tilde{\theta})[\theta - \beta(\tilde{\theta})]
\]

\[
= \Psi(\tilde{\theta})\tilde{\theta} - \Psi(\tilde{\theta})E[\vartheta_1|\vartheta_1 < \tilde{\theta}]
\]

\[
= \Psi(\tilde{\theta})\tilde{\theta} - \Psi(\tilde{\theta})\tilde{\theta} + \int_{0}^{\tilde{\theta}} \Psi(\vartheta_1)d\vartheta_1 \quad \text{(integration by parts)}
\]

\[
= \Psi(\tilde{\theta})(\theta - \tilde{\theta}) + \int_{0}^{\theta} \Psi(\vartheta_1)d\vartheta_1.
\]

Then, no matter \(\theta \geq \tilde{\theta}\) or \(\theta \leq \tilde{\theta}\), we have:

\[
\bar{U}_i(\theta, \theta) - \bar{U}_i(\theta, \tilde{\theta}) = \Psi(\tilde{\theta})(\tilde{\theta} - \theta) + \int_{\theta}^{\tilde{\theta}} \Psi(\vartheta_1)d\vartheta_1 \geq 0.
\]

Using integration by parts, the Bayesian-Nash equilibrium bidding solution can be rewritten as:

\[
\beta^i(\theta) = \theta - \frac{1}{\Psi(\theta)}\int_{0}^{\theta} \Psi(\vartheta_1)d\vartheta_1 < \theta,
\]

which again means the bidder bids less than his true value. That is, no bidders have the incentive to show their true type.

It is easy to verify that \(\beta^i(\theta)\) is a monotonically increasing function. The equilibrium bid has a positive correlation with the private value, that is, for high private value, the equilibrium bid is also high, but it is always less than the private value. However, since

\[
\Psi(\theta) = \left[\frac{\Phi(\theta)}{\Phi(\tilde{\theta})}\right]^{-1},
\]

when the number of bidders \(N\) increases, which means it is more competitive, the Bayesian-Nash equilibrium solution \(\beta^i(x)\) converges to \(x\).

Example 21.2.1 Suppose that that values are exponentially distributed on \([0, \infty)\), and there are only two bidders. If \(\Phi(\theta) = 1 - \exp(-\lambda\theta), \lambda > 0\), then

\[
\beta^i(\theta) = \theta - \int_{0}^{\theta} \frac{\Phi(\vartheta)}{\Phi(\tilde{\theta})}d\vartheta
\]

\[
= \frac{\theta}{\lambda} \exp(-\lambda\theta).
\]
21.2.3 Second-Price Sealed-Bid Auctions

Now we discuss the second-price (sealed-bid) auction equilibrium. The highest bid for other bidders is \( \bar{b}_{(i)} \equiv \max_{j \neq i} b_j \), where \( b_j \) is the bid of bidder \( j \). For a bidder \( i \) with private value \( \theta_i \), the utility of bid \( b_i \) is

\[ U_i = \theta_i y_i(b_i, \bar{b}_{(i)}) + t_i(b_i, \bar{b}_{(i)}) \]

where

\[ y_i(b_i, \bar{b}_{(i)}) = \begin{cases} 1 & \text{if } b_i > \bar{b}_{(i)} \\ 0 & \text{if } b_i < \bar{b}_{(i)} \end{cases} \]

\[ t_i(b_i, \bar{b}_{(i)}) = \begin{cases} -\bar{b}_{(i)} & \text{if } b_i > \bar{b}_{(i)} \\ 0 & \text{if } b_i < \bar{b}_{(i)} \end{cases} \]

The following proposition gives the properties of Bayesian-Nash equilibrium strategy \( \beta^{II} \) for a second-price sealed-bid auction.

**Proposition 21.2.2** For a second-price (sealed-bid) auction, all bidders’ Bayesian-Nash equilibrium strategy \( \beta^{II}(\theta) = \theta \) is a weakly dominant strategy.

**Proof.** This result can be obtained by applying the Vickrey-Clarke-Groves mechanism directly. We present a direct proof of this result.

When \( \theta \geq \bar{b}_{(i)} \), bidding \( b_i \geq \bar{b}_{(i)} \) and the true value \( \theta_i \) bring the same revenue \( \theta_i - \bar{b}_{(i)} > 0 \), but when \( b_i < \bar{b}_{(i)} \), the opportunity to win is lost, so that the revenue is less than that brought by bidding the true value \( \theta_i \). Thus, when \( \theta_i > \bar{b}_{(i)} \), \( \beta^{II}(\theta_i) = \theta_i \) is a weakly dominant strategy.

When \( \theta_i < \bar{b}_{(i)} \), bidding \( b_i < \bar{b}_{(i)} \) and the true value \( \theta_i \) bring the same revenue 0, but when the bidding price \( b_i \geq \bar{b}_{(i)} \), the revenue \( \theta_i - \bar{b}_{(i)} < 0 \) is smaller than the payoff of bidding \( \theta_i \). Thus, when \( \theta_i < \bar{b}_{(i)} \), \( \beta^{II}(\theta_i) = \theta_i \) is also a weakly dominant strategy.

Therefore, for any \( \bar{b}_{(i)} \), to truthfully reveal \( \theta_i \) is a weakly dominant strategy.

As such, in the second-price sealed-bid auction, all the bidders will bid truthfully, that is, their true values are reported; while in the first-price sealed-bid auction, the bidders’ bids are lower than the true values. The intuition of this conclusion is simple. When using the first-price sealed-bid auction, if the bidder bids the true value of the object according to his own value, then even winning the auction is unprofitable. In order to obtain the potential profits, bidders have the incentive to submit prices lower than their true values, while this issue can be well solved in the second-price sealed-bid auction.

21.2.4 Revenue Comparison of First and Second Price Auctions

As mentioned above, Vickrey (1961) compared the four auction forms that are most widely used in single-object auctions, and obtained the landmark
21.2. SINGLE OBJECT AUCTIONS WITH PRIVATE VALUES

theory of auction - the “Revenue Equivalence Theorem”. We now compare the expected revenue of auctioneers with the first-price auction and the second-price auction under symmetric conditions.

First of all, for a first-price auction, the interim expected expenditure of bidder 1 with private value $\theta$ to the auctioneer is:

$$m^f(\theta) \equiv -\mathbb{E}_{\theta_1} t^f_1(\theta) = \Psi(\theta) \beta^f(\theta) = \Psi(\theta) \mathbb{E}[\vartheta_1 \mid \vartheta_1 < \theta],$$

i.e., $m^f(\theta)$ has the sign difference of interim expected transfer payment $\bar{t}^f_1(\theta) \equiv \mathbb{E}_{\theta_1} t^f_1(\theta)$. Here $\Psi(\theta)$ is the probability that bidder 1 wins the auction.

Therefore, the auctioneer’s ex ante expected revenue from bidder 1 is:

$$E \mathbb{E}_{\theta} m^f(\theta) = \int_0^w \frac{\vartheta_1}{(n-1)}(1 - \Phi(\vartheta_1))(\vartheta_1 - \vartheta_1(1 - \Phi(\vartheta_1))) \psi(\vartheta_1) d\vartheta_1.$$
Thus, in a symmetric environment with private value, the expected revenues of auctioneer in the first-price auction and second-price auction are the same. In fact, we can get a more general conclusion that all auction formats that satisfy the same allocation rule and have the same expected utility of the bidder at the lowest private value generate the same expected revenue for the auctioneer.

We further compare the expected revenue for the auctioneer with the first-price auction and the second-price auction.

Consider the payment of bidder 1. When his value \( \theta < \vartheta_1 \), no matter what kind of auction formats, the interim expected payment of bidder 1 is zero. When \( \theta > \vartheta_1 \), under first-price auction, the interim expected payment of bidder 1 is:

\[
m^I(\theta) = -\bar{t}_1^I(\theta) = \Psi(\theta)\beta^I(\theta) = E_\theta[\vartheta_1 | \vartheta_1 < \theta]
\]

For the second-price auction, bidder 1’s interim expected payment is:

\[
m^{II}(\theta) = -E_\theta(t_1^{II}(\theta) | \vartheta_1 < \theta) = E_\theta[\vartheta_1 | \vartheta_1 < \theta],
\]

which means \( \beta^{II} \) is \( \beta^I \)'s mean value spreading. Thus, we have \( m^I(\theta) = m^{II}(\theta) \).

Since interim expected payment is the same, the ex ante expected payment would also be the same. The reason is that when \( m^I(\theta) = m^{II}(\theta) \), under first-price auction, the ex ante expected expected payment the auctioneer receives from bidder 1 is:

\[
E_\theta(m^I) = E_\theta[\Psi(\theta)m^I(\theta)].
\]

Under second-price auction, the ex ante expected payment the auctioneer receives from bidder 1 is:

\[
E_\theta(m^{II}) = E_\theta[\Psi(\theta)m^{II}(\theta)].
\]

Hence, \( E_\theta(m^I) = E_\theta(m^{II}) \). Thus we have verified in two ways that the ex ante expected payment to the auctioneer is the same for both the first-price auction and the second-price auction.

In order to have the specific expression of auctioneer’s expected revenue, from (21.2.5) we give the first-price auction bid:

\[
\beta^I(\theta) = \theta - \frac{1}{\Psi(\theta)} \int_0^\theta \Psi(\vartheta_1) d\vartheta_1 < \theta.
\]

By applying integration by parts to the second part of the above equation, we have

\[
E\beta^I(\theta) = \int_0^\theta \nu(\theta') d\Phi^u(\theta'),
\]
where
\[ \nu(\theta) = \theta - \frac{1 - \Phi(\theta)}{\phi(\theta)}, \]
which is the bidder’s virtual valuation function.

In this way, we obtain the specific expression of the auctioneer’s expected revenue:
\[ E_{\theta}(m^I) = E_{\theta}(m^{II}) = \int_{0}^{\theta} \nu(\theta') d\Phi^n(\theta'). \]

In the auction theory and mechanism design theory, \( \nu(\theta) \) given above plays a very important role. A buyer (bidder) considers the object’s value at most \( \theta \), which is his personal information, and he will tend to underreport this information when dealing with the seller. How much the buyer can under-report depends on the seller’s knowledge on \( \theta \), which is the distribution function of \( \theta \). From the seller’s point of view, \( \nu(\theta) \) is the highest price the buyer is willing to pay due to information asymmetry and adjustment. In the subsection on the design of the optimal auction below, one will see that \( \nu(\theta) \) can be regarded as the seller’s (or auctioneer’s) marginal revenue.

### 21.2.5 Efficient Allocation and Revenue Equivalence Principle

It can be seen from the previous discussion that if all the bidders’ bids are given independently in all the four auctions, the auctioneer can obtain the same expected revenue regardless of the auction form. The four auction formats all have some common features in that bidders are asked to give the bidding price they are willing to pay, which determines who will receive the object and how much they will pay. This results in efficient outcomes as the bidder with the highest value wins the object. Yet another notable feature is that while the distribution functions are very different, the bids are varied and even not strategically-equivalent, the auctioneer receives the same expected revenue. As such, to what extent are the results still valid?

Myerson (for his biography, see Section 19.9.2) used the mechanism design approach to study this issue. On this basis, he extended Vickrey’s theory and proved that: under the assumption that the bidders’ value of the object are independent, the bidders only care about their own expected payment, all standard auctions, in which the highest bidder gets the object, will bring the same expected revenue to the auctioneer. The result is significant, making the auction theory one great step further. Before discussing this result, we will discuss the efficiency of the first-price and second-price auctions.

Since the bidder’s Bayesian-Nash equilibrium bidding functions are both monotonic and continuous in the first-price and second-price auctions, and both are the ones with the highest bids win, the allocation result is efficient and we have the following allocative efficiency theorem.
Theorem 21.2.1 (Allocative Efficiency Theorem) In Symmetric Independent Private Value (SIPV) environments, the first-price and second-price auctions result in efficient allocation of object (the one with the highest value gets the object).

We will see that when the symmetry is not satisfied, the second-price auction is still efficient, but the first-price auction may not be efficient.

Now we discuss the Myerson revenue equivalence theorem. We call an auction standard if the highest bidder gets the object. A non-standard form of auction is to buy a lottery. The chance of a bidder’s winning is related to the ratio of the bid amount to the total bid amount of all the bidders. It is non-standard because it does not necessarily guarantee to obtain the object despite the highest bidding. In addition to these four common standard auctions, another example of a standard auction is the “all-pay auction”, where every bidder pays according to his bid, but only the bidder with highest bid wins the object. For example, lobbying for the government to adopt a policy or bribing a government official who holds power belongs to such auction formats. In those cases, are the auctions still the same for buyers’ payoffs? Myerson’s equivalence theorem gives an affirmative answer.

We now examine the question of whether the auctioneer’s expected revenue under the standard mechanism is equivalent. Consider a linear model that allocates an indivisible object among \( n \) risk-neutral buyers: \( Y = \{y \in \{0, 1\}^n : \sum_i y_i = 1\} \), buyer \( i \)’s revenue is \( \theta_i y_i + t_i \), and the value is a random variable with individually independent density \( \phi_i(\cdot) > 0 \) defined on \( [\bar{\theta}_i, \bar{\theta}_i] \).

Let \( U_i(\theta) = \theta_i y_i(\theta) + t_i(\theta) \). Under an auction rule (or more general social choice in a linear environment) \((y(\cdot), t_1, \cdots, t_n)\), the auctioneer’s expected revenue can be written as

\[
- \sum_i E_{\theta_i} t_i(\theta) = \sum_i E_{\theta_i} [\theta_i y_i(\theta) - U_i(\theta)] = \sum_i E_{\theta_i} [\theta_i y_i(\theta)] - \sum_i E_{\theta_i} E_{\theta-i} [U_i(\theta)].
\]

In the above equation, the first item is the expected total surplus for all participants, while the second item is the expected utility for all participants. According to the Bayesian incentive compatibility characterization theorem (Proposition 19.4.2) in Chapter 19, the second item is given by \( E_{\theta-i} U_i(\theta) = E_{\theta-i} [U_i(\theta_i, \theta_{-i}) + \int_{\bar{\theta_i}}^{\bar{\theta_i}} E_{\theta-i} y(\tau, \theta_{-i}) d\tau] \), so it can be completely determined by efficient (allocation) decision rules \( y(\cdot) \) and the lowest type of interim expected utility \( E_{\theta-i}[U_i(\theta_i, \theta_{-i})] \). Since the total surplus remains unchanged (all mechanisms implement the same decision rule), we have the following Revenue Equivalence Theorem.

**Theorem 21.2.2 (Myerson’s Revenue Equivalence Theorem)** Suppose that two different auction formats both have Bayesian-Nash equilibrium, and at the equilibrium: (i) the same decision (object allocation) rule \( y(\cdot) \) is implemented; (ii) all buyers have the same interim expected utilities at the lowest value \( \bar{\theta}_i \). Then
Bayesian-Nash equilibrium outcomes of these two auction formats result in the same revenue to the auctioneer.

This theorem shows that there are many ways to obtain interim expected transfer payment $t_i(\theta_i)$ by using ex post transfer payment $t_i(\theta)$. Even though the rule of decision and the utility at the lowest-type is fixed, the seller has a great freedom in designing the auction scheme.

For example, suppose the buyers are symmetric (that is, they have the same distribution function). The seller wants to implement efficient decision rule $y(\cdot)$ and the expected utilities of buyers with the lowest value are zero. We can use the second-price sealed-bid auction with dominant strategy to truthfully implement the efficient decision rule or the first-price auction to implement the efficient decision rule. More generally, consider $k$th price sealed-bid auction, where $1 \leq k \leq n$, the highest bidder gets the object and pays the $k$th highest bid. Suppose that the value of every buyer on the object is an independently and identically distributed random variable (i.i.d.), then we can prove that the auction has equilibrium that is unique and symmetric, and the bid of each participant $b(\theta_i)$ is an increasing function of his value (see Fudenberg-Tirole Game Theory, pp: 223-225.) Since the bidder with highest bidding price gets the object, the $k$th price sealed-bid auction implements the efficient outcome and thus is an efficient decision rule. Meanwhile, the probability of buyer with lowest value $\theta$ getting the object is zero, so his expected utility is zero. Thus, by the revenue equivalence theorem, for any $k$, $k$th price sealed-bid auction brings the same revenue to sellers. All-pay auction and $k$th price sealed-bid auction also bring the same revenue to sellers.

For a given bidding scheme, it seems that the auctioneer will get a higher revenue when $k$ is smaller. So how will the above results be established? The answer is that the bidder’s bid would be smaller when $k$ is small. For example, we know that the bidder’s bid is the true value in second-price auction. In first-price auction, the buyer’s bid will be less than his true value because the bid equal to his true value would bring zero expected utility. Myerson’s Revenue Equivalence Theorem shows that for both auctions, the seller’s expected revenue is equal. In particular, we can obtain this without solving the auction equilibrium. When $k > 2$, the revenue equivalence theorem shows that the bidder’s bid will be greater than his true value (but since the price paid is the $k$th ($1 \leq k \leq n$) highest price, it will not be greater than his true value).

### 21.2.6 Applications of Revenue Equivalence Principle

There are many auction formats in theory and reality. In addition to the four common auction forms discussed above, there are also all-pay auctions, $k$th price auction, auction with uncertain number of bidders and oth-
ER AUCTION FORMATS. These auctions have many applications in reality. For example, the all-pay auction can be applied to attrition competition in the industry, whoever holds to the final will win. However, these enterprises, including the enterprises that quit midway, have paid certain cost in the process. In addition, the all-pay auction can also be applied to the lobbying activities of interest groups in politics and so on. Although $k \geq 3$ price auction is rarely used in reality, such form of auctions has theoretical significance. We find that bidders may bid more than their own value in a third-price auction. There are some fine balances of interests. In addition, the actual auctions in reality, such as online auctions where the number of bidders is uncertain, how will each bidder choose his own strategy? Therefore, these auction formats have great theoretical and practical significance. However, in these auctions, the solution to the bidding equilibrium tends to be relatively complicated, whereby applying Myerson’s Revenue Equivalence Theorem provides some shortcuts to the solution process. The following subsection gives such examples.

**All-Pay Auctions**

The all-pay auction is similar to the first-price and second-price auctions where the bidder with the highest bid gets the object, but the difference lies in that each bidder, regardless of whether obtaining the object eventually, pays the auctioneer at his own bidding price. Let $\theta = (\theta_1, \ldots, \theta_n)$ be $n$ types of bidders independently distributed according to $\varphi(\cdot)$ over $[0, w]$, let $b = (b_1, \ldots, b_n)$ be the bidding vector, and let bidder $i$’s utility be $U_i = \theta_i y_i(b_i, b_{-i}) + t_i(b_i, b_{-i})$, where:

$$y_i(b_i, b_{-i}) = \begin{cases} 1 & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases} \quad (21.2.9)$$

where, if $b_i = \max_{j \neq i} b_j$, with the same probability the object is randomly allocated and

$$t_i(b_i, b_{-i}) = -b_i. \quad (21.2.10)$$

As before, $\Psi(\cdot)$ is the distribution of $\vartheta_1$ (the first-order statistics of other participant types). We focus on the case that symmetric equilibrium $\beta(\cdot) = \beta^{AP}(\cdot)$ is a monotonic function (which will be further proved). When $\theta = 0$, it is obvious that bidders will choose the bidding price 0, and at the same time the bidder with the highest private value wins the object. According to the Revenue Equivalence Theorem, at the equilibrium, this auction has the same expected utility as the first-price auction and the second-price auction. Thus, when bidder $i$’s type is $\bar{\theta}$, his interim expected utility under the all-pay auction at equilibrium is:

$$\Pi_i^{AP}(\bar{\beta}(\bar{\theta}), \theta) = \Psi(\bar{\theta})\theta - \beta^{AP}(\theta).$$
The interim expected utility at equilibrium under second-price auction is:

\[ \Pi_{II}^I(\beta(\tilde{\theta}), \theta) = \Psi(\tilde{\theta}) \theta - \Psi(\tilde{\theta}) \int_0^\theta \vartheta \psi(\vartheta) d\vartheta. \]

Thus, making the above two equations equal, we obtain a symmetric and balanced bidding strategy for an all-pay auction:

\[ \beta_{AP}(\theta) = \int_0^\theta \vartheta \psi(\vartheta) d\vartheta, \]

which is obviously a strictly increasing function.

**Third-price Auctions**

Consider a third-price auction. We use the previous approach to find the equilibrium function for the third-price auction. Let \( \theta = (\theta_1, \ldots, \theta_n) \) be \( n \) types of bidders independently distributed according to \( \varphi(\cdot) \) on \([0, w]\), and \( b = (b_1, \ldots, b_n) \) be the bidding vector. Then bidder \( i \)'s payoff is \( U_i = \theta_i y_i(b_i, b_{-i}) + t_i(b_i, b_{-i}) \), which satisfies:

\[
y_i(b_i, b_{-i}) = \begin{cases} 
1 & \text{if } b_i > \max_{j \neq i} b_j \\
0 & \text{if } b_i < \max_{j \neq i} b_j,
\end{cases} \quad (21.2.11)
\]

\[
t_i(b_i, b_{-i}) = \begin{cases} 
-b(3) & \text{if } b_i > \max_{j \neq i} b_j \\
0 & \text{if } b_i < \max_{j \neq i} b_j,
\end{cases} \quad (21.2.12)
\]

where \( b(3) \) is the third highest bidding price.

In the symmetric Bayesian-Nash equilibrium under the third-price auction, like the first-price and second-price auctions, the highest bidder obtains the object; meanwhile the bid of bidder with \( \theta = 0 \) is zero, and the payoff is zero.

Let \( \beta^{III}(\cdot) \) be the third-price Bayesian-Nash equilibrium strategy function. It is assumed to be an increasing function. In a symmetric environment, consider bidder 1’s strategy choices. It is obvious that only when \( \theta_1 = \theta > \vartheta_1 \), bidder 1 can win the object. Consider \( \vartheta_2 \) to be the second order statistics of \( \{\theta_2, \ldots, \theta_n\} \), at equilibrium, bidder 1’s expected payment for winning the object is \( E_{\vartheta_2}[\beta^{III}(\vartheta_2)|\vartheta_1 = \theta] \). By the Revenue Equivalence Theorem, we have:

\[
\Psi_1(\theta)^{(n-1)} E_{\vartheta_2}[\beta^{III}(\vartheta_2)|\vartheta_1 = \theta] = \int_0^\theta \vartheta \Psi_1(\vartheta) d\vartheta.
\]

Let \( \psi_k^{(m)} \) be the distribution function of the \( k \)th order statistics in \( m \) independently and identically distributed random variables, and \( \psi_k^{(m)} \) be the corresponding density function. \( (\vartheta_2|\vartheta_1 = \theta) \)'s density function is:

\[
\psi_2^{(n-1)}(\vartheta_2 = \tau|\vartheta_1 = \theta) = \frac{1}{\Psi_1(\theta)^{(n-1)}(n-1)[\Phi(\theta) - \Phi(\tau)]\psi_1(\tau)^{(n-2)},}
\]
where \((n - 1)[\Phi(\theta) - \Phi(\tau)]\) is the probability of \(\psi_1\)’s value in \([\tau, \theta]\), and 
\(\psi_1(\tau)^{(n-2)}\) is the density function of the first order statistics in \(n - 2\) independently and identically distributed random variables.

Thus, for bidder with \(\theta_1 = \theta\), the expected payment is:

\[
\Psi_1(\theta)^{(n-1)} \int_0^\theta \beta^{III}(\tau)\psi_2^{(n-1)}(\theta_2 = \tau | \theta_1 < \theta) \, d\tau = \int_0^\theta \beta^{III}(\theta_2)(n - 1)[\Phi(\theta) - \Phi(\tau)]\psi_1(\tau)^{(n-2)} \, d\tau.
\]

We then have by the Revenue Equivalence Theorem:

\[
\int_0^\theta \beta^{III}(\tau)(n - 1)[\Phi(\theta) - \Phi(\tau)]\psi_1(\tau)^{(n-2)} \, d\tau = \int_0^\theta \vartheta_1 \psi_1(\vartheta_1) \, d\vartheta_1,
\]

where \(\psi(\cdot)\) is \(\theta_1\)’s density function.

Taking derivative on both sides with respect to \(\theta\) leads to:

\[
(n - 1)\varphi(\theta) \int_0^\theta \beta^{III}(\theta_2)\psi_1(\theta_2)^{(n-2)} \, d\theta = \theta\psi(\theta).
\]

Since \(\psi(\theta) = (n - 1)\Phi(\theta)^{n-2}\varphi(\theta)\), we then have

\[
\int_0^\theta \beta^{III}(\theta)\psi_1(\theta)^{(n-2)} \, d\theta = \theta\Phi(\theta)^{n-2}.
\]

Taking derivative on both sides with respect to \(\theta\) again, we get:

\[
\beta^{III}(\theta)\psi_1(\theta)^{(n-2)} = \Phi(\theta)^{n-2} + \theta(n - 2)\Phi(\theta)^{n-3}\varphi(\theta).
\]

Since \(\psi_1^{(n-2)}(\theta) = \Phi(\theta)^{n-2}\), we have

\[
\beta^{III}(\theta) = \theta + \frac{\Phi(\theta)}{(n - 2)\varphi(\theta)}.
\]

which is an increasing function when \(\frac{\Phi(\theta)}{\varphi(\theta)}\) is increasing or equivalently \(\Phi(\theta)\) is log-concave.

Comparing first-price, second-price and third-price auctions, we find:

\[
\beta^{I}(\theta) < \beta^{II}(\theta) = \theta < \beta^{III}(\theta).
\]

That is to say, bidders under first-price auction will bid lower than their true value (otherwise the expected utility is less than or equal to zero), bidders under the second-price auction will bid their true value, and bidders under the third-price auction would bid more than their true value. The reason why bidders under the third-price auction will bid exceed their value is: when other bidders bid with \(\beta^{III}\), if bidder 1 bids \(b > \theta_1\), compared with bidding \(\theta_1\), bidder 1 would be better off under \(\beta^{III}(\vartheta_2) < \theta_1 < \beta^{III}(\vartheta_1) < b\), though he would be worse off under \(\theta_1 < \beta^{III}(\vartheta_2) < \beta^{III}(\vartheta_1) < b\), but for \(b - \theta_1 = \epsilon\), when \(\epsilon\) is sufficiently small, the probability of \(\beta^{III}(\vartheta_2) < \theta_1 < \beta^{III}(\vartheta_1) < b\) is \(\epsilon^2\) order, while \(\theta_1 < \beta^{III}(\vartheta_2) < \beta^{III}(\vartheta_1) < b\) is \(\epsilon^3\) order. Thus there is incentive for bidders to bid beyond their values.
Uncertain Number of Bidders

We now consider auctions with uncertain number of bidders. Suppose that \( \mathcal{N} = \{1, 2, \ldots, N\} \) denote potential bidders. Let \( A \subseteq \mathcal{N} \) be the set of actual bidders. Assume that for each potential bidder, the values of the object are independent and identically distributed with the distribution function \( \Phi(\cdot) \). For participant \( i \in A \), assume the probability of facing \( n \) other bidders is \( p_n \), and the probability does not depend on the identity of the bidder nor on his value. As long as the bidder is consistent with the type and number of opponents he faces, then for the symmetric equilibrium, we can also use the Revenue Equivalence Theorem to obtain the Bayesian-Nash equilibrium bidding function. Suppose that the symmetric Bayesian-Nash equilibrium bidding strategy \( \beta \) is a monotonically increasing function and the probability for bidder 1 of facing \( n \) other bidders is \( p_n \). Let \( \vartheta_1(n) \) be the first order statistics of types in \( n \) other bidders, and the distribution function is \( \Psi(n)(\cdot) = \Phi^n(\cdot) \). When bidder 1’s type is \( \theta \) and bidding price is \( \beta(\hat{\theta}) \), the probability of winning the auction is:

\[
\Psi(\hat{\theta}) = \sum_{n=0}^{N-1} p_n \Psi(n)(\hat{\theta}),
\]

and the interim expected utility/payoff is:

\[
\bar{U}_1(\theta, \hat{\theta}) = \Psi(\hat{\theta})\theta + \bar{t}_1(\hat{\theta}),
\]

where \( \bar{t}_1(\hat{\theta}) \) is the interim expected transfer payment for bidder 1 when choosing \( b_1 = \beta(\hat{\theta}) \).

Let us consider the first-price and second-price auctions with uncertain number of bidders. First, for the second-price auction, despite the number of opponents, \( \beta(\theta) = \theta \) would always be a weakly dominant strategy for bidder 1. Then the interim transfer payment under second-price auction is

\[
\bar{t}_1^{II}(\hat{\theta}) = -\sum_{n=0}^{N-1} p_n \Psi(n)(\theta) E[\beta^{II}(\vartheta_1(n) | \vartheta_1(n) < \theta)].
\]

For the first-price auction, the interim expected payment in equilibrium is:

\[
\bar{t}_1^I(\hat{\theta}) = -\Psi(\hat{\theta})\beta^I(\theta).
\]

According to the Revenue Equivalence Theorem, we get: \( \bar{t}_1^{II}(\hat{\theta}) = \bar{t}_1^I(\hat{\theta}) \), that is:

\[
\beta^I(\theta) = \sum_{n=0}^{N-1} \frac{p_n \Psi(n)(\theta)}{\Psi(\theta)} E[\beta^{II}(\vartheta_1(n) | \vartheta_1(n) < \theta)]
\]

\[
= \sum_{n=0}^{N-1} \frac{p_n \Psi(n)(\theta)}{\Psi(\theta)} \beta^{I,n}(\theta),
\]
where $\beta^{\ell}(n)(\theta)$ denotes the equilibrium bidding price under first-price auction with his own type $\theta$ when the number of opponents is $n$.

Thus, for a first-price auction with an uncertain number of bidders, the bidding strategy of a symmetric equilibrium is the weighted average of first bidding prices under a given number of bidders, whose weight is equal to the probability distribution of the number of opponent bidders.

### 21.2.7 Optimal Auction Mechanism Design

Now we discuss the seller’s optimal auction scheme, which means to consider the optimal auction that satisfies the Bayesian incentive compatibility and participation constraints. Suppose that the seller can retain the object. We call the seller with “reserve object” “participant 0”, the value of the object is denoted by $\theta_0$, and the decision on whether or not to retain the object is written as $y_0 \in \{0, 1\}$. Then we must have $\sum_{i=0}^{n} y_i = 1$.

Thus, the seller’s total expected revenue can be written as

$$\theta_0 E_{\theta_0}(\theta) + \sum_{i=1}^{n} E_{\theta_i}(\theta_i y_i(\theta)) - \sum_{i=1}^{n} E_{\theta_i} E_{\theta_{-i}}[U_i(\theta, \theta_{-i})].$$

As such, the total expected payoff of the seller equals the difference between the total surplus and the expected information rent of the participants.

According to the Bayesian incentive compatibility characterization theorem in Chapter 19, the expected utility of the buyer must satisfy

$$E_{\theta_{-i}}[U_i(\theta, \theta_{-i})] = E_{\theta_{-i}}[U_i(\theta, \theta_{-i})] + \int_{\theta_i}^{\theta_{-i}} E_{\theta_{-i}, \tau}(\tau, \theta_{-i}) d\tau, \forall \theta_i \in \Theta_i.$$

The interim individual rationality constraint of the buyer is $E_{\theta_{-i}}[U_i(\theta, \theta_{-i})] \geq 0, \forall i$. Given the decision rule $y_i(\theta)$, the optimal choice for a rational buyer is to set the transfer payment as $E_{\theta_{-i}}[U_i(\theta, \theta_{-i})] = 0, \forall i$. Integrating by parts for the equation above, we can write buyer $i$’s information rent as

$$E_{\theta_i} E_{\theta_{-i}} \left[ -\frac{1}{h_i(\theta_i)} y_i(\theta) \right],$$

where $h_i(\theta_i) = \frac{\varphi_i(\theta_i)}{1 - \Phi_i(\theta_i)}$ is participant $i$’s hazard rate. Substituting into the seller’s revenue formula, we can write it as expected virtual surplus:

$$E_{\theta_i} \left[ \theta_0 y_0(\theta) + \sum_{i=1}^{n} \left( \theta_i - \frac{1}{h_i(\theta_i)} \right) y_i(\theta) \right].$$

Finally, for $i \geq 1$, let $\nu_i(\theta_i) = \theta_i - 1/h_i(\theta_i)$, and we call it participant $i$’s “virtual valuation”. Then let $\nu_0(\theta_0) = \theta_0$. The seller’s optimization problem
can be written as
\[
\max_{x(\cdot)} E_\theta \left[ \sum_{i=0}^{n} \nu_i(\theta_i)y_i(\theta_i) \right] \quad \text{s.t.} \quad \sum_{i=0}^{n} y_i(\theta) = 1, \\
E_{\theta^{-i}}y_i(\theta_i, \theta_{-i}) \text{ is non-decreasing in } \theta_i, \forall i \geq 1 \quad (BM).
\]

Let us ignore temporarily the Bayesian monotonicity (BM) constraint. For each status $\theta$, we maximize the above expectation. For all individuals $j \neq i$, when $\nu_i(\theta_i) > \nu_j(\theta_j)$, $y_i(\theta) = 1$. The corresponding decision rule is $y(\theta) = y(\nu_0(\theta_0), \nu_1(\theta_1), \cdots, \nu_n(\theta_n))$, where $y(\cdot)$ is the efficient decision rule. Intuitively, since the seller (principal) cannot extract all of buyer’s information rent, the principal will adopt the virtual valuation rather than the true value of the participant. The virtual valuation of the participant is less than his true value because the latter includes the information rent of the participant and the principal cannot extract the rent. The maximization of seller’s revenue should allocate the object to the participant with the highest virtual valuation.

Then, under what conditions can we ignore monotonicity constraint? Notice that when virtual valuation function $\nu_i(\theta_i) = \theta_i - 1/h_i(\theta_i)$ is an increasing function\(^1\), in the solution to the reduced problem, the increase in the value of the participant will make him more likely to get the object. Therefore, for all $\theta_{-i}$, $y_i(\theta_i, \theta_{-i})$ is non-decreasing with regard to $\theta_i$. Thus according to the dominant incentive compatibility characterization theorem (Proposition 19.4.1), optimal allocation rules can be implemented by not only the Bayesian-Nash equilibrium, but also the dominant strategy equilibrium. We can also have a DIC (dominant-strategy incentive compatible) transfer payment by integrating the DICFOC (first-order condition for dominant-strategy incentive compatibility), and the resulting transfer payment is:
\[
t_i(\theta) = -p_i(\theta_{-i})y_i(\theta),
\]
where
\[
p_i(\theta_{-i}) = \inf\{\hat{\theta}_i \in [\theta_i, \bar{\theta}_i] : y_i(\hat{\theta}_i, \theta_{-i}) = 1\}.
\]

Therefore, for each buyer (agent) $i$, there exists pricing rule $p_i(\theta_{-i})$, which is a function of other bidders’ true values. If the participant’s bid is greater than the price $p_i(\theta_{-i})$, then the bidder gets the object, and pays at price $p_i(\theta_{-i})$. This means telling the truth (truthful revelation) is the dominant strategy. The logic of the argument is the same as that of Vickrey auction: misreporting does not affect the price paid, but only affects the acquisition of the object. Bidders want to get the object just when their value $\theta_i$ is higher than the price $p_i(\theta_{-i})$.

\(^1\)Such a sufficiency condition is that the hazard rate $h_i(\theta_i)$ is a non-decreasing function. The process of proof is the same as that of a single participant (Notice that for linear utility function $\nu(y, \theta) = \theta y$, we have $\nu_{y\theta} = 0$).
Suppose that for all $i$, $\psi_i = \psi$ and $\nu_i = \nu$. That is, the distributions of bidders’ types are the same, then their virtual valuation functions are the same, denoted by $\nu(\cdot)$. Assume the function is increasing. When the principal sells the object, since bidder $i$ with the highest value has the highest virtual valuation $\nu(\theta_i)$, he obtains the object. When $\nu(\max_{i \geq 1} \theta_i) > \theta_0$, the principal sells the object. Then we have

$$p_i(\theta_{-i}) = \max\{\nu^{-1}(\theta_0), \max_{j \neq i} \theta_j\}.$$ 

Thus the optimal mechanism that is truthfully implementable in dominant strategy is a second price auction with a reserve price $r^* = \nu^{-1}(\theta_0)$. The optimal reserve price has two characteristics. First, if it is greater than the value of the object for the principal, it is equivalent to monopoly pricing, the price of which is higher than the marginal cost. In this sense, if the seller can determine the optimal reserve price, the standard auction does not achieve Pareto efficiency, and the object cannot be sold in a socially optimal manner because when the highest buyer’s private value is between $\theta_0$ and $r^*$, the seller will not resell the object to this buyer. The intuition is that principal can reduce the information rent by reducing individual spending. This is similar to the case of only one individual.

Why is it optimal to allocate the object based on the optimal reserve price (virtual valuation)? It is not hard to understand. We give the following explanation. Suppose that the seller adopts a take-it-or-leave-it offer to sell the object to buyer at price $p$. If the seller decides on a reserve price $r$, then only when the buyer’s private value $\theta$ is higher than or equal to $r$, he would buy the object. The probability that the buyer will accept this price level is $1 - \Phi(p)$, which is also the probability of the value of buyer exceeding $p$. If we consider the probability of purchasing as the buyer $i$’s demand, then the demand function is $q(p) \equiv 1 - \Phi(p)$, where the inverse demand function is $p(q) \equiv \Phi^{-1}(1 - q)$. Then, the seller’s revenue function is

$$p(q) \times q = q\Phi^{-1}(1 - q),$$

where the derivative on $q$ is

$$\frac{d}{dq}(p(q) \times q) = \Phi^{-1}(1 - q) - \frac{q}{\Phi'(\Phi^{-1}(1 - q))}.$$ 

Since $\Phi^{-1}(1 - q) = p$, we have

$$MR(p) \equiv p - \frac{1 - \Phi(p)}{\phi(p)} = \nu(p),$$

which means marginal revenue equals buyer’s virtual valuation at $p(q) = p$. Thus, buyer’s virtual valuation $\nu(p)$ can be interpreted as marginal revenue. Since $\psi$ is strictly increasing, the seller can set monopoly price $r^*$ according to the marginal cost equalling the marginal revenue $MR(p) = MC$. Since the latter is assumed to be $\theta_0$, $MR(r^*) = \nu(r^*) = \theta_0$ or $r^* = \nu^{-1}(\theta_0)$. 


When the seller faces different types of buyers, the optimal mechanism is to adopt discriminatory reserve price \( r^*_i = \nu_i^{-1}(\theta_0) \). If no buyer’s value of the object exceeds the reserve price \( r^* \), the seller retains this object. Otherwise, the seller sells the object at marginal revenue, and the buyer who wins the object is asked to pay \( p_i = p_i(\theta_{-i}) \), which is the lowest value of winning the object for the buyer.

The second characteristic of the optimal reserve price is that it is independent of the number of bidders but depends on the distribution of bidders’ private values. This result does not hold when bidders’ private values are not independent or when the bidders are not risk-neutral.

In addition, if the virtual valuation function is not increasing, BM may still be binding, so we need some “ironing” (see discussion on ironing in Chapter 16) scheme to identify the domain in which BM is binding. In this situation, as the principal will underestimate the true values of the participants, the object is unlikely to be sold in the socially optimal way.

21.2.8 Factors Affecting Revenue Equivalence and Efficiency

According to Myerson’s Revenue Equivalence Theorem, since all possible forms of auction yield the same expected payoff to the auctioneer, can he choose an arbitrary auction format? Unfortunately, although Myerson’s conclusion is theoretically significant, the conditions are too strong to hold in reality. If one of the following situations happens, Myerson’s Revenue Equivalence Theorem may not be true. In this sense, Melson’s conclusion does not end the auction theory, but provides a benchmark theory and a new starting point for its development of more realistic auctions. Indeed, most of the subsequent research on auction theory was based on the work of Myerson. We will discussed the remain of this chapter.

Here we do not provide the detailed discussion for the cases Myerson’s Revenue Equivalence Theorem may not hold, instead, we give the basic conclusions.

**Limited Liability**

In the previous discussions, we assume that the bidder is able to pay up to his highest value \( \theta_i \). Suppose that unlimited liability does not hold, and the bidder’s budget \( \tilde{w} \) is a random variable distributed on \([0, \tilde{w}]\). \( w \) is a realization of \( \tilde{w} \).

It can be proved that the second-price auction with limited liability still has a dominant equilibrium, whose equilibrium bidding function is given by \( B^{II}(\theta, w) = \min\{\theta, w\} \). The Bayesian-Nash equilibrium bidding function of first-price auction is given by \( B^I(\theta, w) = \min\{\beta^I(\theta), w\} \), and the auctioneer’s expected payoff under first-price auction is greater than the expected payoff under second-price auction.
Asymmetry between Bidders

An important hypothesis in the SIPV model is symmetry. The distribution functions of all bidders are the same. It is hard to imagine that this symmetry is always satisfied in the real auction environment. What impact will the asymmetry between bidders have on the auction?

First of all, since truth-telling in second-price auction is the dominant equilibrium, regardless of whether the auctioneer’s distribution function is symmetric or not, the highest bidder gets the object, so that the second-price auction is still efficient. However, as the truth-telling may not be a dominant equilibrium in other mechanisms, they are not efficient. For example, the first-price auction is actually not efficient. To see this, suppose there are two bidders, and the Bayesian equilibrium bidding functions are continuously and monotonically increasing, denoted by $\beta_1$ and $\beta_2$. Assume for some $\theta$, $\beta_1(\theta) < \beta_2(\theta)$. Thus, for sufficiently small $\epsilon > 0$, we still have $\beta_1(\theta + \epsilon) < \beta_2(\theta - \epsilon)$. This means that bidder 2 gets the object despite his true value is smaller.

Secondly, for the auctioneer, the profits of different auctions may not be the same. From the previous discussion on the optimal auction mechanism design, $\nu(\theta_i)$ is the virtual valuation function of bidder $i$, which is equivalent to the seller (or auctioneer)’s marginal revenue. As such, the seller’s optimal mechanism should sell the object to the bidder with the highest marginal revenue $\nu(\theta_i)$, provided that his marginal revenue is not less than the seller’s true value or marginal cost. Thus, when the bidder is not symmetric, the optimal auction is discriminatory and the bidder who wins the object is not necessarily the one with the highest private value. This also means that the seller’s optimal mechanism is not efficient. Furthermore, the seller will also set different reserve prices for different bidders. This mechanism is similar to monopoly price discrimination, where monopolist sets different prices according to the different needs of consumers.

Which of the first-price and second-price auctions is beneficial to the seller under asymmetric conditions? First of all, the asymmetry does not affect the bidder’s strategy in the second-price auction, and bidding his true value is still a dominant strategy. However, in the first-price auction, the weaker bidders may bid more aggressively. Therefore, for the seller, first-price auction may be better than the ascending-price auction or second-price auction. For a detailed discussion, see Krishna (2010).

Risk Non-neutrality

For bidders, an auction may bring risk. When the bidder wins, he gains profits; when he loses, his payoff is zero. If the bidder dislikes the risk or has different degrees of risk aversion, his bidding strategy is likely to be different.
21.2. SINGLE OBJECT AUCTIONS WITH PRIVATE VALUES

Under first-price auction, if a bidder increases his bid, it will increase the probability of winning but at the same time his profit is reduced after winning. A risk-averse bidder will be more inclined to raise his bid than a risk-neutral bidder. Therefore, if all the bidders are risk averse, the first-price auction will bring higher profits to the seller. See Krishna (2010) for the proof.

Under the second-price auction, whether a bidder is risk-neutral or risk-averse, he always bids his true value. Therefore, if other assumptions in the SIPV model remain unchanged but all bidders are risk-averse, then the Revenue Equivalence Theorem no longer holds. The first-price auction yields a higher expected revenue than the second-price auction or ascending-price auction.

Collusive Bid

In many auction situations, bidders often bid together to reduce competition and lower the price paid to the seller. This collusive bidding behavior exists in several ways, depending on auction formats and informational structures. For example, some bidders may first take pre-auctions together to choose the best bidders for the formal auction; sometimes bidders take turns to participate in multiple auctions; in some cases, while all bidders participate in the auction, they make an appointment in advance on low bids, allowing one bidder to win and later (or in advance) sharing the revenue.

The analysis of collusive bidding strategy is complicated. The biggest difficulty is that bidders may change their strategy. Thus, in order to prevent participants from changing their strategy, the collusive bidding mechanism must satisfy the collusion-proof incentive compatible constraint. In order to bring more bidders into collusion, but not mandatory, this mechanism must meet the participation constraint. Moreover, collusive bidding is best worked out by letting the bidder who is willing to bid the most win the object.

In the SIPV model, McAfee and McMillan (AER, 1992) demonstrated that there exists a direct revelation collusion mechanism that includes all bidders, satisfying both efficiency and collusion-proof incentive compatibility. If the seller uses a standard auction such as the first-price or the second-price auction, every bidder is willing to participate in this collusive mechanism. The basic procedure for such a mechanism is that before participating in the formal auction, all the participants hold their own auction to elect the only bidder for the formal auction. The rule is that each bidder will bid separately and the highest bidder will be selected and he must pay his own bid, whereby the profits are equally divided among all other bidders. The selected bidder participates in the formal auction, and his best bid is equal to the seller’s reserve price, thus winning the auction and
paying his bid. This collusive bidding mechanism is equivalent to the case that every bidder wants to bribe others to quit the competition, and the bidder who is willing to pay the highest bribe wins and shares the profits of bribery with others.

**Participation Costs**

When bidders have participation costs, low-value bidders are more motivated to participate in collusion mechanism than high-value bidders. High-value bidders send out a signal of higher value by denying collusion mechanism, which is more credible. Tan and Yilankaya (2007) proved that when the auctioneer uses the second-price auction, such information will reduce the participation in the bidding mechanism for other bidders, making it harder to form collusion mechanism. Thus McAfee-McMillan’s conclusion no longer holds. Cao, Hsueh and Tian (2020) proved that when the auctioneer adopts the first-price auction, McAfee-McMillan’s conclusion remains to fail. In addition, Tan and Yilankaya (2006), Cao and Tian (2010, 2013) and Cao, Tan, Tian and Yilankaya (2018) discussed the existence of equilibrium in first-price and second-price auctions with participation costs. In general, in the presence of participation costs, the Revenue Equivalence Theorem will be challenged.

In addition to the above factors, the interdependent value that will be discussed below also affects the validity of Revenue Equivalence Theorem.

### 21.3 Single Object Auctions with Interdependent Values

In the previous section, we obtain the expected revenue equivalence result to the auctioneer. For any two auctions, if the bidders’ expected payments under the lowest value are the same and the allocation rule is the same, then the auctioneer’s expected revenue is the same. In a symmetric economic environment with independent private values, the auctions, such as first-price (sealed-bid) auction, second-price (sealed-bid) auction, the English auction (the auction with ascending price) and the Dutch auction (the auction with descending price), all have the same allocation rule (i.e., the highest bidder gets the object) and the same expected payoff (usually zero) when having the same minimum private value. As such, they are indifferent to the auctioneer.

However, the Myerson’s Revenue Equivalence Theorem relies on a series of assumptions, and the most crucial one is that all bidders’ values of the auctioned object are independent but not interdependent to others. In reality, this assumption is apparently not true, and the bidder’s valuation depends not only on himself but also on those of other bidders. For exam-
ple, in an auction of artworks, bidders take into account not only their own preference for the artwork, but also the potential profits from resale, which is obviously affected by others’ valuations.

After taking into account the behavior of their competitors, the valuation of the object is called “interdependent valuation”. Myerson’s theory no longer holds when there is “interdependent valuation”, and the auctioneer may be able to improve their expected payoff through the design of trading mechanisms. Milgrom and Weber (1982) were the first to study auctions with interdependent values. They built an analytical framework that deals with information, pricing and auctioneer’s profits when there are “interdependent valuations”.

Based on the observation of auction practice, they found that bidders’ valuations may be affiliated: a higher bid by a bidder can easily increase the rating of other participants. Thus, auction can be understood as any buyer’s bid will not only show his own information about the value of object, but also partially reveal the private information of other buyers. In this sense, the payoff of a bidder will depend on the degree of privacy of the information.

Once the information is revealed in the auction, bidders can guess each other’s possible bids, and in order to win the auction, they have to bid a higher price. Therefore, for the auctioneer, the auction that will bring him the highest expected revenue must be the one that most efficiently reveals the private information of bidders. In the literature on auction theory, Milgrom and Weber called this case as interdependent values.

In reality, the auctioneer devotes a great deal of energy to designing a revenue-maximizing auction. By applying the auction theory, many well-known auction theorists such as Larry Ausubel, Ken Binmore, Paul Klemperer, Preston McAfee, John McMillan, Paul Milgrom, and Robert Wilson have brought their talents in real world auction designs, such as the spectrum auction design for the Federal Communications Commission, 3G license in the United Kingdom, etc.

All of these objects are not private values, but with interdependent values, which are commonplace. Another typical example is oilfields. Although the amount of oilfield storage is unknown until it is mined, it has impact for all bidders on the value of the oilfield.

As such, the assumption of private value is not applicable for the auction of such objects, and thus the auction design for such economies with interdependent values will have a big impact on the auction results, which shall be discussed in this section.

21.3.1 Basic Analytical Framework

In the following, we use the mechanism design approach to discuss the auctions under the interdependent value, as well as the comparison of revenue
under different auction formats. As mentioned above, the auction theory of interdependent value was first proposed by Milgrom and Weber (1982), which has a significant influence on the development of auction theory.

We first provide some notations and definitions. There are \(n\) bidders, and bidder \(i\)'s signal for the object value is \(\theta_i\), which is private information. The object value \(V_i\) can be expressed as a function of bidders' private signals, that is, \(V_i = u_i(\theta_1, \ldots, \theta_n)\). If \(V_i = u_i(\theta_i)\), then the object has the characteristics of private value, and \(\theta_i = E(V_i|\theta_i)\) is the private value of bidder \(i\) as described in the previous section. So, the bidder's value of the object is entirely determined by the signal he/she knows. If \(V_i = u(\theta_1, \ldots, \theta_n) = V\), then the object has the characteristics of common value. If \(j \neq i\), \(\theta_j\) affects \(V_i\), then the object has the characteristics of interdependent value, so private value and common value can be seen as special cases of interdependent values.

Suppose that bidder \(i\)'s bidding strategy is \(b_i\), the probability for the bidder to get the object is \(y_i(b_i, b_{-i})\), and the payment is \(t_i(b_i, b_{-i})\). As before, the bidder \(i\)'s utility in bidding mechanism can be written as:

\[
U(\theta_i, \theta_{-i}, b_i, b_{-i}) = u_i(\theta_1, \ldots, \theta_n)y_i(b_i, b_{-i}) + t_i(b_i, b_{-i}).
\]

For the four common auctions with interdependent values, bidder \(i\)'s probability to get the object, \(y_i(b_i, b_{-i})\), and interim expected payment after winning, \(t_i(b_i, b_{-i})\), are the same as in the case of private value.

For the first-price sealed-bid auction and the Dutch auction, the \(y_i(b_i, b_{-i})\) is given by equation (21.2.1), and the ex post transfer payment \(t_i(b_i, b_{-i})\) is given by equation (21.2.2); for the second-price sealed-bid auction and the English auction, the \(y_i(b_i, b_{-i})\) is given (21.2.6), and the ex post transfer payment \(t_i(b_i, b_{-i})\) is given by (21.2.7).

For object with common value, the common value is denoted by \(V\). Assume that the private signal \(\theta_i\) under given common value \(V\) is independently distributed, which is also an unbiased estimate of value, that is, \(E(\theta_i|V = v) = v\). Consider bidder 1 with signal \(\theta_1 = \theta\), then he evaluates the value of the auctioned object as \(E(V|\theta_1 = \theta)\). Let \(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}\) be the largest, second largest, ..., and smallest signal values of \(\theta_2, \theta_3, \ldots, \theta_n\), or be the first-order, second-order, ..., \(n - 1\)th order statistics of \(\{\theta_2, \ldots, \theta_n\}\). Assume that the bidders are symmetric, and they choose the same bidding strategy function.

Milgrom and Weber also showed the possibility of "winner’s curse". In auction practice, there often exists a phenomenon that the bidder feels it unworthy after winning the auction, which is called "winner’s curse" in auction theory. If all bidders are symmetric and follow the same strategy \(\beta\), then this fact reveals to bidder 1 that the highest of the other \(n - 1\) signals is less than \(\theta\). As such, after he wins, his estimate of the value is \(E(V|\theta_1 = \theta, \vartheta_1 < \theta)\) that is less than the initial estimate \(E(V|\theta_1 = \theta)\), which leads
to a decrease in the estimated value and is lower than his initial estimate so that winning brings “bad news”. To put it succinctly, if the successful bid comes from an overly optimistic estimate of the auctioned object, the bidder would regret to win the object, which is called the “winner’s curse”.

For example, in reality it is often seen that some bidders will boldly raise their bids to a certain level when they recognize other competitors’ intention of raising prices. When they find that the competitors are not going to follow the price rise anymore, they realize that they just bid too much, but at this moment, being regretful is too late. This is the “curse” with winning (e.g., Didius Julianus paid nearly two kilos of gold to each Praetorian Guard soldier, winning their support and gaining the emperor’s throne. However, this resulted not only in too high price, but also cost his life; corruption brings promotion, but also leads to prison). In fact, this is caused by bidder’s wrong calculation of his expected payoff according to $E(V|\theta_1 = \theta)$.

Obviously, the traditional assumption on all bidders’ independent valuations of the auction object cannot explain the possibility of “winner’s curse”, while it becomes easy to understand with the introduction of interdependent values. The winning bidder also receives private information about other bidders’ valuations when winning against other bidders, which causes him to lower his rating of the trophy he has just acquired. In the actual bidding, if the bidder is rational, in order to avoid the “winner’s curse”, he should lower his estimation on the value of the object, and the extent of the reduction will be influenced by the auction rules. Therefore, an appropriate auction needs to minimize the impact of “winner’s curse” for bidders. This is true based on the following discussions. By correct computation according to $E(V|\theta_1 = \theta, \theta_1 < \theta)$, this will not happen at equilibrium.

**Basic Assumptions**

Bidders have the profile of private signals, denoted by $\theta = (\theta_1, \ldots, \theta_n)$. If the density function of $\theta$ does not satisfy $\varphi(\theta) = \Pi_i^n \varphi_i(\theta_i)$, where $\varphi_i(\cdot)$ is the density function of $\theta_i$, then the distribution of private signals is not independent. We hereby introduce the concept of “affiliation” between signals.

**Definition 21.3.1** The random vector $\theta = (\theta_1, \ldots, \theta_n)$ on $\Theta \subseteq \mathbb{R}^n$ is affiliated if for any $\theta, \theta' \in \Theta$, 

$$\varphi(\theta \lor \theta') \varphi(\theta \land \theta') \geq \varphi(\theta) \varphi(\theta'),$$

where 

$$\theta \lor \theta' = (\max\{\theta_1, \theta'_1\}, \ldots, \max\{\theta_n, \theta'_n\})$$

and 

$$\theta \land \theta' = (\min\{\theta_1, \theta'_1\}, \ldots, \min\{\theta_n, \theta'_n\}).$$
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Let \( \vartheta_1, \ldots, \vartheta_{n-1} \) be the first, second, ..., and \( n-1 \)th order statistics of \( \{\theta_2, \ldots, \theta_n\} \). We have (see the proof in Milgrom and Weber (1982)): if \( \{\theta_1, \ldots, \theta_n\} \) is affiliated, the variables in any of their subsets are affiliated, and \( \{\theta_1, \vartheta_1, \ldots, \vartheta_{n-1}\} \) are also affiliated. If \( \theta \)’s density function \( \varphi(\cdot) \) is positive on \( \Theta \), and twice continuously differentiable, then \( \{\theta_1, \ldots, \theta_n\} \) are affiliated if and only if \( \frac{\partial^2 \ln(\varphi)}{\partial \theta_i \partial \theta_j} \geq 0, \forall i \neq j \).

In simple terms, the affiliation between variables means that if the value of a random variable is larger, the probability that any other random variables associated with it have a larger value is also higher. For example, in an auction of oilfields, if a bidder learns in advance that a certain oilfield’s reserve is large, other bidders also have a high probability of obtaining similar (private) information.

Define \( \Psi(\cdot|\theta) \) as the conditional distribution function of \( \vartheta_1 \) given \( \theta_1 = \theta \), and \( \psi(\cdot|\theta) \) as the corresponding conditional density function. The affiliation between \( \theta_1 \) and \( \vartheta_1 \) implies that, if \( \theta' > \theta \), then \( \Psi(\cdot|\theta') \) dominates \( \Psi(\cdot|\theta) \) in terms of the reverse hazard rate, that is:

\[
\frac{\psi(\vartheta|\theta')}{\Psi(\vartheta|\theta')} \geq \frac{\psi(\vartheta|\theta)}{\Psi(\vartheta|\theta)}
\]

It is easy to verify that for increasing functions \( h(\cdot) \), we have \( E(h(\vartheta_1)|\theta') \geq E(h(\vartheta_1)|\theta) \).

Applying the concept of affiliation, Milgrom and Weber analyzed several standard auctions. In English auctions, bids from bidders who exit early show their information about the value of the object, and the auction price is affiliated to the valuations of all losing bidders, resulting in higher profits to the seller. In a second-price auction, the auction price is only affiliated to the bidder who values the second-highest, thus generating lower profits. In the Dutch auction and first-price auction, there will be minimal expected payoff to the auctioneer as there is no affiliation between the prices. This finding provides a good explanation for the prevalence of English auctions in reality.

The Differences of Four Basic Auctions under Interdependent Value

In the case of private value, the second-price (sealed-bid) auction and the English auction are outcome-equivalent as private information of other participants does not affect the bidder’s valuation of the object auctioned. This is because a bidder’s valuation relies solely on his own signal, while the exit decisions of others will not affect the bidder’s expected valuation revision, and the rules that the bidder wins the auction and pays (the second highest price) are also the same. Similarly, first-price (sealed-bid) auction and Dutch auction are also outcome and in fact strategically equivalent.
However, in the case of interdependent value and affiliated information, if the number of bidders is three or more, there is no equivalence between the second-price (sealed-bid) auction and the English auction. This is due to the fact that bidders that exit the auction at different prices will change the bidding behavior for the remaining bidders, in particular the bidder winning the auction object, by providing information on revising the estimated value of the item. Thus, the auction results are different under these two auction rules. However, if there are only two bidders, once one bidder exits, the other would win the auction. In this case, the exit of a bidder under a bidding price does not change the winner’s payment, whereby the outcome of the English auction and the second-price (sealed-bid) auction is also equivalent.

Auction rules for Dutch and first-price (sealed-bid) auctions, however, still have no effect on the results because the value and payment to the winner are the same in both cases. However, because the English auction reveals more information than the Dutch auction and the first price auction, its revenue may be higher.

As such, in the following, we mainly compare the first-price (sealed-bid) auction, the second-price auction and the English auction, and focus on the case of symmetry. The symmetry here has two meanings: one is that bidders’ valuation of the object is symmetric, and the other is that bidder’s signal distribution is symmetric, so that we can focus on the discussion of the symmetric bidding (strategy) equilibrium.

We assume that all bidders’ signals are drawn from the same distribution, $\theta_i \in [0, w]$, and their valuations of the object are symmetric. Bidder $i$’s valuation function can be written as $V_i = u_i(\theta) = u(\theta, \theta_{-i})$, which satisfies: $u_i(\theta_i, \theta_{-i}) = u(\theta_i, \tilde{\theta}_{-i})$, where $\tilde{\theta}_{-i}$ is an arbitrary rearrangement of $\theta_{-i}$, such as $u(\theta_i, \theta_j, \theta_k) = u(\theta_i, \theta_k, \theta_j), i \neq j \neq k \neq i$. Of course, the object with symmetric values does not necessarily have a common value. The following example illustrates this point.

**Example 21.3.1 (Symmetric Valuation Function)** Consider three bidders’ value signals for the auction object are $\theta_1, \theta_2, \theta_3$. Suppose that for any bidder $i$, the valuation function is $u(\theta_i, \theta_j, \theta_k) = a\theta_i + b\theta_j + c\theta_k$. Bidders’ valuations are symmetric if and only if $b = c$. When $a \neq b = c$, the valuations of bidders are symmetric, but the object does not have a common value. In this example, the object has common value if and only if $a = b = c$.

We also assume that $u(\theta_i, \theta_{-i})$ is a continuous function which is strictly increasing in $\theta_i$, non-decreasing in $\theta_j, \forall j \neq i$, and satisfies $u(0, \ldots, 0) = 0$. In the symmetric case, we only need to consider bidder 1’s choice.

Define

$$v(\theta, \vartheta) = E(V_1|\theta_1 = \theta, \vartheta_1 = \vartheta),$$

which characterizes bidder 1’s interim expected value of the object when his private signal is $\theta$ and the first order statistics for other bidders’ private
signals is $\vartheta$. Meanwhile we assume that all bidders are risk-neutral, and $v(0, 0) = 0$. $\psi(\vartheta|\theta)$ is the density function of $\vartheta_1 = \vartheta$ under $\theta_1 = \theta$.

In the following we discuss the equilibrium solutions of second-price auction, English auction and first-price auction under interdependent values and their revenue comparison.

### 21.3.2 Second-Price Auctions

In a second-price auction, the bidder with the highest bid gets the object and pays at the second highest price, and pays zero if he does not get the object. We give the symmetric Bayesian-Nash equilibrium results.

**Proposition 21.3.1** Symmetric Bayesian-Nash Equilibrium for second-Price Auction is $\beta_{II}(\theta) = v(\theta, \theta)$.

For the second-price auction, $\beta_{II}(\theta) = v(\theta, \theta) = E(V_1|\theta_1 = \theta, \vartheta_1 = \theta)$ implies that the bidder will use such an interim expected value as the bidding price.

**Proof.** Suppose that bidders $j \neq 1$ all choose the bidding strategy $\beta = \beta_{II}$, bidder 1’s signal is $\theta$, and the bidding price chosen is $b$, then his interim expected payoff is:

$$
\tilde{U}(b, \theta) = \int_{0}^{\beta^{-1}(b)} [v(\theta, b) - \beta(\vartheta)] \psi(\vartheta|\theta) d\vartheta
= \int_{0}^{\beta^{-1}(b)} [v(\theta, \vartheta) - v(\vartheta, \vartheta)] \psi(\vartheta|\theta) d\vartheta.
$$

Obviously under the given assumptions, $v(\theta, \vartheta)$ increases as $\theta$ increases. Thus, if $\vartheta < \theta$, we have $v(\theta, \vartheta) > v(\vartheta, \vartheta)$; If $\vartheta > \theta$, then $v(\theta, \vartheta) < v(\vartheta, \vartheta)$, so when $\beta^{-1}(b) = \theta$, $\tilde{U}(b, \theta)$ reaches the maximum, which means $\beta_{II}$ is a symmetric Bayesian-Nash equilibrium of the second-price auction.

In the above argument, we find that, unlike the second-price sealed-bid price auction with private values, $\beta_{II}$ is not a weakly dominant strategy, so it is not a weakly dominant equilibrium but is a Bayesian-Nash equilibrium.

**Example 21.3.2** Suppose that there are three bidders with a common value $V$ for the object that is uniformly distributed on $[0, 1]$. Given $V$, bidders’ signals $\theta_i$ are uniformly and independently distributed on $[0, 2V]$.

Let $\theta = (\theta_1, \theta_2, \theta_3)$ and $\bar{\theta} \equiv \max\{\theta_1, \theta_2, \theta_3\}$. The density of $\theta_i$ conditional on $V$ is $1/2V$ on the interval $[0, 2V]$, so the joint density of $(V, \theta)$ is $1/8V^3$ on the set

$$
\{(V, \theta)|\theta_i \leq 2V, \forall i = 1, 2, 3\}.
$$
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Note that the only information about $V$ that knowledge of $\theta$ provides is that $V \geq \bar{\theta}/2$. Thus, the joint density of $\theta$ is

$$
\varphi(\theta) = \int_{\bar{\theta}/2}^{1} \frac{1}{8V^3} dV
$$

Thus, the density of $V$ conditional on $\theta$ is the same as the density of $V$ conditional on $\bar{\theta}$, and then we have

$$
\varphi(V|\theta) = \varphi(V|\bar{\theta})
$$

$$
= \frac{1}{\varphi(\theta)} \times \frac{1}{8V^3}
$$

$$
= \frac{1}{8V^3} \times \frac{16\bar{\theta}^2}{4-\bar{\theta}^2}
$$

on the interval $[\bar{\theta}/2, 1]$. Thus,

$$
E(V|\theta) = E(V|\bar{\theta})
$$

$$
= \int_{1/2\bar{\theta}}^{1} V \varphi(V|\theta) dV
$$

$$
= \frac{2\bar{\theta}}{2+\bar{\theta}}.
$$

Notice that since $\vartheta_1 = \max\{\theta_2, \theta_3\}$ and $\bar{\vartheta} = \max\{\theta_1, \theta_2, \theta_3\}$, $\bar{\theta} = \max\{\theta_1, \vartheta_1\}$.

$$
v(\theta_1, \vartheta_1) = E(V|(\theta_1, \vartheta_1))
$$

$$
= E(V|\bar{\theta})
$$

$$
= \frac{2\bar{\theta}}{2+\bar{\theta}}.
$$

Thus we obtain

$$
\beta^{II}(\theta) = v(\theta, \theta) = \frac{2\theta}{2+\theta}.
$$

21.3.3 English Auctions

Under English auction, as the bidding price increases, bidders successively exit from the bidding (they can no longer participate in the bidding after exit), and the remaining bidders are the “winners”. The winner’s bidding price is the price at the time of the last bidder’s exit.

Consider bidder 1’s choice: when there are $k \geq 1$ bidders remaining, let $p_n, p_{n-1}, \ldots, p_{k+1}$ be the prices (random variables) when the first bidder, second bidder, ..., the $n-k$ bidder exit; obviously when $j > i$, then
If bidder 1’s private signal is θ, the minimum price he will exit is
\[ β^k(θ, p_{k+1}, p_{k+2}, \ldots, p_n), \]
which depends on the remaining number of bidders and on the price at the time of the bidder’s exit. This minimum price is equivalent to the bidding price in this case. Due to different information and different beliefs on the expected value of the auction under different numbers of remaining bidders, the bidding strategy of bidders in the English auction is a series of bidding functions
\[ β = (β^n, β^{n-1}, \ldots, β^2). \]

The following proposition characterizes the symmetric Bayesian-Nash equilibrium of the English auction.

**Proposition 21.3.2** The symmetric Bayesian-Nash equilibrium in English auction
\[ β = (β^n, β^{n-1}, \ldots, β^2) \]
is fully determined by (21.3.13) and (21.3.14).

**Proof.** Suppose that all bidders except bidder 1 follow the bidding strategy above. Consider bidder 1’s strategy choice. Let \( ϑ_1, \ldots, ϑ_{n-1} \) be the order statistics of \( θ_2, \ldots, θ_n \). Given any realized values of order statistics \( ϑ_1 ≥ ϑ_2 ≥ \ldots ≥ ϑ_{n-1} \). Let θ be the private signal observed by bidder 1. If bidder 1 also follows the bidding strategy above, \( β \), then only when
\[ θ > ϑ_1, \]
he can win the auction, and the price he shall pay is the exit price when bidders with signal \( ϑ_1 \) exit, denoted by \( p_2 \). Using (21.3.14), we can get
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$p_2 = u(\vartheta_1, \vartheta_1, \vartheta_2, \ldots, \vartheta_n)$. Following the bidding strategy above, $\beta$, when \( \theta > \vartheta_1 \), bidder 1’s utility is:

$$u(\theta, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) - p_2 = u(\theta, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) - u(\vartheta_1, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) > 0.$$  

Obviously, when \( \theta > \vartheta_1 \), other bidding strategies cannot bring bidder 1 a utility higher than does $\beta$.

When \( \theta < \vartheta_1 \), and following $\beta$, bidder 1 cannot win the auction. If under other bidding strategies, bidder 1 wins the auction, his payment is $p_2 = u(\vartheta_1, \vartheta_1, \vartheta_2, \ldots, \vartheta_n)$. However, if bidder $\vartheta_1$ exits at price $p_2$, the bidder’s valuation of the object is revised as: $u(\theta, \vartheta_1, \vartheta_2, \ldots, \vartheta_n)$. When \( \theta < \vartheta_1 \), bidder 1’s utility is:

$$u(\theta, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) - u(\vartheta_1, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) < 0.$$  

So, when \( \theta < \vartheta_1 \), there are no other bidding strategies that can bring bidder 1 a utility higher than does $\beta$. Since $\vartheta_1$ is continuously distributed, the probability of $\vartheta_1 = \vartheta_1$ is zero, which has no real impact on the bidder’s expected utility. In summary, there are no other bidding strategies that can bring bidder 1 a utility higher than $\beta$.

It should be remarked that English auction has a nice property that its symmetric strategy equilibrium given by (21.3.13) and (21.3.14) does not depend on distribution function $\varphi(\cdot)$. As such, it forms an ex post Bayesian-Nash equilibrium. This means that the equilibrium strategy $\beta$ has an important “no regret” feature: for any realization of the signals the bidders shall not regret on the outcome. This characteristic makes English auctions different from second-price auctions, which rely on the signal distribution function, and as the first-price auction discussed below, are not ex post Bayesian-Nash equilibrium.

21.3.4 First-Price Auctions

In the first-price sealed-bid auction, the highest bidder wins the object and pays the bidding price. Suppose that $\beta$ is a symmetric Bayesian-Nash equilibrium, which is an increasing function (to be verified later). Given that other bidders all choose this strategy, for bidder 1, his private signal is $\theta$. If his bid is $\beta(z)$, his interim expected utility is:

$$\bar{U}(z, \theta) = \int_0^z [v(\theta, \vartheta) - \beta(\vartheta)]\psi(\vartheta|\theta)d\vartheta,$$

where $v(\theta, \vartheta) = E(V_1|\vartheta_1 = \theta, \vartheta_1 = \vartheta)$, and $\psi(\cdot|\vartheta_1 = \theta)$ is the density function of the first order statistics of other participants’ signals given the bidder’s private signal $\theta$, and $\Psi(\cdot|\theta)$ is the corresponding distribution function.

The first order condition on $z$ satisfies:

$$[v(\theta, z) - \beta(z)]\psi(z|\theta) - \beta'(z)\Psi(z|\theta) = 0.$$
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If \( \beta \) is a symmetric Bayesian-Nash equilibrium, then it satisfies the first order (necessary) condition:

\[
\beta'(\theta) = v(\theta, \theta) - \beta(\theta) \frac{\psi(\theta|\theta)}{\Psi(\theta|\theta)}.
\]

Since \( v(0, 0) = 0 \), then \( \beta(0) = 0 \). Solving the above first order differential equation yields

\[
\beta^I(\theta) = \int_0^\theta v(\vartheta, \vartheta) dL(\vartheta|\theta), \tag{21.3.15}
\]

where \( L(\vartheta|\theta) = \exp\left(-\int_0^\theta \frac{\psi(\vartheta|\theta)}{\Psi(\vartheta|\theta)} dt\right) \).

The following proposition characterizes the symmetric Bayesian-Nash equilibrium of the first-price sealed-bid auction.

**Proposition 21.3.3**  The symmetric Bayesian-Nash equilibrium of the first-price sealed-bid auction \( \beta^I(\theta) \) is given by (21.3.15).

**Proof.** We first show \( L(\vartheta|\theta) \) can be regarded as a distribution function on \([0, \theta]\). By the assumption of affiliations, for any \( t > 0 \), we have:

\[
\frac{\psi(t|t)}{\Psi(t|t)} > \frac{\psi(t|0)}{\Psi(t|0)}.
\]

So,

\[
-\int_0^\theta \frac{\psi(t|t)}{\Psi(t|t)} dt \leq -\int_0^\theta \frac{\psi(t|0)}{\Psi(t|0)} dt = \int_0^\theta \frac{d}{dt}(\ln \Psi(t|0)) dt = -\infty,
\]

Then \( L(0|\theta) = 0 \). Moreover, \( L(\theta|\theta) = 1 \) and \( L(\cdot|\theta) \) is nondecreasing. Thus, \( L(\vartheta|\theta) \) is a distribution function.

Also, \( L(\vartheta|\theta') \leq L(\vartheta|\theta) \) for \( \theta' > \theta \), which implies \( L(\vartheta|\theta') \) first-order stochastically dominates \( L(\vartheta|\theta) \), and thus for all \( \theta' > \theta \), we have

\[
\beta^I(\theta') = \int_0^{\theta'} v(\vartheta, \vartheta) dL(\vartheta|\theta') \geq \int_0^\theta v(\vartheta, \vartheta) dL(\vartheta|\theta) = \beta^I(\theta).
\]

Therefore, we verify that \( \beta^I(\theta) \) is a monotonically increasing function.

Suppose that bidder 1 bids \( \beta(z) = \beta^I(z) \) when his signal is \( \theta \). Then his interim expected utility is:

\[
U(z, \theta) = \int_0^z [v(\theta, \vartheta) - \beta(z)] \psi(\vartheta|\theta) d\vartheta.
\]

The first order condition on \( z \) is:

\[
\frac{\partial U}{\partial z} = [v(\theta, z) - \beta(z)] \psi(z|\theta) - \beta'(z) \Psi(z|\theta)
\]

\[
= \Psi(z|\theta) [v(\theta, z) - \beta(z)] \psi(\theta|\theta) \frac{\Psi(z|\theta)}{\psi(z|\theta)} - \beta'(z)].
\]
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If \( z < \theta \), then \( v(\theta, z) > v(z, z) \), meanwhile by affiliation, we have:

\[
\frac{\psi(z|\theta)}{\Psi(z|\theta)} > \frac{\psi(z|z)}{\Psi(z|z)},
\]

we get:

\[
\frac{\partial U}{\partial z} > \Psi(z|\theta)\left[\frac{v(z, z) - \beta(z)}{\Psi(z|z)} - \beta'(z)\right] = 0.
\]

Similarly, we can get: if \( z > \theta \), then \( \frac{\partial U}{\partial z} < 0 \). So, when \( z = \theta \), \( U(\theta, z) \) reaches the maximum.

The above bidding function is an extension of the first-price auction with private values. Indeed, under private value, \( v(\theta, \theta) = \theta \). If the signal is distributed independently, \( \Psi(\cdot|\theta) \) does not depend on \( \theta \), that is, it is just a distribution function of \( \vartheta \), then \( L(\vartheta|\theta) = \psi(\theta) \). Therefore, the Bayesian-Nash equilibrium for the symmetric private value first price auction is:

\[
\beta^{I_p}(\theta) = \int_0^\theta \frac{\psi(\theta)}{\Psi(\theta)} d\theta.
\]

21.3.5 Revenue Comparison of Auctions

We now look at the capability of the three forms of auctions above by comparing the seller’s expected revenues. The basic finding will be that under the symmetric equilibrium, the English auction outperforms the second-price auction, while the seller’s expected revenue under the second-price auction is not less than that under the first-price auction.

Let us compare first the expected revenues under the English auction and the second-price auction.

The equilibrium strategy under second-price auction with the symmetric equilibrium is given by \( \beta^{II}(\theta) = v(\theta, \theta) \), where \( v(\theta, \vartheta) \equiv E(V_1|\theta_1 = \theta, \vartheta_1 = \vartheta) \). If \( \theta > \vartheta \), the price paid by bidder 1 (the income of the auctioneer) is:

\[
v(\theta, \vartheta) = E(u(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})|\theta_1 = \theta, \vartheta_1 = \vartheta) = E(u(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})|\theta_1 = \theta, \vartheta_1 = \vartheta) \leq E(u(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})|\theta_1 = \theta, \vartheta_1 = \vartheta) = E(u(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})|\theta_1 = \theta, \vartheta_1 = \vartheta).
\]

In the second-price auction, the expected revenue is:

\[
E(R^{II}) = E(\beta^{II}(\vartheta_1)|\theta_1 > \vartheta_1) = E(\beta^{II}(\vartheta_1)|\theta_1 > \vartheta_1) \leq E(E(u(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})|\theta_1 = \theta, \vartheta_1 = \vartheta)|\theta_1 > \vartheta_1) = E(u(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})|\theta_1 > \vartheta_1) = E(\beta^2(\vartheta_1, \vartheta_1, \ldots, \vartheta_{n-1})) = E(R^{Eng}).
\]
where the second last equation is obtained by the definition of \( \beta^2 \) by (21.3.14),
\( R^{II} \) and \( R^{Eng} \) are the prices paid by the winning bidders of the second-price auction and the English auction, respectively.

Now we compare the payments paid by the winning bidders of the second-price auction and of the first-price auction.

We can also suppose that signal of bidder 1 is the highest, so we only have to consider the expected payment paid by bidder 1 under the symmetric Bayesian-Nash equilibrium.

\[
E(R^{II}(\vartheta_1)|\theta_1 = \theta, \theta_1 > \vartheta_1) = E(v(\vartheta_1, \vartheta_1)|\theta_1 = \theta, \theta_1 > \vartheta_1)
= \int_0^\theta v(\vartheta, \vartheta) dK(\vartheta|\theta),
\]
where \( K(\vartheta|\theta) \equiv \frac{\Psi(\vartheta|\theta)}{\Psi(\theta|\theta)} \) is a distribution function with support \([0, \theta]\).

But in the first-price auction, \( \beta^I = \int_0^\theta v(\vartheta, \vartheta) dL(\vartheta|\theta) \).

Next, we will verify that \( L(\vartheta|\theta) \) is first-order stochastically dominated by \( K(\vartheta|\theta) \).

By affiliation, if \( t < \theta \), then we have
\[
\frac{\psi(t|t)}{\Psi(t|t)} \leq \frac{\psi(t|\theta)}{\Psi(t|\theta)}.
\]

So, for any \( \vartheta < \theta \), we have:
\[
- \int_0^\theta \frac{\psi(t|t)}{\Psi(t|t)} dt \geq - \int_0^\theta \frac{\psi(t|\theta)}{\Psi(t|\theta)} dt
= - \int_0^\theta \frac{d}{dt} \ln \Psi(t|\theta) dt
= \ln \left( \frac{\Psi(\vartheta|\theta)}{\Psi(\theta|\theta)} \right).
\]

We thus have
\[
L(\vartheta|\theta) = \exp \left( - \int_0^\theta \frac{\psi(t|t)}{\Psi(t|t)} dt \right) \geq \frac{\Psi(\vartheta|\theta)}{\Psi(\theta|\theta)} = K(\vartheta|\theta),
\]
which means that \( K(\vartheta|\theta) \) first-order stochastically dominates \( L(\vartheta|\theta) \). Since \( v(\vartheta, \vartheta) \) is an increasing function, by the equivalence condition of first-order dominance, we have
\[
E(R^{II}(\vartheta_1)|\theta_1 = \theta, \theta_1 > \vartheta_1) \geq E(\beta^I(\vartheta_1)|\theta_1 = \theta, \theta_1 > \vartheta_1).
\]

Proposition below summarizes comparisons of different forms of auctions in expected revenue.

**Proposition 21.3.4** Under the symmetric equilibria, the expected revenue of the seller under the English auction is the highest, followed by the second-price auction, while the expected revenue of the first-price auction is the lowest.
21.3.6 Efficiency of Auctions

Under the symmetric economic environment and symmetric Bayesian-Nash equilibria of all the three forms of auctions above, the bidder with the highest bid wins the object. But, can we say that the allocation is efficient? The efficiency means that the bidder with the highest value wins the object. The example below shows that the allocations of all the auction formats above may not be efficient.

Example 21.3.3 Consider a two-bidder auction in which their valuation function is symmetric, \( \theta_1 \) and \( \theta_2 \) are private signals of the two bidders. Suppose that their valuation functions are:

\[
\begin{align*}
    u_1(\theta_1, \theta_2) &= \frac{1}{4}\theta_1 + \frac{3}{4}\theta_2; \\
    u_2(\theta_1, \theta_2) &= \frac{1}{4}\theta_2 + \frac{3}{4}\theta_1.
\end{align*}
\]

Obviously, if \( \theta_1 > \theta_2 \), then \( u_1 < u_2 \). Thus, bidder 1 wins no matter the auction is English auction, second-price auction or first-price auction. So the outcomes of all the three forms of auctions are inefficient.

In this example, the signal \( \theta_i \) of each bidder \( i \) has less influence on his own valuation than it does on the other bidder’s valuation. And thus, the winner of the auction is not the bidder with the highest value. So, we need some extra constraints, such as the single crossing condition.

Definition 21.3.2 (Single Crossing Property) For the valuation system of auctions, we will say that \( u_i, i \in \{1, \ldots, n\} \), satisfies the single crossing condition, if for any \( j \neq i \), and for all \( \theta \),

\[
\frac{\partial u_i}{\partial \theta_i}(\theta) \geq \frac{\partial u_j}{\partial \theta_i}(\theta).
\]

Under the symmetric equilibrium, if the single crossing condition is satisfied, can the three forms of auction allocate efficiently? Or, can we say that the higher the signal of a bidder is, the higher the valuation of the bidder is? The answer to this problem is positive.

Proposition 21.3.5 For economic environments with symmetric, interdependent values and affiliated signals, if the single crossing property is satisfied, then all symmetric Bayesian-Nash equilibria of the second price, English, and first-price auctions result in efficient outcomes.

Proof. In the symmetric model with interdependent values, the value to bidder \( i \) becomes

\[
u_i(\theta) = u(\theta_i, \theta_{-i}).\]
Suppose that \( \theta_i > \theta_j \), and define \( \alpha(s) = (1-s)(\theta_j, \theta_i, \theta_{-ij}) + s(\theta_i, \theta_j, \theta_{-ij}) \). According to the mean value theorem of integrals:

\[
\int_0^1 \nabla u(\alpha(s), \alpha'(s)) ds,
\]

where

\[
\nabla u(\alpha(s), \alpha'(s)) = \frac{\partial}{\partial \theta_i} u(\alpha(s))(\theta_i - \theta_j) + \frac{\partial}{\partial \theta_j} u(\alpha(s))(\theta_j - \theta_i) \geq 0
\]

since \( \theta_i > \theta_j \) and by single crossing condition, \( \frac{\partial}{\partial \theta_j} u(\alpha(s)) \geq \frac{\partial}{\partial \theta_i} u(\alpha(s)) \). Thus, we have

\[
u(\theta_i, \theta_j, \theta_{-ij}) \geq u(\theta_j, \theta_i, \theta_{-ij}).
\]

The proof is completed.

For a general Bayesian-Nash equilibrium such as the non-symmetric equilibrium (if any), does it also result in efficient allocation? First, the single crossing property is also required. The example below illustrates that there is no efficient allocation in an auction if the single crossing condition is not satisfied.

**Example 21.3.4** Suppose that there are two bidders, whose signal are \( \theta_1 \) and \( \theta_2 \), and their valuation functions for an indivisible object are:

\[
u_1(\theta_1, \theta_2) = \theta_1, \\
u_2(\theta_1, \theta_2) = \theta_2^2.
\]

Suppose that \( \theta_1 \in [0, 2] \), the signal of bidder 2 does not affect their valuation to the object. Clearly, the valuations do not satisfy the single crossing condition, such as \( \frac{\partial \nu_1}{\partial \theta_1}(1, \theta_2) < \frac{\partial \nu_2}{\partial \theta_1}(1, \theta_2) \).

It can be easily seen that \( \nu_1 > \nu_2 \) if and only if \( \theta_1 < 1 \). If \( (y(\theta_1), t(\theta_1)) \) is efficient, \( y(\theta_1) = (y_1(\theta_1), y_2(\theta_1)) \) are the probabilities that bidder 1 and bidder 2 get the object; and \( t(\theta_1) = (t_1(\theta_1), t_2(\theta_1)) \) are the transfers of bidder 1 and bidder 2. They need to satisfy that if \( \theta_1 < 1 \), then \( y_1(\theta_1) = 1 \); if \( \theta_1 > 1 \), then \( y_2(\theta_1) = 1 \).

Suppose that \( \vartheta_1 < 1 < z_1 \), then efficiency and incentive compatibility together require that when agent 1’s private signal is \( z_1 \), we have:

\[
0 - t_1(z_1) \geq z_1 - t_1(\vartheta_1);
\]

and when his private signal is \( \vartheta_1 \), we have:

\[
\vartheta_1 - t_1(\vartheta_1) \geq 0 - t_1(z_1).
\]

The two inequalities above imply \( \vartheta_1 \geq z_1 \), which is a contradiction. So there is no incentive-compatible auction which can allocate efficiently.
Moreover, we can in fact show that the single crossing condition is also a necessary condition for an auction to be efficient. In general, if there exists an efficient mechanism with an ex post equilibrium, then by the revelation principle, there must exist an efficient direct mechanism in which “truth-telling” is an ex post equilibrium. Suppose that all the bidders other than bidder \( i \) have observed a signal \( \theta_{-i} \), and the bidder \( i \)'s private signal is \( \theta_i \). If no matter what value of his signal \( \theta_i \) is, bidder \( i \) either always wins or always loses, then the single crossing condition means nothing for \( \theta_i \).

So we are only concerned for the situation that if buyer \( i \)'s signal will determine whether he can get the object or not. Thus, we will say that buyer \( i \) is pivotal, if there exist signals \( \vartheta_i \) and \( z_i \), such that:

\[
\begin{align*}
&u_i(\vartheta_i, \theta_{-i}) > \max_{j \neq i} u_j(\vartheta_i, \theta_{-i}) \\
&u_i(z_i, \theta_{-i}) < \max_{j \neq i} u_j(z_i, \theta_{-i}).
\end{align*}
\]

Incentive compatibility requires that when bidder \( i \)'s signal is \( \vartheta_i \), he would not report \( z_i \), which means that:

\[
0 - t_i(z_i, \theta_{-i}) \geq u_i(z_i, \theta_{-i}) - t_i(\vartheta_i, \theta_{-i}).
\]

And, when he observes \( z_i \) he would not report \( \vartheta_i \), which means that:

\[
0 - t_i(z_i, \theta_{-i}) \geq u_i(\vartheta_i, \theta_{-i}) - t_i(\vartheta_i, \theta_{-i}).
\]

From the two inequalities above, we have:

\[
u_i(\vartheta_i, \theta_{-i}) \geq u_i(z_i, \theta_{-i})
\]

That is to say, bidder \( i \)'s value when he wins the object must be at least as high as when he does not. In other words, keeping others’ signals unchanged, an increase in bidder \( i \)'s private signal cannot reduce the probability of winning the object.

So, by the incentive compatibility, the efficient mechanism must satisfy the monotonicity condition (just like the Maskin monotonicity). This requires that, at \( \theta_i \), if \( u_i(\theta_i, \theta_{-i}) = u_j(\theta_i, \theta_{-i}) \), we can have:

\[
\frac{\partial u_i}{\partial \theta_i}(\theta_i, \theta_{-i}) \geq \frac{\partial u_j}{\partial \theta_i}(\theta_i, \theta_{-i}),
\]

so that the monotonicity condition is satisfied. Thus, if the allocation that satisfies incentive compatibility is efficient, it must satisfy the single-cross property.

21.3.7 The Generalized VCG Mechanism

Under an economic environment with symmetric and interdependent values, the symmetric equilibrium of the auction is efficient if the single crossing condition is satisfied. But, can we find an efficient mechanism under general environments with interdependent values? In the previous chapters on general mechanism design, we have shown that VCG is an efficient
mechanism if the players’ valuation function only depends on their own types.

However, it will lose efficiency when a bidder’s valuation function is interdependent on others’ signals. Indeed, if bidder \( i \) who gets the object is asked to pay the second highest value at all the reported signals, \( \max_{j \neq i} u_j(\theta_i, \theta_{-i}) \), which depends on \( \theta_i \), then player \( i \) would have the incentive to report a lower signal in order to reduce the payment. As such, the usual VCG mechanism is not a “truth-telling” mechanism. But, the so-called generalized VCG mechanism that is revised from the VCG is efficient. We first describe generalized VCG mechanism.

Suppose that there are \( n \) players in the mechanism \((\Theta, y(\theta), t(\theta))\), where \( \theta = (\theta_1, \ldots, \theta_n) \in \Theta \) are the private signals of the bidders, \( y(\theta) = (y_1(\theta), \ldots, y_n(\theta)) \) are the probabilities of bidders allocated to the object, \( t(\theta) = (t_1(\theta), \ldots, t_n(\theta)) \) are the transfers of the players.

If the mechanism is efficient, we have:

\[
y_i^*(\theta) = \begin{cases} 
1 & \text{if } u_i(\theta) > \max_{j \neq i} u_j(\theta) \\
0 & \text{if } u_i(\theta) < \max_{j \neq i} u_j(\theta).
\end{cases}
\]

If there are more than one bidder who have the highest value, the object is allocated to each of these players with an equal probability. But with continuous distribution, the probability of this situation equals to 0.

Let the transfer of the bidder who gets the object be:

\[
t_i^* = -u_i(\vartheta_i(\theta_{-i}), \theta_{-i}),
\]

where

\[
\vartheta_i(\theta_{-i}) = \inf \{ \tilde{\theta}_i : u_i(\tilde{\theta}_i, \theta_{-i}) \geq \max_{j \neq i} u_j(\tilde{\theta}_i, \theta_{-i}) \}.
\]

This implies that, given other bidders’ signal, \( \vartheta_i(\theta_{-i}) \) is the lower bound of the signal of bidder \( i \) that still wins the object. The bidder who does not win the object pays nothing. In such a way, since the transfer that one pays does not depend on the signal of himself directly, just as in the private value model, it can result in efficient outcomes.

We call the mechanism \((\theta, y^*, t^*)\) defined above a generalized VCG mechanism, because it adapted VCG mechanism under the interdependent values setting. As mentioned above, the usual VCG mechanism under the interdependent values is not a “truth-telling” mechanism. To guarantee that the winning player \( i \) have incentive to tell the truth, we should ask player \( i \) to pay only \( \max_{j \neq i} u_j(\vartheta_i(\theta_{-i}), \theta_{-i}) \) (which does not depend on \( \theta_i \) directly), rather than \( \max_{j \neq i} u_j(\theta_i, \theta_{-i}) \). When \( u_i(\theta_i, \theta_{-i}) > \max_{j \neq i} u_j(\theta_i, \theta_{-i}) \), we have \( \vartheta_i(\theta_{-i}) \leq \theta_i \), so \( \max_{j \neq i} u_j(\vartheta_i(\theta_{-i}), \theta_{-i}) \leq \max_{j \neq i} u_j(\theta_i, \theta_{-i}) \).

The following proposition shows that the generalized VCG mechanism is an efficient mechanism under the affiliated valuation system.
Proposition 21.3.6 Suppose that the valuations of bidders to the object satisfy the single crossing property. The generalized VCG mechanism is an efficient and incentive compatible (truth telling) mechanism in dominant strategy.

Proof. Obviously, if everyone tells the true signal, the generalized VCG mechanism is efficient. Now we have to prove that it satisfies dominant-strategy incentive compatibility. When \( u_i(\theta_i, \theta_{-i}) > \max_{j \neq i} u_j(\theta_i, \theta_{-i}) \), player \( i \) who gets the object pays \( u_i(\theta_i(\theta_{-i}), \theta_{-i}) \). As the payments do not depend on the signal \( \theta_i \) he reported, and \( \theta_i \geq \vartheta_i(\theta_{-i}) \), the expected utility of reporting true signal is: \( u_i(\theta_i(\theta_{-i}) - u_i(\vartheta_i(\theta_{-i}), \theta_{-i}) \geq 0 \). When he reports other signal \( z_i \), if \( z_i > \vartheta_i(\theta_{-i}) \), his expected utility does not change; if \( z_i < \vartheta_i(\theta_{-i}) \), he would not win the object and his expected utility will be 0. So, in an efficient allocation, bidder \( i \) gets the object, and there is no other strategy that can bring a higher expected utility than to tell the truth.

When \( u_i(\theta_i, \theta_{-i}) < \max_{j \neq i} u_j(\theta_i, \theta_{-i}) \), if he tells the truth, bidder \( i \)'s expected utility equals to 0; if he reports \( z_i \) to alter the outcome, by the single crossing property: \( z_i > \vartheta_i(\theta_{-i}) > \theta_i \), player \( i \) wins the object, but his expected utility is \( u_i(\theta_i(\theta_{-i}) - u_i(\vartheta_i(\theta_{-i}), \theta_{-i}) < 0 \). Thus we have shown that the generalized VCG mechanism is an efficient mechanism that is dominant-strategy incentive compatible.

For private values, the generalized VCG mechanism above returns to the VCG mechanism. However, when values are interdependent, the generalized VCG mechanism is different from the VCG mechanism since the transfer payment \( u_i(\vartheta_i(\theta_{-i})) \) depends on the valuation functions of all players. As such, the generalized VCG mechanism is not an anonymous mechanism but depends on the characteristics (such as the preferences) of the players.

21.3.8 Optimal Auction Mechanism Design

We now discuss the optimal mechanism design when values are interdependent. Under the optimal mechanism of independent private value, the object is allocated to the bidder whose value is the highest. In order to provide incentives to bidders to tell the truth, the bidder with the highest value is able to obtain some positive information rents. As such, the optimal mechanism only results in second-best outcomes. However, as for interdependent values, since the auctioneer can get some information about the signals of the others bidders from a bidder’s bidding, the optimal mechanism can be first-best. Indeed, just like the Full Surplus Extraction Theorem of Cremer-Mclean discussed in Chapter 16, we can get an optimal mechanism when values are interdependent and the auctioneer will extract all the surplus.

In order to understand the Full Surplus Extraction Theorem easily, we set the space of signals to be discrete. Define \( \Theta_i = \{ \theta_{1i}, \theta_{2i}, \ldots, \theta_{ni} \} \) as the
Suppose of other bidders will change according to different $\theta$, the signals of all bidders are written as $\theta$, and the valuation function of bidder $i$ is $u_i = u_i(\theta_i, \theta_{-i})$. For the discrete values, we can define the single crossing condition as: for $j \neq i$, if $u_i(\theta_i, \theta_{-i}) \geq u_j(\theta_i, \theta_{-i})$, then $u_i(\theta'_i, \theta_{-i}) \geq u_j(\theta'_i, \theta_{-i})$ holds for any $\theta_i < \theta'_i \in \Theta$; at the same time, the conclusion does not change when the inequality is strict.

For any bidder $i$, let $\Pi_i$ be a matrix with $m_i$ rows and $\prod_{j \neq i} m_j$ columns and its element be denoted by $\pi_i(\theta_{-i} | \theta_i)$. Each row corresponds to bidder $i$’s beliefs regarding the signal distribution of other bidders given $\theta_i$. If the signals are independent, then all rows are the same, which means the rank of $\Pi_i$ is 1. If the signals are correlated, the beliefs represented by different rows are not the same. In the proposition below, we suppose the $\Pi_i$ is of rank $m_i$, which means, the bidder’s beliefs regarding the signal distribution of other bidders will change according to different $\theta_i$.

We then have the following Full Surplus Extraction Theorem under interdependent values.

**Theorem 21.3.1 (Full Surplus Extraction Theorem on Interdependence)** Suppose that signals are discrete and the valuations satisfy the single crossing condition. If for any bidder $i$, $\Pi_i$, the matrix of beliefs about other players, is of full rank (that is $m_i$), then there exists a mechanism in which truth-telling is an efficient equilibrium and the expected information rent (that is expected utility in the event) of every bidder is equal to zero. As a result, we have first-best outcomes, i.e., all the surplus is extracted by the auctioneer.

**Proof.** First, consider the generalized VCG mechanism $(\Theta, y(\theta), t(\theta))$. From the above discussions, we know that it is an efficient truth-telling mechanism. In this mechanism, the expected utility of bidder $i$ with signal $\theta_i$ at the equilibrium is:

$$
\bar{U}^*_i(\theta_i) = \sum_{\theta_{-i}} \pi_i(\theta_{-i} | \theta_i)[y_i(\theta)u_i(\theta) + t_i(\theta)].
$$

Let $\bar{u}_i^* = [\bar{U}^*_i(\theta^1_i), \bar{U}^*_i(\theta^2_i), \ldots, \bar{U}^*_i(\theta^{m_i}_i)]'$. As the rank of the matrix $\Pi_i$ is full row, there is a column vector $c_i = (c_i(\theta_{-i}))_{\theta_{-i} \in \Theta_{-i}}$ of size $\sum_{j \neq i} m_j$ such that:

$$
\Pi_i c_i = -\bar{u}_i^*;
$$

for $\forall \theta_i$, and then we have:

$$
\sum_{\theta_{-i}} \pi_i(\theta_{-i} | \theta_i)c_i(\theta_{-i}) = \bar{U}^*_i(\theta_i).
$$

Comparing the generalized VCG mechanism $(\Theta, y(\theta), t(\theta))$ with the Cremer-McLean mechanism $(\Theta, y(\theta), t(\theta) + c_i(\theta_{-i}))$, we find that the allocation rule is the same, but the transfer of every bidder has reduced by $c_i(\theta_{-i})$ which
does not depend on bidder $i$’s report. Thus, the Cremer-McLean mechanism is still a truth-telling mechanism and it is efficient. At the same time, every bidder’s expected utility in the event is reduced to zero, which means the auctioneer may get all the possible surplus.

This conclusion is robust. As long as signals of individuals are correlated, no matter how close they are, the information rent will be reduced to zero. And this conclusion does not depend if values are interdependent, but depends on the correlations of signals.

21.4 Multi-Unit Simultaneous Auctions with Private Values

The rest of this chapter discusses multi-unit auctions especially under private values. In reality, most of the auctions are multi-unit auctions. Multi-unit auctions are of many different forms that depend on the relationship of objects as well as the timing and formats of auctions.

Objects can be either identical or heterogeneous. When the objects are identical, a buyer may only need one or some of them. Under this situation, the marginal willingness to pay will decrease with the increase in the number of objects bought. When the objects are heterogeneous, the objects are different but may affect each other. Then, the willingness to pay of the buyer will change with the relationship of these objects. For instance, the value of consuming two objects simultaneously may be higher than the sum of value from consuming them separately, which means there exists complement within the objects. Of course, there also may be substitutability. The complement and substitutability are the factors worth paying special attention to when we design an auction.

Multi-unit auctions also depend on the timing and formats of auctions. When multiple objects are to be sold, many options can be chosen by the seller. The seller can sell the objects jointly in a single auction or separately in multiple auctions. The former case is called simultaneous auction, in which all the objects are sold in a single auction at one time, but not necessarily all to the same bidder, and the bids on the various objects collectively influence the overall allocation. The latter case is called sequential auction, in which the objects are sold one at a time in separate auctions in a way that the bids in the auction for one of the objects do not directly influence the outcome of the auction for another.

The seller can also choose different auction formats. As for single object auctions, formats can be sealed-bid such as discriminatory auction, uniform-price auction and Vickrey auction or open-bid actions such as Dutch auction, English auction and Ausubel auction.

We will first consider multiple identical-objects simultaneous auctions and then consider multiple identical-objects sequential auctions, followed
by considering heterogeneous multiple-object auctions in private values.

### 21.4.1 Basic Model

Suppose that there are \( K \) units of identical objects for sale and \( n \) bidders. The set of bidders are \( N = \{1, \ldots, n\} \). The marginal value of bidder \( i \) to different units of goods are \( V_i(\theta) = (V_i^1(\theta), \ldots, V_i^K(\theta)) \), where \( V_i^k \) is bidder \( i \)'s value to item \( k \). We often suppose: \( V_i^k \leq V_i^{k'}, \forall k > l \), which means diminishing marginal utility, i.e., the marginal utility decreases with the increase in the number of identical objects; \( \theta_i \) is the signal observed by bidder \( i \) and \( \theta \) is the profile of signals observed by all the bidders. The bidding function of bidder \( i \) is \( b_i = (b_1^i(V_i), \ldots, b_K^i(V_i)) \), where \( b_k^i \) is the bid of bidder \( i \) to the \( k \)-th unit of objects.

As multi-unit auctions are relatively complex, we focus on the private value models, \( V_i(\theta) = V_i(\theta_i) \equiv V_i \). We will discuss the allocation of objects and transfers under different mechanisms. All the formats of auctions obey the rule that objects are given to the \( K \) highest of these bids.

In a simultaneous auction environment with \( K \) identical objects to be auctioned, each bidder \( i \) is asked to submit \( K \) bids \( b_i = (b_1^i, \ldots, b_K^i) \), satisfying \( b_1^i \geq b_2^i \geq \ldots \geq b_K^i \), to indicate how much he is willing to pay for each additional unit. If the bidder wants to buy \( k \) units, he has to pay \( b_1^i + \ldots + b_k^i \).

Given the bid vector \( b_i \) of bidders, we will get the demand function \( d_i \) of bidder \( i \):

\[
d_i(p) = \max\{k : b_k^i \geq p\},
\]

where \( d_i(p) \) is a non-increasing function. If the price \( p \) is between \( b_k^i \) and \( b_{k+1}^i \), bidder \( i \) will buy \( k \) units. In turn, if we know the demand for the objects, we can get the bid vector of the bidders. These two methods are equivalent.

Given a bidding set \( \{b_k^i, i = 1, \ldots, n; k = 1, \ldots, K\} \), assume that \( K \) units of objects are awarded to the \( K \) highest bids among all bids by bidders. For the discriminatory auction, uniform-price auction and Vickrey auction we discuss below, the allocation of objects is the same, but the transfers to win the objects are different in these three auctions.

We will illustrate the allocation and pricing rules of the three auctions through a simple example below.

**Example 21.4.1** Suppose that there are six units of identical objects and three bidders in the auction. The bid vectors are:

\[
\begin{align*}
\mathbf{b}_1 &= (54, 46, 43, 40, 33, 15) \\
\mathbf{b}_2 &= (60, 55, 47, 32, 27, 13) \\
\mathbf{b}_3 &= (48, 45, 35, 24, 14, 9)
\end{align*}
\]
Thus, the six highest bids are:

$$(b_2^2, b_3^2, b_1^3, b_3^1, b_2^1, b_1^2) = (60, 55, 54, 48, 47, 46).$$

As such, bidder 1 gets two units, bidder 2 gets three units and bidder 3 gets one unit.

### 21.4.2 Multiple Identical-Objects Simultaneous Sealed-Bid Auctions

**Discriminatory Auctions**

In a *discriminatory auction*, each bidder pays an amount equal to the sum of his bids that are among the $K$ highest of the $N \times K$ bids submitted in all. That is, if bidder $i$ is allocated to $k_i$ units of objects, his total payment is $b_1^i + \ldots + b_{k_i}^i$. This amounts to perfect price discrimination relative to the submitted demand functions, hence having the name of the auction.

In Example 21.4.1, bidder 1 gets two units of the object, and his payment equals to $54 + 46 = 100$. Bidder 2 and bidder 3 get three units and one unit of the object, respectively, and their payments equal to $60 + 55 + 47 = 162$ and $48$, respectively. Thus, the total revenue received by the auctioneer is $100 + 162 + 48 = 310$.

The discriminatory pricing rule can also be described by the residual supply function. At price $p$, the residual supply faced by bidder $i$ denoted as $s_{-i}(p)$, is equal to the total units of objects subtracting the sum of the amounts demanded by other bidders, so:

$$s_{-i}(p) \equiv \max\{K - \sum_{j \neq i} d_j(p), 0\}.$$

Obviously, as $d_i(p)$ is a non-increasing function, $s_{-i}(p)$ is a non-decreasing function. In the discriminatory auction, the total payment of each bidder equals to the area under his demand function up to the point where it intersects with the residual supply curve. In Example 21.4.1, the demand function of each bidder is:

$$d_1(p) = \begin{cases} 
0 & \text{if } p > 54; \\
1 & \text{if } 54 \geq p > 46; \\
2 & \text{if } 46 \geq p > 43; \\
3 & \text{if } 43 \geq p > 40; \\
4 & \text{if } 40 \geq p > 33; \\
5 & \text{if } 33 \geq p > 15; \\
6 & \text{if } 15 \geq p > 0.
\end{cases}$$
The residual supply function of bidder 1 is:

\[ s_{-1}(p) = \begin{cases} 
6 & \text{if } p > 60; \\
5 & \text{if } 60 \geq p > 55; \\
4 & \text{if } 55 \geq p > 48; \\
3 & \text{if } 48 \geq p > 47; \\
2 & \text{if } 47 \geq p > 45; \\
1 & \text{if } 45 \geq p > 35; \\
0 & \text{if } 35 \geq p \geq 0. 
\end{cases} \]

When \( p \in [45, 46] \), \( d_1(p) = s_{-1}(p) = 2 \), the payment of bidder 1 when allocated with the two units of objects equals to

\[
\int_0^\infty \min\{d_1(p), 2\} dp = \int_0^2 (d_1)^{-1}(q) dq,
\]

which is the area under his demand function up to the point where it intersects with the residual supply curve, while \((d_1)^{-1}\) is the inverse function of \(d_1\).

When \( K = 1 \), discriminatory auction turns out to be first-price auction. Thus, the discriminatory price is just an extension of the first-price auction with multiple objects.

**Uniform-Price Auctions**

In the *uniform-price auction*, all the objects are sold to the bidders at a market-clearing price. In the discrete model studied here, we take the highest losing bid that is the \( K + 1 \) highest of all the bids as the market-clearing price, as the price that clears the market. In Example 21.4.1, the seventh-highest bid is 45.
In a general situation, if the bid set of \( n \) bidders is \( \{b^k_i, i = 1, \ldots, n; k = 1, \ldots, K \} \), and the bidder \( i \) gets \( k_i \) units of objects from the \( K \) highest of these bids, then the market-clearing price is:

\[
p = \max_i \{b^k_i + 1\}
\]

For the above example, \( p = \max_i \{b^k_i + 1\} = \max\{43, 32, 45\} = 45 \). The total revenue received by the auctioneer is \( 6 \times 45 = 270 \).

Suppose that \( c_{-i} \) is the \( K \)-vector of competing bids of bidder \( i \) facing other bidders, which is constructed by the \( K \) highest bids of \( \{b^k_j, j \neq i, k = 1, \ldots, K\} \), as these bids will affect whether bidder \( i \) can get the objects. \( c_{-i} = (c^1_{-i}, \ldots, c^K_{-i}) \) is a price vector for rearranging the component in decreasing order. As such, bidder \( i \) wins exactly \( k^i > 0 \) units if and only if

\[
b^k_i > c^{K-k^i+1}_{-i} \text{ and } b^{k^i+1}_i < c^{K-k^i}_{-i}.
\]

Thus, the residual supply function of bidder \( i \) is:

\[
s_{-i}(p) = K - \max\{k : c^k_{-i} \leq p\},
\]

which is the inverse function of \( c_{-i} \). Then, the market-clearing price can be also determined by

\[
p = \max\{b^k_i + 1, c^{K-k^i+1}_{-i}\},
\]

which can be obtained from any bidder \( i \).

In Example 21.4.1, since

\[
b_1 = (54, 46, 43, 40, 33, 15) \\
b_2 = (60, 55, 47, 32, 27, 13) \\
b_3 = (48, 45, 35, 24, 14, 9)
\]

and

\[
c_{-1} = (60, 55, 48, 47, 45, 35) \\
c_{-2} = (54, 48, 46, 45, 43, 40) \\
c_{-3} = (60, 55, 54, 47, 46, 43),
\]

the market-clearing price can be obtained from bidder 1: \( p = \max\{b^3_1, c^5_{-1}\} = \max\{43, 45\} = 45 \). We can also obtain the same price \( p = \max\{b^2_2, c^4_{-2}\} = \max\{32, 45\} = 45 \) from bidder 2 or \( p = \max\{b^3_3, c^6_{-3}\} = \max\{45, 43\} = 45 \) from bidder 3.

When \( K = 1 \), the uniform-price auction reduces to the second-price auction. When \( K > 1 \), the uniform-price auction is different from the second-price auction in several ways, especially in the bidding strategy, which may not be truth-telling. So we cannot extend the second-price auction to the uniform-price auction directly. The Vickrey auction discussed below is more like an extension of the second-price auction with single-object.
Vickrey Auctions

In a second-price auction with single-object, the winner’s price is the highest of all the losing bids. In the Vickrey auction with multi-unit objects, a bidder who wins \(k_i\) units pays the \(k_i\) highest losing bids of the other bidders. The basic principle of the Vickrey auction is the same as the VCG mechanism: Each bidder is asked to pay an amount equal to the externality he exerts on other competing bidders.

Suppose that bidder \(i\) wins \(k_i\) units, to get his first unit of objects, it must be true that \(b_{k_i}^1 > c_{K-i}^K\). The bidder is then asked to pay an amount equal to \(c_{K-i}^K\) units of externality, where \(c_{K-i}^K\) is the \(K\)-highest bid among other bidders. As such, the bids of bidders only affect the allocation of the objects, but do not affect the payment of the bidders when they win. When the bidder gets the second unit, it must be true that \(b_{k_i}^2 > c_{K-i}^{K-1}\), and thus he pays \(c_{K-i}^{K-1}\) units of externality to other bidders. Thus, if the bidder wins \(k_i\) units of objects, he will bring \(\sum_{k=1}^{k_i} c_{K-k+1}^{K-k+1}\) units of externality to other bidders.

Thus, under the Vickrey, if bidder \(i\) wins \(k_i\) units, then the amount that he pays is equal to

\[
\sum_{k=1}^{k_i} c_{K-k+1}^{K-k+1}.
\]

In Example 21.4.1, bidder 1 wins two units, \(c_{-1} = (60, 55, 48, 47, 35)\), his payment is \(45 + 35 = 80\). Bidder 2 gets three units of objects, \(c_{-2} = (54, 48, 46, 45, 43, 40)\), and thus bidder 2’s payment is \(45 + 43 + 40 = 128\); bidder 3 gets one unit of objects, \(c_{-3} = (60, 55, 54, 47, 46, 43)\), and thus his payment is \(43\). The total revenue of the auctioneer is 251.

When \(K = 1\), the Vickrey auction is reduced to the second-price auction just like the uniform-price auction. But when \(K > 1\), the Vickrey auction is an appropriate extension of the second-price auction to the case of multiple units. Indeed, as we will show, unlike the uniform price auction, it shares many important properties with the second-price auction while the other auction formats do not. For instance, in the uniform-price auction, the bidder \(i\)’s payment may depend on his own bid because the market-clearing price is \(p = \max\{b_{k_i}^{k_i+1}, c_{K-k_i+1}^{K-k_i+1}\}\), which may be determined by \(b_{k_i}^{k_i+1}\).

### 21.4.3 Simultaneous Open-Bid Auctions

In the single-object auction, under the private value, we know that each sealed-bid auction has a corresponding open format. For example, the Dutch auction is strategically equivalent to the first-price auction and the English auction is equivalent to the second-price auction in outcomes. We will have the similar but not the same features in multi-unit auctions. That is, the Dutch auction is outcome but not strategically equivalent to the discriminatory auction, the English auction is outcome equivalent to the
uniform-price auction, and the Ausubel auction is outcome equivalent to the Vickrey auction in the sense that each pair of them result in the same equilibrium outcome.

**Dutch Auctions**

In the *multi-unit Dutch auction* (or *open descending price auction*), the auctioneer starts the auction by calling out a price that is so high that no bidder wants to accept it. The price then keeps going down until there is a bidder who wants to buy a unit at the current price. The bidder gets one unit and the auction continues. Then, the price continues to go down until there is another bidder who wants to buy a unit at the bidding price. The bidder gets the second unit and the auction continues until the price is reduced to a level at which some bidder wants to buy the $K$th unit.

The multi-unit Dutch auction is outcome equivalent to the discriminatory auction. If the bidder $i$ accepts the price according to his bid vector $b_i$ in the Dutch auction, he will purchase one unit when the price is reduced to $p = b_i^1$ and buy the second unit when the price is $p = b_i^2$, and so on. But the Dutch auction is not strategically equivalent to the multi-unit extension of the discriminatory auction. This is because a bidder’s value to different units may be interdependent under information asymmetry. By observing one bidder’s willingness to get one unite, other bidders can calculate his willingness to buy other units of objects. But in the discriminatory auction, as the bidders have to report their bids to different amounts of objects at the same time, there is no time for other bidders to update their beliefs. So they are not strategically equivalent.

**English Auctions**

The process of the multi-unit *English auction* (or *open ascending-price auction*) is opposite to the Dutch auction. The auctioneer changes the price from low to high, and each bidder reports the number he wants to buy at each reported price. Usually, the total amount of the objects the bidders want to buy is larger than $K$ at the initially low price. With the price going up, the bidders will adjust the amounts they want to buy. At a certain price, the amount all the bidders want to buy is equal to $K$, this price will be the market-clearing price. All the bidders will buy their desired units at this price.

Just like the relationship of the Dutch auction and the discriminatory auction, the multi-unit English auction is outcome equivalent to the uniform-price auction, but they are not strategically equivalent.
Ausubel Auctions

The Ausubel auction is an alternative ascending-price format that is outcome equivalent to the Vickrey auction. Similar to the English auction, the price also changes from low to high. The bidders need to report their demand \( d_i(p) \) at price \( p \), and the demand will decrease as the price goes up. But, being different from the English auction, in an Ausubel auction, the price of the objects is determined by the following procedure. At the beginning price \( p \), if each bidder \( i \)'s demand \( d_i(p) \) is large enough, the residual supply function of each bidder is

\[
s_{-i}(p) \equiv \max\{K - \sum_{j \neq i} d_j(p), 0\} = 0.
\]

With the price going up, the demand of each bidder goes down. The price continues to increase until it reaches a level \( p' \) such that at least one bidder \( i \)'s residual supply function satisfies \( s_{-i}(p') > 0 \). Then sell \( s_{-i}(p') \) units to any bidder \( i \) for whom \( s_{-i}(p') > 0 \) at a price of \( p' \). The price continues to increase as long as the sale is less than \( K \).

When the price rises to \( p'' \) such that at least one bidder’s residual supply function satisfies \( s_{-i}(p'') - s_{-i}(p') > 0 \). Then sell \( s_{-i}(p'') - s_{-i}(p') > 0 \) units to a bidder, say bidder \( i \) for whom \( s_{-i}(p'') - s_{-i}(p') > 0 \) at a price of \( p'' \). The price continues to go up until the market is cleared, and then the auction is ended.

We still use Example 21.4.1 again to describe the rule of Ausubel auction. Given the strategy of each bidder, that is,

\[
\begin{align*}
b_1 &= (54, 46, 43, 40, 33, 15) \\
b_2 &= (60, 55, 47, 32, 27, 13) \\
b_3 &= (48, 45, 35, 24, 14, 9).
\end{align*}
\]

the residual supply functions of each bidder are:

\[
s_{-1}(p) = \begin{cases} 
6 & \text{if } p > 60; \\
5 & \text{if } 60 \geq p > 55; \\
4 & \text{if } 55 \geq p > 48; \\
3 & \text{if } 48 \geq p > 47; \\
2 & \text{if } 47 \geq p > 45; \\
1 & \text{if } 45 \geq p > 35; \\
0 & \text{if } 35 \geq p \geq 0.
\end{cases}
\]
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\[ s_{-2}(p) = \begin{cases} 
6 & \text{if } p > 54; \\
5 & \text{if } 54 \geq p > 48; \\
4 & \text{if } 48 \geq p > 46; \\
3 & \text{if } 46 \geq p > 45; \\
2 & \text{if } 45 \geq p > 43; \\
1 & \text{if } 43 \geq p > 40; \\
0 & \text{if } 40 \geq p \geq 0.
\end{cases} \]

\[ s_{-3}(p) = \begin{cases} 
6 & \text{if } p > 60; \\
5 & \text{if } 60 \geq p > 55; \\
4 & \text{if } 55 \geq p > 54; \\
3 & \text{if } 54 \geq p > 47; \\
2 & \text{if } 47 \geq p > 46; \\
1 & \text{if } 46 \geq p > 43; \\
0 & \text{if } 43 \geq p \geq 0.
\end{cases} \]

Suppose that the initial price is \( p^0 \leq 35, s_{-i}(p^0) = 0, \forall i \in N = \{1, 2, 3\} \). When the price rises to \( p^1 = 35 + \epsilon, s_{-1}(35 + \epsilon) = 1, s_{-j}(35 + \epsilon) = 0, \forall j \neq 1 \). When \( \epsilon \to 0 \), bidder 1 buys the first unit of commodity at price 35.

When the price rises to \( p^2 = 40 + \epsilon, s_{-1}(40 + \epsilon) - s_{-1}(35 + \epsilon) = 0, s_{-2}(40 + \epsilon) - s_{-2}(35 + \epsilon) = 1, s_{-3}(40 + \epsilon) - s_{-3}(35 + \epsilon) = 0 \). When \( \epsilon \to 0 \), bidder 2 buys the second unit of commodity at price 40.

When the price rises to \( p^3 = 43 + \epsilon, s_{-1}(43 + \epsilon) - s_{-1}(40 + \epsilon) = 0, s_{-2}(43 + \epsilon) - s_{-2}(40 + \epsilon) = 1, s_{-3}(45 + \epsilon) - s_{-3}(40 + \epsilon) = 1 \). When \( \epsilon \to 0 \), bidder 2 buys the third unit and bidder 3 buys the fourth unit at price 43. Total sales amount to 4.

When the price further rises to \( p^4 = 45 + \epsilon, s_{-1}(45 + \epsilon) - s_{-1}(43 + \epsilon) = 1, s_{-2}(45 + \epsilon) - s_{-2}(43 + \epsilon) = 1, s_{-3}(45 + \epsilon) - s_{-3}(43 + \epsilon) = 0 \). When \( \epsilon \to 0 \), bidder 1 buys the fifth unit and bidder 2 buys the sixth unit at price 45. Total sales amount to 6, and the auction ends.

In the end of this Ausubel auction: bidder 1 gets two units of objects with a payment equal to 35 + 45 = 80; bidder 2 gets three units of objects with the payment equal to 40 + 43 + 45 = 128; bidder 3 gets one unit of objects with the payment equal to 43. The revenue of the auctioneer is 80 + 128 + 43 = 251, which is the same as the Vickrey auction. Similarly, the Ausubel auction is outcome equivalent but not strategically equivalent to the Vickrey auction.

In the following we will discuss equilibria of the above six formats of auctions with identical objects and private values. As each sealed-bid auction has an equivalent open form, we will focus on the three sealed-bid auctions. In the single-object auction, when the bidders’ values are private and independently identically distributed and bidders are risk neutral and have no budget constraints, the symmetric equilibria of both the first-price auction and the second-price auction are efficient. As we will show,
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this conclusion may not be true for multi-unit auctions. Only the Vickrey auction is efficient, while the discriminatory auction and the uniform-price auction may not.

21.4.4 Equilibrium of Vickrey Auctions

We discuss the multi-unit Vickrey auction first. Suppose that the auctioneer has $K$ units of identical objects and the marginal value of bidder $i$ to different units of objects is $V_i = (V_i^1, \ldots, V_i^K)$, where $V_i^k$ is the marginal value of bidder $i$ to the $k$th unit of object that satisfies the law of diminishing marginal utility, i.e., $V_i^k \geq V_i^{k+1}, \forall k < K$. Suppose that the value $V_i$ of bidder $i$ is independently and identically distributed on the value space $\chi = \{v_i \in [0, w]^K : v_i^k \geq v_i^{k+1}, \forall k < K\}$ according to the density function $\phi(v_i)$. Given the value of the bidder, the true demand function of the bidder is $\delta_i(p)$ and can be defined as:

$$\delta_i(p) = \max\{k : v_i^k \geq p\}.$$

From the previous subsection, we know the demand function reported by bidder $i$ is $d_i(p) = \max\{k : b_i^k \geq p\}$. Thus, the reported demand equals to the true demand if and only if the bidder bids according to his true value.

From the previous discussion, we know that, in the multi-unit Vickrey auction, each bidder simultaneously submits a $K$-vector $b_i$ that satisfies $b_i^k \geq b_i^{k+1}$. All the bid vectors of the bidders construct the bid set $\{b_j^k, i \in N, k = 1, \ldots, K\}$, in which the $K$ highest bids win the objects and the bidder $i$ wins $k_i$ units of objects so that $\sum_i k_i = K$. The $K$-vector $c_{-i}$ of competing bids faced by bidder $i$ is obtained by rearranging in decreasing order the bids of the bid set $\{b_j^k, j \neq i, j \in N, k = 1, \ldots, K\}$ and choosing the first $K$ of these. In the Vickrey auction, the bidder $i$ getting $k_i$ units of objects means $b_i^{k_i} > c_{-i}^{K-k_i+1}$ and $b_i^{k_i+1} < c_{-i}^{K-k_i}$. At the same time, the total payment of the bidder $i$ to $k_i$ units of objects is equal to $\sum_{k=1}^{k_i} c_{-i}^{K-k}$. Now we show that truth-telling, namely $b_i(v_i) = v_i$, is a weakly dominant strategy in the Vickrey auction.

**Proposition 21.4.1** For any bidder $i$, it is a weakly dominant strategy to bid according to $b_i(v_i) = v_i$ in a multi-unit Vickrey auction.

**Proof.** Let $b_i = v_i$ be the truthful pricing strategy of bidder $i$ and $c_{-i}$ be the vector of competing bids of any other bidders faced by him. $k_i$ is the units of objects bidder $i$ wins when he chooses to report truthfully, which means $v_i^{k_i} > c_{-i}^{K-k_i+1}$, $v_i^{k_i+1} < c_{-i}^{K-k_i}$ and the total payment is $\sum_{k=1}^{k_i} c_{-i}^{K-k+1}$. Then his interim expected utility is

$$\sum_{k=1}^{k_i} [v_i^k - c_{-i}^{K-k+1}].$$
If bidder \( i \) were to submit a different bid profile \( b' \) and still get the \( k_i \) units of objects, his interim expected utility would be the same. If bidder \( i \) were to submit a different bid profile \( b' \), but the units of objects he gets were \( k'_i \neq k_i \).

When \( k'_i > k_i \), he gets the \( k \)th units of goods such that \( k'_i \geq k \geq k_i + 1 \). His marginal net revenue then is

\[
v_k^i - c_{k-i}^{K-k+1} \leq v_{k+1}^i - c_{k-i}^{K-k+1} < 0.
\]

As such, his interim expected utility of bidding \( b' \) is lower than the expected utility of bidding \( b_i \).

If \( k'_i < k_i \), he cannot get \( k'_i + 1, \ldots, k_i \)th unit of object, and his marginal net revenue of getting \( k < k_i \)th unit of goods is

\[
v_k^i - c_{k-i}^{K-k+1} \geq v_{k+1}^i - c_{k-i}^{K-k+1} > 0,
\]

which is unchanged and therefore so would the surplus derived from these. But the surplus from any unit \( k < k_i \) was positive and is now forgone. As such, his expected utility of bidding \( b' \) is lower than the interim expected utility of bidding \( b_i \).

Hence bidder \( i \)'s truthful pricing strategy \( b_i \) is a weakly dominant strategy.

In the Vickrey auction, the payment that bidder \( i \) makes to get each unit of object is equal to the opportunity cost caused by this unit of goods to other bidders. And that is the externality caused by bidder \( i \) to other bidders.

Ausubel and Milgrom (2006) define the payments of participants in the Vickery auction by the VCG mechanism. Let \( v_i \) be the vector of truthful values of bidder \( i \) to the multiple objects. Let \( \hat{v}_i \) be the vector of reported values of bidder \( i \). Let \( k = (k_1, \ldots, k_n) \) be an allocation of the objects and \( k^* \) be the optimal allocation given bidders’ reported values.

\[
k^* = \arg \max_k \sum_i \hat{v}_i(k_i), \text{ s.t. } \sum_i k_i \leq K.
\]

The payment of the bidder who gets \( k_i \) units of objects is

\[
t_i(k_i, \hat{v}) = \sum_{j \neq i} \hat{v}_j(k^*_j) - \bar{v}_{-i},
\]

where \( \bar{v}_{-i} = \max\{\sum_{j \neq i} \hat{v}_j(k_j) : \sum_{j \neq i} k_j \leq K\} \). Hence the Vickery auction is a form of VCG mechanism with multiple objects.

In the Vickery auction, as each bidder reports his true value and the objects are given to the \( K \) highest bids, it is an efficient auction.

**Proposition 21.4.2** The Vickrey auction allocates the objects efficiently.
Ausubel and Milgrom (2006) also show that among all auctions for identical multiple objects, the Vickery auction is the only one where truth-telling is a dominant strategy and the allocation is efficient and the losing bidders’ payments are zero.

The conclusions above reveal that the Vickrey auction has nice properties. However, if objects are not identical, especially in the presence of complementarity, there will be a number of shortcomings in the Vickrey auction which we will discuss in the following sections. One of the shortcomings of the Vickrey auction with identical multiple objects is that the prices of identical objects are not the same. This will cause arbitrage and other problems. The example below shows this that identical objects do not have the same price in the Vickrey auction.

Example 21.4.2 Suppose that there are two units of identical objects, and the values of two bidders to them are \( v_1 = (12, 2) \) and \( v_2 = (8, 1) \), respectively. In the Vickrey auction, each bidder obtains one unit of commodity and their payments are 1 and 2, respectively. So they pay different prices for the same object. If the bidders are two listed companies, such bids and payments will cause external pressure to the company that pays a higher price.

In the following, we discuss the other formats of multi-unit auctions: the uniform-price auction and the discriminatory auction. In these mechanisms, bidders often bid untruthfully and the allocations may not be efficient.

21.4.5 Equilibrium of Uniform Price Auctions

We now focus on the equilibrium of uniform-price auction in which there is the feature of underreporting demand.

We have already discussed that given the bid profile \( b = (b_i)_{i \in N} \) in the uniform-price auction, how the price is determined and how the objects are allocated. Suppose that there are \( K \) units of identical objects, and bidder \( i \)'s \( K \)-dimensional bid vector is \( b_i \). The (marginal) value vector of the bidder to the object is \( v_i \). Let \( c_{-i} \) be the \( K \)-vector of competing bids of the other bidders faced by bidder \( i \). If bidder \( i \) wins \( k_i \) units of objects, we must have:

\[
b_{k_i}^{i} > c_{-i}^{K-k_i+1} \quad \text{and} \quad b_{k_i+1}^{i} < c_{-i}^{K-k_i}.
\]

The price of the uniform-price auction is then given by:

\[
p = \max \{b_{k_i+1}^{i}, c_{-i}^{K-k_i+1}\}.
\]

In general, we do not have the explicit equilibrium solution in the uniform-price auction. As such, we try to characterize some features of the equilibrium strategy of this auction.
The equilibrium strategy of the uniform-price auction has two important features: one is that the bid will not exceed the marginal value; the other is that the bid for the first unit is equal to its marginal value.

**Fact 21.4.1** \( \forall i, k, b^k_i \leq v^k_i \); i.e., the bids cannot exceed marginal values.

**Proof.** Suppose that the bid \( b_i \) of bidder \( i \) satisfies \( b^k_i > v^k_i \). We show that it will be weakly dominated by the bidding strategy that satisfies \( b'^k_i = v^k_i, b'^{k'}_i = b'^{k'}, k' \neq k \). We discuss it in four cases. 1) \( b^k_i = p \). The bidder \( i \) can get \( k - 1 \) units of objects under both bidding strategies. But when he chooses \( b^k_i = v^k_i \), \( p \) may decrease. 2) \( b^k_i < p \). The two bidding strategies make no difference to bidder \( i \). 3) \( b^k_i > p > v^k_i \). The bidder will suffer a loss from winning the \( k \)th unit as the price exceeds the marginal value. But he cannot obtain the \( k \)th unit of object if he chooses \( b'_i \). 4) \( b^k_i > v^k_i > p \). The two bidding strategies make no difference. Thus, in all four cases, the bidder’s utility cannot be increased by not truth-telling. Hence in the uniform-price auction, the bidder’s bid will not exceed marginal value.

**Fact 21.4.2** For all \( i, b^1_i = v^1_i \); i.e., the bid on the first unit must be the same as its true value.

**Proof.** We want to show that, if the bidder bids the first unit according to his marginal value, that is, \( b^1_i = v^1_i \), then this bid is a weakly dominant strategy, or any \( b'^1_i < v^1_i \) is a weakly dominated strategy. We will discuss it in three cases: 1) \( p \geq v^1_i > b'^1_i \). The bidder can get nothing under the two strategies, so they make no difference for him. 2) \( v^1_i \geq p > b'^1_i \). If he bids the first unit truthfully, the bidder \( i \) will win it and have a positive revenue. But if he chooses \( b'^1_i \), he can get nothing and no revenue. 3) \( v^1_i > b'^1_i > p \). The two bidding strategies make no difference to bidder \( i \).

Thus, in the uniform-price auction, the bidder’s strategy satisfies that the bid will not exceed the true value. Despite that the bid to the first unit is equal to its value, the bids to the other units are less than the true value. This phenomenon is called strategic demand reduction.

Let us use a simple example to describe the logic behind the strategic demand reduction of the uniform-price auction. Suppose that there are two units of identical objects, that is \( K = 2 \). Suppose that there are two bidders whose marginal value vector \( V_i \) is distributed on the value space \( \chi = \{ v_i \in [0, w]^2 : v^1_i \geq v^2_i \} \) according to the density function \( \varphi(\cdot) \) and the distribution function \( \Phi(\cdot) \). Let \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \) be the marginal distribution functions of \( V^1_i \) and \( V^2_i \), respectively. We will focus on bidder 1’s strategies of uniform-price auction in the symmetric equilibrium (subscript omitted).

Suppose that the marginal values of bidder 1 are \((v^1_1, v^2_1) \equiv (v^1, v^2)\), bids are \((b^1_1, b^2_1) \equiv (b^1, b^2)\), and the competing bids faced by him are \((b^1_2, b^2_2) = (c^1_{1-1}, c^2_{2-1}) \equiv (c^1, c^2)\). In the symmetric equilibrium, as \( b_i(v_i) = \beta(v_i) \), \( b_i \) is random and its distribution function is \( H(\cdot) \) (its density function is \( h(\cdot) \),
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$H_1(\cdot)$ and $H_2(\cdot)$ are the marginal distribution functions of $b_1^2$ and $b_2^2$, respectively, so they are marginal distribution functions of $c_1$ and $c_2$. Thus, $H_1(b_2^2) = \text{Prob}[C_1 < b_2^2]$ is the probability that bidder 1 will defeat both competing bids and win two units of objects.

Similarly, $H_2(b_1^1) = \text{Prob}[C_2 < b_1^1]$ is the probability that he will defeat the lower competing bid, and win at least one unit. The probability that he will win exactly one unit is then the difference $H_2(b_1^1) - H_1(b_2^2) > 0$. Also, $H_2(b_2^2) - H_1(b_2^2) = \text{Prob}[C_2 < b_2^2 < C_1]$ is the probability that the highest losing bid at which the units are sold is $b_2^2$.

Using these facts, bidder 1’s expected utility when he bids $b_1 = (b_1^1, b_1^2)$ is:

$$Eu_1(b, v) = \int_{c_1 < b_2^2} (v_1 + v_2 - 2c_1^1) h(c) dc + \int_{c_1 > b_2^2, c_2 < b_2^2} (v_1 - \max\{c_2^1, b_2^2\}) h(c) dc$$

$$= H_1(b_2^2)(v_1 + v_2) - 2 \int_0^{b_2^2} c_1^1 h_1(c_1^1) dc^1 + [H_2(b_1^1) - H_1(b_2^2)] v_1 - [H_2(b_1^1) - H_1(b_2^2)] b_2^2 - \int_{b_2^2}^{b_1^2} c_2^2 h_2(c_2^2) dc^2.$$

We know that the bid for the first unit of object satisfies $b_1^1 = v_1^1$. The first-order condition for maximizing $b_2^2$ results in:

$$\frac{dEu_1}{db_2^2} = h_1(b_2^2)(v_2^2 - b_2^2) - [H_2(b_2^2) - H_1(b_2^2)] = 0.$$

Since $H_2(b_2^2) - H_1(b_2^2) > 0$, we have

$$b_2^2 = v_2^2 - \frac{H_2(b_2^2) - H_1(b_2^2)}{h_1(b_2^2)} < v_2^2.$$

Thus, in the symmetric equilibrium of the uniform-price auction, there is strategic demand reduction except for the first unit. In a more general case, Ausubel and Cramton (2002) show that as long as $K > 1$, in the equilibrium of the uniform-price auction, the bid of each bidder satisfies $\beta_1^1(v_1^1) = v_1^1, \beta_k^1(v_1^1) < v_k^1, \forall k > 1$.

Milgrom (2004) relates the strategic demand reduction of the uniform-price auction with the properties of the demand of the buyer monopoly. Consider a simple example. Suppose that $K = 2$ and bidder 1’s demand is unit demand, that is $V_2^2 = 0$. Suppose that his value to one unit of commodity is $V_1^1$, which is distributed on $[0,1]$ according to the uniform distribution. By the features of the uniform-price auction, we have $b_1(v_1^1) = v_1^1.$
Bidder 2’s marginal value vectors are \( v^1 \) and \( v^2 \) that satisfy \( 1 > v^1 \geq v^2 > 0 \), each of which is distributed on \([0,1]\) according to the uniform distribution. In bidding strategy, he takes \( b^1_2 = v^1 \) and \( b^2_2 = b^2 \). If \( b^2 > v^1 \), bidder 2 will win two units of objects. If \( 0 \leq b^2 < v^1 \), he will win one unit. Then the expected utility of bidder 2 choosing \( b^2_2 \) is:

\[
E_{u_2}(b^2_2, v^1, v^2) = \int_{v^1 < b^2_2} (v^1 + v^2 - 2v^1) dv^1 + \int_{v^1 > b^2_2} (v^1 - b^2_2) dv^1 = v^1 - b^2_2 (1-v^2).
\]

Thus the optimal bidding strategy is \( b^2_2 = 0 \). This means that no matter what value \( v^2 \) takes, bidder 2’s optimal bid for the second unit is equal to zero.

The reason for strategic demand reduction is similar to the logic behind the buyer monopoly market. In a buyer monopoly market, the monopolist pays \( TE(q) = qp(q) \) to buy \( q \) units of objects, where \( p(q) \) is the market supply function. The marginal expenditure of the monopolist buying the \( q \)th unit is \( TE'(q) = p(q) + qp'(q) > p(q) \) (as the market supply function satisfies \( p'(q) > 0 \)).

Thus, there are two kinds of effects of buying one more unit of commodity, one is the expenditure \( p(q) \) of buying the object, the other is the increased marginal expenditure \( qp'(q) \). The latter reduces the demand incentive of the monopolist. Suppose that the value of the monopolist to the object is \( v(q) \), the amount of goods purchased to achieve maximum utility is:

\[
v'(q) - TE'(q) = v'(q) - p(q) - qp'(q) = 0,
\]

which implies the demand \( q \) of the monopolist satisfies \( v'(q) - p(q) > 0 \). This is the logic behind the strategic demand reduction.

In the example from Milgrom (2004), when the bid \( b^2_2 = 0 \), bidder 2 will get one unit of goods for free. But if he wants to have the second unit, not only does the acquisition cost \( v^1 \) of the second unit have to be paid, but the cost of the first unit also increases. Then, even if \( v^2 > v^1 \), bidder 2 does not have the incentive to have the second unit. Thus, multi-unit uniform-price auction is in general not an efficient mechanism. However, if every bidder is of unit demander, by Fact 21.4.2, all the bidders report truthfully, and uniform-price auction is also efficient.

Summarizing our discussions, we have the following propositions.

**Proposition 21.4.3** When \( K > 1 \), at an undominated equilibrium of the uniform-price auction, the bid on the first unit is equal to the value of the first unit, but bids on other units are lower than the respective marginal values.

**Proposition 21.4.4** When \( K > 1 \), every undominated equilibrium outcome of the uniform price auction is inefficient. However, when every bidder has a unit demand, it is efficient.
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21.4.6 Equilibrium of Discriminatory Auctions

Now we discuss the equilibrium of the discriminatory auction with multiple objects. In this auction, a bidder pays his bids when he wins some units of the objects. Let \( b_i \) be the bid vector of the bidder \( i \). When he wins \( k_i \) units of objects, his total payment is \( \sum_{k=1}^{k_i} b_i^k \). As the general close-form solution is impossible to get, like the situation of the uniform-price auction, we focus on the features of the symmetric equilibrium here.

To have the features of equilibrium bids, consider a simple economy with two units of objects and two bidders. Suppose that the vector of bidders’ marginal values is \( v_i = (v_i^1, v_i^2) \). Consider the symmetric equilibrium \((\beta^1, \beta^2)\), i.e. \( b_i^1 = \beta^1(v_i) \) and \( b_i^2 = \beta^2(v_i) \).

First, if the highest bid on the second unit is \( \max_v \beta^2(v) \), the bidder does not bid more than \( \max_v \beta^2(v) \) on the first unit. This is because if \( b_i^1 \) on the first unit is greater than \( \max_v \beta^2(v) \), the bidder will win with probability 1 and he could do better by reducing it slightly. Indeed, let \( b = \max_v \beta^2(v) \). For the bidder \( i \), \( b_i^1 > b \) is strictly dominated by \( b_i^2 = b_i^1 - \epsilon > b \), where \( \epsilon > 0 \). As \( b_i^2 \leq b \), for bidder \( i \), \( b_i^1 \) and \( b_i^2 \) are both successful bids, but paying \( b_i^1 \) will lower the acquisition cost.

Thus, we have the first feature that for any symmetric equilibrium bidding strategy:

\[
\max_v \beta^1(v) = \max_v \beta^2(v).
\]

Second, consider a bidder, say bidder 1, and let \( c_{-1} = (b_2^1, b_2^2) = c_{-1} = (c^1, c^2) \) denote the competing bids of bidder 1, that is, the bids of bidder 2. Similarly, let \( H_1(\cdot) \) and \( H_2(\cdot) \) be the marginal distribution functions of \( c^1 \) and \( c^2 \), respectively.

Thus,

\[
H^k(c) = \text{Prob}[\beta^k(v) \leq c].
\]

Since, for all \( v \), \( \beta^1(v) \geq \beta^2(v) \), it is clear that the distribution \( H_1 \) stochastically dominates the distribution \( H_2 \). As usual, let \( h_1 \) and \( h_2 \) denote the corresponding densities.

Suppose that bidder 1 has values \((v_1^1, v_1^2) = (v^1, v^2) \) and bids \((b_1^1, b_1^2) = (b^1, b^2) \). He wins both units if \( c^1 < b^2 \) and the probability of this event is \( H_1(b^2) \). But if \( c^2 < b^1 \) and \( c^1 > b^2 \), he may get one unit with the probability \( H_2(b^1) - H_1(b^2) \). Thus, given bidder 1’s marginal value \( v_1 = (v_1^1, v_1^2) = (v^1, v^2) \) and bid vector \( b_1 = (b_1^1, b_1^2) = (b^1, b^2) \), his expected utility is:

\[
Eu_1(b_1, v_1) = H_1(b^2)(v^1 + v^2 - b^1 - b^2) + [H_2(b^1) - H_1(b^2)](v^1 - b^1)
\]

\[
= H_2(b^1)(v^1 - b^1) + H_1(b^2)(v^2 - b^2).
\]

The bidder’s optimization problem is to choose \( b_1 \) to maximize \( Eu_1(b_1, v_1) \) subject to the constraint:

\[
b^1 \geq b^2.
\]
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When the constraint does not bind at the optimum, that is $b^1 > b^2$, solving the first-order conditions with respect to $b^1$ and $b^2$, we have

\[
\begin{align*}
    h_2(b^1)(v^1 - b^1) &= H_2(b^1), \\
    h_1(b^1)(v^2 - b^2) &= H_1(b^2).
\end{align*}
\]

(21.4.16) (21.4.17)

Thus, the bids are independent, which means $\beta^1$ does not depend on $v^2$, and $\beta^2$ does not depend on $v^1$ when $b^1 > b^2$.

When the constraint binds at the optimum, that is $b^1 = b^2 = b$, the first-order conditions are:

\[
    h_2(b)(v^1 - b) + h_1(b)(v^2 - b) = H_2(b) + H_1(b).
\]

In this case, the bidder submits a “flat demand” function, i.e. bidding the same amount for each unit when $b^1 = b^2 = b$. The examination of the first-order conditions above shows that if the bidder submits the bid $b$ for $v = (v^1, v^2)$ and $v' = (v'_1, v'_2)$, then for any $\lambda \in [0, 1]$, he will submit the same bid $b$ for the value $\lambda v + (1 - \lambda)v'$.

21.4.7 Efficiency in Multi-Unit Auctions

For the uniform-price auction, we know that it is not efficient except that all the bidders’ demand is unit demand. Is the discriminatory auction efficient? We will offer a conclusion on how to judge whether a multi-unit auction is efficient before talking about the efficiency of discriminatory auction.

As there are different forms of auctions, we will focus on the standard auctions in which the objects are given to the bidders whose bids are the $K$ highest. For a standard auction, there are $K$ units of objects. Let $\beta = (\beta_1, \ldots, \beta_n)$ be the profile of bidding functions, where $\beta_i = (\beta^1_i, \ldots, \beta^K_i)$ is the vector of bidding functions of the bidder $i$. Suppose that the bidder $i$’s marginal value vector is $v_i$. If $\beta^k_i(v_i)$ is a winning bid, it must belong to the $K$ highest bids. If the auction is efficient and $v^k_i$ belongs to the $K$ highest marginal values, the bidder $i$ at least wins the $k$th unit of objects. If the auction is efficient, the equilibrium of this auction must satisfy the condition below:

\[
    \beta^k_i(v_i) > \beta^j_l(v_j) \text{ if and only if } v^k_i > v^l_j.
\]

Then, it must have the following features for efficient allocation:

(1) To have an efficient auction, bidders’ bidding strategies for different units must be independent, that is, the bid of any bidder $i$ to the $k$th unit only depends on his marginal value $v^k_i$ to $k$th unit:

\[
    \beta^k_i(v_i) = \beta^k_i(v^k_i).
\]
If the bid on the $k$th unit depends on marginal value of the other units, say $v_k^i$, then $v_k^i = v_j^l$ and $v_k^i \neq v_j^l$, implies $\beta_k^i \neq \beta_j^l$. That means that if $v_k^i$ and $v_j^l$ belong to the $K$ highest marginal values of bidders, but $\beta_k^i$ and $\beta_j^l$ might not belong to $K$ highest bids, then this auction is not efficient.

(2) To have an efficient auction, the bidding strategy must be the same for any bidder and any object, that is,

$$\beta_k^i(\cdot) = \beta_j^l(\cdot) = \hat{\beta}(\cdot), \forall i, j, k, l.$$ 

Otherwise, we may have $b_k^i \neq b_j^l$ when $v_k^i = v_j^l$, which will make the allocation of the auction inefficient.

In review of the three multi-unit auctions above, the Vickrey auction is efficient as all the bidders report their true values, that is, $b_k^i = \beta_k^i(v_i) = v_k^i$. For the uniform-price auction, suppose that at least one bidder’s demand is not unit demand. Then, each bidder’s bid on the first unit is true, but their bids on the other units are less than the true marginal value, so the uniform-price auction is not efficient.

For the discriminatory multi-unit auction, we consider the bidding strategy equilibrium of two units of object and two bidders. In case of $b^1 > b^2$, by the first-order conditions (21.4.16) and (21.4.17), despite the bids on different units are independent, they are not the same bidding strategies unless $\frac{H_2}{H_1} = \frac{H_3}{H_4}$. And if $\beta^1(\cdot) = \beta^2(\cdot)$, that means $H_1(\cdot)$ will first-order stochastically dominate $H_2(\cdot)$, where $\frac{H_2}{H_1} \neq \frac{H_3}{H_4}$. But if $b^1 = b^2$, then $v^1_k > v^2_k$, which is not efficient, either. Thus, of all the three static multi-unit auctions, only the Vickrey auction is efficient.

The above two features imply that the bidding functions $\beta$ are separable and symmetric across both bidders and objects. Thus, if an equilibrium outcome is efficient, the bidding function must be a single-valued increasing function. The converse is also true. We then have the following proposition.

**Proposition 21.4.5** An equilibrium allocation of a standard auction is efficient if and only if the bidding functions $\beta$ are separable and symmetric across both bidders and objects, i.e.,

$$\beta_k^i(v_i) = \beta(v_k^i).$$

As a corollary, one can see a Vickrey auction is efficient while uniform-price and discriminatory auctions in general are not efficient.

### 21.4.8 Revenue Equivalence Principle in Multi-Unit Auctions

For single-object auction, if it is private value, independent, symmetric, without external (budget) constraints and with risk neutrality, the stan-
standard auctions, such as the first-price auctions, second-price auctions, English auctions, Dutch auctions, and all-pay auctions, are not only efficient but also interim revenue equivalent to the auctioneer. When we extend the single-object auction to the multi-unit auction, as discussed above, even if all these assumptions are satisfied, not all these auctions are efficient. The question here is whether the revenue equivalence principle also holds in a multi-unit auction. We now discuss the issue.

The revenue equivalence principle of single-object auction applies to auctions resulting in the same equilibrium allocation. In the multi-unit auction, the discussion below also applies to the auctions resulting in the same equilibrium allocation. Of course, we know that some standard auctions such as discriminatory auction and uniform price auction are not efficient. Nevertheless, we cannot exclude the possibility that outcomes of these auctions and consequently the expected revenue of the seller may be the same in some economic environments. When two multi-unit auctions allocate in the same way, the revenue equivalence principle will become a useful tool.

Now we investigate that whether the revenues of two auctions are the same when the allocations of the auctions are the same.

Suppose that there are $K$ units of identical objects to be sold to $n > K$ bidders. The set of bidders is $N = \{1, \ldots, n\}$. Each bidder’s marginal value (random) vector is $V_i$ with $V_i \in \chi \equiv \{v \in [0, w]^K : v^k \geq v^{k+1}, \forall k\}$. They are independent but do not need to be identically distributed. Let $\Phi_i(\cdot)$ be the distribution function and $\varphi_i(\cdot)$ be the density function.

We can use the mechanism design approach to discuss revenue equivalence principle of the multi-unit auction, especially, apply the Bayesian incentive compatibility characterization theorem (Proposition 19.4.2) in Chapter 19 to obtain the revenue equivalence principle. Here, instead, we give a direct proof on the principle. Let $(\beta_1, \ldots, \beta_n)$ be the bidding equilibrium strategy of an auction, say, auction A. In equilibrium, the bidder with $v_i$ will choose $\beta_i(v_i)$ as his strategy. When the bidder $i$ with marginal value $v_i$ report $\hat{v}_i$, which means he chooses $\beta_i(\hat{v}_i)$ as strategy, his probability of getting the $k$th unit of objects is $y^k_i(\hat{v}_i)$, and his transfer is $t_i(\hat{v}_i)$. So bidder $i$’s expected utility is:

$$u_i(\hat{v}_i, v_i) = y_i(\hat{v}_i)v_i + t_i(\hat{v}_i),$$

where $y_i = (y^1_i, \ldots, y^K_i)$ is the probability vector of bidder $i$ getting the objects.

In equilibrium, for any $\hat{v}_i$, we have

$$y_i(v_i)v_i + t_i(v_i) \geq y_i(\hat{v}_i)v_i + t_i(\hat{v}_i). \quad (21.4.18)$$

Let $U_i(v_i) \equiv u_i(v_i, v_i) = \max_{\hat{v}_i} y_i(\hat{v}_i)v_i + t_i(\hat{v}_i)$. (21.4.18) can be written as:

$$U_i(v_i) \geq U_i(\hat{v}_i) + y_i(\hat{v}_i)(v_i - \hat{v}_i). \quad (21.4.19)$$
In (21.4.19), $y_i(\hat{v}_i)$ can be seen as the subgradient of the convex function $U_i(\cdot)$ at the point $\hat{v}_i$.

Compared to the single-object auction, there are multidimensional private values in the multi-unit auction, i.e. the private marginal value vector. In the single-object auction, we are able to integrate to obtain $U_i$. By reducing the multidimensional problem to a single dimension, we can also express $U_i$ as a form of integration.

For any point $v_i$ of the $K$ dimensional space, define a one dimensional function $W_i : [0, 1] \rightarrow \mathbb{R}$:

$$W_i(s) = U_i(sv_i).$$

Thus, $W_i(0) = U_i(0)$ and $W_i(1) = U_i(v_i)$. As $U_i(\cdot)$ is a continuous convex function, $W_i(\cdot)$ is also continuous and convex.

The convex function of one variable is absolutely continuous, and thus it is differentiable in the interior of its domain. Moreover, since any absolutely continuous function is the integral of its derivative, we have:

$$W_i(1) = W_i(0) + \int_0^1 \frac{dW_i(s)}{ds} ds. \quad (21.4.20)$$

By the (21.4.19), we have

$$W_i(s + \Delta) - W_i(s) = U_i((s + \Delta)v_i) - U_i(sv_i) \geq y_i(sv_i) \Delta v_i.$$ 

When $\Delta > 0$, we get:

$$\frac{W_i(s + \Delta) - W_i(s)}{\Delta} \geq y_i v_i.$$ 

As $\Delta \downarrow 0$, we obtain:

$$\frac{dW_i(s)}{ds} \geq y_i(sv_i)v_i.$$ 

When $\Delta < 0$, we similarly have:

$$\frac{dW_i(s)}{ds} \leq y_i(sv_i)v_i.$$ 

Since $W_i(s)$ is differentiable, we must have:

$$\frac{dW_i(s)}{ds} = y_i(sv_i)v_i.$$ 

Thus, by (21.4.20), we have:

$$U_i(v_i) = U_i(0) + \int_0^1 y_i(sv_i)v_i ds. \quad (21.4.21)$$ 

Hence, at $v_i$, the equilibrium expected utility $U_i(\cdot)$ can be determined by $y_i$, and thus

$$t_i(y_i) = U_i(y_i) - y_i(v_i)v_i.$$
which is the same provided \( y_i \) is the same. Then, we have the following proposition that summarizes the revenue equivalence in multi-unit auction.

**Theorem 21.4.1** When the expected utilities \( U_i(0) \) are the same, the interim expected utility and transfer of any bidder in any two multi-unit auctions that have the same allocation rule are the same, and therefore, the revenue of auctioneer is the same for these two auctions.

### 21.4.9 Application of Revenue Equivalence Principle

For multi-unit auctions, although the uniform-price auction and discriminatory auction in general are not efficient, they may be efficient in some special cases. For instance, if the demands of bidders are unit demand, or the uniform-price auction is reduced to Vickrey auction. Then every bidder will bid truthfully and consequently it is efficient.

Usually, solving the equilibrium of the uniform-price auction and discriminatory auction is very hard. But the revenue equivalence principle can be used to derive their equilibrium bidding strategies in some situations.

Consider an economic environment where there are three units of identical objects and two bidders, each of whom wants at most two units, which means \( V^3_i = 0, i = 1, 2 \). The vector of bidders’ values is \( V_i = (V^1_i, V^2_i), i = 1, 2 \). Suppose that on the support \( \chi = \{ v \in [0, 1]^2 : v_1 \geq v_2 \} \), the marginal valuation functions are identically and independently distributed according to the density function \( \varphi(\cdot) \) and the distribution function \( \Phi(\cdot) \). Let \( \varphi_1(\cdot) \) and \( \varphi_2(\cdot) \) be the marginal density functions of \( V^1_i \) and \( V^2_i \), respectively, the marginal distribution functions be \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \), respectively.

We show that the allocation of the uniform-price auction and the discriminatory auction is as efficient as the Vickrey auction.

We will first show that the allocation of the uniform-price auction is efficient. Recall that, for the uniform-price auction, each bidder bids truthfully on the first unit, that is, \( b^1_i = v^1_i, i = 1, 2 \). As there are three units of identical objects and two bidders, each of whom wants at most two units, each bidder wins at least one unit of object. If there exists a symmetric and increasing equilibrium bidding strategy \( \beta^2(\cdot) \) on the second unit so that \( \beta^2(v^2_i) > \beta^2(v^2_j), i \neq j \) which means \( v^2_i > v^2_j \), so the bidder \( i \) will win the second unit. Suppose that the equilibrium bidding strategy is symmetric and increasing. Then for any \( (v^1_1, v^1_2) \) and \( (v^2_1, v^2_2) \), if \( v^1_2 > v^2_2 \), bidder 1 will win two units and bidder 2 will win one unit. The opposite is similar. Obviously, now the uniform-price function is efficient.

Then we show that the allocation of the discriminatory auction is also efficient. Once again, each bidder is assured of winning at least one unit. Suppose that the bid vector of one bidder is \( (b^1, b^2) \) such that \( b^1 > b^2 \). Reducing the bid on the first unit to \( b^1 - \epsilon > b^2 \) does not affect a bidder winning
the first unit (as each bidder will get at least one unit). This does not affect the bidder winning the second unit either, but will reduce his payoff. As such, there must be \( b^1 = b^2 \) at the equilibrium, which means each bidder will report a “flat demand”. Moreover, the equilibrium bidding strategy is determined merely by the marginal value \( v^2_i \) of the second unit.

In the discriminatory auction, if the bid on the second unit is an increasing function \( \beta^2 : [0, 1] \rightarrow R_+ \), each bidder will choose the bid vector \((\beta^2(v^1_i), \beta^2(v^2_i))\) in symmetric equilibrium. As each bidder gets at least one unit, \( \beta^2(v^2_i) > \beta^2(v^1_j) \) implies \( v^2_i > v^2_j \), and bidder \( i \) will get two units. Obviously, the discriminatory auction is efficient.

Next, we will use the revenue equivalence principle to derive the symmetric equilibrium bidding strategy on the second unit for the uniform-price auction and the discriminatory auction.

In the three auction formats above, the bidder with \( v_i = 0 \) will not win the object, and \( U_i(0) = 0 \). In the case above, the three auctions are efficient. Thus the conditions of the revenue equivalence principle are satisfied.

First, we calculate the expected revenue in the Vickrey auction. As each bidder will choose the true bids \( b^k_i = v^k_i, \ i = 1, 2, k = 1, 2 \) in the Vickrey auction, the competing bid vector faced by bidder \( i \) is \( c_{-i} = (v^1_j, v^2_j, 0) \). If \( v^2_i > v^2_j \), bidder \( i \) will win two units and pay \(-t_i = 0 + v^2_j\). If \( v^2_i < v^2_j \), bidder \( i \) will win one unit and pay \(-t_i = 0\), then the expected payment of bidder \( i \) with value \( v_i \) is:

\[
-Et^v_i(v_i) = \int_0^{v_i^2} v^2_2\varphi_2(v_2)dv_2. \quad (21.4.22)
\]

We then want to solve the symmetric equilibrium bidding strategy \( \beta^{u2}(v^2_i) \) of the uniform-price auction on the second unit. In the equilibrium, if \( v^2_i > v^2_j \), bidder \( i \) wins two units and pays \( 2\beta^{u2}(v^2_j) \). If \( v^2_i < v^2_j \), bidder \( i \) wins one unit and pays \( \beta^{u2}(v^2_i) \). So in the uniform-price auction, the expected payment of the bidder \( i \) with value \( v_i \) is:

\[
-Et^u_i(v_i) = \int_0^{v_i^2} 2\beta^{u2}(v_2)\varphi_2(v_2)dv_2 + (1 - \Phi_2(v^2_i))\beta^{u2}(v^2_i). \quad (21.4.23)
\]

According to the revenue equivalence principle, \( Et^v_i(v_i) = Et^u_i(v_i) \), that is:

\[
\int_0^{v_i^2} v^2_2\varphi_2(v_2)dv_2 = \int_0^{v_i^2} 2\beta^{u2}(v_2)\varphi_2(v_2)dv_2 + (1 - \Phi_2(v^2_i))\beta^{u2}(v^2_i).
\]

Differentiating with respect to \( v^2_i \), we have:

\[
v^2_i\varphi_2(v^2_i) = \beta^2(v^2_i)\varphi_2(v^2_i) + (1 - \Phi_2(v^2_i))\beta^{u2}(v^2_i),
\]

and thus we obtain the differential equation:

\[
\beta^{u2}(v^2_i) = (v^2_i - \beta^2(v^2_i))\lambda_2(v^2_i),
\]
where
\[ \lambda_2(v^2_i) = \frac{\varphi_2(v^2_i)}{1 - \Phi_2(v^2_i)}. \]

Then, by the initial condition \( \beta^2(0) = 0 \), the symmetric equilibrium bidding strategy on the second unit is:

\[ \beta^{u2}(v^2_i) = \int_0^{v^2_i} v^2_2 \lambda_2(v^2) dL(v^2|v^2_i), \]

where \( L(v^2|v^2_i) = \exp\left(-\int_{v^2_i}^{v^2} \lambda_2(s) ds\right) \).

Obviously \( \beta^{u2}(\cdot) \) is an increasing function. In the uniform-price auction, the equilibrium bidding strategy of the bidder whose marginal value is \( v_i \) is \( (\beta^{u1}(v^1_i), \beta^{u2}(v^2_i)) \).

In the end, we now solve the symmetric equilibrium bidding strategy of the discriminatory auction on the second unit. We already know \( b^1_i = b^2_i \). Let \( \beta^{d2}(\cdot) \) be the symmetric equilibrium strategy. Then, at the equilibrium, if \( v^2_i > v^2_j \), bidder \( i \) will win two units and pay \( 2\beta^{d2}(v^2_i) \). If \( v^2_i < v^2_j \), bidder \( i \) will win one unit and pay \( \beta^{d2}(v^2_i) \), and thus the expected payment of the bidder whose value is \( v_i \) is:

\[ -E\ell^d_i(v_i) = \beta^{d2}(v^2_i) + \Phi_2(v^2_i)\beta^{d2}(v^2_i). \]  

(21.4.24)

According to the revenue equivalence principle, we have

\[ \int_0^{v^2_i} v^2_2 \varphi_2(v^2) dv^2 = \beta^{d2}(v^2_i) + \Phi_2(v^2_i)\beta^{d2}(v^2_i). \]

Then, the equilibrium bidding strategy on the second unit is:

\[ \beta^{d2}(v^2_i) = \frac{1}{1 + \Phi_2(v^2_i)} \int_0^{v^2_i} v^2_2 \varphi_2(v^2) dv^2. \]

It is easy to verify that \( \beta^{d2}(\cdot) \) is an increasing function. Thus, in the discriminatory auction, the equilibrium bidding strategy of the bidder whose marginal value is \( v_i \) is \( (\beta^{d2}(v^2_i), \beta^{d2}(v^2_i)) \).

Through the example above, we find that by using the revenue equivalence principle, we can easily obtain the equilibrium bidding strategy of some auctions that are hard to solve directly. But we have to ensure the decision rule (allocations) of the two auctions are the same when we use this method.

### 21.5 Multi-Unit Sequential Auctions with Private Values

In this section we study the sequential auctions of multiple identical objects. A sequential auction is an auction in which the units are sold one at
each time in separate auctions that are conducted sequentially till all objects are sold. In particular, we mainly consider two formats of auctions: the sequential first-price sealed-bid auctions and the sequential second-price sealed-bid auctions. In these two auctions, at each unit of objects, all bidders simultaneously submit sealed bids, and then the price at which it is sold – the winning bid – is announced.

The sequential auctions of multiple objects are very different from the simultaneous multi-unit auctions discussed earlier, in which all the units are sold at one time, while in the former situation the units are sold one at a time in separate auctions that are conducted sequentially. Sequential auctions bring some new strategic issues where bidders may choose different bidding strategies at different stages, and thus this dynamic consideration can make the analysis of the bidder’s strategy quite complicated. For the sake of discussion, we focus our attention on the simplest situations in which each bidder has demand for only a single unit. That is, for any bidder $i$, we have $V^k_i = 0, \forall k \geq 2$.

We first discuss the sequential first-price sealed-bid auctions, and then the sequential second-price sealed-bid auctions.

### 21.5.1 Equilibrium of Sequential First-Price Auctions

Consider an economic environment in which $K$ identical items are sold to $n$ bidders ($n > K$), and each bidder has a unit demand. We assume that bidder $i$ has private value $V_i$, and that each bidder’s value $V_i$ is drawn independently from the same distribution $\Phi(\cdot)$, corresponding density $\varphi(\cdot)$ on $[0, \omega]$. A particular bidder may be still active in the $K$ round auction (if he wins an object, he will stop bidding). So a bidding strategy for a bidder consists of $K$ functions, denoted by $\beta^I_1(\cdot), \cdots, \beta^I_{K-1}(\cdot)$, with the superscript $I$ standing for the first-price auction, and the superscript $k$ representing the strategy of the $k$ round auction.

We look for a symmetric equilibrium that is an equilibrium where all bidders use the same strategy. As in each round, there is different information, in $k \leq K$ round, the bidding strategy is $\beta^I_k(v; p^1, \cdots, p^{k-1})$, where $p^1, \cdots, p^{k-1}$ were the prices in the previous $k - 1$ auctions, respectively. Moreover, the sequential auctions are assumed to be held in a short time, so we disregard the time discount issue.

If the equilibrium strategies $\beta^I_k(v; \cdot)$ for $k \leq K$ are increasing functions of $v$, the first item will go to the bidder with the highest value, and the $k$th item to the bidder with the $k$th highest value. In this sense, the sequential first-price auction is efficient. Before discussing the sequential auctions of $K$ units of objects, we begin by considering the simplest situation in which only two units are sold so as to get some basic characteristics of equilibrium bidding strategies in the sequential first price auctions.
The Sequential Auctions for Two Units of Identical Objects

We begin by looking at an economy in which only two units are sold. Let \((\beta^1, \beta^2)\) (the superscript \(I\) is omitted here) be the symmetric equilibrium bidding strategy profile. In the first round, the bidding strategy is \(\beta^1(\cdot): [0, w] \to \mathbb{R}_+\). If it is a strictly increasing function, there is an inverse function, which is the price paid in the first-price auction, \(p^1\).

Before the start of the second round of bidding, the bidder can infer that the private value of the bidder who won the auction in the first round according to the inverse function \((\beta^1)^{-1}\) at \(p^1\). As it is a symmetric equilibrium, we can simply consider only the bidding strategy of bidder 1. Denote by \(W_1 \equiv W_1^{(n-1)}\) the first order statistics of \(n - 1\) stochastic variables \(V_2, \ldots, V_n\) by \(W_2 \equiv W_2^{(n-1)}\) the second order statistics of \(n - 1\) stochastic variables \(V_2, \ldots, V_n\) and by \(w_1\) and \(w_2\) their realizations, respectively.

Let \(\psi_1\) and \(\psi_2\) be the corresponding densities, and \(\Psi_1\) and \(\Psi_2\) be the distributions of \(W_1 \equiv W_1^{(n-1)}\) and \(W_2 \equiv W_2^{(n-1)}\), respectively, where \(\Psi_1 = \Phi^{(n-1)}(\cdot)\). We assume that the considered concept of bidding equilibrium is the perfect Bayesian equilibrium that requires sequentially rational, so starting from the first bidding price and then the second round of bidding strategy should be a Bayesian-Nash equilibrium. Below we start with the analysis of the second round of auction.

In the second round of auction, suppose the auction price of the first round is \(p^1\). If bidder 1 does not get the auctioned object in the first round, he can infer that \(w_1 = (\beta^1)^{-1}(p^1)\). Suppose that all the other bidders follow the equilibrium strategy \(\beta^2(\cdot; w_1)\). Since \(W_2 < w_1\), \(\beta^2(\cdot; w_1)\) is an increasing function, it makes no sense for bidder 1 to bid an amount greater than \(\beta^2(w_1; w_1)\). Noting that the private value of bidder 1 is \(v_1\), if he bids \(\beta^2(\hat{v}_1; w_1)\) where \(\hat{v}_1 \leq w_1\), his interim expected utility in the second auction is:

\[
E u_1(\hat{v}_1, v_1; w_1) = \Psi_2(\hat{v}_1|w_1)[v_1 - \beta^2(\hat{v}_1; w_1)].
\]

Differentiating with respect to \(\hat{v}_1\), we obtain the first-order condition in equilibrium:

\[
\psi_2(v_1|w_1)[v_1 - \beta^2(v_1; w_1)] - \Psi_2(v_1|w_1)\beta^2(v_1; w_1) = 0.
\]

Rearranging this results in the differential equation and then we have

\[
\beta^2(v_1; w_1) = \frac{\psi_2(v_1|w_1)}{\Psi_2(v_1|w_1)}[v_1 - \beta^2(v_1; w_1)],
\]

where the initial value condition satisfies \(\beta^2(0; w_1) = 0\).

Given the winner of the first round, the highest of \(n - 1\) values, except that of bidder 1, equals \(w_1\). The probability that bidder 1 will win the second auction, if he chooses \(\hat{v}_1\), is \(\text{prob}(W_2^{(n-1)} < \hat{v}_1)\) that is equivalent
\[
\text{prob}(W_1^{(n-2)} < \hat{v}_1), \text{ where the } W_1^{(n-2)} \text{ is the private value of the second-highest of } n-2 \text{ values, except that of bidder 1 and the winner at the first round. Actually, } W_2^{(n-1)} \equiv W_1^{(n-2)}. \\

\Psi_2(v_1|w_1) = \Psi_1^{(n-2)}(v_1|W_1^{(n-2)} < w_1) = \frac{\Phi(v_1)^{n-2}}{\Phi(w_1)^{n-2}}, \quad (21.5.26)
\]

and thus we have

\[
\psi_2(v_1|w_1) = \frac{(n-2)\Phi(v_1)^{n-1}\varphi(v_1)}{\Phi(w_1)^{n-2}}. \quad (21.5.27)
\]

Using (21.5.26) and (21.5.27), the differential equation (21.5.25) can be written as

\[
\beta_2'(v_1; w_1) = \frac{(n-2)\varphi(v_1)}{\Phi(v_1)}[v_1 - \beta_2(v_1; w_1)],
\]

or equivalently,

\[
\frac{\partial}{\partial v_1} (\Phi(v_1)^{n-2} \beta_2(v_1; w_1)) = (n-2)\Phi(v_1)^{n-3}\varphi(v_1)v_1.
\]

Together with the initial conditions, the solution to the differential equation is:

\[
\beta_2(v_1) = \frac{1}{\Phi(v_1)^{n-2}} \int_{v_1}^{v_1} vd(\Phi(v)^{n-2}) = E[W_1^{(n-2)}|W_1^{(n-2)} < v_1] = E[W_2|W < v_1 < W_1]. \quad (21.5.28)
\]

Thus, for bidder 1, the complete bidding strategy for the second period is to bid \( \beta_2(v_1) \) following (21.5.28) if \( v_1 \leq w_1 \) and to bid \( \beta_2(w_1) \) if \( v_1 > w_1 \). In case of \( v_1 > w_1 \), if bidder 1 chooses an equilibrium strategy in the first auction, he should win the auction in the first round. Even though this represents off-equilibrium behavior on the part of bidder 1 himself, as per the concept of a strategy, we must count in this possibility.

Now we discuss the first-round equilibrium strategy \( \beta_1(\cdot) \). Again let us assume that the private value of bidder 1 is \( v_1 \) and that all the other bidders follow the first-period strategy \( \beta_1(\cdot) \). Further, suppose that all bidders, including bidder 1, will follow the equilibrium bidding strategy (21.5.28) in the second period. We now study the conditions that the equilibrium calls on bidder 1 to bid \( \beta_1(\cdot) \) in the first stage.

In equilibrium, bidder 1 should bid \( \beta_1(v_1) \) in the first stage, but consider what is his expected utility if he decides to bid \( \beta_1(\hat{v}_1) \) instead. If \( \hat{v}_1 \geq v_1 \), he wins the first auction when \( \hat{v}_1 > W_1 \). But when \( \hat{v}_1 < W_1 \), he can not get
the item in the first round, and he will enter the second round of bidding. When \( W_2 < v_1 \leq \hat{v}_1 < W_1 \), he wins the second auction and cannot win the auction in any other cases. The expected utility of choosing \( \beta^1(\hat{v}_1) \) is:

\[
Eu_1(\hat{v}_1, v_1) = \Psi_1(\hat{v}_1)[v_1 - \beta^1(\hat{v}_1)] + (n - 1)(1 - \Phi(\hat{v}_1))\Phi(v_1)^{n-2}[v_1 - \beta^1(v_1)].
\] (21.5.29)

On the other hand, if \( \hat{v}_1 < v_1 \), when \( \hat{v}_1 > W_1 \), he wins the first auction with a bid. When \( \hat{v}_1 < W_1 \), if \( W_2 < v_1 < W_1 \), he loses the first auction but wins the second at the price of \( \beta^2(v_1) \). If \( \hat{v}_1 < W_1 < v_1 \), he will pay \( \beta(W_1) \), and then his expected utility is:

\[
Eu_1(\hat{v}_1, v_1) = \Psi_1(\hat{v}_1)[v_1 - \beta^1(\hat{v}_1)] + [\Psi_2(v_1) - \Psi_1(v_1)][v_1 - \beta^2(v_1)]
+ \int_{\hat{v}_1}^{v_1} [v_1 - \beta^2(w_1)]\psi_1(w_1)dw_1.
\] (21.5.30)

Differentiating equations (21.5.29) and (21.5.30) with respect to \( \hat{v}_1 \), we obtain the first-order conditions in the two cases

\[
0 = \psi_1(\hat{v}_1)[v_1 - \beta^1(\hat{v}_1)] - \Psi_1(\hat{v}_1)\beta^1(\hat{v}_1)
- (n - 1)\varphi(\hat{v}_1)\Phi(v_1)^{n-2}[v_1 - \beta^1(v_1)],
\] (21.5.31)

\[
0 = \psi_1(\hat{v}_1)[v_1 - \beta^1(\hat{v}_1)] - \Psi_1(\hat{v}_1)\beta^1(\hat{v}_1)
- \psi_1(\hat{v}_1)[v_1 - \beta^1(v_1)].
\] (21.5.32)

Since \( \psi_1(v) = \Phi(v)^{n-1}, \psi_1(v) = (n - 1)\varphi(v)\Phi(v)^{n-2} \). Therefore, the two equations (21.5.31) and (21.5.32) are the same. In equilibrium it is optimal to bid \( \beta^1(\cdot) \) and setting \( \hat{v}_1 = v_1 \) in either first-order condition results in the differential equations ((21.5.31) and (21.5.32)). From the equation (21.5.31) and the equation (21.5.32), we have

\[
\beta^1(v_1) = \frac{\psi_1(v_1)}{\Psi_1(v_1)}[\beta^2(v_1) - \beta^1(v_1)].
\] (21.5.33)

Together with the initial value condition \( \beta^1(0) = 0 \), we can get the solution to the above differential equation:

\[
\beta^1(v_1) = \frac{1}{\Psi_1} \int_0^{v_1} \beta^2(w_1)\psi_1(w_1)dw_1
= E[\beta^2(W_1)|W_1 < v_1]
= E[E[W_2|W_2 < W_1]|W_1 < v_1]
= E[W_2|W_1 < v_1].
\] (21.5.34)

Thus, together with the equations (21.5.34) and (21.5.28), we obtain the following proposition:
Proposition 21.5.1 Suppose that all the bidders have a unit demand. Then, the symmetric equilibrium strategies for the sequential first-price sealed-bid auction with two units are

\[
\beta_{I1}(v) = E[W_2|W_1 < v],
\]

\[
\beta_{I2}(v) = E[W_2|W_2 < v < W_1],
\]

where \( W_1 \equiv W_1^{(n-1)} \) is the highest, and \( W_2 \equiv W_2^{(n-1)} \) is the second highest, of \( n-1 \) independently drawn values.

Obviously \( \beta_{I1}(v_1) \) and \( \beta_{I2}(v_1) \) are strictly increasing functions in \( v_1 \). In the following we provide two features of the dynamic bidding equilibrium behavior for the case of \( K = 2 \). First, from (21.5.34), we have

\[
\beta^1(v_1) = E[\beta^2(W_1)|W_1 < v_1],
\]

which means that if bidder 1 with \( v_1 \) is the bidder with the highest value, whether he chooses to win the item in the first round or in the second round, his interim expected payment will be the same. To be precise, the (random) dynamic payment price is martingale (i.e., the conditional expected value of the next observation, given all the past observations, is equal to the most recent observation). At this point, the bidder with the highest value has no incentive to delay winning the item, thus equilibrium results can prevent strategic delays.

Secondly, from the (21.5.33) and the fact that \( \beta^1(\cdot) \) is a strictly increasing function, we have:

\[
\beta^2(v_1) - \beta^1(v_1) > 0,
\]

which means that the bidding strategy in the second round will be more active than in the first round. This is because, after the second round, the bidder will no longer have the opportunity to win the auction, the bidding in the latter stages will be more intense. If we extend \( K = 2 \) to a more general case, the above two features are still true.

The Sequential Auctions for \( K \) Units of Identical Objects

Consider a more general economy in which the \( K > 1 \) identical items are sold to \( n > K \) bidders. Assume that \( K \) units of identical objects are sold in a sequential first-price sealed-bid auction. Also, assume that, at round \( k \), the prices in the first \( k-1 \) auctions are \( p^1, \cdots, p^{k-1} \). Let \( W_k = W_k^{(n-1)} \) be the \( k \)th highest independent and stochastically drawn variables from \( n-1 \) identical distributions, \( \Psi_k(\cdot) \) be the distribution of \( W_k \), and \( \psi_k(\cdot) \) be the corresponding density. We will derive symmetric perfect Bayesian equilibrium bidding strategies \((\beta^1, \cdots, \beta^K)\), and similarly, only the strategies of bidder 1 are considered.

We now derive symmetric perfect Bayesian equilibrium bidding strategies by backward from the last round auction. So first consider the \( K \)th
auction conducted in the last period. Using the similar inference process as before, the equilibrium bidding strategy \( \beta^K \) in the last period is

\[
\beta^K(v_1) = E[W_K|W_K < v_1 < W_{K-1}]. \tag{21.5.35}
\]

We then consider the auction for the \( k < K \) round. Again let us take the perspective of bidder 1. Suppose that all bidders follow symmetric equilibrium bidding strategies \( \beta^{k+1}, \ldots, \beta^K \) in the sequential auctions. When all other bidders choose the bidding strategy \( \beta^k \) in the round \( k \), we examine the strategy choice of bidder 1. Suppose that all the other bidders choose the equilibrium bidding strategy.

If \( v_1 > W_1^{(n-1)} \), bidder 1 wins the first round auction. If \( W_{k-1}^{(n-1)} < v_1 < W_k^{(n-1)} \), he wins the \( k \)-th round auction. If \( v_1 < W_k^{(n-1)} \), he cannot get any item in equilibrium. If \( W_k^{(n-1)} > v_1 \), he wins the \( k \)-th round auction with a bid of \( W_k^{(n-1)} \). If \( W_k^{(n-1)} < v_1 \), he wins the \( k+1 \)-th round auction with a bid of \( W_k^{(n-1)} \).

In addition, the other bidders cannot obtain the item. Therefore, if the symmetric equilibrium bidding strategy \((\beta^1, \ldots, \beta^K)\) is a strictly increasing function, the sequential auction is efficient. Besides, in the first \( k-1 \) round auctions, if the prices are \( p^1, \ldots, p^{k-1} \), the bidders can infer that the bidders’ private value of winning the first \( k-1 \) round auctions is: \( w_1 = (\beta^1)^{-1}(p^1), \ldots, w_{k-1} = (\beta^{k-1})^{-1}(p^{k-1}) \).

The equilibrium calls on bidder 1 with private value \( v_1 \) to bid \( \beta^k(v_1) \) in the \( k \)-th stage but consider what happens if he decides to bid slightly higher, say, \( \beta^k(v_1 + \Delta) \). (His choice of a slightly lower tender price \( \beta^k(v_1 - \Delta) \) is similar. As for the previous case \( K = 2 \), in the analysis of the first round of bids, we know that the first-order conditions are the same regardless of whether the bid price is higher or lower. If \( v_1 > W_k = W_k^{(n-1)} \), he obtains the item in the \( k \) round and the expected payment increases by

\[
\Psi_k(v_1|w_{k-1})[\beta^k(v_1 + \Delta) - \beta^k(v_1)].
\]

If \( v_1 < W_k < v_1 + \Delta \), he would have lost the \( k \)-th auction with his equilibrium bidding strategy whereas bidding higher \( \beta^k(v_1 + \Delta) \) results in his winning. Now there are two subcases.

Case 1. When \( W_{k+1} < v_1 < W_k < v_1 + \Delta \), in equilibrium, he would have won the \((k + 1)\)st round auction.

Case 2. When \( v_1 < W_{k+1} < W_k < v_1 + \Delta \), however, in equilibrium, he would have lost both the \( k \)-th and the \( k + 1 \)st auctions, and possibly won a later auction, for \( l > k + 1 \). When \( \Delta \) is small, however, the probability of \( v_1 < W_{k+1} < W_k < v_1 + \Delta \) is sufficiently small — it is of second order in magnitude (proportional to \( \Delta^2 \)). The probability of \( W_{k+1} < v_1 < W_k < v_1 + \Delta \) is sufficiently small — it is of first order in magnitude (proportional to \( \Delta \)). Therefore, when \( \Delta \) is sufficiently small, the main effect is Case 1, and
then, when bidder 1’s bidding strategy changes from \( \beta^k(v_1) \) to \( \beta^k(v_1 + \Delta) \), his expected utility increases approximately by:

\[
[\Psi_k(v_1 + \Delta|w_{k-1}) - \Psi_k(v_1|w_{k-1})][(v_1 - \beta^k(v_1)) - (v_1 - \beta^{k+1}(v_1))].
\]

Thus the total expected change in utility is approximately:

\[
\Psi_k(v_1|w_{k-1})[\beta^k(v_1 + \Delta) - \beta^k(v_1)]
+ [\Psi_k(v_1 + \Delta|w_{k-1}) - \Psi_k(v_1|w_{k-1})]
\times [(v_1 - \beta^k(v_1)) - (v_1 - \beta^{k+1}(v_1))].
\]

Equation (21.5.36), divided by \( \Delta \), and taking the limit as \( \Delta \to 0 \), we obtain the differential equation:

\[
\beta^k(v_1) = \frac{\psi_k(v_1|w_{k-1})}{\Psi_k(v_1|w_{k-1})}[\beta^{k+1}(v_1) - \beta^k(v_1)],
\]

(21.5.37)

where

\[
\Psi_k(v_1|w_{k-1}) = \Psi_k^n(v_1|w_{k-1})
= \frac{\Phi(v_1)^{n-k}}{\Phi(w_{k-1})^{n-k}},
\]

because of \( W_k^{(n-1)} = W_1^{(n-k)} \).

Together with the initial condition \( \beta^k(0) = 0 \), we can obtain the solution to the differential equation (21.5.37):

\[
\beta^k(v_1) = \frac{1}{\Phi^{n-k}(v_1)} \int_0^{v_1} \beta^{k+1}(v)d(\Phi^{n-k}(v))
= E[\beta^{k+1}(W_1^{(n-k)})|W_1^{(n-k)} < v_1]
= E[\beta^{k+1}(W_k)|W_k < v_1 < W_{k-1}].
\]

(21.5.38)

The equation (21.5.38) is a first-order recursion equation, given the bidding equilibrium equation of the last round (21.5.35), we can obtain the equilibrium bidding strategy in the \( K - 1 \)th auction:

\[
\beta^{K-1}(v_1) = E[\beta^K(W_{K-1})|W_{K-1} < v_1 < W_{K-2}]
= E[E[W_K|W_K < W_{K-1}]|W_{K-1} < v_1 < W_{K-2}]
= E[W_K|W_{K-1} < v_1 < W_{K-2}].
\]

(21.5.39)

By backward from the last around auction proceeding inductively in this fashion results in the equilibrium bidding strategy for all \( k < K \) rounds:

\[
\beta^k(v_1) = E[\beta^{k+1}(W_k)|W_k < v_1 < W_{k-1}]
= E[E[W_K|W_{k+1} < W_k]|W_k < v_1 < W_{k-1}]
= E[W_K|W_k < v_1 < W_{k-1}].
\]

(21.5.40)
Let $W_0 = \infty$, we get a symmetrical equilibrium of sequential first-price sealed-bid auction.

**Proposition 21.5.2**. Suppose that $K$ identical items are sold to $n > K$ bidders by sequential first-price sealed-bid auctions and each bidder has a unit demand and private values. Then, symmetric equilibrium strategies $(\beta^{11}(v), \ldots, \beta^{1K}(v))$ are given by

$$
\beta^{1k}(v) = \mathbb{E}[W_K | W_k < v < W_{k-1}], k = 1, \ldots, K,
$$

where $W_k \equiv W^{(n-1)}_k$ is the $k$th highest of independent and stochastically drawn variables from $n - 1$ identical distributions.

Again, two features of the sequential auctions should be mentioned. First, as the auction goes on, the bidding prices become more and more active, that is, $\beta^k(v) > \beta^m(v)$ with $k > m$. It can be shown that $\beta^{1k}(v)$ is an increasing function and can also be induced by the equation (21.5.37).

Secondly, the equilibrium price path is a martingale. Suppose that in equilibrium, bidder 1 with value $v_1$ wins the $k < K$th round auction. Then, there must be that $W_K < \cdots < W_k < v_1 < W_{k-1} < \cdots < W_1$. We know that the (equilibrium) realized price in period $k$ is $p^k_k = \beta^k(v_1)$. Moreover, the price in the $k + 1$th round is a random variable $p^{k+1}_k = \beta^{k+1}(W_k)$ and from (21.5.40), we have:

$$
\mathbb{E}(p^{k+1}|p^k_k) = \mathbb{E}[\beta^{k+1}(W_k) | W_k < v_1 < W_{k-1}]
$$

$$
= \beta^k(v_1)
$$

$$
= p^k_k.
$$

This establishes that the price path in sequential auctions is a martingale. The logic behind this conclusion is very simple: if $\mathbb{E}(p^{k+1}|p^k_k) \neq p^k_k$, for example, $\mathbb{E}(p^{k+1}|p^k_k) < p^k_k$, the bidder who wins the auction in the $k$ round would be unwilling to use the equilibrium strategy of $\beta^k$, as the expected auction price will be lower and the expected utility for his winning the auction in the next round will be greater.

In the model, we assume that the bidder is risk neutral, but the conclusion will be similar even for risk aversion.

### 21.5.2 Equilibrium of Sequential Second-Price Auctions

We now discuss another kind of sequential auctions: the items are sold in sequential second-price sealed-bid auctions. If there is a symmetric increasing bidding strategies $(\beta^{II1}, \ldots, \beta^{IIK})$ in the sequential second-price sealed-bid auctions, then such second-price sealed-bid auctions will also be efficient. As such, the sequential first-price and second price auctions are revenue equivalent. We will simply find the equilibrium strategies by making use of this revenue equivalence principle.
Specifically, let \( m_I(v) \) and \( m_{II}(v) \) denote the interim expected payments by a bidder with value \( v \) in \( k \) sequential first- and second-price auctions, respectively. Now define \( m^k_I(v) \) to be the expected payment made in the \( k \)th auction by a bidder when the items are sold by means of the first-price sealed-bid auctions. So \( m_I(v) = \sum_{k=1}^{K} m^k_I(v) \). Define \( m_{II}(v) = \sum_{k=1}^{K} m^k_{II}(v) \) in an analogous fashion for the sequential second-price sealed-bid auctions. We first use the revenue equivalence principle to get a strong version of revenue equivalence:

\[
m^k_I(v) = m^k_{II}(v), \quad \forall k \in \{1, \ldots, K\}.
\]

Equation (21.5.41) implies that each round of auction satisfies the revenue equivalence principle. We verify by induction, starting with the \( K \)th auction. Prior to the last round auction, as the equilibrium strategies of two auctions are monotonic, the remaining \( n - K + 1 \) bidders can infer the private value of the previous winning bidders. For instance, due to symmetry, we assume that bidder 1 knows his own value \( v_1 = v \), while his competitors have values \( W_{K-1} \), \( \ldots \), \( W_{n-1} \). At the round \( K-1 \), the winner’s value is \( W_{K-1} = w_{K-1} \). The revenue equivalence principle implies that \( m_I(v) = m_{II}(v) \). Since bidders with a private value of \( v \) does not win the auction in the previous \( K-1 \) rounds, \( m^k_I(v) = m^k_{II}(v) = 0, \forall k \leq K - 1 \), so we have:

\[
m^K_I(v) = m^K_{II}(v).
\]

Now consider the start of auction \( K-1 \). The remaining \( n-K+2 \) bidders bid the remaining items. Suppose that bidder 1 knows his own value \( v_1 = v \) and his competitors have values \( W_{K-1}, \ldots, W_{n-1} \). At the round of \( K-2 \), the winner’s value is \( W_{K-2} = w_{K-2} \). Once again, the revenue equivalence principle implies that

\[
m^{K-1}_I(v) + m^K_I(v) = m^{K-1}_{II}(v) + m^K_{II}(v).
\]

And since \( m^K_I(v) = m^K_{II}(v) \), we have \( m^{K-1}_I(v) = m^{K-1}_{II}(v) \). Proceeding inductively in this way we will get:

\[
m^k_I(v) = m^k_{II}(v), \quad \forall k \in \{1, \ldots, K\}.
\]

With this strong version of revenue equivalence principle, we are ready to find the equilibrium bidding strategies of sequential second-price auctions.

Now notice that for the previous \( K-1 \) rounds, bidder 1 does not win the auction. He wins the \( K \)th auction with a private value of \( v_1 = v \), then apparently in the \( K \)th auction his (weakly dominant) bidding strategy must be:

\[
\beta^{IIK}(v) = v.
\]
Suppose that in the $k < K$ round, the bidder wins the auction, then at this time there must be:

$$W_K < \cdots < W_k < v < W_{k-1} < \cdots < W_1.$$  

In the second-price auctions, his payment is $\beta^{IIk}(W_k)$, and for the bidder with a private value of $v$, his expected payment in the $k$ round is:

$$m^{II}_{k}(v) = \text{Prob}[W_k < v < W_{k-1}] \times E[\beta^{IIk}(W_k) | W_k < v < W_{k-1}].$$

On the other hand, in the sequential first-price auctions, the payment for a winning bidder with private value $v$ in the $k$ round is:

$$m^{I}_{k}(v) = \text{Prob}[W_k < v < W_{k-1}] \times \beta^{Ik}(v)$$

$$= \text{Prob}[W_k < v < W_{k-1}] \times E[\beta^{I(k+1)}(W_k) | W_k < v < W_{k-1}], \quad (21.5.42)$$

where the second line of the equation (21.5.42) comes from

$$\beta^{Ik}(v) = E[\beta^{I(k+1)}(W_k | W_k < v < W_{k-1})].$$

According to the revenue equivalence principle for each period we mentioned above, that is, $m^{II}_{k}(v) = m^{I}_{k}(v)$, we have

$$E[\beta^{IIk}(W_k) | W_k < v < W_{k-1}] = E[\beta^{I(k+1)}(W_k) | W_k < v < W_{k-1}].$$

Differentiating both sides of the equality with respect to $v$ results in the identity:

$$\beta^{IIk}(v) = \beta^{I(k+1)}(v).$$

Thus, we have

$$\beta^{IIk}(v) = E[W_K | W_{k+1} < v < W_k].$$

Obviously, $\beta^{IIk}(\cdot)$ is an increasing function. Summarizing the above discussion, we have the following proposition.

**Proposition 21.5.3** Suppose that $K$ identical items are sold to $n > K$ bidders by sequential second-price sealed-bid auctions and that each bidder has a unit demand and private values. Then, symmetric equilibrium strategies are given by

$$\beta^{IIK}(v) = v$$

and for all $k < K$,

$$\beta^{IIk}(v) = \beta^{I(k+1)}(v)$$

where $\beta^{I(k+1)}(v)$ is the $(k + 1)$st period equilibrium bidding strategy in the sequential first-price auction format, derived in Proposition 21.5.2.
Comparing the strategies for the sequential first-price auctions with the strategies for the sequential second-price auctions, we find that due to $\beta^{I(k+1)}(v) > \beta^{I}(v)$, we have $\beta^{IIk}(v) > \beta^{Ik}(v)$. That is, every bidder bids more actively in sequential second-price auctions than in sequential first-price auctions. In addition, as with sequential first-price auctions, the equilibrium price process in a sequential second-price auction is also a martingale. Suppose that the equilibrium price process in a sequential second-price auction is $P^{I II}, \ldots, P^{IIK}$. Since $\beta^{IIk}(v) = \beta^{I(k+1)}(v)$, we only need to verify that the prices between the last two periods have the property of a martingale.

Suppose that bidder 1 with a value of $v$ wins in round $K - 1$, so we have $W_{k_1} < v < W_{k_2}$. Thus, the auction (random) price of the $K - 1$ round is $P^{K-1} = \beta^{II(K-1)}(W_{K-1})$. Let the realized price be $p^{K-1} = \beta^{II(K-1)}(w_{K-1})$, where $w_{K-1}$ is the realized value of $W_{K-1}$. In the last round the bidder with a value of $w_{K-1}$ will win and his payment will be $p^K = \beta^{IIK}(W_K) = W_K$. Now we have:

$$E[p^{IIK}|p^K] = E[W_K|W_K < w_{K-1}] = \beta^{II(K-1)}(w_{K-1}) = p^{K-1}.$$

Thus, the (random) equilibrium price process in a sequential second-price auction is also a martingale. No participant would like to win the auction through a strategic delay.

### 21.6 Combinatorial Auctions for Heterogeneous Objects with Private Values

We have so far only discussed homogeneous objects in considering multi-term auctions. In reality, objects are often heterogeneous: they can be substitutable, complementary, or sometimes not related, and the relevance of items to different bidders may also vary. Thus, there is a new problem with multi-term auctions, where the auction of different combinations of objects creates new complications. For example, if there is complementarity between two objects, the bidder would be cautious about the bidding if it was anticipated that the complementary items would not be obtained in the auction, which is known as exposure problem. Many auctions are dedicated to solving such incentive issues in multi-term auctions. As the issue of combinatorial auctions of heterogeneous objects will be more complex, we mainly introduce some basic conclusions in this section. We again focus on the case of private values.
21.6. COMBINATORIAL AUCTIONS FOR HETEROGENEOUS OBJECTS WITH PRIVATE VALUES

21.6.1 The Basic Model

Let $K = \{a, b, c, \ldots\}$ be a finite set of distinct objects to be auctioned and $N = \{1, 2, \ldots, n\}$ be the set of bidders. Bidder $i$’s private values of combination $S \subseteq K$ is $v_i(S)$, and bidder $i$’s value vector of all combinations is $v_i = (v_i(S))_{S \subseteq K}$. Suppose that $v_i(\emptyset) = 0$ and $v_i(S) \leq v_i(T), \forall S \subseteq T \subseteq K$. The set of all value vectors for bidder $i$ is $\chi_i$, which is a nonnegative closed convex set with $0 \in \chi_i$.

For bidder $i$, there may be some specific relationship between objects, such as substitutability. Any two of multiple identical objects are perfect substitutes, and generally have diminishing marginal nature, that is, with one object already in hand, the marginal value of the other item will decrease. For heterogeneous multi objects, there is a similar substitution: if a set of combinational objects in hand is larger than another, the marginal value of additional object is smaller. We then have the following definition.

**Definition 21.6.1 (Substitutable Multiple Items)** Objects are said to be substitutable for bidder $i$ if for any $a \in K, a \notin T, S \subseteq T,$

$$v_i(S \cup a) - v_i(S) \geq v_i(T \cup a) - v_i(T), \forall v_i \in \chi_i. \quad (21.6.43)$$

It can be shown that (21.6.43) is equivalent to:

$$v_i(S) + v_i(T) \geq v_i(S \cup T) + v_i(S \cap T), \forall v_i \in \chi_i. \quad (21.6.44)$$

If (21.6.44) holds, it implies bidder $i$’s value has submodular. If $S \cap T = \emptyset$, the inequality (21.6.44) reduces to

$$v_i(S) + v_i(T) \geq v_i(S \cup T), \forall v_i \in \chi_i. \quad (21.6.45)$$

In this case, we say that bidder $i$’s value is subadditive.

The following defines the complementarity of objects.

**Definition 21.6.2 (Complementary Multiple Items)** Objects are said to be complementary for bidder $i$ if for any $a \in K, a \notin T, S \subseteq T,$

$$v_i(S \cup a) - v_i(S) \leq v_i(T \cup a) - v_i(T), \forall v_i \in \chi_i. \quad (21.6.46)$$

Then analogously, bidder $i$’s value has supermodular if it satisfies

$$v_i(S) + v_i(T) \leq v_i(S \cup T) + v_i(S \cap T), \forall v_i \in \chi_i. \quad (21.6.47)$$

Moreover, bidder $i$’s value is said to be superadditive, if it satisfies:

$$v_i(S) + v_i(T) \leq v_i(S \cup T), \forall v_i \in \chi_i, S \cap T = \emptyset. \quad (21.6.48)$$

In multi-item auctions, let $y_i \in K$ denote the set of items obtained by bidder $i$ in allocation $y$. The allocation $y = (y_1, \ldots, y_n)$ is feasible, if $y_i \cap$
$y_j = \emptyset, \forall i \neq j$, and $\bigcup_i y_i = K$. As such, a feasible allocation is actually a partition of the set of all objects $K$. Let $Y$ denote the set of all feasible allocations.

Next we use the mechanism design approach to discuss multi-item auctions. Suppose that the value profile of all bidders is $v = (v_1, \cdots, v_n)$. Define the allocation rules of an auction as $y(v) = y_1(v), \cdots, y_n(v)$, where $y_i(v)$ is the set of items obtained by participant $i$ at a value of $v$. We consider the direct mechanism, where each bidder reports his value $\hat{v}_i$, $i \in N$. Let $y_i(\hat{v}_i) = (y_i(S, \hat{v}_i)_{S \subseteq K})$, where $y_i(S, \hat{v}_i)$ is the probability that bidder $i$ wins the combination $S$ when reporting his values of $\hat{v}_i$. $y_i(\hat{v}_i)$ is the probability vector that bidder $i$ wins the item combination, the payment of which is $m_i(\hat{v}_i)$ or in terms of transfer, it is $t_i(\hat{v}_i) = -m_i(\hat{v}_i)$.

A direct mechanism $(y(\cdot), t(\cdot))$ is said to be incentive compatible, if it satisfies

$$U_i(v_i) = y_i(v_i).v_i + t_i(v_i) = \max_{\hat{v}_i} y_i(\hat{v}_i).v^1 + t_i(\hat{v}_i).$$

Similarly, we have the following revenue equivalence theorem for combinatorial auctions in the following.

**Theorem 21.6.1** Suppose that the allocation rule $y(\cdot)$ of two auctions are the same, and every bidder’s expected utility is zero at the lowest value. Then the auctioneer’s revenues from the two auctions are the same.

The proof is similar to that of Proposition 21.4.1. The only difference is that reported value vectors are of size $2^{|K|}$ instead of size $K$, thus we have the following equation:

$$U_i(v_i) = U_i(0) + \int_0^1 y_i(sv_i)v_ids. \tag{21.6.49}$$

### 21.6.2 A Benchmark of Combinatorial Auction: VCG Mechanism

In the following we discuss a benchmark mechanism of combinatorial auctions, i.e., the Vickrey-Clarke-Groves (VCG) mechanism. In private value settings, similar to the cases of single item and homogeneous multiple-object auction, the heterogeneous multiple-object combinatorial VCG auction is also an efficient mechanism.

An allocation rule is efficient if for any value vector $v$ of any combination of participants, the allocation $y(v)$ maximizes social welfare, that is,

$$y(v) \in \arg\max_{y_1, \cdots, y_n} \sum_{i \in N} v_i(y_i). \tag{21.6.50}$$
Given the value vector \( v \), the social welfare (social surplus) from an allocation \( y(v) \) is
\[
W(v) = \sum_{i \in N} v_i(y_i(v)).
\]  
(21.6.51)

The total welfare of individuals other than \( i \) from an allocation \( y(v) \) is
\[
W_{-i}(v) = \sum_{j \neq i} v_j(y_j(v)).
\]  
(21.6.52)

Now we define the VGG mechanism of combinatorial auctions, which is an efficient mechanism. Given the value vector \( v \), and the allocation rule \( y(v) \) as defined by (21.6.50), the transfer rule is
\[
t_i(v_i, v_{-i}) = \sum_{j \neq i} v_j(y(v)) - \sum_{j \neq i} v_j(y(v_{-i})) = W_{-i}(v) - W_{-i}(v_{-i}).
\]  
(21.6.53)

In fact, from Chapter 18, this mechanism is also called the pivotal or Clarke mechanism, which is a special case of a general VCG mechanism. We can interpret the transfers of the VCG mechanism defined by the formula (21.6.53) as the externality that bidder \( i \) exerts on the other bidders. When bidder \( i \) does not bid, the social allocation rule is \( y(v_{-i}) \). When bidder \( i \) bids, the social allocation rule is \( y(v) \), while the change in others’ welfare is characterized by (21.6.53). Like the usual VCG mechanism, the transfer \( t_i(v_i, v_{-i}) \) of bidder \( i \) is independent of \( v_i \). Thus, truth-telling is a weakly dominant strategy in the VCG mechanism. The following proposition describes the VCG mechanism under the combinatorial auctions.

**Proposition 21.6.1** Under combinatorial auction VCG mechanism, truth-telling is a weakly dominant strategy to every bidder. The VCG mechanism is an efficient auction when each bidder truthfully reports his own value type.

### 21.6.3 Properties of VCG Mechanism

VCG mechanism is an efficient combinatorial auction, under which it is a weakly dominant strategy of each bidder to report his true value. The VCG mechanism can strengthen the robustness of the conclusion of efficiency, as the efficiency of VCG mechanism does not depend on the distribution of the value types of participants so that it is ex post implementable. As we mentioned in Chapter 18, Green and Laffont (1979) and Holmstrom (1979) showed that, under relatively weak assumptions, VCG mechanism is the unique weakly dominant mechanism with truthful reporting, efficient outcomes, and zero utility by losing bidders who do not win the object (see Proposition 18.7.3).

However, this conclusion needs one extra assumption on the bidder’s value type, that is, smooth path connection. This assumption that is weaker
than differentiability used in the proof of Proposition 18.7.3 has been used in the proof of revenue equivalence earlier. Here’s a formal definition.

**Definition 21.6.3** The set of bidder’s valuation functions is said to be smoothly path connected if, for any two functions \( v_i(\cdot), \hat{v}_i(\cdot) \in \chi_i, \forall i \in N \), there is a family of valuation functions \( v_i(\cdot, s) \in \chi_i \) for \( s \in [0, 1] \), such that \( v_i(\cdot, 0) = v_i(\cdot), v_i(\cdot, 1) = \hat{v}_i(\cdot) \), \( v_i(\cdot, s) \) is differentiable in \( s \), and \( \int_0^1 \sup_{S \subseteq K} |\frac{\partial V_i(S, s)}{\partial s}| < \infty \).

Under the above conditions, Ausubel and Milgrom (2006) proved the uniqueness of the VCG mechanism.

**Proposition 21.6.2** If the set of all valuation functions is smoothly path connected and \( 0 \in \chi_i, \forall i \in N \), then the VCG mechanism is the unique direct revelation mechanism in which truthful reporting is a weakly dominant strategy equilibrium, the resulting equilibrium outcomes are efficient, and there are no transfers to losing bidders.

**Proof.** Given a value combinations \( v_{-i} \) of the other bidders and consider any mechanism satisfying the above assumptions. If bidder \( i \) reports \( \hat{v}_i = 0 \), then his VCG allocation is zero and his transfer is also zero. Suppose that bidder \( i \) reports some valuation function \( v_i(\cdot, 1) \) and let \( v_i(\cdot, 0) = 0 \), \( \{v_i(\cdot, s)| s \in [0, 1]\} \) be a family of smooth valuation functions. Denote the socially optimal allocation when bidder \( i \) reports \( v_i(\cdot, s) \) by \( y(s) \), and let \( U_i(s) = \max_{s'} \{v_i(y(s'), s) - t_i(s')\} \), where \( y_i(s') = y(v_i(\cdot, s'), v_{-i}) \) and \( t_i(s') = t_i(v_i(\cdot, s'), v_{-i}) \) is the allocation and transfer received by bidder \( i \) under the VCG mechanism. By the envelope theorem in integral form (Milgrom and Segal (2002)),

\[
U_i(1) - U_i(0) = \int_0^1 \frac{\partial v_i(y(s), s)}{\partial s} ds.
\]

Let \( \hat{t}(s) \) be the transfer made under any direct revelation mechanism for which truthful reporting is a dominant strategy equilibrium, the resulting equilibrium outcomes are always efficient, and there are no transfers by or to losing bidders.

Let \( \hat{U}_i(s) = \max_{s'} \{v_i(y(s'), s) + \hat{t}_i(s')\} \). By the envelope theorem,

\[
\hat{U}_i(1) - \hat{U}_i(0) = \int_0^1 \frac{\partial v_i(y(s), s)}{\partial s} ds = U_i(1) - U_i(0).
\]

Since \( U_i(0) = \hat{U}_i(0) = 0 \), we have

\[
v_i(y(1), 1) + t_i(1) = U_i(1) = \hat{U}_i(1) = v_i(y(1), 1) + \hat{t}_i(1).
\]

Hence, \( t_i(1) = \hat{t}_i(1) \), so the VCG mechanism is the direct revelation mechanism for which truthful reporting is a dominant strategy, the outcomes are efficient, and there are no payments by or to losing bidders. \( \square \)
Another important property of the VCG mechanism is that when the solution concept of implementation is Bayesian-Nash equilibrium, combined with the previous revenue equivalence theorem (21.6.1), we can obtain that the expected revenues under VCG mechanism are not less than that from any other efficient combinatorial auction that satisfies Bayesian incentive compatibility.

21.6.4 Defects of Combinatorial Auction VCG Mechanism

Despite the above desired properties of VCG mechanism, it is seldom adopted in practical multiple objects auctions. There are many reasons for this.

First, VCG mechanism is complicated. For instance, the US Federal Communications Commission’s auction of spectrum No.31 in 1994 contained 12 different frequency spectrums and bidders were required to report a value of possible combinations of \(2^{12} = 4,096\). This will not only entail a significant computational burden to the auctioneer, but as the number of items to be auctioned increases, the calculations required for the auction show an exponential growth.

Second, the VCG mechanism involves a considerable amount of private information, and such information disclosure may affect future bidding. Bidders may rationally be reluctant to report their true values, fearing that the information they revealed will later be used against them.

Third, the VCG mechanism sometimes may trigger controversy because two bidders may pay different prices for identical items. In addition to these issues mentioned above, we will also discuss VCG mechanism from the perspective of auction revenues: excessively low seller’s revenues, non-monotonicity of the seller’s revenues in the set of bidders and the bids, vulnerability to collusion by a coalition of bidders and vulnerability to shill bidding. This subsection mainly refers to Ausubel and Milgrom (2002, 2006).

We now discuss these defects of combinatorial VCG mechanism by a series of examples.

Example 21.6.1 Consider an auction of two items \(\{a, b\}\) to three bidders. The values of the three bidders are:

\[
\begin{align*}
  v_1 &= (v^a_1, v^b_1, v^{a,b}_1) = (0, 0, 2), \\
  v_2 &= (v^a_2, v^b_2, v^{a,b}_2) = (2, 2, 2), \\
  v_3 &= (v^a_3, v^b_3, v^{a,b}_3) = (2, 2, 2), \\
\end{align*}
\]

and \(v^\emptyset_i = 0, \forall i\). It is easy to see that these two items are complements to bidder 1 but substitutes to bidders 2 and 3. At this point, there are two efficient allocations: one is \((y_1, y_2, y_3) = (\emptyset, \{a\}, \{b\})\); the other is \((y_1, y_2, y_3) = (\emptyset, \{b\}, \{a\})\).
In the VCG mechanism, the transfers of bidders 2 and 3 are both zero regardless of which allocation is. The reason is that:

\[ t_2(v) = W_{-2}(v) - W_{-2}(v_{-2}) = 2 - 2 = 0. \]

The same conclusion applies symmetrically to \( t_3(v) = 0 \). Thus the total auctioneer’s revenue is zero in the multi-item VCG mechanism.

Notice that, in this example, if the auctioneer sells the two items as bound sale, and every bidder values the combined items \( \{a, b\} \) as 2, then the seller revenues are 2. The multi-item VCG mechanism brings excessively low seller revenue.

In this example, we also find that for multi-item VCG mechanism, the seller’s revenue is not an increasing function of the number of bidders.

**Example 21.6.2** The following example is a variant of Example 21.6.1. Suppose that bidder 3 does not participate in bidding, that is, only bidders 1 and 2 participate. Also consider the VCG mechanism where there are four possible efficient allocations and corresponding VCG transfer payments:

1. \((y_1, y_2) = (\{a, b\}, \emptyset), \text{ and } t_1 = -2, t_2 = 0;\)
2. \((y_1, y_2) = (\{a\}, \{b\}), \text{ and } t_1 = 0, t_2 = -2;\)
3. \((y_1, y_2) = (\{b\}, \{a\}), \text{ and } t_1 = 0, t_2 = -2;\)
4. \((y_1, y_2) = (\emptyset, \{a, b\}), \text{ and } t_1 = 0, t_2 = -2.\)

Under the VCG mechanism where only bidder 1 and bidder 2 participate, the seller’s revenue is 2. The more bidders may bring a less auction revenue.

In addition, the multi-item VCG mechanism is prone to bidder collusion. The following example shows the VCG mechanism’s collusion incentive.

**Example 21.6.3** Consider a variant of Example 21.6.1. Suppose that the value of bidder 1 is unchanged, but bidders 2 and 3’s value of the item is only 0.5. That is, the values of the three bidders of the item are:

\[ v_1 = (v_1^a, v_1^b, v_1^{\{a,b\}}) = (0, 0, 2), \]
\[ v_2 = (v_2^a, v_2^b, v_2^{\{a,b\}}) = (0.5, 0.5, 0.5), \]
\[ v_3 = (v_3^a, v_3^b, v_3^{\{a,b\}}) = (0.5, 0.5, 0.5). \]

Then the only efficient allocation is \((y_1, y_2, y_3) = (\{a, b\}, \emptyset, \emptyset)\). In the VCG mechanism, \(t_1 = -1, t_2 = t_3 = 0\), and the auction revenue is 1.

However, if bidders 2 and 3 collude, and the values they report are \(\hat{v}_2 = \hat{v}_3 = (2, 2, 2)\), according to the analysis of Example 21.6.1, the auction outcome becomes \((\emptyset, \{a\}, \{b\})\) or \((\emptyset, \{b\}, \{a\})\), and \(t_2 = t_3 = 0\). Thus, the VCG mechanism is not efficient under collusion, and the auctioneer’s revenue turns into zero again.
A multi-item VCG mechanism may not be a mechanism that can prevent shill bidding. The so-called “shill” in the auction means that the bidder participates in bidding by introducing a dummy bidder to obtain higher profits. Here, the dummy bidder is called the shill of the bidder. The example below shows that in a multi-item VCG mechanism, there will be a dummy bidder.

**Example 21.6.4** Consider a variant of Example 21.6.3. Suppose that there are two bidders for the auction of two items \{a, b\}. Bidder 1 has the same value as Example 21.6.3, and bidder 2 can be considered as the sum of bidder 2 and bidder 3 in Example 21.6.3. That is, the values of bidders 1 and 2 are

\[
\begin{align*}
v_1 &= (v^a_1, v^b_1, v^{a,b}_1) = (0, 0, 2), \\
v_2 &= (v^a_2, v^b_2, v^{a,b}_2) = (1, 1, 1).
\end{align*}
\]

In the VCG mechanism, the allocation rule is \((y_1, y_2) = (\{a, b\}, \emptyset)\), \(t_1 = -1, t_2 = 0\), and the auctioneer’s revenue is 1.

With a dummy bidder 3, suppose the values of bidder 2 and dummy bidder 3 are reported as \(v_2 = v_3 = (2, 2, 2)\). As in the case of Example 21.6.3, the auction allocation outcome become \((\emptyset, \{a\}, \{b\})\) or \((\emptyset, \{b\}, \{a\})\) under VCG mechanism, and \(t_2 = t_3 = 0\). Bidders can obtain higher profits through dummy bidder 3, but the auction allocation is no longer efficient, and at the same time the auctioneer’s revenue has been reduced to zero.

In addition, the VCG mechanism may prevent some efficient decisions. The following example reveals that the VCG mechanism will inhibit the adjustment of efficient organizational structure.

**Example 21.6.5** This example is a variant of Example 21.6.1. The values of three bidders are the same as Example 21.6.1. We can consider bidders 2 and 3 as two firms. If the two firms merge and assume the merged firm as bidder 4, and then suppose bidder 4 has the value of \(v_4 = (v^a_4, v^b_4, v^{a,b}_4) = (4+x, 4+x, 4+x)\). \(x\) can be seen as the benefits of the merger, it is clear that as long as \(x > 0\), the merger can produce efficiency improvement (to the value of the item). However, if the two firms merge, the merged firm gets the item in the VCG mechanism and needs to pay 2 at the same time. The merged firm’s net income is \(2 + x\). If two firms do not merge, bidders 2 and 3 can get a total of 4 net gains. Thus, if and only if \(x \geq 2\), the two firms will have an incentive to merge. This shows that VCG mechanism may distort the merger decision before the auction.

Note that in the above five examples, bidder 1’s value of the item is not substitutable. If the value of bidder 1 is also substitutable as that of bidders 2 and 3, e.g., the value of bidder 1 is changed to:

\[
v_1 = (v^a_1, v^b_1, v^{a,b}_1) = (1, 1, 2),
\]
the above discussion of VCG defects may change. In Example 21.6.1, we can verify that the auctioneer can get revenue 2 under the VCG mechanism. In the meantime, we can verify that the auctioneer’s revenue is an increasing function of the number of bidders in the cases from Examples 21.6.2 to 21.6.5, and the bidders have no conspiracy nor incentive to find a dummy bidder and will not distort the pre-auction merger decision. Thus, the substitutability of auction items is very important and affects the operational efficiency of the auction. When the assumption of substitutability of bidder’s value of items is no longer satisfied, we need to introduce other multi-item auctions. Ausubel and Milgrom (2002) devised a series of ascending combinatorial auctions. When the substitutability is satisfied, the ascending combinatorial auctions are similar to the VCG mechanism, and when the substitutability assumption is not satisfied, they have better properties.

21.6.5 Ascending Combinatorial Auction Mechanism

We now focus on the ascending combinatorial auction of Ausubel and Milgrom (2002). We first introduce the mechanism: ascending combinatorial bidding proceeds in multiple rounds, and in each round, bidders place bids in terms of a certain amount of money, say $c$, (i.e. the bids are discrete); in any round, say $s$, each bidder can place a bid $b^s_i(S)$ on any package $S \subseteq K$, $b^s_i$ is the bid vector of bidder $i$ in the round $s$, and within a round, all bids are placed simultaneously. At the end of a round, the auctioneer designates a set of provisionally winning bids that maximize the total revenue and the allocation corresponding to the set of winning bids: given that the bids of all bidders in the round $s$ is $b^s$, the allocation rule is $y^s = (y^s_1, \ldots, y^s_n)$ which corresponds to the winning bids designated by the auctioneer, so that $y^s$ is feasible and

$$y^s = \arg\max_y \sum_{i \in N} b^s_i(y_i). \quad (21.6.54)$$

Then if $y^s_i \neq \emptyset$, bidder $i$ will be the provisionally winning bidder in the round $s$. As long as there is a higher bid on a certain package in round $s + 1$, i.e. there is $b^{s+1}_i(S) > \max_{j \in N} b^s_j(S)$, the auctioneer will re-select a new set of provisionally winning bids and its corresponding allocation in round $s + 1$. If no higher bids on any package appear in round $s^* + 1$, the auction comes to an end in the round $s^*$. Let $y^{s*}$ denote the allocation that corresponds to the set of winning bids, then if $i$ is the winner, that is, $y^{s*}_i \neq \emptyset$, his expected utility is

$$u^{s*}_i = v_i(y^{s*}_i) - b^{s*}_i(y^{s*}_i).$$

If $i$ is not the winner, his utility is zero.

The ascending combinatorial auction of Ausubel and Milgrom (2002) uses proxy bidding, being a direct mechanism in which each bidder submits a value vector to a proxy agent who then bids in the bidder’s interest.
21.6. COMBINATORIAL AUCTIONS FOR HETEROGENEOUS OBJECTS WITH PRIVATE VALUES

The goal of the ascending proxy auction of Ausubel and Milgrom (2002) is to ensure that each bidder has no incentive to lie to the proxy agent. The following is the illustration of the operation of the ascending proxy auction mechanism.

Suppose that the true value of bidder $i$ is $v_i$ and the value reported to the proxy agent is $\hat{v}_i$. 

1. In round 0, the auctioneer sets an initial price vector as the bidding price vector $b_0^0$; at the same time, let $b_0^0(S) = b_0^0$, $\forall i \in N, S \subseteq K$, the provisional winner of this round be the seller, and $y_0^0 = K, y_0^i = \emptyset, \forall i \in N$. Here superscript 0 denotes the seller. Define $\hat{N} = N \cup \{0\}$, i.e. the set of all the bidders and the seller.

2. Let $b_{s-1}^i$ be bidder $i$’s bidding price vector in round $s - 1$. The minimum bid that $i$ can place in round $s$ is $b_s^i(S) = b_{s-1}^i(S) + \epsilon$ if $S \neq S_{s-1}^i$; otherwise, $b_s^i(S) = b_{s-1}^i(S)$. The $\epsilon$ here is the minimum bidding increment required if the bidder wants to change the provisional allocation of the previous round.

The proxy $i$ determines the optimal package $\hat{y}_i$ based on $\hat{y}_i$ given by

$$\hat{y}_i = \arg\max_S v_i(S) - b_s^i(S).$$

The proxy $i$ then places a bid of $\beta_s^i(S|\hat{y}_i)$, that is

$$\beta_s^i(S|\hat{y}_i) = \begin{cases} b_s^i(S), & \text{if } S = \hat{y}_i \text{ and } \hat{y}_i(S) \geq b_s^i(S); \\ b_{s-1}^i(S), & \text{otherwise} \end{cases}$$

At the end of the round, the auctioneer chooses a set of provisionally winning bids that maximize the total revenue, that is, (21.6.54), and the allocation $y_t$ that corresponds to the set of winning bids.

If no proxy places a new bid in the next round following $s^*$, the winner of round $s^*$ will pay for the package according to the bidding price placed by his proxy. If next round a proxy places a new bid, then repeat (2) until the auction ends. As the bid price is ascending, the auction will end in finite rounds. It is noted that in some cases, some auction items are held by the seller unless the auctioneer sets the initial bid price vector as zero.

When the auction items are substitutable for each other, the ascending proxy auction has the same result as the VCG mechanism. Before the formal proof of this conclusion, we need to introduce a definition and a lemma.

Given a value vector $v$ and a coalition of some buyers and seller $I \subseteq N$, define the coalitional valuation function

$$w(I) = \max_y \sum_{i \in I} v_i(y_i). \tag{21.6.55}$$

to be the maximum surplus attainable by the coalition $I$ from an optimal allocation of all the objects.
Definition 21.6.4 The coalitional valuation function is bidder-submodular if for all \( \{0\} \in I \subseteq J \subseteq \tilde{N} \) and \( i \notin J \), and all coalitions satisfy

\[
\begin{align*}
    w(I \cup i) - w(I) & \geq w(J \cup i) - w(J).
\end{align*}
\]

Describing the coalitional valuation function as bidder-submodular means that the surplus marginal contribution of each individual to the coalition decreases as the set of the coalition gets larger. In the definition, symbol \( \{0\} \) denotes the auctioneer. If the coalition does not include the buyer, it is clear that the value of the coalition is zero.

The property of coalitional valuation function in the auction is closely related to the relationship between the goods, and the following is a lemma proved by Ausubel and Milgrom (2002).

Lemma 21.6.1 If the items are substitutes for all the bidders, the valuation function of coalition is submodular.

The proof of the lemma is relatively complicated, and readers interested can refer to Ausubel and Milgrom (2002).

Next, we use the coalitional valuation function to characterize the expected utility of the bidder in a VCG auction. Given the bidder’s value \( v \) of items, based on the definition of \( W(\cdot) \), i.e. the equation (21.6.51), we have \( w(\tilde{N}) = W(v) \) and \( w(\tilde{N}\setminus i) = W(v_{-i}) \). By the definition of transfers in the VCG mechanism, that is, the equation (21.6.53), the expected utility of bidder \( i \in N \) under the VCG mechanism can be written as

\[
\bar{u}_i = w(\tilde{N}) - w(\tilde{N}\setminus i). \tag{21.6.56}
\]

Ausubel and Milgrom (2002) show that the result of ascending proxy auction is equivalent to that of the VCG mechanism when the items are substitutes.

As discussed earlier, the so-called ex post Nash equilibrium is such a strategic combination that when all participants in the game know the private information of other participant types, the participants will not change their strategic choices. Compared with Bayesian–Nash equilibrium, the ex post Nash equilibrium is a more robust equilibrium that does not depend on the distribution of bidders’ types. Obviously, the ex post Nash equilibrium is Bayesian–Nash equilibrium, but the reverse is not necessarily true.

We then have the following the proposition. To prove it, we first need to establish that truthful reporting in the ascending proxy auction will lead to VCG outcomes, and then show that truthful reporting is indeed an ex post equilibrium.

Proposition 21.6.3 When objects to be auctioned are substitutes for all bidders, truthful reporting to the proxy is an ex post equilibrium of the ascending proxy auction, in which the equilibrium outcome is the same as in the VCG mechanism.
PROOF. Suppose that all bidders report their values truthfully to the proxy, we claim that the utility of the ascending proxy auction is the same as in the VCG mechanism. Let \( s^* \) be the round where the ascending proxy auction ends, and the utility obtained by each bidder is \( u_i^{s^*} \). We claim that for any \( i \), there is \( u_i^{s^*} \geq \bar{u}_i \) (up to \( \epsilon \)). Suppose by way of contradiction that in some round \( s \leq s^* \), \( u_i^s < \bar{u}_i \). Then bidder \( i \) must be a provisional winner of round \( s \).

Let \( \hat{I} = I \setminus 0 \) be the set of provisional winners in round \( s \). If \( i \notin I \), then the seller’s provisional revenue in round \( s \) is

\[
w(I) - \sum_{j \in I} u_j^s < w(I) - \sum_{j \in I} u_j^s + \bar{u}_i - u_i^s
\]

\[
= w(I) - \sum_{j \in I \cup i} u_j^s + w(\tilde{N}) - w(\tilde{N} \setminus i)
\]

\[
\leq w(I) - \sum_{j \in I \cup i} u_j^s + w(I \cup i) - w(I)
\]

\[
= w(I \cup i) - \sum_{j \in I \cup i} u_j^s.
\]

Here the inequality in the third line follows Lemma 21.6.1. Thus, for the auctioneer, including \( i \) in the set of provisional winners in round \( s \) will generate larger provisional revenue. However, in round \( s \), the utility of bidder \( u_j^s < \bar{u}_i \) shows that the proxy of bidder \( i \) does not bid in a way that maximizes bidder \( i \)'s interest, unless \( \bar{u}_i - u_i^s < \epsilon \). Hence, we have that for all \( i \), \( u_i^{s^*} \geq \bar{u}_i \) (up to \( \epsilon \)).

Now suppose that there is a bidder \( i \in N \) such that \( u_i^{s^*} > \bar{u}_i \). Then

\[
w(\tilde{N}) = u_0^{s^*} + u_i^{s^*} + \sum_{j \neq i} u_j^{s^*}
\]

\[
> u_0^{s^*} + w(\tilde{N}) - w(\tilde{N} \setminus i) + \sum_{j \neq i} u_j^{s^*},
\]

which implies that

\[
u_0^{s^*} + \sum_{j \neq i} u_j^{s^*} < w(\tilde{N} \setminus i).
\]

(21.6.57)
On the other hand, the seller’s revenue
\[ u_0^* = \max_y \sum_{j \in N} b_j^*(y_j | v_j) \]
\[ = \max_y \sum_{j \in N} \max(v_j(y_j) - u_j^*), 0) \]
\[ = \max_{I \subseteq N} \sum_{j \in I} (v_j(y_j) - u_j^*) \]
\[ = \max_{I \subseteq N} \sum_{j \in I} (v_j(y_j) - u_j^*) \]
\[ = \max_{I \subseteq N} (w(I) - \sum_{j \in I} u_j^*). \]

and thus, in particular, for \( I = \tilde{N} \setminus i \), this implies that
\[ u_0^* + \sum_{j \neq i} u_j^* \geq w(\tilde{N} \setminus i), \quad \text{(21.6.58)} \]

which contradicts to (21.6.57). Thus we have argued that with truthful reporting, ascending proxy mechanism will have the same outcome as the VCG mechanism.

In order to show that truthful reporting constitutes an ex post equilibrium, consider bidder \( i \), and suppose that all bidders \( j \neq i \) report truthfully. Then we discuss the reporting incentives of bidder \( i \). We know from (21.6.58) that independent of what bidder \( i \) submits as his value vector, the seller’s revenue satisfies
\[ u_0^* > w(\tilde{N} \setminus i) - \sum_{j \neq i} u_j^*. \]

The right-hand side is a lower bound on the seller’s revenue because it can be obtained by ignoring bidder \( i \) and including all other bidders in the set of provisionally winning bidders. At the same time, since the total payoff of all the participants (bidders and the seller) can never exceed \( w(\hat{N}) \), which means
\[ u_0^* \leq w(\hat{N}) - \sum_{j \in N} u_j^*. \]

For bidder \( i \), his arbitrary report will not make his utility exceed \( w(\tilde{N}) - w(\tilde{N} \setminus i) = \bar{u}_i \). The right hand of the equation is the expected utility obtained in ascending proxy auction if bidder \( i \) chooses to report truthfully, so bidder \( i \) will have the incentive to report truthfully and thus truthful reporting constitutes an ex post equilibrium. \( \square \)
21.6.6 Sun-Yang Auction Mechanism for Multiple Complements

Ausubel and Milgrom (2002, 2006) further discussed when the auction items are not substitutes for each other, the ascending proxy auction will be better than the VCG mechanism and can overcome the drawbacks and problems faced by the VCG mechanism mentioned earlier.

The ascending combinatorial auction is similar to the dynamic adjustment mechanism in general equilibrium. When the goods are over-demanded, the price will rise. The slight difference is that in ascending combinatorial auctions there is only one-way price change (i.e. increment), the equilibrium price of ascending combinatorial auction is actually the general equilibrium price (of discrete commodity), and the demand corresponding to the aggregate bidding prices is consistent with the supply corresponding to the auctioneer’s optimal income. However, when the items are complementary, the ascending proxy mechanism will face an exposure problem, that is, there may be no general equilibrium price. We now discuss the exposure problem in multiple objects auctions by an example (adapted from Milgrom (2000, p. 257, Footnote 12)).

Example 21.6.6 (Exposure problem in auction for complements) Suppose that the set of objects is \( K = \{a, b, c\} \), there are three bidders \( N = \{1, 2, 3\} \), the objects are mutual complements for all bidders (valuation function is superadditivity), and the values attached by the bidders to these objects (package) are

\[
\begin{align*}
  v_1(a) &= 1; \\
  v_1(b) &= 1; \\
  v_1(c) &= 0; \\
  v_1(\{a, b\}) &= 3; \\
  v_2(a) &= 0; \\
  v_2(b) &= 0.9; \\
  v_2(c) &= 1; \\
  v_2(\{a, b\}) &= 1; \\
  v_3(a) &= 1, \\
  v_3(b) &= 0; \\
  v_3(c) &= 1; \\
  v_3(\{a, b\}) &= 1; \\
  v_1(\{a, c\}) &= 1; \\
  v_1(\{b, c\}) &= 1; \\
  v_1(\{a, b, c\}) &= 3; \\
  v_2(\{a, c\}) &= 1; \\
  v_2(\{b, c\}) &= 3; \\
  v_2(\{a, b, c\}) &= 3; \\
  v_3(\{a, c\}) &= 3.5; \\
  v_3(\{b, c\}) &= 1; \\
  v_3(\{a, b, c\}) &= 3.5.
\end{align*}
\]

If a competitive equilibrium does exist, its allocation would be efficient. In this example, there is an efficient allocation

\[
y_1 = \{b\}, y_2 = \emptyset, y_3 = \{a, c\}.
\]

To support the efficient allocation, the prices must satisfy

\[
p^b \leq 1, p^a + p^b \geq 3, p^b + p^c \geq 3, p^a + p^c \leq 3.5.
\]

However, these together imply that \( p^a + 2p^b + p^c \geq 6 \) and \( p^b \leq 1 \), which in turn implies that \( p^a + p^c \geq 4 \), contradicting to \( p^a + p^c \leq 3.5 \). So, there is no competitive equilibrium.

The problem associated with the existence of general equilibrium of multi-item auctions is called the exposure problem. The new problems appear when auction items are complements. If bidders submit their bids
accoring to actual needs, they may be exposed to a possible risk. When the bidder is included in the list of provisional winners by the auctioneer, he will face the risk of not getting some complementary items, and he will not be willing to pay for the other complementary items according to the bidding price. It is quite difficult to design an efficient auction for complements.

In the following, we briefly discusses an auction for multiple complements. Sun and Yang (JPE, 2014) constructed a dynamic auction to solve the design of an efficient auction for multiple complements. This dynamic mechanism is called Sun–Yang auction.

Sun–Yang auction introduces the concept of nonlinear general equilibrium, where the general equilibrium of nonlinearity exists when the items are complementary to all the bidders, although the general linear equilibrium may not exist.

Suppose that there are a set of objects $K = \{a, b, \cdots \}$ and a group of bidders $N = \{1, 2, \cdots, n\}$. Bidder $i \in N$ attaches a monetary value to the objects, which is $u_i : 2^K \to \mathbb{Z}_+$, where $2^K$ denotes the set of all subsets of items $K$ and $\mathbb{Z}_+$ is the set of all nonnegative integers. The seller (denoted by superscript 0) has a reserve price function $u_0 : 2^K \to \mathbb{Z}_+$ with $u_i(\emptyset) = 0, \forall i \in N \cup \{0\}$. Let $\bar{N} = N \cup 0$ represent the set of all agents.

Suppose that the bidder’s utility function is quasi-linear and satisfies the superadditivity for the valuation function of the item. Here we first introduce the concept of non-linear general equilibrium.

A price function $p : 2^K \to \mathbb{R}_+$, with $p(\emptyset) = 0$, is feasible if $p(S) \geq u_0(S), \forall S \subseteq K$. For the given pricing function $p$, bidder’s demand correspondence (which may be multi-valued) is

$$ D_i(p) = \text{argmax}_{S \subseteq K} u_i(S) - p(S). $$

Let $\pi = \{\pi^1, \cdots, \pi^m\}, m \leq n$ be a partition of $K$ and $y = (y_0, y_1, \cdots, y_n)$ be a feasible allocation, that is, for any $y_0$, or $y_i = \emptyset$, or $y_i \in \pi, y_i \cap y_j = \emptyset, i \neq j$, $\bigcup_{i \in \bar{N}} y_i = K$. The corresponding partition of allocation $y$ is $\pi = \{y_i, i \in \bar{N} : y_i \neq \emptyset\}$. Given the price function $p$, the seller’s supply correspondence is

$$ Y(p) = \text{argmax}_\pi \sum_{S \in \pi} p(S). $$

An allocation $y^* = (y_0^*, y_1^*, \cdots, y_n^*)$ is efficient, if for every allocation $y$, we have

$$ \sum_{i \in \bar{N}} u_i(y_i^*) \geq \sum_{i \in \bar{N}} u_i(y_i). $$

**Definition 21.6.5 (Nonlinear general equilibrium)** A nonlinear general equilibrium consists of a price function $p^*$ and an allocation $y^*$ such that

1. for the seller, $y^* \in Y(p^*)$;
2. for every bidder $i \in N$, $y_i^* \in D_i(p^*)$. 

In the above definition, similar to general equilibrium, condition (1) characterizes the supply under the nonlinear equilibrium price function; condition (2) depicts the demand under the nonlinear equilibrium price function, in which at the equilibrium price $p^*$, the quantity supplied equals the quantity demanded.

The following example on complements (from Sun and Yang) shows that a nonlinear general equilibrium exists, but the general equilibrium does not exist.

**Example 21.6.7** Suppose that objects $K = \{a, b, c\}$ are complements for three bidders $N = \{1, 2, 3\}$. Bidders’ values and seller’s reserve prices are given as follows:

\[
\begin{align*}
n_1(a) &= 2; & n_1(b) &= 2; & n_1(c) &= 0; & n_1(\{a, b\}) &= 7; \\
n_2(a) &= 2; & n_2(b) &= 0; & n_2(c) &= 2; & n_2(\{a, b\}) &= 3; \\
n_3(a) &= 0; & n_3(b) &= 2; & n_3(c) &= 2; & n_3(\{a, b\}) &= 4; \\
u_0(a) &= 1; & u_0(b) &= 2; & u_0(c) &= 1; & u_0(\{a, b\}) &= 3; \\

v_1(\{a, c\}) &= 4; & v_1(\{b, c\}) &= 4; & v_1(\{a, b, c\}) &= 7; \\
v_2(\{a, c\}) &= 6; & v_2(\{b, c\}) &= 3; & v_2(\{a, b, c\}) &= 6; \\
v_3(\{a, c\}) &= 3; & v_3(\{b, c\}) &= 6; & v_3(\{a, b, c\}) &= 7; \\
u_0(\{a, c\}) &= 3; & u_0(\{b, c\}) &= 4; & u_0(\{a, b, c\}) &= 5.
\end{align*}
\]

In this example, there are two efficient allocations: $y_1 = \{a, b\}, y_2 = \{c\}, y_3 = \emptyset$, and $y^1 = \{a, b\}, y^2 = \emptyset, y^3 = \{c\}$. At the same time, there is a nonlinear general equilibrium price: $p^*(a) = p^*(b) = p^*(c) = 2, p^*(\{a, b\}) = p^*(\{a, c\}) = p^*(\{b, c\}) = 6$, and $p^*(\{a, b, c\}) = 7$.

On the demand side, $D_1(p^*) = \{a, b\}, \{c\} \in D_2(p^*), \emptyset \in D_3(p^*)$; on the supply side, $(\{a, b\}, \{c\}, \emptyset) \in Y(p^*)$. Also, price $p^*$ and $y^*$ also constitute a nonlinear general equilibrium.

However, there is no general equilibrium under the linear price. Suppose there is a price vector $p(a), p(b), p(c)$. If the price $p^*$ and the allocation $y^*$ consist an equilibrium, then:

for the seller, we must have $p(a) \geq 1, p(b) \geq 2, p(c) \geq 1$;

for bidder 1, we must have: $p(a) + p(b) \leq 7$;

for bidder 2, we must have: $p(c) \leq 2, p(a) + p(c) \geq 6$;

for bidder 3, we must have: $p(c) \geq 2, p(b) + p(c) \geq 6$.

From $p(c) = 2$, combining the inequalities $p(a) + p(c) \geq 6$ and $p(b) + p(c) \geq 6$, we have $p(a) \geq 4$ and $p(b) \geq 4$, which contradicts $p(a) + p(b) \leq 7$.

Sun and Yang (2014) show that if objects are complementary to all bidders and the bidders have quasi-linear utility functions, there always is a nonlinear general equilibrium. For readers who are interested in the proof...
of the theorem, see the Proof Appendix of Sun and Yang’s Working Paper (2012).

Sun and Yang (2014) construct an ascending auction when bidders choose to bid truthfully (which will be defined below). An elaborate dynamic auction is then constructed to ensure that each bidder has the incentive to bid truthfully, and more precisely speaking, truthful bidding was a refined ex post Nash equilibrium.

**Definition 21.6.6** In a dynamic auction, we say that bidder $i$ bids **truthfully**, if at every round and for any price function $p^t(\cdot)$, bidder $i$ reports demand

$$d^t_i \in D_i(p^t) = \arg\max_{S \subseteq K \{ u_i(S) - p^t(S) \}}$$

with $d^t_i = \emptyset$ when $\emptyset \in D_i(p^t)$.

When bidders choose to bid truthfully, Sun and Yang construct the following ascending auction, by which the final result of the auction is a non-linear general equilibrium.

**Definition 21.6.7 (Ascending auction under truthful reporting)** There are several steps:

1. The seller announces her reserve price $u_0$ and the auctioneer sets the initial pricing functions $p^0 = u_0$.
2. In each round $t = 0,1,2,\cdots$, given the the initial pricing functions $p^t$ at the round $t$, the auctioneer chooses a supply set $y_t \in Y(p^t)$, and every bidder $i$ reports his demand $d^t_i \in D^t_i(p(t))$ against prices $p^t$. When there is at least a pair of $i,j$ with $i \neq j$ such that $d^t_i = d^t_j = S, S \subseteq K$ or there is only one $i$ such that $d^t_i = S$ but no $k$ that makes $S = y^t_k$, then we call this bundle $S$ over demanded. For any over demanded bundle $S$, the auctioneer adjusts the price function through the formulas $p^{t+1}(S) = p^t(S) + 1$ and repeats (2) into round $t+1$. If no bundle at the $t$ round is overly demanded, written as $t^*$.
3. In round $t^*$, the auctioneer allocates $d^{t^*}_i = y^{t^*}_i$ to bidder $i$ and asks him to pay the price $p^{t^*}(y^{t^*}_i)$. If there is no $i$ of bundle $S$ such that $S = d^{t^*}_i$ and $p^{t^*}(S) = u_0(S)$, the bundle $S$ is left to the seller. If $p^{t^*}(S) > u_0(S)$, the bundle $S$ is left to the bidder who finally gives up the choice of bundle $S$. Then the auction ends.

Here we show that the results of the ascending auction under truthful reporting are non-linear general equilibria.
Given

\[ p^i = (p^i(a), p^i(b), p^i(c), p^i(\{a, b\}), p^i(\{a, c\}), p^i(\{b, c\}), p^i(\{a, b, c\})) , \]

in the initial round, the price, supply and demand are as follows:

\[ p^0 = (1, 2, 1, 3, 3, 4, 5), y_0 = \{\{b, c\}, a\}, d^0_1 = \{a, b\}, d^0_2 = \{a, c\}, d^0_3 = \{a, b, c\}. \]

In rounds 1~6, the price of supply and demand are as follows:

\[ p^1 = (1, 2, 1, 4, 4, 4, 6); \quad \pi^1 = \{\{a, c\}, b\}; \quad d^1_1 = \{a, b\}, d^1_2 = \{a, c\}, d^1_3 = \{b, c\}; \]
\[ p^2 = (1, 2, 1, 5, 4, 5, 6); \quad \pi^2 = \{\{a, b\}, c\}; \quad d^2_1 = \{a, b\}, d^2_2 = \{a, c\}, d^2_3 = \{b, c\}; \]
\[ p^3 = (1, 2, 1, 5, 5, 6, 6); \quad \pi^3 = \{\{a, c\}, b\}; \quad d^3_1 = \{a, b\}, d^3_2 = \{a, c\}, d^3_3 = \{a, b, c\}; \]
\[ p^4 = (1, 2, 1, 6, 5, 6, 7); \quad \pi^4 = \{\{a, b\}, c\}; \quad d^4_1 = \{a, b\}, d^4_2 = \{a, c\}, d^4_3 = \{c\}; \]
\[ p^5 = (1, 2, 2, 6, 6, 6, 7); \quad \pi^5 = \{\{a, c\}, b\}; \quad d^5_1 = \{a\}, d^5_2 = \{a\}, d^5_3 = \emptyset; \]
\[ p^6 = (2, 2, 2, 6, 6, 6, 7); \quad \pi^6 = \{\{a, b\}, c\}; \quad d^6_1 = \{a, b\}, d^6_2 = \emptyset, d^6_3 = \{c\}. \]

Then no bundle is overdemanded, \( t_\ast = 6 \), and the equilibrium prices are \( p^* = (2, 2, 2, 6, 6, 6, 7) \). Bidder 1 gets \( \{a, b\} \) and pays 6. Bidder 3 gets \( \{c\} \) and pays 2.

Comparing with Example 21.6.7, the results of an ascending auction are non-linear general equilibria.

Sun and Yang (2014) build a dynamic Sun-Yang auction that induces each bidder to report truthfully. Since this mechanism involves much technical details and symbols, here we only briefly discuss the basic structure of it.

In the incentive-compatible dynamic auction, Sun and Yang construct \( n + 1 \) markets and each corresponds to two price functions: the first price function and the second price function. All participants, including the seller, have different prices for each market at each round. Let \( p^\Delta_0 \) and \( p^0 \) denote the sellers’ the first price function and the second price function in each round \( t \) of market \( M_0 \) with the set \( N \) of bidders. Let \( p^\Delta_i \) and \( p^i \) denote bidder \( i \)’s the first price function and the second price function in each round \( t \) of market \( M_i \) that stands for the market without bidder \( i \).

Each bidder \( i \) reports a demand bundle against the second price function \( p^i \), and the auctioneer announces a revenue-maximizing supply set of each market \( n + 1 \) at the first price function \( p^{-k}(k \in \hat{N}) \). For bidders, because of different prices in each market, the prices faced in making demand decisions are also different. When the market overly demands certain bundle, the seller adjusts the price depending on how much the bidders influence the demand for the bundle, so the adjustment of each market and each bidder is different. If bidder \( i \) is the crucial demander of a bundle (i.e. the bidder who reports the highest price), bidder \( i \) will face a price increase of 1 unit while the price of the others will be unchanged. This way of price
adjustment can internalize the externality of price changing by bidder’s demand report.

Sun and Yang (2014) show that under this dynamic mechanism, the bidder’s payment is a generalized VCG payment. Sun–Yang auction has many elaborate designs: the over demand is divided into first overdemand and second overly demanded; the price adjustment process is bidirectional, raising the price when overly demanded and decreasing the price when over supplied; the bidder is allowed to withdraw some of their past bids, but will be punished if they nullify their bids for too many times; the Sun–Yang auction can always end within a finite time and so on. Interested in more details, please refer to their paper.

The Sun–Yang auction has many favorable features. For example, it requires bidder to report only one demand against the market price so it is privacy preserving and informationally efficient; it can solve the exposure problem easily that is caused by complementarity; it tolerates mistakes made by bidders in the bidding process and so on.

For the exposure problem, a more complex situation is that a bundle of objects can be complements to one bidder but substitutes to another. In this case, the nonlinear general equilibrium may not exist either. Sun and Yang (2009, 2015) put forward some other efficient auctions according to the substitution and complementarity of items.

21.7 Biographies

21.7.1 William Vickrey

William Vickrey (1914-1996) was the pioneer of auction theory and had made profound contributions in information economics, incentive theory and so on. In 1996, Vickrey and James Mirrlees were awarded the Nobel Memorial Prize in Economic Sciences for their great contributions to the economic theory of incentives under asymmetric information. Unfortunately, Professor Vickrey died three days after winning the Nobel Prize when he was on the way to a conference.

At the end of the 1940s, Vickrey started to become well-known in academia, especially in the field of optimal taxation. His doctoral dissertation, “Agenda for Progressive Taxation”, was reprinted as an “economic classic” in 1972. Vickrey was well-known for the practice of economic theory. He was not only outstanding in taxation, transportation, public services and pricing, but also well-known for his pioneering research on incentive theory and auction theory. His deep thoughts on incentive theory in his early papers did not obtain the attention of economics community until the 1970s, which greatly stimulated the development of auction theory, information economics, incentive theory and other fields.
As mentioned in the introduction to this chapter, although the practice of auctions has lasted for thousands of years and even some auctions have caused a huge historical influence, the research on auction based on economic theory did not begin until the 1960s. Vickrey began to study specific market mechanisms of auction in the 1960s. In 1960, Vickrey published an article in *The Quarterly Journal of Economics* to discuss public pricing and sealed bidding strategies. In 1961, his classic paper entitled “Counter speculation, auctions, and competitive sealed tenders”, which was published in the *The Journal of Finance*, discussed the relationship between the auction rules and public pricing. In this paper he also analyzed private information about auctions, strategic pricing and other issues. Vickrey discussed the four auction forms most widely used in single-item auctions as we described in this chapter and obtained the landmark theory of auction—the “Revenue Equivalence Theorem”. That is, if all bidders have independent private values for the auctioned items, then all four of the standard single-item auctions generate the same expected seller’s revenue. In addition, Vickrey invented the second-price sealed-bid auction which provides incentives for all bidders to bid truthfully, that is, they will bid the price at which they value the item. Indeed, the second-price auction is a special case of the general Vickrey-Clarke-Groves mechanism. Although Vickrey did not consider the efficient provision of items from the perspective of the mechanism design approach, VCG mechanism is the unique incentive compatible mechanism that can provide public goods efficiently. In other words, the Vickrey-Clarke-Groves mechanism is the only mechanism that truthfully implements socially efficient decision rules (the maximization of the sum of individual surpluses) in dominant strategy equilibrium.

These two papers are path breaking about auction and built the foundation for this research area. Vickrey’s research about bidding methods solved the problem of how to allocate resources most efficiently under incomplete information or asymmetric information, not necessarily limited to auction problems. His research created a precedent for the study of information economics. Under incomplete information or asymmetric information, those with more information can strategically use their information to obtain benefits. And information economics needs to explore how to design contracts or mechanisms to deal with various incentive and regulatory issues. Vickrey’s research about bidding and pricing had brought lots of relevant studies to help people understand better on such problems as observed in the insurance market, the credit market, internal organizations of firms, salary structures, tax systems, social insurance, and political organizations.

Vickrey also made great contributions to the optimal pricing theory of public utilities and transportation. His research areas include marginal cost pricing, responsive pricing, urban congestion pricing, simulated futures markets, inflation effects on utility adjustment and pricing methods, etc.
Most of Vickrey’s researches were about specific market mechanisms, but his researches had a great value for us to understand more general market mechanisms and establish a general theory of market microstructure. The Nobel Memorial Prize in Economic Sciences is also an acknowledgement of his theory.

21.7.2 Paul Milgrom

Paul Milgrom (1948- ) is a professor at Stanford University. He is also a fellow of the American Academy of Arts and Sciences and also a fellow of the National Academy of Sciences. Milgrom has a great reputation for making pioneering contributions to auction theory, pricing strategy and mechanism design theory.

Paul Milgrom was born in Detroit, Michigan, the United States in 1948. He received a bachelor of Mathematics from the University of Michigan in 1970, and a master degree of statistics and a Ph.D. degree in Business from Stanford University in 1978 and 1979, respectively. After graduation, he first taught at the Kellogg School of Management at Northwestern University from 1979 to 1983, and then taught at Yale University from 1983 to 1987. Milgrom has been a professor of economics at Stanford University since 1987 and the Shirley R. and Leonard W. Ely Jr. Professor of Humanities and Sciences since 1993. While studying at Stanford University, Milgrom studied under the guidance of Wilson and completed his Ph.D. dissertation on auction theory. Since then, he has formed an indissoluble relationship with the auction theory. His proposals of “interdependent value” and “the design of the simultaneous ascending auctions” have greatly enriched the content of auction theory. His books entitled *The Structure of Information in Competitive Bidding* and *Putting Auction Theory to Work* have become classical works in this field. In addition to his deep research on auction theory, Milgrom was also devoted to applying theory to practice. His proud was to successfully design an auction for the US telecommunications market. Besides auction theory, he has published many papers in areas of incentives in organizations, mathematical economics, game theory, network economy, pricing strategy and actuarial science.

After Vickrey’s landmark conclusion on auction theory–Revenue Equivalence Theorem, a large number of scholars began to focus on auction theory, including Roger B. Myerson who was awarded the Nobel Memorial Prize in Economic Sciences in 2007. Myerson used the newly developed mechanism design theory to restudy the auction theory. And he generalized the theory of Vickrey based on mechanism design theory. Under a series of assumptions that bidder’s value of an item are independent and that bidders only care about their own expected returns, he proved that all possible standard auctions would bring auctioneer the same expected revenue. Obviously, this conclusion surpassed previous research such as
those of Vickrey and among others who focused on specific forms of auction, which hence enabled the study of all possible auctions and progressed the auction theory a great step further.

For an auctioneer, since all possible auction formats bring him the same expected revenue, can he arbitrarily choose a specific auction format? Unfortunately, although Myerson’s conclusion is very beautiful in theory, the preconditions for his conclusion to hold are very strict so that it is almost impossible to meet these assumptions in reality, which makes Myerson’s revenue equivalence theorem no longer valid, especially when the a bidder’s value to the auction item depends not only on himself, but also on the values of other bidders. Milgrom took the lead in the research of auctions with interdependent values. In 1982, Milgrom wrote a paper with Weber in which they constructed an analytical framework that deals with information, prices, and auctioneer revenue when there are interdependent values. According to their observations of auction practices, they proposed that bidders’ values may be interdependent. And a bidder’s higher value to the auction item may also easily increase the values of other participants.

Milgrom has a wide range of research interests, including auction design for the real world, organizational economics, bounded rationality, and economic history. His book entitled *Economics, Organization, and Management* published in 1992 with John Roberts is a well-known textbook. In 1993, Milgrom was commissioned by the former U.S. President Clinton to design the auction protocol for the Federal Communications Commission (FCC) and completed the major design of the auction. Because of his contribution, the FCC auction made a great success. As such, Milgrom has become one of the most well-known figures in the world of auctions and industrial economics. His team also designed channel and public auctions for Germany, Mexico, Canada and many other countries.

21.8 Exercises

**Exercise 21.1 (All-Pay Auctions under Uniform Distribution)** An all-pay auction is similar to the first-price sealed bid auction. The difference is that bidders are required to pay their bids to the seller regardless of winning or losing. Suppose that there are \( n \) bidders. The distribution of each bidder’s value is independent and identical on \([0, 1]\).

1. For only two bidders whose values are a uniform distribution over the unit interval, characterize the symmetric equilibrium under the all-pay auction in a symmetric environment. (Hint: we can guess the equilibrium price function as an increasing quadratic function.)

2. When there are \( n \) bidders, derive the corresponding symmetric equilibrium bidding function. (Hint: The equilibrium bidding function
must have the form of $\beta(\theta_i) = \alpha(\theta_i)^n$ in this case.)

3. Analyze the relationship between the equilibrium bidding function and the equilibrium bidding function under the first-price sealed-bid auction.

**Exercise 21.2 (First-Price Auctions under Binary Distribution)** Consider a first-price auction in a symmetric environment where prices are distributed according to a binary distribution. That is, bidder $i$’s value satisfies $\theta_i \in \{ \theta_l, \theta_h \}$, where $0 \leq \theta_l < \theta_h < \infty$. For any bidder $i$, the ex-ante probability satisfies $\Pr(\theta_i = \theta_h) = \alpha$. Consider the case with only two bidders.

1. Characterize the equilibrium of this first-price auction.

2. When the values follow the binary distribution, does the Revenue Equivalence Theorem between the first-price auction and the second-price auction still hold true?

**Exercise 21.3** Assume that there are $n$ buyers in the first-price auction, whose private values are uniformly distributed on $[0, 1]$. We further assume that all buyer’s utility function is $U(\theta) = \theta^c$, for $c < 1$.

1. Under what values of $c$, the buyer is risk-averse, risk-neutral, or risk-seeking?

2. Prove that the symmetric equilibrium $\gamma$ must satisfy the following differential equation:

   $$\gamma'(\theta) = \frac{n-1}{c} \frac{f(\theta)}{F(\theta)} (\theta - \gamma(\theta)).$$

3. Prove that the following strategy is a symmetric equilibrium:

   $$\gamma(\theta) = \theta - \int_0^\theta \frac{F^{\frac{n-1}{c}}(x)dx}{F^{\frac{n-1}{c}}}. $$

4. When $F$ is a uniform distribution on $[0, 1]$, calculate the symmetric equilibrium $\gamma$.

5. What is the functional form of the seller’s expected revenue in $c$? Is it an increasing function?

**Exercise 21.4** Consider a first-price sealed-bid auction with two bidders. In order to avoid the case that bidder 1 bids the price $\frac{1}{4} + \epsilon$, where $\epsilon > 0$ is small, we define the function $p$ (who shall get the item) as follows: if both bidders’ prices are $\frac{1}{4}$, then bidder 1 is the winner; if both bidders’ prices are the same and lower than $\frac{1}{4}$, then each would win with the same probability. Answer the following questions and prove your conclusions.
21.8. EXERCISES

1. Verify whether the following strategy combination \((\beta_1, \beta_2)\) constitutes an equilibrium:

\[
\beta_1(\theta_1) = \frac{\theta_1}{2}, \quad \beta_2(\theta_2) = \min\left\{\frac{\theta_2}{2}, \frac{1}{4}\right\}.
\]

2. Denote by \((\beta_1^*, \beta_2^*)\) a non-decreasing equilibrium. What are the values of \(\beta_1^*(0)\) and \(\beta_2^*(0)\)? Is it true that \(\beta_1^*(1) = \beta_2^*(1)\)?

3. Verify whether the Revenue Equivalence Theorem holds.

4. Is it possible to verify in this question whether the direct revelation mechanism that maximizes seller’s expected revenue under individual rationality and incentive compatibility is a second-price auction with a reserve price?

Exercise 21.5 Suppose that there are two buyers with independent private values distributed uniformly on \([0, 1]\). Buyer 2 has budget constraint, and the maximum value he can bid is \(\frac{1}{4}\). Buyer 1 does not have budget constraint. Answer the following questions:

1. If buyers are involved in a second-price sealed-bid auction, find an equilibrium strategy.

2. Calculate the seller’s expected revenue under the above equilibrium strategy.

Exercise 21.6 Consider a variant of the following second-price auction: there is a single indivisible item for sale, and the values of \(n\) bidders are respectively given by an independent random variable \(\theta_i\) distributed over the same interval. The number of bidders is greater than 3. If bidder \(i\) wins the item, his utility is \(\theta_i + t_i\); if bidder \(i\) loses the item, his utility is \(t_i\). Here \(t_i\) refers to theauctioneer’s money transfer to the bidder, and if \(t_i < 0\), it means that the bidder pays the auctioneer money. Note that the money the auctioneer receives from the bidder will not be refunded.

The rules of the auction are the following: all bidding prices must not be negative; the item will be allocated to the bidder of the highest price. The money is paid as follows: the highest price bidder pays the second highest price to the auctioneer; at the same time, the highest price bidder and second highest price bidder will receive the \(\frac{1}{n}\) rebate of third highest bid price from the auctioneer. In addition, the highest bidder also needs to pay the auctioneer the difference between the second highest bid and the third highest bid; the third highest bidder and the lower bidders will receive a rebate equal to the second highest bid of \(\frac{1}{n}\) from the auctioneer.

1. Prove that there is a dominant strategy for each bidder.
2. Prove that the auctioneer does not lose money.

3. Prove that any bidder will participate in the auction.

4. Since the auction seems to have all the good characteristics of the second-price auction, and the bidders get more surplus, why do people pay more attention to the second-price auction instead of this mechanism?

**Exercise 21.7 (Uncertain Number of Buyers)** Suppose that there are \( N \) potential buyers, whose private values \( \theta_1, \theta_2, \ldots, \theta_n \) are independent and have the same distribution on \([0, 1]\). The cumulative distribution function is \( \Phi \).

Each bidder participating in the auction does not know the number of participants in the auction. Each bidder thinks that except for himself, there are \( n \) individuals who participate in the auction with a probability \( p_n \), satisfying \( \sum_{n=0}^{N-1} p_n = 1 \). Note that everyone has the same belief about the distribution of participants in the auction.

1. When using a second-price sealed-bid auction, find the symmetric equilibrium in this situation. Explain why this constitutes an equilibrium.

2. Denote \( \Psi^{(n)}(\theta) = (\Phi(\theta))^n \). Prove that the expected return of a buyer who participates in this auction with private value \( \theta \) is:

\[
\sum_{n=0}^{N-1} p_n \Psi^{(n)}(\theta) E[\hat{\theta}^{(n)}_1 | \hat{\theta}^{(n)}_1 < \theta],
\]

where \( \hat{\theta}^{(n)}_1 \) is the highest of \( n \) random variables with the same cumulative distribution function \( \Phi \).

3. Prove the Revenue Equivalence Theorem in the following situation: Consider a symmetric sealed-bid auction with independent private values, and denote by \( \beta \) a monotonically increasing symmetric equilibrium that satisfies the following assumptions: (a) The winner is the highest bidder; (b) When private value is 0, the expected return of buyer is 0. Then the expected return of a buyer with private value \( \theta \) is given by the formula in question 2.

4. When using a first-price sealed-bid auction, calculate the symmetric equilibrium strategy.

**Exercise 21.8** Consider a single-item auction. There are \( n \) bidders, whose value is \( \theta_i \) that is uniformly distributed on the interval \([0, 1]\). Supposed that the participants are risk-averse, and the utility of bidder \( i \) to win the item on price \( p \) is given by \( \sqrt{\theta_i - p} \), where \( p \leq \theta_i \). We assume that no bidder’s bid or payment exceeds its value.
21.8. EXERCISES

1. Write down the bidder’s expected payment function at the first-price and second-price sealed-bid auctions.

2. Prove that the bidder with \( \theta_i \) in the first-price sealed-bid auction chooses the bidding strategy \( \beta(\theta_i) = \frac{n(n-1)}{n(n-1)+1} \theta_i \) that forms a symmetric Bayesian Nash equilibrium.

3. Prove that in the second-price sealed-bid auction, reporting true value is a (weakly) dominant strategy for each bidder.

4. Compare seller’s expected revenues derived from the first-price sealed-bid auction and the second-price sealed-bid auction. In what way is this example different from the case with risk-neutral bidders?

Exercise 21.9 (Number of Participants is Endogenously Determined) There are \( n \) potential bidders, whose private values are independently distributed on \([0, \theta]\). The cumulative distribution function is \( \Phi \), and the probability density function is \( \varphi \). In order to participate in the auction, a bidder must pay a non-refundable participation cost of \( c \). All bidders make decisions at the same time and do not know about others’ decisions when they are bidding. Assume that everyone adopts the same strategy.

1. If the seller adopts a first-price auction, derive the bidder’s participation decision and the equilibrium bidding price.

2. If the seller adopts a second-price auction, derive the bidder’s participation decision and the equilibrium bidding price.

3. Prove: Under symmetric equilibrium, the first-price auction and the second-price auction produce the same expected revenue.

Exercise 21.10 (Number of participants is endogenously determined, Cao and Tian (2010)) Consider first-price auction where there are \( n \) potential bidders whose private values are independently distributed on \([0, 1]\), and the cumulative distribution function is \( \Phi \). The probability density function is \( \varphi \). In order to participate in the auction, a bidder must pay a non-refundable participation cost of \( c \) (for example, some preparations for participating in the auction). All bidders make participation decisions at the same time and they are able to observe the number of other people participating in the auction when bidding. We focus on the strategy of threshold value participation, namely, one participant participates in the auction if and only if his valuation \( \theta \) exceeds some critical value \( \theta^* \).

1. What is the critical value if everyone chooses the same critical value and adopts the same bidding strategy? What is the corresponding bidding price function?
CHAPTER 21. AUCTION THEORY

2. For symmetric equilibrium, write down the seller’s expected revenue and compare it with that when the number of participants in the above question is unobservable.

3. If potential bidders are divided into two groups, each group uses a different threshold strategy, then write down the equation solving for the critical value and make judgment on when such a critical value equilibrium exists.

Exercise 21.11 Use the Revenue Equivalence Theorem to find the following equilibrium strategies for the auction. Consider a symmetric and independent private value auction with \( n \) bidders whose values are drawn from the distribution function \( \Phi(\theta_i) \) on \([0, 1]\). Assume that the density function corresponding to \( \Phi \) is continuously differentiable. Answer the following questions:

1. Find the equilibrium strategy for all-pay auction.
2. Find the equilibrium strategy for second-price all-pay auction.

Exercise 21.12 (An Application of Revenue Equivalence Theorem) In 1991, the US Vice President Quayle proposed that the losing party in the lawsuit should give the prevailing party a fee equal to losing party’s legal expenses. Quayle claimed that this can reduce the expenditure on legal services (current U.S. law requires each party to pay their own legal fees). Suppose that the two parties \( i = 1, 2 \) have private values \( \theta_i \) for winning. \( \theta_i \) follows an independent and identical distribution, and the cumulative distribution function is \( \Phi(\nu) \) (we assume that the lowest value is \( \theta \), namely, \( \Phi(\nu) = 0 \)). Both parties decided at the same time on how much they would spend on legal fees, and the party who spent more legal fees would win the case.

1. By using the Revenue Equivalence Theorem, find the expression of the equilibrium cost of the two parties under the current U.S. law. What other assumptions might you need in order to apply the Revenue Equivalence Theorem?
2. Use our model to evaluate Quayle’s statement without doing more calculations.
3. In the European legal system, the losing party is usually responsible for paying a part of the legal costs of the winning party. Without more calculations, do you think whether this rule will increase or decrease the expected legal fee expenses?

Exercise 21.13 Use the distribution characteristics given in Table 4 below to characterize the corresponding revenue maximization mechanism in the Bayesian Nash equilibrium solution concept. From this, we can summarize what kind of conclusions and give corresponding explanations.
21.8. EXERCISES

Exercise 21.14 Consider a first-price auction with \( n \geq 2 \) bidders and a reserve price \( r \). Suppose that each bidder \( i \)'s value \( \theta_i \) of the auction item is drawn from the distribution \( \Phi(\theta_i) \) (not necessarily a uniform distribution) on the interval \([0, 1]\), which satisfies \( i.i.d. \) (independent and identical distribution).

1. Explain why bidding with values higher than \( r \) is strictly better than not bidding for all bidders, namely, the equilibrium bidding strategy \( \beta_i(\theta_i) \) is in the \([r, 1]\) interval.

2. Prove that the equilibrium bidding function is

\[
\beta_i(\theta_i) = \frac{r\Phi(r)^{n-1} + \int_{\theta_i}^{\Phi(r)^{n-1}} \Phi(x)\Phi^{n-2}(x)dx}{\Phi(\theta_i)^{n-1}}.
\]

3. Using integration by part to prove that the above equilibrium bidding function can be expressed in the following form

\[
\beta_i(\theta_i) = \theta_i - \int_{r}^{\theta_i} \left( \frac{\Phi(x)}{\Phi(\theta_i)} \right)^{n-1} dx.
\]

Exercise 21.15 Under the first-price auction setting with the reserve price \( r \) in the above question, using the results of the above question to answer the following two questions:

1. If the cumulative distribution function \( \Phi_A \) stochastically dominates \( \Phi_B \), is the equilibrium bidding price under \( \Phi_A \) higher than that under \( \Phi_B \)? Please explain.

2. Does the equilibrium bidding price increase in the number of bidders \( n \)? Please explain.

Exercise 21.16 Consider a first-price auction with two bidders. The utility of bidder \( i \) is \( u(\theta_i - b_i) \), which is assumed to be concave, that is, bidders are risk-averse. The valuation of bidder \( i \) follows distribution \( \Phi(v_i) \) on the interval \([0, 1]\), which satisfies \( i.i.d. \) (independent and identical distribution).
1. If utility function \( u(\cdot) : R \rightarrow R \) is increasing and strictly concave, and satisfies \( u(0) = 0 \), prove that the following formula must hold:

\[
\frac{u(x)}{u'(x)} > x, \forall x > 0.
\]

2. Let \( b_{i}^{RN}(v_i) \) and \( b_{i}^{RA}(v_i) \) represent equilibrium bidding functions for risk-neutral and risk-averse bidders, respectively. Prove that they satisfy the following differential equations:

\[
\frac{db_{i}^{RN}(\theta_i)}{d\theta_i} = \frac{\varphi(\theta_i)}{\Phi(\theta_i)} (\theta_i - b_{i}^{RN});
\]

\[
\frac{db_{i}^{RA}(\theta_i)}{d\theta_i} = \frac{\varphi(\theta_i)}{\Phi(\theta_i)} \frac{u(\theta_i - b_{i}^{RA})}{u'(\theta_i - b_{i}^{RA})}.
\]

3. Apply the concavity expression in question 1 to prove the following formula:

\[
\frac{db_{i}^{RA}(\theta_i)}{d\theta_i} = \frac{\varphi(\theta_i)}{\Phi(\theta_i)} (\theta_i - b_{i}^{RA}).
\]

4. Prove for all \( \theta \), \( b_{i}^{RA}(\theta) \geq b_{i}^{RN}(\theta) \).

**Exercise 21.17 (Complete Information First-Price Auctions)** Consider the first-price auction with bidders 1 and 2. The values of an item are \( \theta_1 = 5 \) and \( \theta_2 = 3 \), respectively. Suppose that bidders know the values of each other, but they can only submit one of the following three bids: \( b_i = 0 \), \( b_i = 2 \), \( b_i = 4 \).

1. Write down the payoff matrix.

2. Which strategy combinations can survive the iterated elimination of strictly dominated strategies?

3. Which strategy combinations can form a Nash equilibrium?

**Exercise 21.18 (Expected Utility of Two-Bidder Auctions)** Consider an auctioneer auctioning an item to two bidders 1 and 2. The value \( \theta_i \) of bidder \( i \) is drawn from a uniform distribution on \([0, 10]\), which is the common knowledge to the auctioneer and all bidders. Assume that bidder 1’s value is \( \theta_1 = 8 \). Find the expected utility of bidder 1 under the following two auctions:

1. English Auctions.

2. First-price auction. The bidder’s equilibrium strategy is a linear function of valuation, namely, \( \beta_i(\theta_i) = \alpha_i \theta_i + \beta_i \) where \( \alpha_i, \beta_i > 0 \).
21.8. EXERCISES

Exercise 21.19 (Private Value First-Price Auctions) Consider a first-price auction between two risk-neutral bidders. Each bidder \( i (i = 1, 2) \) chooses to bid price \( b_i \geq 0 \) at the same time. The highest bidder gets the item and pays his bidding price. If the bidding prices are the same, then they get the item with the same probability. Prior to the auction, each bidder \( i \) privately observed a random variable \( \theta_i \) which is uniformly distributed on \([0, 2]\). The actual value of the bidder \( i \) is equal to \( \theta_i + 1 \), so that the bidder \( i \)'s payoff function is:

\[
u_i = \begin{cases} 
\theta_i + 1 - b_i, & \text{if } b_i > b_j; \\
\frac{1}{2}(\theta_i + 1 - b_i), & \text{if } b_i = b_j; \\
0, & \text{if } b_i < b_j. 
\end{cases}
\]

1. Derive the symmetric linear Bayesian-Nash Equilibrium of this game (i.e., each bidder uses the same equilibrium strategy of the form \( \beta_i = \alpha \theta_i + \beta \)).

2. In this equilibrium, what is the conditional expected payoff for bidder \( i \) of type \( \theta_i \)?

Exercise 21.20 (Common Value First-Price Auctions) Consider a similar first-price auction as in the previous question. The difference now is that the true value of the bidder \( i \) is \( \theta_i + \theta_j (j \neq i) \), so that the bidder \( i \)'s payoff function is:

\[
u_i = \begin{cases} 
\theta_i + \theta_j - b_i, & \text{if } b_i > b_j; \\
\frac{1}{2}(\theta_i + \theta_j - b_i), & \text{if } b_i = b_j; \\
0, & \text{if } b_i < b_j. 
\end{cases}
\]

Notice that given the bidder \( i \)'s private signal \( \theta_i \), his expected value for the auction item is \( \theta_i + 1 \), which is the same as in the above question.

1. Derive the symmetric linear Bayesian-Nash Equilibrium of this game.

2. In this equilibrium, compare the equilibrium bidding price of bidder \( \theta_i \) with that of the above question. Explain your conclusions.

Exercise 21.21 (Common Value Auctions) Given the following joint density function of the two bidders' signals \( \theta_1 \) and \( \theta_2 \) for the value of the auction item:

\[ \varphi(\theta_1, \theta_2) = 1, \forall(\theta_1, \theta_2) \in [0, 1]^2 \]

and that the common value of the auction item is \( u = \theta_1 + \theta_2 \). For first-price and second-price auctions, find out their Bayesian-Nash equilibrium strategies and calculate equilibrium expected revenues, respectively.
Exercise 21.22 (Comment Value Auctions) Given the following joint density function of the of two bidders’ signals $\theta_1$ and $\theta_2$ for the value of the auction item

$$\varphi(\theta_1, \theta_2) = \frac{4(1 + \theta_1 \theta_2)}{5}, \forall (\theta_1, \theta_2) \in [0, 1]^2,$$

and that the common value of the auction item is $u = \theta_1 + \theta_2$.

1. What is the Bayesian-Nash equilibrium strategy of bidders when using second-price auction? And what is the expected revenue of the auctioneer?

2. What is the Bayesian-Nash equilibrium strategy of bidders if we use first-price auction? And what is the expected revenue of the auctioneer? Compared with second-price auction, which format will the auctioneer prefer?

Exercise 21.23 (Three Formats of Multiple Auctions) Consider the multiple object auction model with independently private values, in which there are $K$ identical objects to be sold to $n$ bidders. Three sealed-bid auction formats are considered: 1) discriminatory auction; 2) uniform-price auction, and 3) Vickrey auction. In each of these auctions, a bidder is asked to submit $K$ bids $b^k_i$, satisfying $b^1_i \geq b^2_i \geq \ldots \geq b^K_i$, to indicate how much he is willing to pay for each additional unit. We refer to $b_i = (b^1_i, b^2_i, \ldots, b^K_i)$ as a bid vector. Thus, a total of $n \times K$ bids $\{b^k_i : i = 1, 2, \ldots, n; k = 1, 2, \ldots, K\}$ are collected and the $K$ units are awarded to the $K$ highest of these bids.

1. Define the discriminatory auction, uniform-price auction, and Vickrey auction, respectively.

2. Now consider a situation in which there are seven units ($K = 7$) to be sold to three bidders and the submitted bid vectors are

$$b_1 = (64, 46, 43, 40, 32, 15, 5)$$
$$b_2 = (60, 55, 47, 35, 27, 13, 8)$$
$$b_3 = (50, 45, 38, 24, 14, 9, 6)$$

(a) What are the seven highest bids? How many units does each bidder win?

(b) What is the $K$-vector of competing bids $c_{-i}$ of bidder $i$ for $i = 1, 2, 3$?

(c) What is the payment of each bidder under a discriminatory auction?

(d) What is the market-cleaning price and the payment of each bidder under a uniform-price auction?
(e) What is the payment of each bidder under a Vickrey Auctions?

(f) What is ranking in terms of the seller’s expected revenue among the English, second-price, and first-price auctions?

Exercise 21.24 (Uniform-Price Auctions) Consider the uniform-price auction with 3 items and 2 bidders as follows. The price vector $V^i = (V^i_1, V^i_2, V^i_3)$ of each bidder $i$ is independently and identically distributed on the set $\{v \in [0, 1]^3 | v_1 \geq v_2 \geq v_3\}$. The marginal distribution is given as follows:

\[
\Phi_1(v_1) = (v_1)^2, \\
\Phi_2(v_2) = (2 - v_2)v_2, \\
\Phi_3(v_3) \text{ is not given.}
\]

Prove that the bidding strategy $\beta(v_1, v_2, v_3) = (v_1, (v_2)^2, 0)$ constitutes a symmetric equilibrium for this uniform-price auction.

Exercise 21.25 (Sequential Second-Price Auctions) There are 2 homogenous items to be auctioned. There are two bidders, whose types are independently distributed. The type of bidder $i$ is $v_i \in [0, 1], i = 1, 2$, which follows the uniform distribution $\Phi(v)$. Suppose that the bidder $i$’s utility for the first unit item is $u_1(v_i) = v_i$, while for second unit item is $u_2(v_i) = \frac{1}{2}v_i$. Assuming we adopt the sequential second price auction, answer the following questions:

1. Find the equilibrium strategy for bidders.
2. Derive seller’s expected revenue in this sequential second price auction.
3. Is the Revenue Equivalence Theorem still valid in this auction with 2 units of an item? Why?

Exercise 21.26 (Discriminatory Auctions) Consider the economic environment that consists of 2 items and 2 bidders. Assumptions are the same as the previous question. We now consider discriminatory price auction. In a simultaneous discriminatory auction, bidder $i$ who has won $k_i$ units needs to pay his bid, i.e., total payment price is $b_1^i + \cdots + b_{k_i}^i$. Answer the following questions:

1. For the increasing equilibrium, i.e., the higher type $v$ is, the higher the bid is, what is the equilibrium strategy in this economic environment?
2. Solving for seller’s expected revenue in this discriminatory price auction.
Exercise 21.27 (Uniform-Price Auctions) The economic environment is the same as the above two questions. Now consider the case of using a uniform price multi-unit auction. Each bidder bids the same price for all items. The highest bids whose number is equal to the number of items win, but the price paid is the same. For this uniform price auction with two items and two bidders, answer the following questions:

1. If the price paid is equal to the lowest bid among winning bids, what is the equilibrium strategy for bidding? What is the expected revenue of the seller?

2. If the price paid is equal to the highest bid among winning bids, what is the equilibrium strategy for bidding? What is the expected revenue of the seller? Is the Revenue Equivalence Theorem still applicable here?

3. If \( u_2(v_i) = v_i \), that is, the value of item does not decrease, what are the corresponding equilibrium and revenue?

Exercise 21.28 (Sequential Second-Price Auctions, Katzman (1999)) Consider the case where 2 units of homogenous goods are sold to two bidders in two sequential second price auctions. The bidder has demand for multiple items and details are given as follows. Each bidder draws independently \( \theta_1 \) and \( \theta_2 \) from the uniform distribution \( \Phi(\theta) = \theta, \theta \in [0, 1] \). The bidder’s value for the first item is \( V_1 = \max\{\theta_1, \theta_2\} \), and his value for the second item is \( V_2 = \min\{\theta_1, \theta_2\} \).

1. Prove that the following strategy is a symmetric equilibrium of this sequential second price auction:
   (a) In the first round of auction, bidding \( \beta_1(\theta_1, \theta_2) = \frac{1}{2}V_1 \);
   (b) Bid truthfully in the second round of auction. That is, if he wins the auction in the first round of auction, then \( \beta_2(\theta_1, \theta_2) = V_2 \); otherwise, \( \beta_2(\theta_1, \theta_2) = V_1 \).

2. Prove that the sequence of the equilibrium bidding price is a submartingale, namely, \( E(P_2|P_1 = p_1) \geq p_1 \) and it must hold in the strict inequality with a positive probability.

Exercise 21.29 Consider the setting of the previous question. Answer the following questions:

1. Prove that the following strategy is another symmetric equilibrium of this sequential second price auction:
   (a) In the first round of auction, bidding \( \beta_1(\theta_1, \theta_2) = V_2 \);
   (b) In the second round of auction, bidding the true value.
2. Comment on the sequence consists of symmetric equilibrium prices \((P_1, P_2)\) generated as above.

**Exercise 21.30** Consider following optimal auction model. There are 2 bidders, and each bidder has two types. For the value of the item, two bidders have the following distribution of prior beliefs.

<table>
<thead>
<tr>
<th></th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta_1)</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{6})</td>
</tr>
<tr>
<td>(\theta_2)</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{3})</td>
</tr>
</tbody>
</table>

Table 1 Type Space I

Assume \(\theta_1 = 1, \theta_2 = 2\). The seller has two units of this item, and it is supposed that each buyer only needs one unit at most. Therefore, this is a multi-unit auction model. Answer the following questions:

1. Given the distribution of prior beliefs, calculate the expected social surplus.

2. Use Bayesian-Nash equilibrium solution concept to give a revenue maximizing mechanism. Under this mechanism, what is the maximum revenue of the seller?

3. Use ex-post equilibrium solution concept to give a revenue maximizing mechanism. Under this mechanism, what is the maximum revenue?

**Exercise 21.31** Consider a multi-unit auction model with two bidders and the following type-distribution characteristics, where \(t^j_i\) represents the participant type and \(\theta_i\) represents the reward type of the corresponding participant.

<table>
<thead>
<tr>
<th></th>
<th>(t^1_1)</th>
<th>(t^2_1)</th>
<th>(t^1_2)</th>
<th>(t^2_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta_1)</td>
<td>(\frac{1}{30})</td>
<td>(\frac{1}{10})</td>
<td>(\frac{1}{10})</td>
<td>(\theta_1)</td>
</tr>
<tr>
<td>(\theta_1)</td>
<td>(\frac{1}{10})</td>
<td>(\frac{1}{30})</td>
<td>(\frac{1}{30})</td>
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<tr>
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<td>(\frac{1}{30})</td>
<td>(\frac{1}{20})</td>
<td>(0)</td>
<td>(\theta_1)</td>
</tr>
<tr>
<td>(\theta_1)</td>
<td>(\frac{1}{20})</td>
<td>(\frac{1}{20})</td>
<td>(\frac{1}{20})</td>
<td>(\theta_2)</td>
</tr>
<tr>
<td>(\theta_2)</td>
<td>(\frac{1}{20})</td>
<td>(\frac{1}{20})</td>
<td>(\frac{1}{20})</td>
<td>(\theta_2)</td>
</tr>
</tbody>
</table>

Table 2 Type Space II

1. Calculate the distribution of reward types. What is the difference compared with the above question? Can you solve the expected social surplus at this moment? If you can, how much is it?
2. Characterize the revenue maximizing mechanism in Bayesian-Nash equilibrium. Under this mechanism, what is the maximum revenue?

Exercise 21.32 What is the difference between the distribution of reward types corresponding to type space III and Exercise 21.30? In this case, is it possible to characterize a similar revenue maximizing mechanism that corresponds to the Table in Exercise 21.31? Why?

<table>
<thead>
<tr>
<th>$t_1^1$</th>
<th>$t_2^1$</th>
<th>$t_2^2$</th>
<th>$\theta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1^2$</td>
<td>$t_2^1$</td>
<td>$t_2^2$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>$t_1^3$</td>
<td>$t_2^1$</td>
<td>$t_2^2$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>$t_1^4$</td>
<td>$t_2^1$</td>
<td>$t_2^2$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>$t_1^5$</td>
<td>$t_2^1$</td>
<td>$t_2^2$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>$t_1^6$</td>
<td>$t_2^1$</td>
<td>$t_2^2$</td>
<td>$\theta_1$</td>
</tr>
</tbody>
</table>

Table 3 Type Space III

Exercise 21.33 (Auctions for Complementary Items) Consider the case of auctioning 4 items $K = \{a_1, a_2, b_1, b_2\}$ to 5 buyers. Buyer 1 is only interested in receiving $a_1$ and $b_1$ simultaneously; buyers 2 and 3 are only interested in receiving $a_2$ and $b_2$ simultaneously; buyer 4 is only interested in receiving $b_1$ and $b_2$ simultaneously; and buyer 5 is only interested in receiving both $a_1$ and $a_2$. Specifically, the values are as follows:

$x^1(a_1b_1) = 10$;
$x^2(a_2b_2) = 20$;
$x^3(a_2b_2) = 25$;
$x^4(b_1b_2) = 10$;
$x^5(a_1a_2) = 10$.

The values of all other combinations are 0s. Using VCG mechanism to find the efficient allocation and the corresponding payment.

Exercise 21.34 (Combinatorial Auctions, Mochon and Saez, 2015) Sell three items, $A, B, C$, to four bidders 1, 2, 3, and 4. The bids are as follows:

<table>
<thead>
<tr>
<th>$S$</th>
<th>$b_1(S)$</th>
<th>$b_2(S)$</th>
<th>$b_3(S)$</th>
<th>$b_4(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>200</td>
<td>150</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>$B$</td>
<td>100</td>
<td>250</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>$C$</td>
<td>100</td>
<td>100</td>
<td>300</td>
<td>100</td>
</tr>
<tr>
<td>$AB$</td>
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<td>200</td>
<td>200</td>
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</tr>
<tr>
<td>$AC$</td>
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<td>275</td>
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<tr>
<td>$ABC$</td>
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</tr>
<tr>
<td>$BC$</td>
<td>150</td>
<td>150</td>
<td>150</td>
<td>200</td>
</tr>
</tbody>
</table>
21.9. **REFERENCES**

Given the above information, answer the following questions:

1. What is the revenue-maximizing sale combination for seller?

2. Find the allocation outcome of the VCG mechanism. Is it a core allocation? If not, give a blocking coalition.

**Exercise 21.35** The following table shows the bids for a combinatorial auction with two items and two bidders.

<table>
<thead>
<tr>
<th></th>
<th>$b_1(S)$</th>
<th>$b_2(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>AB</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

According to the above information, answer the following questions:

1. List all possible allocation results. What is the allocation of the highest value?

2. If we adopt a uniform price rule, how much will the winning bidder pay? How much revenue will the seller get?

3. If we adopt the VCG mechanism, how much will the winning bidder pay? How much revenue will the seller get?

21.9 **References**

**Books and Monographs:**


**Papers:**


