Lecture Notes

Mathematical Economics

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1The most materials of this lecture notes are drawn from Chiang’s classic textbook Fundamental Methods of Mathematical Economics, which are used for my teaching and convenience of my students in class. Please not distribute it to any others.
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Chapter 1

The Nature of Mathematical Economics

The purpose of this course is to introduce the most fundamental aspects of the mathematical methods such as those matrix algebra, mathematical analysis, and optimization theory.

1.1 Economics and Mathematical Economics

Economics is a social science that studies how to make decisions in face of scarce resources. Specifically, it studies individuals’ economic behavior and phenomena as well as how individuals, such as consumers, households, firms, organizations and government agencies, make trade-off choices that allocate limited resources among competing uses.

Mathematical economics is an approach to economic analysis, in which the economists make use of mathematical symbols in the statement of the problem and also draw upon known mathematical theorems to aid in reasoning.

Since mathematical economics is merely an approach to economic anal-
The difference between these two approaches is that in the former, the assumptions and conclusions are stated in mathematical symbols rather than words and in the equations rather than sentences so that the interdependent relationship among economic variables and resulting conclusions are more rigorous and concise by using mathematical models and mathematical statistics/econometric methods.

I propose the "three-dimension and six-nature" methodology in studying and solving realistic social economic problems as well as arising your managemental skills and leadership well.

**Three dimensions**: theoretical logic, practical knowledge, and historical perspective;

**Six Characteristics/natures**: scientific rigorous, preciseness, reality, pertinence, foresight and ideological.

Only by accomplishing the above-mentioned "three-dimensional" and "six characteristics", can we put forward logical, factual and historical insights, high perspicacity views and policy recommendations for the economic development of a nation or and the world, as well as theoretical innovations and works that can be tested by data, practice and history. Only through the three dimensions, can your assessments and proposals be guaranteed to have these six natures and become a good economist.

The study of economic social issues cannot simply involve real world in its experiment, so it not only requires theoretical analysis based on inherent logical inference and, more often than not, vertical and horizontal comparisons from the larger perspective of history so as to draw experience and lessons, but also needs tools of statistics and econometrics to do empirical quantitative analysis or test, the three of which are all indispens-
1.2. ADVANTAGES OF MATHEMATICAL APPROACH

able. When conducting economic analysis or giving policy suggestions in the realm of modern economics, the theoretical analysis often combines theory, history, and statistics, presenting not only theoretical analysis of inherent logic and comparative analysis from the historical perspective but also empirical and quantitative analysis with statistical tools for examination and investigation. Indeed, in the final analysis, all knowledge is history, all science is logics, and all judgment is statistics.

As such, it is not surprising that mathematics and mathematical statistics/econometrics are used as the basic and most important analytical tools in every field of economics. For those who study economics and conduct research, it is necessary to grasp enough knowledge of mathematics and mathematical statistics. Therefore, it is of great necessity to master sufficient mathematical knowledge if you want to learn economics well, conduct economic research and become a good economist.

## 1.2 Advantages of Mathematical Approach

Mathematical approach has the following advantages:

1. It makes the language more precise and the statement of assumptions more clear, which can deduce many unnecessary debates resulting from inaccurate verbal language.

2. It makes the analytical logic more rigorous and clearly states the boundary, applicable scope and conditions for a conclusion to hold. Otherwise, the abuse of a theory may occur.

3. Mathematics can help obtain the results that cannot be easily attained through intuition.

4. It helps improve and extend the existing economic theories.
CHAPTER 1. THE NATURE OF MATHEMATICAL ECONOMICS

It is, however, noteworthy a good master of mathematics cannot guarantee to be a good economist. It also requires fully understanding the analytical framework and research methodologies of economics, and having a good intuition and insight of real economic environments and economic issues. The study of economics not only calls for the understanding of some terms, concepts and results from the perspective of mathematics (including geometry), but more importantly, even when those are given by mathematical language or geometric figure, we need to get to their economic meaning and the underlying profound economic thoughts and ideals. Thus we should avoid being confused by the mathematical formulas or symbols in the study of economics.

All in all, to become a good economist, you need to be of original, creative and academic way of thinking.
Chapter 2

Economic Models

2.1 Ingredients of a Mathematical Model

A economic model is merely a theoretical framework, and there is no inherent reason why it must mathematical. If the model is mathematical, however, it will usually consist of a set of equations designed to describe the structure of the model. By relating a number of variables to one another in certain ways, these equations give mathematical form to the set of analytical assumptions adopted. Then, through application of the relevant mathematical operations to these equations, we may seek to derive a set of conclusions which logically follow from those assumptions.

2.2 The Real-Number System

Whole numbers such as 1, 2, · · · are called positive numbers; these are the numbers most frequently used in counting. Their negative counterparts −1, −2, −3, · · · are called negative integers. The number 0 (zero), on the other hand, is neither positive nor negative, and it is in that sense unique. Let us lump all the positive and negative integers and the number zero
into a single category, referring to them collectively as the set of all integers.

Integers of course, do not exhaust all the possible numbers, for we have fractions, such as \(\frac{2}{3}, \frac{5}{4},\) and \(\frac{7}{5}\) which – if placed on a ruler – would fall between the integers. Also, we have negative fractions, such as \(-\frac{1}{2}\) and \(-\frac{3}{5}\). Together, these make up the set of all fractions.

The common property of all fractional number is that each is expressible as a ratio of two integers; thus fractions qualify for the designation rational numbers (in this usage, rational means ratio-nal). But integers are also rational, because any integer \(n\) can be considered as the ratio \(n/1\). The set of all fractions together with the set of all integers from the set of all rational numbers.

Once the notion of rational numbers is used, however, there naturally arises the concept of irrational numbers – numbers that cannot be expressed as ratios of a pair of integers. One example is \(\sqrt{2} = 1.4142\ldots\). Another is \(\pi = 3.1415\ldots\).

Each irrational number, if placed on a ruler, would fall between two rational numbers, so that, just as the fraction fill in the gaps between the integers on a ruler, the irrational number fill in the gaps between rational numbers. The result of this filling-in process is a continuum of numbers, all of which are so-called “real numbers.” This continuum constitutes the set of all real numbers, which is often denoted by the symbol \(\mathbb{R}\).

### 2.3 The Concept of Sets

A set is simply a collection of distinct objects. The objects may be a group of distinct numbers, or something else. Thus, all students enrolled in a particular economics course can be considered a set, just as the three integers 2, 3, and 4 can form a set. The object in a set are called the elements of the set.
2.3. THE CONCEPT OF SETS

There are two alternative ways of writing a set: by enumeration and by description. If we let \( S \) represent the set of three numbers 2, 3 and 4, we write by enumeration of the elements, \( S = \{2, 3, 4\} \). But if we let \( I \) denote the set of all positive integers, enumeration becomes difficult, and we may instead describe the elements and write \( I = \{x \mid x \text{ is a positive integer}\} \), which is read as follows: “\( I \) is the set of all \( x \) such that \( x \) is a positive integer.” Note that the braces are used enclose the set in both cases. In the descriptive approach, a vertical bar or a colon is always inserted to separate the general symbol for the elements from the description of the elements.

A set with finite number of elements is called a finite set. Set \( I \) with an infinite number of elements is an example of an infinite set. Finite sets are always denumerable (or countable), i.e., their elements can be counted one by one in the sequence 1, 2, 3, \( \cdots \). Infinite sets may, however, be either denumerable (set \( I \) above) or nondenumerable (for example, \( J = \{x \mid 2 < x < 5\} \)).

Membership in a set is indicated by the symbol \( \in \) (a variant of the Greek letter epsilon \( \epsilon \) for “element”), which is read: “is an element of.”

If two sets \( S_1 \) and \( S_2 \) happen to contain identical elements,

\[
S_1 = \{1, 2, a, b\} \text{ and } S_2 = \{2, b, 1, a\}
\]

then \( S_1 \) and \( S_2 \) are said to be equal \( (S_1 = S_2) \). Note that the order of appearance of the elements in a set is immaterial.

If we have two sets \( T = \{1, 2, 5, 7, 9\} \) and \( S = \{2, 5, 9\} \), then \( S \) is subset of \( T \), because each element of \( S \) is also an element of \( T \). A more formal statement of this is: \( S \) is subset of \( T \) if and only if \( x \in S \) implies \( x \in T \). We write \( S \subseteq T \) or \( T \supseteq S \).

It is possible that two sets happen to be subsets of each other. When
this occurs, however, we can be sure that these two sets are equal.

If a set have \( n \) elements, a total of \( 2^n \) subsets can be formed from those elements. For example, the subsets of \( \{1, 2\} \) are: \( \emptyset, \{1\}, \{2\} \) and \( \{1, 2\} \).

If two sets have no elements in common at all, the two sets are said to be disjoint.

The union of two sets \( A \) and \( B \) is a new set containing elements belong to \( A \), or to \( B \), or to both \( A \) and \( B \). The union set is symbolized by \( A \cup B \) (read: “\( A \) union \( B \)”).

**Example 2.3.1** If \( A = \{1, 2, 3\} \), \( B = \{2, 3, 4, 5\} \), then \( A \cup B = \{1, 2, 3, 4, 5\} \).

The intersection of two sets \( A \) and \( B \), on the other hand, is a new set which contains those elements (and only those elements) belonging to both \( A \) and \( B \). The intersection set is symbolized by \( A \cap B \) (read: “\( A \) intersection \( B \)”).

**Example 2.3.2** If \( A = \{1, 2, 3\} \), \( A = \{4, 5, 6\} \), then \( A \cup B = \emptyset \).

In a particular context of discussion, if the only numbers used are the set of the first seven positive integers, we may refer to it as the universal set \( U \). Then, with a given set, say \( A = \{3, 6, 7\} \), we can define another set \( \bar{A} \) (read: “the complement of \( A \)”) as the set that contains all the numbers in the universal set \( U \) which are not in the set \( A \). That is: \( \bar{A} = \{1, 2, 4, 5\} \).

**Example 2.3.3** If \( U = \{5, 6, 7, 8, 9\} \), \( A = \{6, 5\} \), then \( \bar{A} = \{7, 8, 9\} \).

**Properties of unions and intersections:**

\[
A \cup (B \cup C) = (A \cup B) \cup C \\
A \cap (B \cap C) = (A \cap B) \cap C \\
A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\
A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
\]
2.4 Relations and Functions

An ordered pair \((a, b)\) is a pair of mathematical objects. The order in which the objects appear in the pair is significant: the ordered pair \((a, b)\) is different from the ordered pair \((b, a)\) unless \(a = b\). In contrast, a set of two elements is an unordered pair: the unordered pair \(\{a, b\}\) equals the unordered pair \(\{b, a\}\). Similar concepts apply to a set with more than two elements, ordered triples, quadruples, quintuples, etc., are called ordered sets.

Example 2.4.1 To show the age and weight of each student in a class, we can form ordered pairs \((a, w)\), in which the first element indicates the age (in years) and the second element indicates the weight (in pounds). Then \((19, 128)\) and \((128, 19)\) would obviously mean different things.

Suppose, from two given sets, \(x = \{1, 2\}\) and \(y = \{3, 4\}\), we wish to form all the possible ordered pairs with the first element taken from set \(x\) and the second element taken from set \(y\). The result will be the set of four ordered pairs \((1,2)\), \((1,4)\), \((2,3)\), and \((2,4)\). This set is called the Cartesian product, or direct product, of the sets \(x\) and \(y\) and is denoted by \(x \times y\) (read “\(x\) cross \(y\)”).

Extending this idea, we may also define the Cartesian product of three sets \(x\), \(y\), and \(z\) as follows:

\[x \times y \times z = \{(a, b, c) | a \in x, b \in y, c \in z\}\]

which is the set of ordered triples.

Example 2.4.2 If the sets \(x\), \(y\), and \(z\) each consist of all the real numbers, the Cartesian product will correspond to the set of all points in a three-dimension space. This may be denoted by \(\mathcal{R} \times \mathcal{R} \times \mathcal{R}\), or more simply, \(\mathcal{R}^3\).
Example 2.4.3 The set \( \{ (x, y) | y = 2x \} \) is a set of ordered pairs including, for example, (1,2), (0,0), and (-1,-2). It constitutes a relation, and its graphical counterpart is the set of points lying on the straight line \( y = 2x \).

Example 2.4.4 The set \( \{ (x, y) | y \leq x \} \) is a set of ordered pairs including, for example, (1,0), (0,0), (1,1), and (1,-4). The set corresponds the set of all points lying on below the straight line \( y = x \).

As a special case, however, a relation may be such that for each \( x \) value there exists only one corresponding \( y \) value. The relation in example 2.4.3 is a case in point. In that case, \( y \) is said to be a function of \( x \), and this is denoted by \( y = f(x) \), which is read: ”\( y \) equals \( f \) of \( x \).” A function is therefore a set of ordered pairs with the property that any \( x \) value uniquely determines a \( y \) value. It should be clear that a function must be a relation, but a relation may not be a function.

A function is also called a mapping, or transformation; both words denote the action of associating one thing with another. In the statement \( y = f(x) \), the functional notation \( f \) may thus be interpreted to mean a rule by which the set \( x \) is “mapped” (“transformed”) into the set \( y \). Thus we may write

\[ f : x \rightarrow y \]

where the arrow indicates mapping, and the letter \( f \) symbolically specifies a rule of mapping.

In the function \( y = f(x) \), \( x \) is referred to as the argument of the function, and \( y \) is called the value of the function. We shall also alternatively refer to \( x \) as the independent variable and \( y \) as the dependent variable. The set of all permissible values that \( x \) can take in a given context is known as the domain of the function, which may be a subset of the set of all real numbers. The \( y \) value into which an \( x \) value is mapped is called the image of that \( x \) value. The set of all images is called the range of the function, which is the...
set of all values that the \( y \) variable will take. Thus the domain pertains to the independent variable \( x \), and the range has to do with the dependent variable \( y \).

### 2.5 Types of Function

A function whose range consists of only one element is called a *constant function*.

**Example 2.5.1** The function \( y = f(x) = 7 \) is a constant function.

The constant function is actually a “degenerated” case of what are known as *polynomial functions*. A polynomial function of a single variable has the general form

\[
y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n
\]

in which each term contains a coefficient as well as a nonnegative-integer power of the variable \( x \).

Depending on the value of the integer \( n \) (which specifies the highest power of \( x \)), we have several subclasses of polynomial function:

- **Case of \( n = 0 \) :** \( y = a_0 \) \hspace{1cm} [constant function]
- **Case of \( n = 1 \) :** \( y = a_0 + a_1 x \) \hspace{1cm} [linear function]
- **Case of \( n = 2 \) :** \( y = a_0 + a_1 x + a_2 x^2 \) \hspace{1cm} [quadritic function]
- **Case of \( n = 3 \) :** \( y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) \hspace{1cm} [cubic function]

A function such as

\[
y = \frac{x - 1}{x^2 + 2x + 4}
\]

in which \( y \) is expressed as a ratio of two polynomials in the variable \( x \), is known as a *rational function* (again, meaning *ratio-nal*). According to the definition, any polynomial function must itself be a rational function,
because it can always be expressed as a ratio to 1, which is a constant function.

Any function expressed in terms of polynomials and or roots (such as square root) of polynomials is an algebraic function. Accordingly, the function discussed thus far are all algebraic. A function such as \( y = \sqrt{x^2 + 1} \) is not rational, yet it is algebraic.

However, exponential functions such as \( y = b^x \), in which the independent variable appears in the exponent, are nonalgebraic. The closely related logarithmic functions, such as \( y = \log_b x \), are also nonalgebraic.

Rules of Exponents:

- Rule 1: \( x^m \times x^n = x^{m+n} \)
- Rule 2: \( \frac{x^m}{x^n} = x^{m-n} (x \neq 0) \)
- Rule 3: \( x^{-n} = \frac{1}{x^n} \)
- Rule 4: \( x^0 = 1 (x \neq 0) \)
- Rule 5: \( x^{\frac{1}{n}} = \sqrt[n]{x} \)
- Rule 6: \( (x^m)^n = x^{mn} \)
- Rule 7: \( x^m \times y^m = (xy)^m \)

2.6 Functions of Two or More Independent Variables

Thus far, we have considered only functions of a single independent variable, \( y = f(x) \). But the concept of a function can be readily extended to the case of two or more independent variables. Given a function

\[ z = g(x, y) \]
2.7. LEVELS OF GENERALITY

A given pair of \( x \) and \( y \) values will uniquely determine a value of the dependent variable \( z \). Such a function is exemplified by

\[
z = ax + by \quad \text{or} \quad z = a_0 + a_1x + a_2x^2 + b_1y + b_2y^2
\]

Functions of more than one variables can be classified into various types, too. For instance, a function of the form

\[
y = a_1x_1 + a_2x_2 + \cdots + a_nx_n
\]

is a linear function, whose characteristic is that every variable is raised to the first power only. A quadratic function, on the other hand, involves first and second powers of one or more independent variables, but the sum of exponents of the variables appearing in any single term must not exceed two.

**Example 2.6.1** \( y = ax^2 + bxy + cy^2 + dx + ey + f \) is a quadratic function.

2.7 Levels of Generality

In discussing the various types of function, we have without explicit notice introducing examples of functions that pertain to varying levels of generality. In certain instances, we have written functions in the form

\[
y = 7, \quad y = 6x + 4, \quad y = x^2 - 3x + 1 \quad (etc.)
\]

Not only are these expressed in terms of numerical coefficients, but they also indicate specifically whether each function is constant, linear, or quadratic. In terms of graphs, each such function will give rise to a well-defined unique curve. In view of the numerical nature of these functions, the solutions of the model based on them will emerge as numerical values also.
The drawback is that, if we wish to know how our analytical conclusion will change when a different set of numerical coefficients comes into effect, we must go through the reasoning process afresh each time. Thus, the result obtained from specific functions have very little generality.

On a more general level of discussion and analysis, there are functions in the form

\[ y = a, \ y = bx + a, \ y = cx^2 + bx + a \ (\text{etc.}) \]

Since parameters are used, each function represents not a single curve but a whole family of curves. With parametric functions, the outcome of mathematical operations will also be in terms of parameters. These results are more general.

In order to attain an even higher level of generality, we may resort to the general function statement \( y = f(x) \), or \( z = g(x, y) \). When expressed in this form, the functions is not restricted to being either linear, quadratic, exponential, or trigonometric – all of which are subsumed under the notation. The analytical result based on such a general formulation will therefore have the most general applicability.
Chapter 3

Equilibrium Analysis in Economics

3.1 The Meaning of Equilibrium

Like any economic term, equilibrium can be defined in various ways. One definition here is that an equilibrium for a specific model is a situation that is characterized by a lack of tendency to change. Or more generally, it means that from the various feasible and available choices, choose “best” one according to a certain criterion, and the one being finally chosen is called an equilibrium. It is for this reason that the analysis of equilibrium is referred to as statics. The fact that an equilibrium implies no tendency to change may tempt one to conclude that an equilibrium necessarily constitutes a desirable or ideal state of affairs.

This chapter provides two typical examples of equilibrium. One that is from microeconomics is the equilibrium attained by a market under given demand and supply conditions. The other that is from macroeconomics is the equilibrium of national income model under given conditions of consumption and investment patterns. We will use these two models as
CHAPTER 3. EQUILIBRIUM ANALYSIS IN ECONOMICS

running examples throughout the course.

3.2 Partial Market Equilibrium - A Linear Model

In a static-equilibrium model, the standard problem is that of finding the set of values of the endogenous variables which will satisfy the equilibrium conditions of the model.

Partial-Equilibrium Market Model

Partial-equilibrium market model is a model of price determination in an isolated market for a commodity.

Three variables:

\[ Q_d = \text{the quantity demanded of the commodity}; \]
\[ Q_s = \text{the quantity supplied of the commodity}; \]
\[ P = \text{the price of the commodity}. \]

The Equilibrium Condition: \( Q_d = Q_s \).

The model is
\[ Q_d = Q_s \]
\[ Q_d = a - bp \quad (a, b > 0) \]
\[ Q_s = -c + dp \quad (c, d > 0) \]

The slope of \( Q_d = -b \), the vertical intercept = \( a \).

The slope of \( Q_d = d \), the vertical intercept = \( -c \).

Note that, contrary to the usual practice, quantity rather than price has been plotted vertically in the figure.

One way of finding the equilibrium is by successive elimination of variables and equations through substitution.
From $Q_s = Q_d$, we have

$$a - bp = -c + dp$$

and thus

$$(b + d)p = a + c.$$  

Since $b + d \neq 0$, the equilibrium price is

$$\bar{p} = \frac{a + c}{b + d}.$$  

The equilibrium quantity can be obtained by substituting $\bar{p}$ into either $Q_s$ or $Q_d$:

$$\bar{Q} = \frac{ad - bc}{b + d}.$$  

Since the denominator $(b + d)$ is positive, the positivity of $\bar{Q}$ requires that the numerator $(ad - bc) > 0$. Thus, to be economically meaningful, the model should contain the additional restriction that $ad > bc$. 
3.3 Partial Market Equilibrium - A Nonlinear Model

The partial market model can be nonlinear. Suppose a model is given by

\[ Q_d = Q_s; \]
\[ Q_d = 4 - p^2; \]
\[ Q_s = 4p - 1. \]

As previously stated, this system of three equations can be reduced to a single equation by substitution.

\[ 4 - p^2 = 4p - 1, \]

or

\[ p^2 + 4p - 5 = 0, \]

which is a quadratic equation. In general, given a quadratic equation in the form

\[ ax^2 + bx + c = 0 \quad (a \neq 0), \]

its two roots can be obtained from the quadratic formula:

\[ \bar{x}_1, \bar{x}_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

where the “+” part of the “±” sign yields \( \bar{x}_1 \) and “−” part yields \( \bar{x}_2 \). Thus, by applying the quadratic formulas to \( p^2 + 4p - 5 = 0 \), we have \( \bar{p}_1 = 1 \) and \( \bar{p}_2 = -5 \), but only the first is economically admissible, as negative prices are ruled out.
3.4 General Market Equilibrium

In the above, we have discussed methods of an isolated market, wherein the \( Q_d \) and \( Q_s \) of a commodity are functions of the price of that commodity alone. In the real world, there would normally exist many substitutes and complementary goods. Thus a more realistic model for the demand and supply function of a commodity should take into account the effects not only of the price of the commodity itself but also of the prices of other commodities. As a result, the price and quantity variables of multiple commodities must enter endogenously into the model. Thus, when several interdependent commodities are simultaneously considered, equilibrium would require the absence of excess demand, which is the difference
between demand and supply, for each and every commodity included in the model. Consequently, the equilibrium condition of an $n$–commodity market model will involve $n$ equations, one for each commodity, in the form

$$E_i = Q_{di} - Q_{si} = 0 \ (i = 1, 2, \ldots, n),$$

where $Q_{di} = Q_{di}(P_1, P_2, \ldots, P_n)$ and $Q_{si} = Q_{si}(P_1, P_2, \ldots, P_n)$ are the demand and supply functions of commodity $i$, and $(P_1, P_2, \ldots, P_n)$ are prices of commodities.

Thus, solving $n$ equations for $P$:

$$E_i(P_1, P_2, \ldots, P_n) = 0,$$

we obtain the $n$ equilibrium prices $\bar{P}_i$ – if a solution does indeed exist. And then the $\bar{Q}_i$ may be derived from the demand or supply functions.

**Two-Commodity Market Model**

To illustrate the problem, let us consider a two-commodity market model with linear demand and supply functions. In parametric terms, such a model can be written as

$$Q_{d1} - Q_{s1} = 0;$$

$$Q_{d1} = a_0 + a_1 P_1 + a_2 P_2;$$

$$Q_{s1} = b_0 + b_1 P_1 + b_2 P_2;$$

$$Q_{d2} - Q_{s2} = 0;$$

$$Q_{d2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2;$$

$$Q_{s2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2.$$

By substituting the second and third equations into the first and the
fifth and sixth equations into the fourth, the model is reduced to two equations in two variables:

\[(a_0 - b_0) + (a_1 - b_1)P_1 + (a_2 - b_2)P_2 = 0\]

\[(a_0 - \beta_0) + (a_1 - \beta_1)P_1 + (a_2 - \beta_2)P_2 = 0\]

If we let

\[c_i = a_i - b_i \quad (i = 0, 1, 2),\]

\[\gamma_i = \alpha_i - \beta_i \quad (i = 0, 1, 2),\]

the above two linear equations can be written as

\[c_1P_1 + c_2P_2 = -c_0;\]

\[\gamma_1P_1 + \gamma_2P_2 = -\gamma_0,\]

which can be solved by further elimination of variables.

The solutions are

\[\bar{P}_1 = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1};\]

\[\bar{P}_2 = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}.\]

For these two values to make sense, certain restrictions should be imposed on the model. First, we require the common denominator \(c_1\gamma_2 - c_2\gamma_1 \neq 0\). Second, to assure positivity, the numerator must have the same sign as the denominator.

Numerical Example

Suppose that the demand and supply functions are numerically as follows:

\[Q_{d1} = 10 - 2P_1 + P_2;\]
CHAPTER 3. EQUILIBRIUM ANALYSIS IN ECONOMICS

\[ Q_{s1} = -2 + 3P_1; \]
\[ Q_{d2} = 15 + P_1 - P_2; \]
\[ Q_{s2} = -1 + 2P_2. \]

By substitution, we have

\[ 5P_1 - P_2 = 12; \]
\[ -P_1 + 3P_2 = 16, \]

which are two linear equations. The solutions for the equilibrium prices and quantities are \( \bar{P}_1 = 52/14, \bar{P}_2 = 92/14, \bar{Q}_1 = 64/7, \bar{Q}_2 = 85/7. \)

Similarly, for the \( n \)-commodities market model, when demand and supply functions are linear in prices, we can have \( n \) linear equations. In the above, we assume that an equal number of equations and unknowns has a unique solution. However, some very simple examples should convince us that an equal number of equations and unknowns does not necessarily guarantee the existence of a unique solution.

For the two linear equations,

\[
\begin{align*}
  x + y &= 8, \\
  x + y &= 9
\end{align*}
\]

we can easily see that there is no solution.

The second example shows a system has an infinite number of solutions:

\[
\begin{align*}
  2x + y &= 12; \\
  4x + 2y &= 24
\end{align*}
\]

These two equations are functionally dependent, which means that one
can be derived from the other. Consequently, one equation is redundant and may be dropped from the system. Any pair \((\bar{x}, \bar{y})\) is the solution as long as \((\bar{x}, \bar{y})\) satisfies \(y = 12 - x\).

Now consider the case of more equations than unknowns. In general, there is no solution. But, when the number of unknowns equals the number of functional independent equations, the solution exists and is unique. The following example shows this fact.

\[
2x + 3y = 58; \\
y = 18; \\
x + y = 20.
\]

Thus for simultaneous-equation model, we need systematic methods of testing the existence of a unique (or determinate) solution. There are our tasks in the following chapters.

### 3.5 Equilibrium in National-Income Analysis

The equilibrium analysis can be also applied to other areas of economics. As a simple example, we may cite the familiar Keynesian national-income model,

\[
Y = C + I_0 + G_0 \quad (\text{equilibrium condition});
\]

\[
C = a + bY \quad (\text{the consumption function}),
\]

where \(Y\) and \(C\) stand for the endogenous variables national income and consumption expenditure, respectively, and \(I_0\) and \(G_0\) represent the exogenously determined investment and government expenditures, respectively.

Solving these two linear equations, we obtain the equilibrium national
income and consumption expenditure:

\[ \bar{Y} = \frac{a + I_0 + G_0}{1 - b}, \]

\[ \bar{C} = \frac{a + b(I_0 + G_0)}{1 - b}. \]
From the last chapter we have seen that for the one-commodity model, the solutions for $\bar{P}$ and $\bar{Q}$ are relatively simple, even though a number of parameters are involved. As more and more commodities are incorporated into the model, such solutions formulas quickly become cumbersome and unwieldy. We need to have new methods suitable for handling a large system of simultaneous equations. Such a method is found in matrix algebra.

Matrix algebra can enable us to do many things, including: (1) It provides a compact way of writing an equation system, even an extremely large one. (2) It leads to a way of testing the existence of a solution by evaluation of a determinant – a concept closely related to that of a matrix. (3) It gives a method of finding that solution if it exists.
4.1 Matrix and Vectors

In general, a system of \( m \) linear equations in \( n \) variables \((x_1, x_2, \ldots, x_n)\) can be arranged into such formula

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1, \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2, \\
  \cdots & \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m,
\end{align*}
\]

(3.1)

where the double-subscripted symbol \( a_{ij} \) represents the coefficient appearing in the \( i \)th equation and attached to the \( j \)th variable \( x_j \) and \( d_j \) represents the constant term in the \( j \)th equation.

**Example 4.1.1** The two-commodity linear market model can be written – after eliminating the quantity variables – as a system of two linear equations.

\[
\begin{align*}
  c_1P_1 + c_2P_2 &= -c_0, \\
  \gamma_1P_1 + \gamma_2P_2 &= -\gamma_0.
\end{align*}
\]

**Matrix as Arrays**

There are essentially three types of ingredients in the equation system (3.1). The first is the set of coefficients \( a_{ij} \); the second is the set of variables \( x_1, x_2, \ldots, x_n \); and the last is the set of constant terms \( d_1, d_2, \ldots, d_m \). If we arrange the three sets as three rectangular arrays and label them, respectively, as \( A \), \( x \), and \( d \), then we have

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix},
\]
4.1. MATRIX AND VECTORS

\[ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad (3.2) \]

\[ d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}. \]

**Example 4.1.2** Given the linear-equation system

\[
\begin{align*}
6x_1 + 3x_2 + x_3 &= 22 \\
x_1 + 4x_2 - 2x_3 &= 12 \\
4x_1 - x_2 + 5x_3 &= 10
\end{align*}
\]

We can write

\[ A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}, \]

\[ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \]

\[ d = \begin{pmatrix} 22 \\ 12 \\ 10 \end{pmatrix}. \]

Each of three arrays given above constitutes a matrix.

A *matrix* is defined as a rectangular array of numbers, parameters, or variables. As a shorthand device, the array in matrix \( A \) can be written...
more simple as

\[ A = [a_{ij}]_{m \times n} \quad (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n). \]

**Vectors as Special Matrices**

The number of rows and number of columns in a matrix together define the dimension of the matrix. For instance, \( A \) is said to be of dimension \( m \times n \). In the special case where \( m = n \), the matrix is called a *square matrix*.

If a matrix contains only one column (row), it is called a *column (row) vector*. For notation purposes, a row vector is often distinguished from a column vector by the use of a primed symbol.

\[ x' = [x_1, x_2, \ldots, x_n]. \]

**Remark 4.1.1** A vector is merely an ordered \( n \)-tuple and as such it may be interpreted as a point in an \( n \)-dimensional space.

With the matrices defined in (3.2), we can express the equations system (3.1) simply as

\[ Ax = d. \]

However, the equation \( Ax = d \) prompts at least two questions. How do we multiply two matrices \( A \) and \( x \)? What is meant by the equality of \( Ax \) and \( d \)? Since matrices involve whole blocks of numbers, the familiar algebraic operations defined for single numbers are not directly applicable, and there is need for new set of operation rules.
4.2. MATRIX OPERATIONS

4.2 Matrix Operations

The Equality of Two Matrices

A = B if and only if $a_{ij} = b_{ij}$ for all $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$.

Addition and Subtraction of Matrices

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}],$$

i.e., the addition of $A$ and $B$ is defined as the addition of each pair of corresponding elements.

Remark 4.2.1 Two matrices can be added (equal) if and only if they have the same dimension.

Example 4.2.1

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}.$$

Example 4.2.2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}.$$

The Subtraction of Matrices:

$A - B$ is defined by

$$[a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}].$$

Example 4.2.3

$$\begin{bmatrix} 19 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 1 & -3 \end{bmatrix}.$$
Scalar Multiplication:

\[ \lambda A = \lambda [a_{ij}] = [\lambda a_{ij}], \]

i.e., to multiply a matrix by a number is to multiply every element of that matrix by the given scalar.

Example 4.2.4

\[
7 \begin{bmatrix}
3 & -1 \\
0 & 5
\end{bmatrix}
= \begin{bmatrix}
21 & -7 \\
0 & 35
\end{bmatrix}.
\]

Example 4.2.5

\[
-1 \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{bmatrix}
= \begin{bmatrix}
-a_{11} & -a_{12} & -a_{13} \\
-a_{21} & -a_{22} & -a_{23}
\end{bmatrix}.
\]

Multiplication of Matrices:

Given two matrices \( A_{m \times n} \) and \( B_{p \times q} \), the conformability condition for multiplication \( AB \) is that the column dimension of \( A \) must be equal to the row dimension of \( B \), i.e., the matrix product \( AB \) will be defined if and only if \( n = p \). If defined, the product \( AB \) will have the dimension \( m \times q \).

The product \( AB \) is defined by

\[ AB = C \]

with \( c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} = \sum_{l=1}^{n} a_{il} b_{lj} \).

Example 4.2.6

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
= \begin{bmatrix}
a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\
a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22}
\end{bmatrix}.
\]
Example 4.2.7

\[
\begin{bmatrix}
3 & 5 \\
4 & 6
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
4 & 7
\end{bmatrix}
= \begin{bmatrix}
-3 + 20 & 35 \\
-4 + 24 & 42
\end{bmatrix}
= \begin{bmatrix}
17 & 35 \\
20 & 42
\end{bmatrix}.
\]

Example 4.2.8

\[u' = [u_1, u_2, \cdots, u_n] \text{ and } v' = [v_1, v_2, \cdots, v_n],\]

\[u'v = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i.\]

This can be described by using the concept of the \textit{inner product} of two vectors \(u\) and \(v\).

\[v \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n = u'v.\]

Example 4.2.9

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\text{ and } x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix},
\]

\[Ax = \begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}.
\]

Example 4.2.10 Given \(u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\) and \(v' = [1, 4, 5]\),

\[uv' = \begin{bmatrix} 3 \times 1 & 3 \times 4 & 3 \times 5 \\ 2 \times 1 & 2 \times 4 & 2 \times 5 \end{bmatrix} = \begin{bmatrix} 3 & 12 & 15 \\ 2 & 8 & 10 \end{bmatrix}.\]
CHAPTER 4. LINEAR MODELS AND MATRIX ALGEBRA

It is important to distinguish the meaning of $uv'$ (a matrix with dimension larger than $1 \times 1$) and $u'v$ (a $1 \times 1$ matrix, or a scalar).

4.3 Linear Dependence of Vectors

A set of vectors $v_1, \cdots, v_n$ is said to be linearly dependent if and only if one of them can be expressed as a linear combination of the remaining vectors; otherwise they are linearly independent.

Example 4.3.1 The three vectors

$$v_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent since $v_3$ is a linear combination of $v_1$ and $v_2$;

$$3v_1 - 2v_2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = v_3,$$

or

$$3v_1 - 2v_2 - v_3 = 0,$$

where $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ represents a zero vector.

Example 4.3.2 The three vectors

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

are linearly dependent since $v_1$ is a linear combination of $v_2$ and $v_3$:

$$v_1 = \frac{1}{2}v_2 + \frac{1}{2}v_3.$$
A equivalent definition of linear dependence is: a set of \( m \)-vectors \( v_1, v_2, \cdots, v_n \) is linearly dependent if and only if there exists a set of scalars \( k_1, k_2, \cdots, k_n \) (not all zero) such that

\[
\sum_{i=1}^{n} k_i v_i = 0.
\]

If this equation can be satisfied only when \( k_i = 0 \) for all \( i \), these vectors are linearly independent.

### 4.4 Commutative, Associative, and Distributive Laws

The commutative and associative laws of matrix can be stated as follows:

**Commutative Law:**

\[ A + B = B + A. \]

**Proof:**

\[
A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A.
\]

**Associative Law:**

\[
(A + B) + C = A + (B + C).
\]

**Proof:**

\[
(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}] = [a_{ij} + b_{ij}] + [c_{ij}] = [a_{ij} + b_{ij} + c_{ij}] = [a_{ij} + (b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij} + c_{ij}]) = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C).
\]

**Matrix Multiplication**

Matrix multiplication is not commutative, that is,

\[ AB \neq BA. \]

Even when \( AB \) is defined, \( BA \) may not be; but even if both products
are defined, $AB = BA$ may not still hold.

Example 4.4.1 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$. Then $AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}$, but $BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$.

The scalar multiplication of a matrix does obey. The commutative law:

$$kA = Ak$$

if $k$ is a scalar.

**Associative Law:** $(AB)C = A(BC)$

if $A$ is $m \times n$, $B$ is $n \times p$, and $C$ is $p \times q$.

**Distributive Law**

$$A(B + C) = AB + AC \quad \text{[premultiplication by $A$];}$$

$$(B + C)A = BA + CA \quad \text{[postmultiplication by $A$].}$$

### 4.5 Identity Matrices and Null Matrices

*Identity matrix* is a square matrix with ones in its principal diagonal and zeros everywhere else. It is denoted by $I$ or $I_n$ in which $n$ indicates its dimension.

**Fact 1:** Given an $m \times n$ matrix $A$, we have

$$I_mA = AI_n = A.$$

**Fact 2:**

$$A_{m\times n}I_nB_{n\times p} = (AI)B = AB.$$
Fact 3:

\[(I_n)^k = I_n.\]

**Idempotent Matrices:** A matrix \(A\) is said to be idempotent if \(AA = A\).

**Null Matrices:** A null–or zero matrix–denoted by 0, plays the role of the number 0. A null matrix is simply a matrix whose elements are all zero. Unlike \(I\), the zero matrix is not restricted to being square. Null matrices obey the following rules of operation.

\[A_{m \times n} + 0_{m \times n} = A_{m \times n};\]
\[A_{m \times n}0_{n \times p} = 0_{m \times p};\]
\[0_{q \times m}A_{m \times n} = 0_{q \times n}.\]

**Remark 4.5.1** (a) \(CD = CE\) does not imply \(D = E\). For instance,

\[
C = \begin{bmatrix}
2 & 3 \\
6 & 9
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}, \quad E = \begin{bmatrix}
-2 & 1 \\
3 & 2
\end{bmatrix}.
\]

We can see that

\[CD = CE = \begin{bmatrix}
5 & 8 \\
15 & 24
\end{bmatrix},\]

even though \(D \neq E\).

(b) Even if \(A\) and \(B \neq 0\), we can still have \(AB = 0\).

**Example 4.5.1** \(A = \begin{bmatrix}
2 & 4 \\
1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
-2 & 4 \\
1 & -2
\end{bmatrix}.\)

### 4.6 Transposes and Inverses

The transpose of a matrix \(A\) is a matrix which is obtained by interchanging the rows and columns of the matrix \(A\). We say that \(B = [b_{ij}]_{n \times m}\) is the
transpose of $A = [a_{ij}]_{m \times n}$ if $a_{ji} = b_{ij}$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$.

Usually transpose is denoted by $A'$ or $A^T$.

**Recipe - How to Find the Transpose of a Matrix:**

The transpose $A'$ of $A$ is obtained by making the columns of $A$ into the rows of $A'$.

**Example 4.6.1** For $A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$, its transpose is

$A' = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix}$

Thus, by definition, if the dimension of a matrix $A$ is $m \times n$, then the dimension of its transpose $A'$ must be $n \times m$.

**Example 4.6.2** For $D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$,

its transpose is:

$D' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix} = D$.

A matrix $A$ is said to be symmetric if $A' = A$.

A matrix $A$ is called anti-symmetric (or skew-symmetric) if $A' = -A$.

A matrix $A$ is called orthogonal if $A'A = I$.

**Properties of Transposes:**

a) $(A')' = A$;
b) \((A + B)' = A' + B'\);

c) \((\alpha A)' = \alpha A'\) where \(\alpha\) is a real number;

d) \((AB)' = B'A'\).

The property d) states that the transpose of a product is the product of the transposes in reverse order.

**Inverses and Their Properties**

For a given square matrix \(A\), \(A'\) is always derivable. On the other hand, its inverse matrix may or may not exist. The *inverse* of \(A\), denoted by \(A^{-1}\), is defined only if \(A\) is a square matrix, in which case the inverse is the matrix that satisfies the condition:

\[ AA^{-1} = A^{-1}A = I. \]

**Remark 4.6.1** The following statements are true:

1. Not every square matrix has an inverse–squareness is a necessary but not sufficient condition for the existence of an inverse. If a square matrix \(A\) has an inverse, \(A\) is said to be *nonsingular*. If \(A\) possesses no inverse, it is said to be a *singular* matrix.

2. If \(A\) is nonsingular, then \(A\) and \(A^{-1}\) are inverse of each other, i.e., \((A^{-1})^{-1} = A\).

3. If \(A\) is \(n \times n\), then \(A^{-1}\) is also \(n \times n\).

4. The inverse of \(A\) is unique.

**Proof.** Let \(B\) and \(C\) both be inverses of \(A\). Then

\[ B = BI = BAC = IC = C. \]
5. $AA^{-1} = I$ implies $A^{-1}A = I$.

**Proof.** We need to show that if $AA^{-1} = I$, and if there is a matrix $B$ such that $BA = I$, then $B = A^{-1}$. Postmultiplying both sides of $BA = I$ by $A^{-1}$, we have $BAA^{-1} = A^{-1}$ and thus $B = A^{-1}$.

**Example 4.6.3** Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

So $B$ is the inverse of $A$.

6. Suppose $A$ and $B$ are nonsingular matrices with dimension $n \times n$.

(a) $(AB)^{-1} = B^{-1}A^{-1}$;

(b) $(A')^{-1} = (A^{-1})'$.

**Inverse Matrix and Solution of Linear-Equation System**

The application of the concept of inverse matrix to the solution of a simultaneous-equation is immediate and direct. Consider

$$Ax = d.$$ 

If $A$ is a nonsingular matrix, then premultiplying both sides of $Ax = d$, we have

$$A^{-1}Ax = A^{-1}d.$$ 

So, $x = A^{-1}d$ is the solution of $Ax = d$ and the solution is unique since $A^{-1}$ is unique. Methods of testing the existence of the inverse and of its calculation will be discussed in the next chapter.
Chapter 5

Linear Models and Matrix Algebra (Continued)

In chapter 4, it was shown that a linear-equation system can be written in a compact notation. Furthermore, such an equation system can be solved by finding the inverse of the coefficient matrix, provided the inverse exists. This chapter studies how to test for the existence of the inverse and how to find that inverse, and consequently give the ways of solving linear equation systems.

5.1 Conditions for Nonsingularity of a Matrix

As was pointed out earlier, the squareness condition is necessary but not sufficient for the existence of the inverse $A^{-1}$ of a matrix $A$.

Conditions for Nonsingularity

When the squareness condition is already met, a sufficient condition for the nonsingularity of a matrix is that its rows (or equivalently, its columns) are linearly independent. In fact, the necessary and sufficient conditions for non-
singularity are that the matrix satisfies the squareness and linear independence conditions.

An $n \times n$ coefficient matrix $A$ can be considered an ordered set of row vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{bmatrix}$$

where $v'_i = [a_{i1}, a_{i2}, \cdots, a_{in}]$, $i = 1, 2, \cdots, n$. For the rows to be linearly independent, for any set of scalars $k_i$, $\sum_{i=1}^n k_i v_i = 0$ if and only if $k_i = 0$ for all $i$.

**Example 5.1.1** For a given matrix,

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix},$$

since $v'_3 = 2v'_1 + 0v'_2$, so the matrix is singular.

**Example 5.1.2** $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is nonsingular.

**Example 5.1.3** $C = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$ is singular.

**Rank of a Matrix**

Even though the concept of row independence has been discussed only with regard to square matrices, it is equally applicable to any $m \times n$ rectangular matrix. If the maximum number of linearly independent rows that can be found in such a matrix is $\gamma$, then matrix is said to be of rank $\gamma$. The rank
also tells us the maximum number of linearly independent columns in the matrix. Then rank of an $m \times n$ matrix can be at most $m$ or $n$, whichever is smaller.

By definition, an $n \times n$ nonsingular matrix $A$ has $n$ linearly independent rows (or columns); consequently it must be of rank $n$. Conversely, an $n \times n$ matrix having rank $n$ must be nonsingular.

### 5.2 Test of Nonsingularity by Use of Determinant

To determine whether a square matrix is nonsingular, we can make use of the concept of determinant.

**Determinant and Nonsingularity**

The determinant of a square matrix $A$, denoted by $|A|$, is a uniquely defined scalar associated with that matrix. Determinants are defined only for square matrices. For a $2 \times 2$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

its determinant is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$ 

In view of the dimension of matrix $A$, $|A|$ as defined in the above is called a second-order determinant.
Example 5.2.1 Given \( A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} \), then

\[
|A| = \begin{vmatrix} 10 & 4 \\ 8 & 5 \end{vmatrix} = 50 - 32 = 18;
\]

\[
|B| = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = -3 - 5 \times 0 = -3.
\]

Example 5.2.2 \( A = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix} \). Then its determinant

\[
|A| = \begin{vmatrix} 2 & 6 \\ 8 & 24 \end{vmatrix} = 2 \times 24 - 6 \times 8 = 48 - 48 = 0.
\]

This example shows that the determinant is equal to zero if and only if its rows are linearly dependent. As will be seen, the value of a determinant \( |A| \) can serve as a criterion for testing the linear independence of the rows (hence nonsingularity) of matrix \( A \) but also as an input in the calculation of the inverse \( A^{-1} \), if it exists.

Evaluating a Third-Order Determinant

For a 3 \( \times \) 3 matrix \( A \), its determinants have the value

\[
|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

\[
= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.
\]

We can use the following diagram to calculate the third-order determi-
Example 5.2.3

\[
\begin{vmatrix}
2 & 1 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix} = 2 \times 5 \times 9 + 1 \times 6 \times 7 + 4 \times 8 \times 3 - 3 \times 5 \times 7 - 1 \times 4 \times 9 - 6 \times 8 \times 2 \\
= 90 + 42 + 96 - 105 - 36 - 96 = -9.
\]

Example 5.2.4

\[
\begin{vmatrix}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8
\end{vmatrix} = 0 \times 4 \times 8 + 1 \times 5 \times 6 + 3 \times 7 \times 2 - 2 \times 4 \times 6 - 1 \times 3 \times 8 - 5 \times 7 \times 0 \\
= 0 + 30 + 42 - 48 - 24 - 0 = 0.
\]
Example 5.2.5

\[
\begin{vmatrix}
-1 & 2 & 1 \\
0 & 3 & 2 \\
1 & 0 & 2
\end{vmatrix} = -1 \times 3 \times 2 + 2 \times 2 \times 1 + 0 \times 0 \times 1 - 1 \times 3 \times 1 - 2 \times 0 \times 2 - 2 \times 0 \times (-1)
\]

\[
= -6 + 4 + 0 - 3 - 0 - 6 = -5.
\]

The method of cross-diagonal multiplication provides a handy way of evaluating a third-order determinant, but unfortunately it is not applicable to determinants of orders higher than 3. For the latter, we must resort to the so-called “Laplace expansion” of the determinant.

**Evaluating an \( n \)th-Order Determinant by Laplace Expansion**

The *minor* of the element \( a_{ij} \) of a determinant \( |A| \), denoted by \( |M_{ij}| \), can be obtained by deleting the \( i \)th row and \( j \)th column of the determinant \( |A| \), the minors of \( a_{11}, a_{12} \) and \( a_{13} \) are

\[
|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad |M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad |M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.
\]

A concept closely related to the minor is that of the cofactor. A cofactor, denoted by \( C_{ij} \), is a minor with a prescribed algebraic sign attached to it. Formally, it is defined by

\[
|C_{ij}| = (-1)^{i+j} |M_{ij}| = \begin{cases} -|M_{ij}| & \text{if } i + j \text{ is odd;} \\
|M_{ij}| & \text{if } i + j \text{ is even.}
\end{cases}
\]

Thus, if the sum of the two subscripts \( i \) and \( j \) in \( M_{ij} \) is even, then \( |C_{ij}| = |M_{ij}| \). If it is odd, then \( |C_{ij}| = -|M_{ij}| \). Using these new concepts, we can
express a third-order determinant as

\[
|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|
= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}|.
\]

The Laplace expansion of a third-order determinant serves to reduce the evaluation problem to one of evaluating only certain second-order determinants. In general, the Laplace expansion of an \(n\)th-order determinant will reduce the problem to one of evaluating \(n\) cofactors, each of which is of the \((n-1)\)th order, and the repeated application of the process will methodically lead to lower and lower orders of determinants, eventually culminating in the basic second-order determinants. Then the value of the original determinant can be easily calculated. Formally, the value of a determinant \(|A|\) of order \(n\) can be found by the Laplace expansion of any row or any column as follows:

\[
|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the \(i\)th row]}
= \sum_{i=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the \(j\)th column]}
\]

Even though one can expand \(|A|\) by any row or any column, as the numerical calculation is concerned, a row or column with largest number of 0’s or 1’s is always preferable for this purpose, because a 0 times its cofactor is simply 0.

**Example 5.2.6** For the \(|A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}\), the easiest way to expand the determinant is by the third column, which consists of the elements 1, 0, and
0. Thus,

\[ |A| = 1 \times (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -6 - 21 = -27. \]

Example 5.2.7

\[
\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix} = 1 \times (-1)^{1+4} \begin{vmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{vmatrix} = -1 \times (-24) = 24.
\]

Example 5.2.8

\[
\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \times (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}
\]

\[
= a_{11} \times a_{22} \times (-1)^{1+1} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}
\]

\[
= \cdots = a_{11} \times a_{22} \times s a_{nn}.
\]

5.3 Basic Properties of Determinants

Property I. The determinant of a matrix \( A \) has the same value as that of its transpose \( A' \), i.e.,

\[ |A| = |A'|. \]
Example 5.3.1

\[ |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \]

\[ |A'| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = |A|. \]

Property II. The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

Example 5.3.2

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \]

but the interchange of the two rows yields

\[ \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc). \]

Property III. The multiplication of any one row (or one column) by a scalar \( k \) will change the value of the determinant \( k \)-fold, i.e., for \( |A| \),

\[ \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{11} & ka_{12} & \cdots & ka_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k|A|. \]

In contrast, the factoring of a matrix requires the presence of a common divisor for all its elements, as in

\[ k \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{bmatrix}. \]
Property IV. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the value of the determinant unaltered.

This is an extremely useful property, which can be used to greatly simplify the computation of a determinant.

Example 5.3.3

\[
\begin{vmatrix}
  a & b \\
  c + ka & d + kb
\end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix}.
\]

Example 5.3.4

\[
\begin{vmatrix}
  a & b & b & b \\
  b & a & b & b \\
  b & b & a & b \\
  b & b & b & a
\end{vmatrix} = \begin{vmatrix}
  a + 3b & b & b & b \\
  a + 3b & a & b & b \\
  a + 3b & b & a & b \\
  a + 3b & b & b & a
\end{vmatrix} = (a + 3b)(a - b)^3.
\]

Example 5.3.5

\[
\begin{vmatrix}
  1 & 2 & 3 & 4 \\
  2 & 3 & 4 & 1 \\
  3 & 4 & 1 & 2 \\
  4 & 1 & 2 & 3
\end{vmatrix} = \begin{vmatrix}
  10 & 2 & 3 & 4 \\
  10 & 3 & 4 & 1 \\
  10 & 4 & 1 & 2 \\
  10 & 1 & 2 & 3
\end{vmatrix} = -168.
\]

Property V. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example 5.3.6

\[
\begin{vmatrix}
  ka & kb \\
  a & b
\end{vmatrix} = kab - kab = 0.
\]
Example 5.3.7
\[
\begin{vmatrix}
-2 & 5 & -1 & 3 \\
1 & -9 & 13 & 7 \\
3 & -1 & 5 & -5 \\
2 & 8 & -7 & -10
\end{vmatrix} = 312
\]

Remark 5.3.1 Property V is a logic consequence of Property IV.

Property VI. If \(A\) and \(B\) are both square matrices, then \(|AB| = |A||B|\).

The above basic properties just discussed are useful in several ways. For one thing, they can be of great help in simplifying the task of evaluating determinants. By subtracting multipliers of one row (or column) from another, for instance, the elements of the determinant may be reduced to much simpler numbers. If we can indeed apply these properties to transform some row or column into a form containing mostly 0’s or 1’s, Laplace expansion of the determinant will become a much more manageable task.

Recipe - How to Calculate the Determinant:

1. The multiplication of any one row (or column) by a scalar \(k\) will change the value of the determinant \(k\)-fold.

2. The interchange of any two rows (columns) will alter the sign but not the numerical value of the determinant.

3. If a multiple of any row is added to (or subtracted from) any other row it will not change the value or the sign of the determinant. The same holds true for columns (i.e. the determinant is not affected by linear operations with rows (or columns)).

4. If two rows (or columns) are proportional, i.e., they are linearly dependent, then the determinant will vanish.

5. The determinant of a triangular matrix is a product of its principal diagonal elements.
Using these rules, we can simplify the matrix (e.g. obtain as many zero elements as possible) and then apply Laplace expansion.

**Determinantal Criterion for Nonsingularity**

Our present concern is primarily to link the linear dependence of rows with the vanishing of a determinant. By property I, we can easily see that row independence is equivalent to column independence.

Given a linear-equation system $Ax = d$, where $A$ is an $n \times n$ coefficient matrix, we have

$$|A| \neq 0 \iff A \text{ is row (or column) independent}$$

$$\iff \text{rank}(A) = n$$

$$\iff A \text{ is nonsingular}$$

$$\iff A^{-1} \text{ exists}$$

$$\iff \text{a unique solution } \bar{x} = A^{-1}d \text{ exists.}$$

Thus the value of the determinant of $A$ provides a convenient criterion for testing the nonsingularity of matrix $A$ and the existence of a unique solution to the equation system $Ax = d$.

**Rank of a Matrix Redefined**

The rank of a matrix $A$ was earlier defined to be the maximum number of linearly independent rows in $A$. In view of the link between row independence and the nonvanishing of the determinant, we can redefine the rank of an $m \times n$ matrix as the maximum order of a nonvanishing determinant that can be constructed from the rows and columns of that matrix. The rank of any matrix is a unique number.

Obviously, the rank can at most be $m$ or $n$ for a $m \times n$ matrix $A$, whichever...
er is smaller, because a determinant is defined only for a square matrix. Symbolically, this fact can be expressed as follows:

\[
\gamma(A) \leq \min\{m, n\}.
\]

**Example 5.3.8**

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 6 & 4 \\
-5 & 7 & 1
\end{bmatrix}
\]

\[\text{rank} = 2.\]

The first two rows are linearly dependent, but the last two are independent, therefore the maximum number of linearly independent rows is equal to 2.

Properties of the rank:

1) The column rank and the row rank of a matrix are equal.

2) \(\text{rank}(AB) \leq \min\{\text{rank}(A); \text{rank}(B)\}\).

3) \(\text{rank}(A) = \text{rank}(AA') = \text{rank}(A'A)\).

### 5.4 Finding the Inverse Matrix

If the matrix \(A\) in the linear-equation system \(Ax = d\) is nonsingular, then \(A^{-1}\) exists, and the solution of the system will be \(\bar{x} = A^{-1}d\). We have learned to test the nonsingularity of \(A\) by the criterion \(|A| \neq 0\). The next question is how we can find the inverse \(A^{-1}\) if \(A\) does pass that test.
Expansion of a Determinant by Alien Cofactors

We have known that the value of a determinant \( |A| \) of order \( n \) can be found by the Laplace expansion of any row or any column as follows;

\[
|A| = \sum_{j=1}^{n} a_{ij} |C_{ij}| \quad \text{[expansion by the } i \text{th row]}
\]

\[
= \sum_{i=1}^{n} a_{ij} |C_{ij}| \quad \text{[expansion by the } j \text{th column]}
\]

Now what happens if we replace \( a_{ij} \) by \( a_{i'j} \) for \( i \neq i' \) or by \( a_{ij'} \) for \( j \neq j' \). Then we have the following important property of determinants.

**Property VI.** The expansion of a determinant by alien cofactors (the cofactors of a “wrong” row or column) always yields a value of zero. That is, we have

\[
\sum_{j=1}^{n} a_{i'j} |C_{ij}| = 0 \ (i \neq i') \quad \text{[expansion by the } i' \text{th row and use of cofactors of } i \text{th row]}
\]

\[
\sum_{j=1}^{n} a_{ij'} |C_{ij}| = 0 \ (j \neq j') \quad \text{[expansion by the } j' \text{th column and use of cofactors of } j \text{th column]}
\]

The reason for this outcome lies in the fact that the above formula can be considered as the result of the regular expansion by the \( i \)th row (\( j \)th column) and \( i' \)th row (\( j' \)th column) are identical.

**Example 5.4.1** For the determinant

\[
|A| = \begin{vmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{vmatrix},
\]
5.4. FINDING THE INVERSE MATRIX

consider another determinant

\[ |A^*| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \]

If we expand \( |A^*| \) by the second row, then we have

\[ 0 = |A^*| = a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}| = \sum_{j=1}^{3} a_{1j}|C_{2j}|. \]

Matrix Inversion

Property VI is of finding the inverse of a matrix. for a \( n \times n \) matrix \( A \):

\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \]

since each element of \( A \) has a cofactor \( |C_{ij}| \), we can form a matrix of cofactors by replacing each element \( a_{ij} \) with its cofactor \( |C_{ij}| \). Such a cofactor matrix \( C = [|C_{ij}|] \) is also \( n \times n \). For our present purpose, however, the transpose of \( C \) is of more interest. This transpose \( C' \) is commonly referred to as the adjoint of \( A \) and is denoted by \( \text{adj} \ A \). Thus,

\[ C' \equiv \text{adj} \ A \equiv \begin{vmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{vmatrix}. \]

By utilizing the formula for the Laplace expansion and Property VI, we
have

\[
AC' = \begin{bmatrix}
\sum_{j=1}^{n} a_{1j} |C_{1j}| & \sum_{j=1}^{n} a_{1j} |C_{2j}| & \cdots & \sum_{j=1}^{n} a_{1j} |C_{nj}| \\
\sum_{j=1}^{n} a_{2j} |C_{1j}| & \sum_{j=1}^{n} a_{2j} |C_{2j}| & \cdots & \sum_{j=1}^{n} a_{2j} |C_{nj}| \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^{n} a_{nj} |C_{1j}| & \sum_{j=1}^{n} a_{nj} |C_{2j}| & \cdots & \sum_{j=1}^{n} a_{nj} |C_{nj}| \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
|A| & 0 & \cdots & 0 \\
0 & |A| & \cdots & 0 \\
\cdots & \cdots & \ddots & \cdots \\
0 & 0 & \cdots & |A| \\
\end{bmatrix}
\]

\[
= |A| I_n.
\]

Thus, by the uniqueness of $A^{-1}$ of $A$, we know

\[
A^{-1} = \frac{C'}{|A|} = \frac{\text{adj } A}{|A|}.
\]

Now we have found a way to invert the matrix $A$.

**Remark 5.4.1** The general procedures for finding the inverse of a square $A$ are:

1. find $|A|$;
2. find the cofactors of all elements of $A$ and form $C = [ |C_{ij}| ]$;
3. form $C'$ to get $\text{adj } A$; and
4. determine $A^{-1}$ by $\frac{\text{adj } A}{|A|}$.

In particular, for a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have the following
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formula:

\[ A^{-1} = \frac{\text{adj} A}{|A|} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \]

which is a very useful formula.

**Example 5.4.2**  \( A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \).

The inverse of \( A \) is given by

\[ A^{-1} = \frac{1}{-2} \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}. \]

**Example 5.4.3**  Find the inverse of \( B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix} \).

Since \( |B| = 99 \neq 0 \), \( B^{-1} \) exists. The cofactor matrix is

\[ \begin{vmatrix} 3 & 2 & 0 & 2 & 0 & 3 \\ 0 & 7 & 3 & 7 & 3 & 0 \\ -1 & -1 & 4 & -1 & -4 & 1 \\ 0 & 7 & 3 & 7 & 3 & 0 \\ 1 & -1 & 4 & -1 & 4 & 1 \\ 3 & 2 & 0 & 2 & 0 & 3 \end{vmatrix} = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}. \]

Then

\[ \text{adj} \ B = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}. \]
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Hence

\[
B^{-1} = \frac{1}{99} \begin{bmatrix}
21 & -7 & 5 \\
6 & 31 & -8 \\
-9 & 3 & 12
\end{bmatrix}
\]

Example 5.4.4 \( A = \begin{bmatrix}
2 & 4 & 5 \\
0 & 3 & 0 \\
1 & 0 & 1
\end{bmatrix} \).

We have \(|A| = -9\) and

\[
A^{-1} = -\frac{1}{9} \begin{bmatrix}
3 & -4 & -15 \\
0 & -3 & 0 \\
-3 & 4 & 6
\end{bmatrix}
\]

5.5 Cramer’s Rule

The method of matrix inversion just discussed enables us to derive a convenient way of solving a linear-equation system, known as Cramer’s rule.

Derivation of the Rule

Given an equation system \( Ax = d \), the solution can be written as

\[
\bar{x} = A^{-1}d = \frac{1}{|A|} (\text{adj} A)d
\]
provided \( A \) is nonsingular. Thus,

\[
x = \frac{1}{|A|}
\begin{vmatrix}
|C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\
|C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\
\vdots & \vdots & \ddots & \vdots \\
|C_{1n}| & |C_{2n}| & \cdots & |C_{nn}|
\end{vmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{bmatrix}
\]

\[
= \frac{1}{|A|}
\begin{bmatrix}
\sum_{i=1}^{n} d_i |C_{i1}| \\
\sum_{i=1}^{n} d_i |C_{i2}| \\
\vdots \\
\sum_{i=1}^{n} d_i |C_{in}|
\end{bmatrix}
\]

That is, the \( \bar{x}_i \) is given by

\[
\bar{x}_j = \frac{1}{|A|} \sum_{i=1}^{n} d_i |C_{ij}|
\]

\[
= \frac{1}{|A|}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn}
\end{vmatrix}
\]

\[
= \frac{1}{|A|} |A_j|,
\]

where \( |A_j| \) is obtained by replacing the \( j \)th column of \( |A| \) with constant terms \( d_1, \cdots, d_n \). This result is the statement of Cramer’s rule.

**Example 5.5.1** Let us solve

\[
\begin{bmatrix}
2 & 3 \\
4 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
10
\end{bmatrix}
\]

for \( x_1, x_2 \) using Cramer’s rule:

\[
|A| = -14, \quad |A_1| = -42, \quad |A_2| = -28,
\]
therefore

\[
\begin{align*}
x_1 &= \frac{-42}{-14} = 3, \\
x_2 &= \frac{-28}{-14} = 2.
\end{align*}
\]

Example 5.5.2

\[
\begin{align*}
5x_1 + 3x_2 &= 30; \\
6x_1 - 2x_2 &= 8.
\end{align*}
\]

We then have

\[
\begin{align*}
|A| &= \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28; \\
|A_1| &= \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84; \\
|A_2| &= \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140.
\end{align*}
\]

Therefore, by Cramer’s rule, we have

\[
\begin{align*}
\bar{x}_1 &= \frac{|A_1|}{|A|} = \frac{-84}{-28} = 3 \quad \text{and} \quad \bar{x}_2 = \frac{|A_2|}{|A|} = \frac{-140}{-28} = 5.
\end{align*}
\]

Example 5.5.3

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
12x_1 + 2x_2 - 3x_3 &= 5 \\
3x_1 + 4x_2 + x_3 &= -4.
\end{align*}
\]
In the form of matrix

\[
\begin{bmatrix}
1 & 1 & 1 \\
12 & 2 & -3 \\
3 & 4 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
=
\begin{bmatrix}
0 \\
5 \\
-4 \\
\end{bmatrix}.
\]

We then have

\[|A| = 35, \quad |A_3| = 35, \text{ and thus } x_3 = 1.\]

Example 5.5.4

\[
\begin{align*}
7x_1 - x_2 - x_3 &= 0 \\
10x_1 - 2x_2 + x_3 &= 8 \\
6x_1 + 3x_2 - 2x_3 &= 7.
\end{align*}
\]

We then have

\[|A| = -61, \quad |A_1| = -61, \quad |A_2| = -183, \quad |A_3| = -244.\]

Thus \(\bar{x}_1 = |A_1|/|A| = 1, \quad \bar{x}_2 = |A_2|/|A| = 3, \quad \bar{x}_3 = |A_3|/|A| = 4.\]

Note on Homogeneous-Equation System

A linear-equation system \(Ax = d\) is said to be a homogeneous-equation system if \(d = 0\), i.e., if \(Ax = 0\). If \(|A| \neq 0\), \(\bar{x} = 0\) is a unique solution of \(Ax = 0\) since \(\bar{x} = A^{-1}0 = 0\). This is a “trivial solution.” Thus, the only way to get a nontrivial solution from the homogeneous-equation system is to have \(|A| = 0\), i.e., \(A\) is singular. In this case, Cramer’s rule is not applicable. Of course, this does not mean that we cannot obtain solutions; it means only that we cannot get a unique solution. In fact, it has an infinite number of solutions.
Example 5.5.5

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 &= 0, \\
  a_{21}x_2 + a_{22}x_2 &= 0.
\end{align*}
\]

If \( |A| = 0 \), then its rows are linearly dependent. As a result, one of two equations is redundant. By deleting, say, the second equation, we end up with one equation with two variables. The solutions are

\[
\tilde{x}_1 = -\frac{a_{12}}{a_{11}} x_2 \text{ if } a_{11} \neq 0
\]

Overview on Solution Outcomes for a linear-Equation System

A necessary and sufficient condition for the existence of solution for a linear-equation system \( Ax = d \) is that the rank of \( A \) and the rank of the added matrix \([A; d]\) are the same, i.e.,

\[
r(A) = r([A; d]).
\]

For a linear-equation system \( Ax = d \), our discussion can be summarized as in the following table.
5.6 Application to Market and National-Income Models

Market Model:

The two-commodity model described in chapter 3 can be written as follows:

\[ c_1 P_1 + c_2 P_2 = -c_0, \]
\[ \gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0. \]

Thus

\[
|A| = \begin{vmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} = c_1 \gamma_2 - c_2 \gamma_1,
\]
\[
|A_1| = \begin{vmatrix} -c_0 & c_2 \\ \gamma_0 & \gamma_2 \end{vmatrix} = c_1 \gamma_2 - c_2 \gamma_1,
\]
\[
|A_2| = \begin{vmatrix} -c_1 & c_0 \\ \gamma_1 & \gamma_0 \end{vmatrix} = c_1 \gamma_0 - c_0 \gamma_1.
\]
Thus the equilibrium is given by

\[ \bar{P}_1 = \frac{|A_1|}{|A|} = \frac{c_2 \gamma_0 - c_0 \gamma_2}{c_1 \gamma_1 - c_2 \gamma_1} \quad \text{and} \quad \bar{P}_2 = \frac{|A_2|}{|A|} = \frac{c_0 \gamma_1 - c_1 \gamma_0}{c_1 \gamma_1 - c_2 \gamma_1}. \]

**General Market Equilibrium Model:**

Consider a market for three goods. Demand and supply for each good are given by:

\[
\begin{align*}
D_1 &= 5 - 2P_1 + P_2 + P_3, \\
S_1 &= -4 + 3P_1 + 2P_2.
\end{align*}
\]

\[
\begin{align*}
D_2 &= 6 + 2P_1 - 3P_2 + P_3, \\
S_2 &= 3 + 2P_2.
\end{align*}
\]

\[
\begin{align*}
D_3 &= 20 + P_1 + 2P_2 - 4P_3, \\
S_3 &= 3 + P_2 + 3P_3,
\end{align*}
\]

where \( P_i \) is the price of good \( i; i = 1; 2; 3 \).

The equilibrium conditions are: \( D_i = S_i; i = 1; 2; 3 \), that is

\[
\begin{align*}
5P_1 + P_2 - P_3 &= 9, \\
-2P_1 + 5P_2 - P_3 &= 3, \\
-P_1 - P_2 + 7P_3 &= 17.
\end{align*}
\]

This system of linear equations can be solved via Cramer’s rule

\[ P_1 = \frac{|A_1|}{|A|} = \frac{356}{178} = 2, \quad P_2 = \frac{|A_1|}{|A|} = \frac{356}{178} = 2, \quad P_1 = \frac{|A_1|}{|A|} = \frac{534}{178} = 3. \]
National-Income Model

\[ Y = C + I_0 + G_0, \]
\[ C = a + bY \ (a > 0, 0 < b < 1). \]

These can be rearranged into the form

\[ Y - C = I_0 + G_0, \]
\[ -bY + C = a. \]

While we can solve \( \bar{Y} \) and \( \bar{C} \) by Cramer’s rule, here we solve this model by inverting the coefficient matrix.

Since \( A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \), then \( A^{-1} = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix} \).

Hence

\[
\begin{bmatrix} \bar{Y} \\ \bar{C} \end{bmatrix} = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix} = \frac{1}{1-b} \begin{bmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{bmatrix}.
\]
5.7 Quadratic Forms

Quadratic Forms

A function $q$ with $n$-variables is said to have the quadratic form if it can be written as

$$q(u_1, u_2, \cdots, u_n) = d_{11}u_1^2 + 2d_{12}u_1u_2 + \cdots + 2d_{1n}u_1u_n$$
$$+ d_{22}u_2^2 + 2d_{23}u_2u_3 + \cdots + 2d_{2n}u_2u_n$$
$$\cdots$$
$$+ d_{nn}u_n^2.$$ 

If we let $d_{ji} = d_{ij}, i < j$, then $q(u_1, u_2, \cdots, u_n)$ can be written as

$$q(u_1, u_2, \cdots, u_n) = d_{11}u_1^2 + d_{12}u_1u_2 + \cdots + d_{1n}u_1u_n$$
$$+ d_{12}u_2^2 + d_{22}u_2^2 + \cdots + d_{2n}u_2u_n$$
$$\cdots$$
$$+ d_{n1}u_nu_1 + d_{n2}u_nu_2 + \cdots + d_{nn}u_n^2$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}u_iu_j$$
$$= u'Du,$$

where

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix},$$

which is called quadratic-form matrix.

Since $d_{ij} = d_{ji}$, $D$ is a symmetric matrix.

**Example 5.7.1** A quadratic form in two variables: $q = d_{11}u_1^2 + d_{12}u_1u_2 +$
5.7. QUADRATIC FORMS

\[ d_{22}u^2. \] The symmetric matrix is

\[
\begin{bmatrix}
  d_{11} & d_{12}/2 \\
  d_{12}/2 & d_{22}
\end{bmatrix}.
\]

Then we have \( q = u'Du. \)

**Positive and Negative Definiteness:**

A quadratic form \( q(u_1, u_2, \ldots, u_n) = u'Du \) is said to be

(a) **positive definite (PD)** if \( q(u) > 0 \) for all \( u \neq 0; \)

(b) **positive semidefinite (PSD)** if \( q(u) \geq 0 \) for all \( u \neq 0; \)

(c) **negative definite (ND)** if \( q(u) < 0 \) for all \( u \neq 0; \)

(d) **negative semidefinite (NSD)** if \( q(u) \leq 0 \) for all \( u \neq 0. \)

Otherwise \( q \) is called indefinite (ID).

Sometimes, we say that a matrix \( D \) is, for instance, positive definite if the corresponding quadratic form \( q(u) = u'Du \) is positive definite.

**Example 5.7.2** \( q = u^2_1 + u^2_2 \) is PD, \( q = (u_1 + u_2)^2 \) is PSD, \( q = u^2_1 - u^2_2 \) is ID.

**Determinantal Test for Sign Definiteness:**

We state without proof that for the quadratic form \( q(u) = u'Du, \) the necessary and sufficient condition for positive definiteness is the principal minors of \( |D|, \) namely,

\[
|D_1| = d_{11} > 0,
\]

\[
|D_2| = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} > 0,
\]

\[ \ldots \]
The corresponding necessary and sufficient condition for negative definiteness is that the principal minors alternate in sign as follows:

\[ |D_1| < 0, \ |D_2| > 0, \ |D_3| < 0, \ \text{etc.} \]

### Two-Variable Quadratic Form

**Example 5.7.3** Is \( q = 5u^2 + 3uv + 2v^2 \) either positive or negative? The symmetric matrix is

\[
\begin{bmatrix}
5 & 1.5 \\
1.5 & 2
\end{bmatrix}
\]

Since the principal minors of \( |D| \) is \( |D_1| = 5 \) and

\[
|D_2| = \begin{vmatrix}
5 & 1.5 \\
1.5 & 2
\end{vmatrix} = 10 - 2.25 = 7.75 > 0,
\]

so \( q \) is positive definite.

### Three-Variable Quadratic Form

**Example 5.7.4** Determine whether \( q = u_1^2 + 6u_2^2 + 3u_3^2 - 2u_1u_2 - 4u_2u_3 \) is positive or negative definite. The matrix \( D \) corresponding this quadratic form is

\[
D = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 6 & -2 \\
0 & -2 & 3
\end{bmatrix},
\]
and the principal minors of \( |D| \) are \( |D_1| = 1 > 0, \)

\[
|D_2| = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 6 - 1 = 5,
\]

and

\[
|D_3| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 11 > 0.
\]

Thus, the quadratic form is positive definite.

**Example 5.7.5** Determine whether \( q = -3u_1^2 - 3u_2^2 - 5u_3^2 - 2u_1u_2 \) is positive or negative definite. The matrix \( D \) corresponding this quadratic form is

\[
D = \begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}.
\]

Leading principal minors of \( D \) are

\[
|D_1| = -3 < 0, \quad |D_2| = 8 > 0, \quad |D_3| = -40 < 0,
\]

and therefore, the quadratic form is negative definite.

### 5.8 Eigenvalues and Eigenvectors

Consider the matrix equation:

\[
Dx = \lambda x.
\]

Any number \( \lambda \) such that the equation \( Dx = \lambda x \) has a non-zero vector-solution \( x \) is called an eigenvalue (or a characteristic root) of the above e-
Any non-zero vector \( x \) satisfying the above equation is called an eigenvector (or characteristic vector) of \( D \) for the eigenvalue \( \lambda \).

**Recipe** - How to calculate eigenvalues:

From \( Dx = \lambda x \), we have

\[
(D - \lambda I)x = 0.
\]

Since \( x \) is non-zero, the determinant of \( (D - \lambda I) \) should vanish. Therefore all eigenvalues can be calculated as roots of the equation (which is often called the characteristic equation or the characteristic polynomial of \( D \))

\[
|D - \lambda I| = 0.
\]

**Example 5.8.1** Let

\[
D = \begin{bmatrix}
3 & -1 & 0 \\
-1 & 3 & 0 \\
0 & 0 & 5
\end{bmatrix}
\]

\[
|D - \lambda I| = \begin{vmatrix}
3 - \lambda & -1 & 0 \\
-1 & 3 - \lambda & 0 \\
0 & 0 & 5 - \lambda
\end{vmatrix} = (5 - \lambda)(\lambda - 2)(\lambda - 4) = 0,
\]

and therefore the eigenvalues are \( \lambda_1 = 2, \lambda_2 = 4, \) and \( \lambda_3 = 5 \).

**Properties of Eigenvalues:**

A quadratic form \( q(u_1, u_2, \cdots, u_n) = u'Du \) is

- **positive definite** if and only if eigenvalues \( \lambda_i > 0 \) for all \( i = 1, 2, \cdots, n \).

- **negative definite** if and only if eigenvalues \( \lambda_i < 0 \) for all \( i = 1, 2, \cdots, n \).
positive semidefinite if and only if eigenvalues \( \lambda_i \geq 0 \) for all 
\[ i = 1, 2, \ldots, n. \]

negative semidefinite if and only if eigenvalues \( \lambda_i \leq 0 \) for all 
\[ i = 1, 2, \ldots, n. \]

A form is indefinite if at least one positive and one negative eigenvalues exist.

Matrix \( A \) is diagonalizable if there exists a non-singular matrix \( P \) and a diagonal matrix \( D \) such that
\[ P^{-1}AP = D. \]

Matrix \( U \) is orthogonal matrix if \( U' = U^{-1} \).

**Theorem 5.8.1 (The Spectral Theorem for Symmetric Matrices)** Suppose that \( A \) is a symmetric matrix of order \( n \) and \( \lambda_1, \ldots, \lambda_n \) are its eigenvalues. Then there exists an orthogonal matrix \( U \) such that
\[ U^{-1}AU = \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix}. \]

Usually, \( U \) is the normalized matrix formed by eigenvectors. It has the property \( U'U = I \). “Normalized” means that for any column \( u \) of the matrix \( U \), \( u'u = 1 \).

**Example 5.8.2** Diagonalize the matrix
\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}. \]
First, we need to find the eigenvalues:

\[
\begin{vmatrix}
1 - \lambda & 2 \\
2 & 4 - \lambda \\
\end{vmatrix} = \lambda(\lambda - 5) = 0,
\]

i.e., \( \lambda_1 = 0 \) and \( \lambda_2 = 5 \).

For \( \lambda_1 = 0 \), we solve

\[
\begin{bmatrix}
1 - 0 & 2 \\
2 & 4 - 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
\end{bmatrix}.
\]

The eigenvector, corresponding to \( \lambda_1 = 0 \) is \( v_1 = C_1 \cdot (2, -1)' \), where \( C_1 \) is an arbitrary real constant. Similarly, for \( \lambda_2 = 5 \), \( v_2 = C_2 \cdot (1, 2)' \).

Let us normalize the eigenvectors, i.e. let us pick constants \( C_i \) such that \( v_i'v_i = 1 \). We get

\[
v_1 = \left( \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right), \quad v_2 = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).
\]

Thus the diagonalization matrix \( U \) is

\[
U = \begin{bmatrix}
\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\
\frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}
\end{bmatrix}.
\]

You can easily check that

\[
U^{-1}AU = \begin{bmatrix}
0 & 0 \\
0 & 5
\end{bmatrix}.
\]

The trace of a square matrix of order \( n \) is the sum of the \( n \) elements on its principal diagonal, i.e., \( tr(A) = \sum_{i=1}^{n} a_{ii} \).
Properties of the Trace:

1) \( tr(A) = \lambda_1 + \cdots + \lambda_n \);
2) if \( A \) and \( B \) are of the same order, \( tr(A + B) = tr(A) + tr(B) \);
3) if \( a \) is a scalar, \( tr(aA) = atr(A) \);
4) \( tr(AB) = tr(BA) \), whenever \( AB \) is square;
5) \( tr(A') = tr(A) \);
6) \( tr(A'A) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \).

5.9 Vector Spaces

A (real) vector space is a nonempty set \( V \) of objects together with an additive operation \( + : V \times V \to V \), \( +(u, v) = u + v \) and a scalar multiplicative operation \( \cdot : \mathcal{R} \times V \to V \), \( \cdot(a, u) = au \) which satisfies the following axioms for any \( u, v, w \in V \) and any \( a, b \in \mathcal{R} \) (\( \mathcal{R} \) is the set of all real numbers):

1. \( (u + v) + w = u + (v + w) \);
2. \( u + v = v + u \);
3. \( 0 + u = u \);
4. \( u + (-u) = 0 \);
5. \( a(u + v) = au + av \);
6. \( (a + b)u = au + bu \);
7. \( a(bu) = (ab)u \);
8. \( 1u = u \).

The objects of a vector space \( V \) are called vectors, the operations \( + \) and \( \cdot \) are called vector addition and scalar multiplication, respectively. The element \( 0 \in V \) is the zero vector and \(-v\) is the additive inverse of \( V \).
Example 5.9.1 (The $n$-Dimensional Vector Space $\mathcal{R}^n$)

Define $\mathcal{R}^n = \{(u_1, u_2, \cdots, u_n)' | u_i \in \mathcal{R}, i = 1, \cdots, n\}$ (the apostrophe denotes the transpose). Consider $u, v \in \mathcal{R}^n$, $u = (u_1, u_2, \cdots, u_n)'$, $v = (v_1, v_2, \cdots, v_n)'$ and $a \in \mathcal{R}$. Define the additive operation and the scalar multiplication as follows:

$$u + v = (u_1 + v_1, \cdots, u_n + v_n)'$$

$$au = (au_1, \cdots, au_n)'$$

It is not difficult to verify that $\mathcal{R}^n$ together with these operations is a vector space.

Let $V$ be a vector space. An inner product or scalar product in $V$ is a function $s : V \times V \rightarrow \mathcal{R}$, $s(u, v) = u \cdot v$ which satisfies the following properties:

1. $u \cdot v = v \cdot u$,
2. $u \cdot (v + w) = u \cdot v + u \cdot w$,
3. $a(u \cdot v) = (au) \cdot v = u \cdot (av)$,
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$.

Example 5.9.2 Let $u, v \in \mathcal{R}^n$, $u = (u_1, u_2, \cdots, u_n)'$, $v = (v_1, v_2, \cdots, v_n)'$. Then $u \cdot v = u_1v_1 + \cdots + u_nv_n$.

Let $V$ be a vector space and $v \in V$. The norm of magnitude is a function $|| \cdot || : V \rightarrow \mathcal{R}$ defined as $||v|| = \sqrt{v \cdot v}$. For any $v \in V$ and any $a \in \mathcal{R}$, we have the following properties:

1. $||au|| = |a|||u||$;
2. (Triangle inequality) $||u + v|| \leq ||u|| + ||v||$;
3. (Schwarz inequality) $|u \cdot v| \leq ||u|| \times ||v||$. 
5.9. VECTOR SPACES

The nonzero vectors $u$ and $v$ are parallel if there exists $a \in \mathbb{R}$ such that $u = av$.

The vectors $u$ and $v$ are orthogonal or perpendicular if their scalar product is zero, that is, if $u \cdot v = 0$.

The angle between vectors $u$ and $v$ is $\arccos \left( \frac{uv}{||u|| ||v||} \right)$.

A nonempty subset $S$ of a vector space $V$ is a subspace of $V$ if for any $u, v \in S$ and $a \in \mathbb{R}$

$$u + v \in S \text{ and } au \in S.$$ 

**Example 5.9.3** $V$ is a subset of itself. $\{0\}$ is also a subset of $V$. These subspaces are called proper subspaces.

**Example 5.9.4** $L = \{(x, y)| y = mx + n\}$ where $m, n \in \mathbb{R}$ and $m \neq 0$ is a subspace of $\mathbb{R}^2$.

Let $u_1, u_2, \cdots, u_k$ be vectors in a vector space $V$. The set $S$ of all linear combinations of these vectors

$$S = \{a_1u_1 + a_2u_2 + \cdots + a_ku_k | a_1, \cdots, a_k \in \mathbb{R} \}$$

is called the subspace generated or spanned by the vectors $u_1, u_2, \cdots, u_k$ and denoted as $sp(u_1, u_2, \cdots, u_k)$. One can prove that $S$ is a subspace of $V$.

**Example 5.9.5** Let $u_1 = (2, -1, 1)'$, $u_2 = (3, 4, 0)'$. Then the subspace of $\mathbb{R}^3$ generated by $u_1$ and $u_2$ is

$$sp(u_1, u_2) = \{(2a + 3b, -a + 4b, a)' | a, b \in \mathbb{R} \}.$$ 

As we discussed in Chapter 4, a set of vectors $\{u_1, u_2, \cdots, u_k\}$ in a vector space $V$ is linearly dependent if there exists the real numbers $a_1, a_2, \cdots, a_k$, not all zero, such that $a_1u_1 + a_2u_2 + \cdots + a_ku_k = 0$. In other words, the set
of vectors in a vector space is linearly dependent if and only if one vector can be written as a linear combination of the others. A set of vectors \( \{u_1, u_2, \ldots, u_k\} \) in a vector space \( V \) is *linearly independent* if it is not linearly dependent.

**Properties:** Let \( \{u_1, u_2, \ldots, u_k\} \) be \( n \) vectors in \( \mathbb{R}^n \). The following conditions are equivalent:

i) The vectors are independent.

ii) The matrix having these vectors as columns is nonsingular.

iii) The vectors generate \( \mathbb{R}^n \).

A set of vectors \( \{u_1, u_2, \ldots, u_k\} \) in \( V \) is a *basis* for \( V \) if it, first, generates \( V \), and, second, is linearly independent.

**Example 5.9.6** Consider the following vectors in \( \mathbb{R}^n \). \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)' \), where 1 is in the \( i \)th position, \( i = 1, \ldots, n \). The set \( E_n = \{e_1, e_2, \ldots, e_n\} \) forms a basis for \( \mathbb{R}^n \) which is called the *standard basis*.

Let \( V \) be a vector space and \( B = \{u_1, u_2, \ldots, u_k\} \) a basis for \( V \). Since \( B \) generates \( V \), for any \( u \in V \) there exists the real numbers \( x_1, x_2, \ldots, x_n \) such that \( u = x_1u_1 + \cdots + x_nu_n \). The column vector \( x = (x_1, x_2, \ldots, x_n)' \) is called the vector of coordinates of \( u \) with respect to \( B \).

**Example 5.9.7** Consider the vector space \( \mathbb{R}^n \) with the standard basis \( E_n \). For any \( u = (u_1, \ldots, u_n)' \), we can represent \( u \) as \( u = u_1e_1 + \cdots + u_ne_n \); therefore, \( (u_1, \ldots, u_n)' \) is the vector of coordinates of \( u \) with respect to \( E_n \).

**Example 5.9.8** Consider the vector space \( \mathbb{R}^2 \). Let us find the coordinate vector of \((-1, 2)' \) with respect to the basis \( B = (1, 1)', (2, -3)' \) (i.e., find \((-1, 2)'^B\)). We have to solve for \( a, b \) such that \((-1, 2)' = a(1, 1)' + b(2, -3)' \). Solving the system \( a + 2b = -1 \) and \( a - 3b = 2 \), we find \( a = \frac{1}{5} \) and \( b = -\frac{3}{5} \). Thus, \((-1, 2)'_B = (\frac{1}{5}, -\frac{3}{5})' \)
The dimension of a vector space $V \dim(V)$ is the number of elements in any basis for $V$.

**Example 5.9.9** The dimension of the vector space $\mathbb{R}^n$ with the standard basis $E_n$ is $n$.

Let $U$ and $V$ be two vector spaces. A linear transformation of $U$ into $V$ is a mapping $T : U \to V$ such that for any $u, v \in U$ and any $a, b \in \mathbb{R}$, we have

$$T(au + bv) = aT(u) + bT(v).$$

**Example 5.9.10** Let $A$ be a $m \times n$ real matrix. The mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(u) = Au$ is a linear transformation.

**Properties:**

Let $U$ and $V$ be two vector spaces, $B = (b_1, \ldots, b_n)$ a basis for $U$ and $C = (c_1, \ldots, c_m)$ a basis for $V$.

1. Any linear transformation $T$ can be represented by an $m \times n$ matrix $A_T$ whose $i$th column is the coordinate vector of $T(b_i)$ relative to $C$.

2. If $x = (x_1, \ldots, x_n)'$ is the coordinate vector of $u \in U$ relative to $B$ and $y = (y_1, \ldots, y_m)'$ is the coordinate vector of $T(u)$ relative to $C$ then $T$ defines the following transformation of coordinates:

$$y = A_T x \text{ for any } u \in U.$$ 

The matrix $A_T$ is called the matrix representation of $T$ relative to bases $B, C$.

**Remark 5.9.1** Any linear transformation is uniquely determined by a transformation of coordinates.
Example 5.9.11 Consider the linear transformation \( T: \mathbb{R}^3 \to \mathbb{R}^2, T((x, y, z)') = (x - 2y, x + z) \) and bases \( B = \{(1, 1, 1)', (1, 1, 0)', (1, 0, 0)'\} \) for \( \mathbb{R}^3 \) and \( C = \{(1, 1)', (1, 0)'\} \) for \( \mathbb{R}^2 \). How can we find the matrix representation of \( T \) relative to bases \( B, C \)?

We have

\[
T((1, 1, 1)') = (-1, 2), \quad T((1, 1, 0)') = (-1, 1), \quad T((1, 0, 0)') = (1, 1).
\]

The columns of \( A_T \) are formed by the coordinate vectors of \( T((1, 1, 1)'), T((1, 1, 0)'), T((1, 0, 0)') \) relative to \( C \). Applying the procedure developed in Example 5.9.9, we find

\[
A_T = \begin{bmatrix} 2 & 1 & 1 \\ -3 & -2 & 0 \end{bmatrix}.
\]

Let \( V \) be a vector space of dimension \( n \), \( B \) and \( C \) be two bases for \( V \), and \( I: V \to V \) be the identity transformation ((\( I(v) = v \) for all \( v \in V \)). The \emph{change-of-basis} matrix \( D \) relative to \( B, C \) is the matrix representation of \( I \) to \( B, C \).

Example 5.9.12 For \( u \in V \), let \( x = (x_1, \ldots, x_n)' \) be the coordinate vector of \( u \) relative to \( B \) and \( y = (y_1, \ldots, y_n)' \) is the coordinate vector of \( u \) relative to \( C \). If \( D \) is the change-of-basis matrix relative to \( B, C \) then \( y = Cx \). The change-of-basis matrix relative to \( C, B \) is \( D^{-1} \).

Example 5.9.13 Given the following bases for \( \mathbb{R}^2 \): \( B = \{(1, 1)', (1, 0)\} \) and \( C = \{(0, 1)', (1, 1)'\} \), find the change-of-basis matrix \( D \) relative to \( B, C \). The columns of \( D \) are the coordinate vectors of \( (1, 1)' \), \( (1, 0)' \) relative to \( C \). Following Example 5.9.9 we find

\[
D = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.
\]
Chapter 6

Comparative Statics and the Concept of Derivative

6.1 The Nature of Comparative Statics

Comparative statics is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables. When the value of some parameter or exogenous variable that is associated with an initial equilibrium changes, we can get a new equilibrium. Then the question posed in the comparative-static analysis is: How would the new equilibrium compare with the old?

It should be noted that in the comparative-statics analysis we don’t concern with the process of adjustment of the variables; we merely compare the initial (prechange) equilibrium state with the final (postchange) equilibrium state. We also preclude the possibility of instability of equilibrium for we assume the equilibrium to be attainable.

It should be clear that the problem under consideration is essentially one of finding a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular pa-
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rameter or exogenous variable. For this reason, the mathematical concept of derivative takes on preponderant significance in comparative statics.

6.2 Rate of Change and the Derivative

We want to study the rate of change of any variable \( y \) in response to a change in another variable \( x \), where the two variables are related to each other by the function

\[
y = f(x)
\]

Applied in the comparative static context, the variable \( y \) will represent the equilibrium value of an endogenous variable, and \( x \) will be some parameter.

The Different Quotient

We use the symbol \( \Delta \) to denote the change from one point, say \( x_0 \), to another point, say \( x_1 \). Thus \( \Delta x = x_1 - x_0 \). When \( x \) changes from an initial value \( x_0 \) to a new value \( x_0 + \Delta x \), the value of the function \( y = f(x) \) changes from \( f(x_0) \) to \( f(x_0 + \Delta x) \). The change in \( y \) per unit of change in \( x \) can be represented by the difference quotient.

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.
\]

Example 6.2.1 \( y = f(x) = 3x^2 - 4 \)

Then \( f(x_0) = 3x_0^2 - 4 \), \( f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4 \),
and thus,
\[
\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}
= \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x}
= \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x}
= 6x_0 + 3\Delta x.
\]

**The Derivative**

Frequently, we are interested in the rate of change of \( y \) when \( \Delta x \) is very small. In particular, we want to know the rate of \( \Delta y/\Delta x \) when \( \Delta x \) approaches to zero. If, as \( \Delta x \to 0 \), the limit of the difference quotient \( \Delta y/\Delta x \) exits, that limit is called the derivative of the function \( y = f(x) \), and the derivative is denoted by
\[
\frac{dy}{dx} \equiv y' \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

**Remark:** Several points should be noted about the derivative: (1) a derivative is a function. Whereas the difference quotient is a function of \( x_0 \) and \( \Delta x \), the derivative is a function of \( x_0 \) only; and (2) since the derivative is merely a limit of the difference quotient, it must also be of necessity a measure of some rate of change. Since \( \Delta x \to 0 \), the rate measured by the derivative is in the nature of an *instantaneous* rate of change.

**Example 6.2.2** Referring to the function \( y = 3x^2 - 4 \) again. Since
\[
\frac{\Delta y}{\Delta x} = 6x + 3\Delta x,
\]
we have \( \frac{dy}{dx} = 6x \).
6.3 The Derivative and the Slope of a Curve

Elementary economics tells us that, given a total-cost function \( C = f(Q) \), where \( C \) is the total cost and \( Q \) is the output, the marginal cost \( MC \) is defined as \( MC = \Delta C / \Delta Q \). It is a continuous variable, \( \Delta Q \) will refer to an infinitesimal change. It is well known that \( MC \) can be measured by the slope of the total cost curve. But the slope of the total-cost curve is nothing but the limit of the ratio \( \Delta C / \Delta Q \) as \( \Delta Q \to 0 \). Thus the concept of the slope of a curve is merely the geometric counterpart of the concept of the derivative.

![Figure 6.1](image)

6.4 The Concept of Limit

In the above, we have defined the derivative of a function \( y = f(x) \) as the limit of \( \Delta y / \Delta x \) as \( \Delta x \to 0 \). We now study the concept of limit. For a given
function \( q = q(v) \), the concept of limit is concerned with the question: What value does \( q \) approach as \( v \) approaches a specific value? That is, as \( v \to N \) (the \( N \) can be any number, say \( N = 0, N = +\infty, -\infty \)), what happens to \( \lim_{v \to N} g(v) \).

When we say \( v \to N \), the variable \( v \) can approach the number \( N \) either from values greater than \( N \), or from values less than \( N \). If, as \( v \to N \) from the left side (from values less than \( N \)), \( q \) approaches a finite number \( L \), we call \( L \) the left-side limit of \( q \). Similarly, we call \( L \) the right-side limit of \( q \) if \( v \to N \) from the right side. The left-side limit and right-side limit of \( q \) are denoted by \( \lim_{v \to N^-} q \) and \( \lim_{v \to N^+} q \), respectively. The limit of \( q \) at \( N \) is said to exit if

\[
\lim_{v \to N^-} q = \lim_{v \to N^+} q,
\]

and is denoted by \( \lim_{v \to N} q \). Note that \( L \) must be a finite number. If we have the situation of \( \lim_{v \to N} q = \infty \) (or \( -\infty \)), we shall consider \( q \) to possess no limit or an “infinite limit.” It is important to realize that the symbol \( \infty \) is not a number, and therefore it cannot be subjected to the usual algebraic operations.

**Graphical Illustrations**

There are several possible situations regarding the limit of a function, which are shown in the following diagrams.

**Evaluation of a limit**

Let us now illustrate the algebraic evaluation of a limit of a given function \( q = g(v) \).

**Example 6.4.1** Given \( q = 2 + v^2 \), find \( \lim_{v \to 0} q \). It is clear that \( \lim_{v \to 0^-} q = 2 \) and \( \lim_{v \to 0^+} q = 2 \) and \( v^2 \to 0 \) as \( v \to 0 \). Thus \( \lim_{v \to 0} q = 2 \).
Note that, in evaluating $\lim_{v \to N} q$, we only let $v$ tends $N$ but, as a rule, do not let $v = N$. Indeed, sometimes $N$ even is not in the domain of the function $q = g(v)$.

**Example 6.4.2** $q = (1 - v^2)/(1 - v)$. For this function, $N = 1$ is not in the domain of the function, and we cannot set $v = 1$ since it would involve division by zero. Moreover, even the limit-evaluation procedure of letting $v - 1$ will cause difficulty since $(1 - v) \to 0$ as $v \to 1$.

One way out of this difficulty is to try to transform the given ratio to a form in which $v$ will not appear in the denominator. Since
\[
q = \frac{1 - v^2}{1 - v} = \frac{(1 - v)(2 + v)}{1 - v} = 1 + v \ (v \neq 1)
\]
and $v \to 1$ implies $v \neq 1$ and $(1 + v) \to 2$ as $v \to 1$, we have $\lim_{v \to 1} q = 2$.

**Example 6.4.3** Find $\lim_{v \to \infty} \frac{2v + 5}{v + 1}$.
Since \( \frac{2v+5}{v+1} = \frac{2(v+1)+3}{v+1} = 2 + \frac{3}{v+1} \) and \( \lim_{v \to \infty} \frac{3}{v+1} = 0 \), so \( \lim_{v \to \infty} \frac{2v+5}{v+1} = 2 \).

Formal View of the Limit Concept

Definition of the Limit

The number \( L \) is said to be the limit of \( q = g(v) \) as \( v \) approaches \( N \) if, for every neighborhood of \( L \), there can be found a corresponding neighborhood of \( N \) (excluding the point \( v = N \)) in the domain of the function such that, for every value of \( v \) in that neighborhood, its image lies in the chosen \( L \)-neighborhood. Here a neighborhood of a point \( L \) is an open interval defined by

\[
(L - a_1, L + a_2) = \{ q | L - a_1 < q < L + a_2 \} \text{ for } a_1 > a_2 > 0
\]
6.5 Inequality and Absolute Values

Rules of Inequalities

Transitivity:
\[ a > b \text{ and } b > c \implies a > c; \]
\[ a \geq b \text{ and } b \geq c \implies a \geq c. \]

Addition and Subtraction:
\[ a > b \implies a \pm k > b \pm k; \]
\[ a \geq b \implies a \pm k \geq b \pm k. \]

Multiplication and Division:
\[ a > b \implies ka > kb \text{ (} k > 0\); \]
\[ a > b \implies ka < kb \text{ (} k < 0\). \]

Squaring:
\[ a > b \text{ with } b \geq 0 \implies a^2 > b^2. \]

Absolute Values and Inequalities

For any real number \( n \), the absolute value of \( n \) is defined and denoted by

\[
|n| = \begin{cases} 
 n & \text{if } n > 0, \\
 -n & \text{if } n < 0, \\
 0 & \text{if } n = 0.
\end{cases}
\]

Thus we can write \( |x| < n \) as an equivalent way \(-n < x < n \) \((n > 0)\).

Also \( |x| \leq n \) if and only if \(-n \leq x \leq n \) \((n > 0)\).

The following properties characterize absolute values:

1) \( |m| + |n| \geq |m + n|; \)
2) \( |m| \cdot |n| = |m \cdot n|; \)
3) \( \frac{|m|}{|n|} = \frac{|m|}{|n|}. \)
Solution of an Inequality

Example 6.5.1 Find the solution of the inequality $3x\! -\! 3 > x\! +\! 1$. By adding $(3 - x)$ to both sides, we have

$3x - 3 + 3 - x > x + 1 + 3 - x$.

Thus, $2x > 4$ so $x > 2$.

Example 6.5.2 Solve the inequality $|1 - x| \leq 3$.

From $|1 - x| \leq 3$, we have $-3 \leq 1 - x \leq 3$, or $-4 \leq -x \leq 2$. Thus, $4 \geq x \geq -2$, i.e., $-2 \leq x \leq 4$.

6.6 Limit Theorems

Theorems Involving a Single Equation

Theorem I: If $q = av + b$, then $\lim_{v \to N} q = aN + b$.

Theorem II: If $q = g(v) = b$, then $\lim_{v \to N} q = b$.

Theorem III: $\lim_{v \to N} v^k = N^k$.

Example 6.6.1 Given $q = 5v + 7$, then $\lim_{v \to 2} = 5 \cdot 2 + 7 = 17$.

Example 6.6.2 $q = v^3$. Find $\lim_{v \to 2} q$.

By theorem III, we have $\lim_{v \to 2} = 2^3 = 8$.

Theorems Involving Two Functions

For two functions $q_1 = g(v)$ and $q_2 = h(v)$, if $\lim_{v \to N} q_1 = L_1$, $\lim_{v \to N} q_2 = L_2$, then we have the following theorems:

Theorem IV: $\lim_{v \to N}(q_1 + q_2) = L_1 + L_2$.

Theorem V: $\lim_{v \to N}(q_1 q_2) = L_1 L_2$.

Theorem VI: $\lim_{v \to N} \frac{q_1}{q_2} = \frac{L_1}{L_2}$ (if $L_2 \neq 0$).
Example 6.6.3 Find \( \lim_{v \to 0} \frac{1 + v}{2 + v} \).

Since \( \lim_{v \to 0} (1 + v) = 1 \) and \( \lim_{v \to 0} (2 + v) = 2 \), so \( \lim_{v \to 0} \frac{1 + v}{2 + v} = \frac{1}{2} \).

Remark 6.6.1 Note that \( L_1 \) and \( L_2 \) represent finite numbers; otherwise theorems do not apply.

Limit of a Polynomial Function

\[
\lim_{v \to N} a_0 + a_1 v + a_2 v^2 + \cdots + a_n v^n = a_0 + a_1 N + a_2 N^2 + \cdots + a_n N^n.
\]

6.7 Continuity and Differentiability of a Function

Continuity of a Function

A function \( q = g(v) \) is said to be continuous at \( N \) if \( \lim_{v \to N} q \) exists and \( \lim_{v \to N} g(v) = g(N) \).

Thus the term continuous involves no less than three requirements: (1) the point \( N \) must be in the domain of the function; (2) \( \lim_{v \to N} g(v) \) exists; and (3) \( \lim_{v \to N} g(v) = g(N) \).

Remark 6.7.1 It is important to note that while – in discussing the limit of a function – the point \((N, L)\) is excluded from consideration, we are no longer excluding it in the present context. Rather, as the third requirement specifically states, the point \((N, L)\) must be on the graph of the function before the function can be considered as continuous at point \(N\).

Polynomial and Rational Functions

From the discussion of the limit of polynomial function, we know that the limit exists and equals the value of the function at \( N \). Since \( N \) is a point in
the domain of the function, we can conclude that any polynomial function is continuous in its domain. By those theorems involving two functions, we also know any rational function is continuous in its domain.

**Example 6.7.1** \( q = \frac{4v^2}{v^2 + 1}. \)

Then
\[
\lim_{v \to N} \frac{4v^2}{v^2 + 1} = \lim_{v \to N} 4v^2 \lim_{v \to N} \frac{1}{v^2 + 1} = \frac{4N^2}{N^2 + 1}.
\]

**Example 6.7.2** The rational function
\[
q = \frac{v^3 + v^2 - 4v - 4}{v^2 - 4}
\]
is not defined at \( v = 2 \) and \( v = -2 \). Since \( v = 2, -2 \) are not in the domain, the function is discontinuous at \( v = 2 \) and \( v = -2 \), despite the fact that its limit exists as \( v \to 2 \) or \( -2 \).

**Differentiability of a Function**

By the definition of the derivative of a function \( y = f(x) \), at \( x_0 \) we know that \( f'(x_0) \) exists if and only if the limit of \( \Delta y/\Delta x \) exists at \( x = x_0 \) as \( \Delta x \to 0 \), i.e.,

\[
f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad \text{(differentiability condition)}
\]

On the other hand, the function \( y = f(x) \) is continuous at \( x_0 \) if and only if

\[
\lim_{x \to x_0} f(x) = f(x_0) \quad \text{(continuity condition)}
\]

We want to know what is the relationship between the continuity and differentiability of a function. Now we show the continuity of \( f \) is a necessary condition for its differentiability. But this is not sufficient.
Since the notation \( x \to x_0 \) implies \( x \neq x_0 \), so \( x - x_0 \) is a nonzero number, it is permissible to write the following identity:

\[
f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} (x - x_0)
\]

Taking the limit of each side of the above equation as \( x \to x_0 \) yields the following results:

Left side \( = \lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} f(x) - f(x_0). \)

Right side \( = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \)
\( = f'(x_0) \lim_{x \to x_0} (x - x_0) = 0. \)

Thus \( \lim_{x \to x_0} f(x) - f(x_0) = 0. \) So \( \lim_{x \to x_0} f(x) - f(x_0) \) which means \( f(x) \) is continuous at \( x = x_0. \)

Although differentiability implies continuity, the converse is not true. That is, continuity is a necessary, but not sufficient, condition for differentiability. The following example shows that fact.

**Example 6.7.3** \( f(x) = |x|. \)

This function is clearly continuous at \( x = 0. \) Now we show that it is not differentiable at \( x = 0. \) This involves the demonstration of a disparity between the left-side and the right-side limits. Since, in considering the right-side limit \( x > 0, \) thus

\[
\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1.
\]
On the other hand, in considering the left-side limit, \(x < 0\); thus

\[
\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1.
\]

Thus, \(\lim_{x \to 0^-} \frac{\Delta y}{\Delta x}\) does not exist, this means the derivative of \(y = |x|\) does not exist at \(x = 0\).
Chapter 7

Rules of Differentiation and Their Use in Comparative Statics

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem finding the derivative of some function \( y = f(x) \), provided only a small change in \( x \) is being considered. Before going into comparative-static models, we begin some rules of differentiation.

7.1 Rules of Differentiation for a Function of One Variable

Constant-Function Rule

If \( y = f(x) = c \), where \( c \) is a constant, then

\[
\frac{dy}{dx} \equiv y' \equiv f' = 0.
\]
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Proof.

\[ \frac{dy}{dx} = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} = \lim_{x' \to x} \frac{c - c}{x' - x} = x' \to x 0 = 0. \]

We can also write \( \frac{dy}{dx} = \frac{df}{dx} \) as

\[ \frac{d}{dx}y = \frac{d}{dx}f. \]

So we may consider \( d/dx \) as an operator symbol.

**Power-Function Rule**

If \( y = f(x) = x^a \) where \( a \) is any real number \(-\infty < a < \infty\),

\[ \frac{d}{dx}f(x) = ax^{a-1}. \]

**Remark 7.1.1**

(i) If \( a = 0 \), then

\[ \frac{d}{dx}x^0 = \frac{d}{dx}1 = 0. \]

(ii) If \( a = 1 \), then \( y = x \). Thus

\[ \frac{dx}{dx} = 1. \]

For simplicity, we prove this rule only for the case where \( a = n \), where \( n \) is any positive integer. Since

\[ x^n - x_0^n = (x - x_0)(x^{n-1} + x_0x^{n-2} + x_0^2x^{n-3} + \cdots + x_0^{n-1}), \]

then

\[ \frac{x^n - x_0^n}{x - x_0} = x^{n-1} + x_0x^{n-2} + x_0^2x^{n-3} + \cdots + x_0^{n-1}. \]
Thus,

\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} \]

\[ = \lim_{x \to x_0} x^{n-1} + x_0x^{n-2} + x_0^2x^{n-3} + \cdots + x_0^{n-1} \]

\[ = x_0^{n-1} + x_0^{n-1} + x_0^{n-1} + \cdots + x_0^{n-1} \]

\[ = nx_0^{n-1}. \]

**Example 7.1.1** Suppose \( y = f(x) = x^{-3} \). Then \( y' = -3x^{-4} \).

**Example 7.1.2** Suppose \( y = f(x) = \sqrt{x} \). Then \( y' = \frac{1}{2}x^{-1/2} \). In particular, we can know that \( f'(2) = \frac{1}{2} \cdot 2^{-\frac{1}{2}} = \frac{\sqrt{2}}{4} \).

**Power-Function Rule Generalized**

If the function is given by \( y = cx^a \), then

\[ \frac{dy}{dx} = \frac{df}{dx} = acx^{a-1}. \]

**Example 7.1.3** Suppose \( y = 2x \). Then

\[ \frac{dy}{dx} = 2x^0 = 2. \]

**Example 7.1.4** Suppose \( y = 4x^3 \). Then

\[ \frac{dy}{dx} = 4 \cdot 3x^{3-1} = 12x^2. \]

**Example 7.1.5** Suppose \( y = 3x^{-2} \). Then

\[ \frac{dy}{dx} = -6x^{-2-1} = -6x^{-3}. \]
Some Special Rules:

\[ f(x) = \text{constant} \Rightarrow f'(x) = 0; \]
\[ f(x) = x^a (a \text{ is constant}) \Rightarrow f'(x) = a x^{a-1}; \]
\[ f(x) = e^x \Rightarrow f'(x) = e^x; \]
\[ f(x) = a^x (a > 0) \Rightarrow f'(x) = a^x \ln a; \]
\[ f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}; \]
\[ f(x) = \log_a x (a > 0; a \neq 1) \Rightarrow f'(x) = \frac{1}{x} \log_a e = \frac{1}{x \ln a}; \]
\[ f(x) = \sin x \Rightarrow f'(x) = \cos x; \]
\[ f(x) = \cos x \Rightarrow f'(x) = -\sin x; \]
\[ f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}; \]
\[ f(x) = \csc x \Rightarrow f'(x) = -\frac{1}{\sin^2 x}; \]
\[ f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}; \]
\[ f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}; \]
\[ f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1-x^2}; \]
\[ f(x) = \arccsc x \Rightarrow f'(x) = -\frac{1}{1-x^2}. \]

7.2 Rules of Differentiation Involving Two or More Functions of the Same Variable

Let \( f(x) \) and \( g(x) \) be two differentiable functions. We have the following rules:
Sum-Difference Rule:

\[
\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x).
\]

This rule can easily be extended to more functions

\[
\frac{d}{dx} \left[ \sum_{i=1}^{n} f_i(x) \right] = \sum_{i=1}^{n} \frac{d}{dx}f_i(x) = \sum_{i=1}^{n} f'_i(x).
\]

Example 7.2.1

\[
\frac{d}{dx}(ax^2 + bx + c) = 2ax + b.
\]

Example 7.2.2 Suppose a short-run total-cost function is given by

\[c = Q^3 - 4Q^2 + 10Q + 75.\]

Then the marginal-cost function is the limit of the quotient \(\Delta C/\Delta Q\), or the derivative of the \(C\) function:

\[
\frac{dC}{dQ} = 3Q^2 - 8Q + 10.
\]

In general, if a primitive function \(y = f(x)\) represents a total function, then the derivative function \(dy/dx\) is its marginal function. Since the derivative of a function is the slope of its curve, the marginal function should show the slope of the curve of the total function at each point \(x\). Sometimes, we say a function is smooth if its derivative is continuous.

Product Rule:

\[
\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}g(x) + g(x) \frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x).
\]
CHAPTER 7. RULES OF DIFFERENTIATION AND THEIR USE IN COMPARATIVE STATICS

Proof.

\[
\frac{d}{dx}[f(x_0)g(x_0)] = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
= \lim_{x \to x_0} f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0) - x - x_0 \\
= \lim_{x \to x_0} f(x)[g(x) - g(x_0)] + g(x_0)[f(x) - f(x_0)] - x - x_0 \\
= \lim_{x \to x_0} f(x) \frac{g(x) - g(x_0)}{x - x_0} + \lim_{x \to x_0} g(x_0) \frac{f(x) - f(x_0)}{x - x_0} \\
= f(x_0)g'(x_0) + g(x_0)f'(x_0).
\]

Since this is true for any \( x = x_0 \), this proves the rule.

Example 7.2.3 Suppose \( y = (2x+3)(3x^2) \). Let \( f(x) = 2x + 3 \) and \( g(x) = 3x^2 \).

Then \( f'(x) = 2, g'(x) = 6x \). Hence,

\[
\frac{d}{dx}[(2x + 3)(3x^2)] = (2x + 3)6x + 3x^2 \\
= 12x^2 + 18x + 6x^2 \\
= 18x^2 + 18x.
\]

As an extension of the rule to the case of three functions, we have

\[
\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).
\]

Finding Marginal-Revenue Function from Average-Revenue Function

Suppose the average-revenue (AR) function is specified by

\[
AR = 15 - Q.
\]

The total-revenue (TR) function is

\[
TR = AR \cdot Q = 15Q - Q^2.
\]
Thus, the marginal-revenue (MR) function is given by

\[ MR \equiv \frac{d}{dQ} TR = 15 - 2Q. \]

In general, if \( AR = f(Q) \), then

\[ TR \equiv AR \cdot Q = Qf(Q). \]

Thus

\[ MR \equiv \frac{d}{dQ} TR = f(Q) + Qf'(Q). \]

From this, we can tell relationship between \( MR \) and \( AR \). Since

\[ MR - AR = Qf'(Q), \]

they will always differ the amount of \( Qf'(Q) \). Also, since

\[ AR \equiv \frac{TR}{Q} = \frac{PQ}{Q} = P, \]

we can view \( AR \) as the inverse demand function for the product of the firm. If the market is perfectly competitive, i.e., the firm takes the price as given, then \( P = f(Q) = \text{constant} \). Hence \( f'(Q) = 0 \). Thus \( MR - AR = 0 \) or \( MR = AR \). Under imperfect competition, on the other hand, the \( AR \) curve is normally downward-sloping, so that \( f'(Q) < 0 \). Thus \( MR < AR \).

**Quotient Rule**

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.
\]

We will come back to prove this rule after learning the chain rule.
Example 7.2.4

\[
\frac{d}{dx} \left( \frac{2x - 3}{x + 1} \right) = \frac{2(x + 1) - (2x - 3)(1)}{(x + 1)^2} = \frac{5}{(x + 1)^2}.
\]

\[
\frac{d}{dx} \left( \frac{5x}{x^2 + 1} \right) = \frac{5(x^2 + 1) - 5x(2x)}{(x^2 + 1)^2} = \frac{5(1 - x^2)}{(x^2 + 1)^2}.
\]

\[
\frac{d}{dx} \left( \frac{ax^2 + b}{cx} \right) = \frac{2ax(cx) - (ax^2 + b)c}{(cx)^2} = \frac{c(ax^2 - b)}{(cx)^2} = \frac{ax^2 - b}{cx^2}.
\]

**Relationship Between Marginal-Cost and Average-Cost Functions**

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total cost function \( C = C(Q) \), the average cost (\( AC \)) function and the marginal-cost (\( MC \)) function are given by

\[
AC \equiv \frac{C(Q)}{Q} \quad (Q > 0)
\]

and

\[
MC \equiv C'(Q).
\]

The rate of change of \( AC \) with respect to \( Q \) can be found by differentiating \( AC \):

\[
\frac{d}{dQ} \left[ \frac{C(Q)}{Q} \right] = \frac{C'(Q)Q - C(Q)}{Q^2} = \frac{1}{Q} \left[ C'(Q) - \frac{C(Q)}{Q} \right].
\]
From this it follows that, for $Q > 0$.

$$\frac{d}{dQ} AC > 0 \text{ iff } MC(Q) > AC(Q);$$
$$\frac{d}{dQ} AC = 0 \text{ iff } MC(Q) = AC(Q);$$
$$\frac{d}{dQ} AC < 0 \text{ iff } MC(Q) < AC(Q).$$

7.3 Rules of Differentiation Involving Functions of Different Variables

Now we consider cases where there are two or more differentiable functions, each of which has a distinct independent variable.
Chain Rule

If we have a function \( z = f(y) \), where \( y \) is in turn a function of another variable \( x \), say, \( y = g(x) \), then the derivative of \( z \) with respect to \( x \) is given by

\[
\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x). \quad [\text{ChainRule}]
\]

The chain rule appeals easily to intuition. Given a \( \Delta x \), there must result in a corresponding \( \Delta y \) via the function \( y = g(x) \), but this \( \Delta y \) will in turn being about a \( \Delta z \) via the function \( z = f(y) \).

**Proof.** Note that

\[
\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.
\]

Since \( \Delta x \to 0 \) implies \( \Delta y \to 0 \), we have

\[
\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x). \quad Q.E.D.
\]

In view of the function \( y = g(x) \), we can express the function \( z = f(y) \) as \( z = f(g(x)) \), where the contiguous appearance of the two function symbols \( f \) and \( g \) indicates that this is a compose function (function of a function). So sometimes, the chain rule is also called the composite function rule. As an application of this rule, we use it to prove the quotient rule.

Let \( z = y^{-1} \) and \( y = g(x) \). Then \( z = g^{-1}(x) = 1/g(x) \) and

\[
\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = -\frac{1}{y^2}g'(x) = -\frac{g'(x)}{g^2(x)}.
\]
Thus,

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{d}{dx} \left[ f(x) \cdot g^{-1}(x) \right] \\
= f'(x)g^{-1}(x) + f(x) \frac{d}{dx} \left[ g^{-1}(x) \right] \\
= f'(x)g^{-1}(x) + f(x) \left[ -\frac{g(x)}{g^2(x)} \right] \\
= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}. \quad Q.E.D.
\]

**Example 7.3.1** If \( z = 3y^2 \) and \( y = 2x + 5 \), then

\[
\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = 6y(2) = 12y = 12(2x + 5).
\]

**Example 7.3.2** If \( z = y - 3 \) and \( y = x^3 \), then

\[
\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = 1 \cdot 3x^2 = 3x^2.
\]

**Example 7.3.3** \( z = (x^2 + 3x - 2)^{17} \). Let \( z = y^{17} \) and \( y = x^2 + 3x - 2 \). Then

\[
\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = 17y^{16} \cdot (2x + 3) \\
= 17(x^2 + 3x - 2)^{16}(2x + 3).
\]

**Example 7.3.4** Suppose \( TR = f(Q) \), where output \( Q \) is a function of labor input \( L \), or \( Q = g(L) \). Then, by the chain rule, the marginal revenue product of labor (\( MRP_L \)) is

\[
MRP_L = \frac{dR}{dL} = \frac{dR}{dQ} \frac{dQ}{dL} = f'(Q)g'(L) = MR \cdot MPP_L,
\]

where \( MPP_L \) is marginal physical product of labor. Thus the result shown above constitutes the mathematical statement of the well-known result in economics that \( MRP_L = MR \cdot MPP_L \).
Inverse-Function Rule

If the function \( y = f(x) \) represents a one-to-one mapping, i.e., if the function is such that a different value of \( x \) will always yield a different value of \( y \), then function \( f \) will have an inverse function \( x = f^{-1}(y) \). Here, the symbol \( f^{-1} \) is a function symbol which signifies a function related to the function \( f \); it does not mean the reciprocal of the function \( f(x) \). When \( x \) and \( y \) refer specifically to numbers, the property of one-to-one mapping is seen to be unique to the class of function known as monotonic function. A function \( f \) is said to be monotonically increasing (decreasing) if \( x_1 > x_2 \) implies \( f(x_1) > f(x_2) \) (\( f(x_1) < f(x_2) \)). In either of these cases, an inverse function \( f^{-1} \) exists.

A practical way of ascertaining the monotonicity of a given function \( y = f(x) \) is to check whether the \( f'(x) \) always adheres to the same algebraic sign for all values. Geometrically, this means that its slope is either always upward or always downward.

**Example 7.3.5** Suppose \( y = 5x + 25 \). Since \( y' = 5 \) for all \( x \), the function is monotonic and thus the inverse function exists. In fact, it is given by \( x = 1/5y - 5 \).

Generally speaking, if an inverse function exists, the original and the inverse functions must be both monotonic. Moreover, if \( f^{-1} \) is the inverse function of \( f \), then \( f \) must be the inverse function of \( f^{-1} \). For inverse functions, the rule of differentiation is

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}
\]

**Example 7.3.6** Suppose \( y = x^5 + x \). Then

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{5x^4 + 1}.
\]
Example 7.3.7 Given $y = \ln x$, its inverse is $x = e^y$. Therefore $\frac{dx}{dy} = \frac{1}{1/x} = x = e^y$.

7.4 Integration (The Case of One Variable)

Let $f(x)$ be a continuous function. The indefinite integral of $f$ (denoted by $\int f(x)dx$) is defined as

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is such that $F'(x) = f(x)$, and $C$ is an arbitrary constant.

Rules of Integration

- $\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx$, $a, b$ are constants (linearity of the integral);

- $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$ (integration by parts);

- $\int f(u(t))\frac{du}{dt}dt = \int f(u)du$ (integration by substitution).
Some Special Rules of Integration:

\[
\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C; \\
\int \frac{1}{x} \, dx = \ln |x| + C; \\
\int e^x \, dx = e^x + C; \\
\int f'(x) e^{f(x)} \, dx = e^{f(x)} + C; \\
\int x^a \, dx = \frac{x^{a+1}}{a+1} + C, \ a \neq -1; \\
\int a^x \, dx = \frac{a^x}{\ln a} + C, \ a > 0.
\]

Example 7.4.1

\[
\int \frac{x^2 + 2x + 1}{x} \, dx = \int x \, dx + \int 2 \, dx + \int \frac{1}{x} \, dx = \frac{x^2}{2} + 2x + \ln |x| + C.
\]

Example 7.4.2

\[
\int xe^{-x^2} \, dx = -\frac{1}{2} \int (-2x)e^{-x^2} \, dx = -\frac{1}{2} \int e^{-z} \, dz = -\frac{e^{-x^2}}{2} + C.
\]

Example 7.4.3

\[
\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.
\]

(The Newton-Leibniz formula) The definite integral of a continuous function \( f \) is

\[
\int_a^b f(x) \, dx = F(x)|_a^b = F(b) - F(a)
\]

for \( F(x) \) such that \( F'(x) = f(x) \) for all \( x \in [a, b] \).
Remark: The indefinite integral is a function. The definite integral is a number.

Properties of Definite Integrals:

\((a, b, c, \alpha, \beta)\) are real numbers\)

\[
\int_{a}^{b} [\alpha f(x) + \beta g(x)] \, dx = \alpha \int_{a}^{b} f(x) \, dx + \beta \int_{a}^{b} g(x) \, dx;
\]

\[
\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx;
\]

\[
\int_{a}^{a} f(x) \, dx = 0;
\]

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx;
\]

\[
\left| \int_{a}^{b} f(x) \, dx \right| \leq \int_{a}^{b} |f(x)| \, dx;
\]

\[
\int_{a}^{b} f(x)g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du, \quad u = g(x) \text{ (change of variable)};
\]

\[
\int_{a}^{b} f'(x)g(x) \, dx = f(x)g(x) \bigg|_{a}^{b} - \int_{a}^{b} f(x)g'(x) \, dx.
\]

Some More Useful Results:

\[
\frac{d}{d\lambda} \int_{a(\lambda)}^{b(\lambda)} f(x) \, dx = f(b(\lambda))b'(\lambda) - f(a(\lambda))a'(\lambda).
\]

Example 7.4.4

\[
\frac{d}{dx} \int_{a}^{x} f(x) \, dx = f(x).
\]

7.5 Partial Differentiation

So far, we have considered only the derivative of functions of a single independent variable. In comparative-static analysis, however, we are likely to
encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than parameter. Because of this, we now consider the derivative of a function of more than one variable.

**Partial Derivatives**

Consider a function

$$y = f(x_1, x_2, \cdots, x_n),$$

where the variables $x_i \ (i = 1, 2, \cdots, n)$ are all independent of one another, so that each can vary by itself without affecting the others. If the variable $x_i$ changes $\Delta x_i$ while the other variables remain fixed, there will be a corresponding change in $y$, namely, $\Delta y$. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_i} = \frac{f(x_1, x_2, \cdots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{\Delta x_i}.$$

If we take the limit of $\Delta y/\Delta x_i$, that limit will constitute a derivative. We call it the partial derivative of $y$ with respect to $x_i$. The process of taking partial derivatives is called partial differentiation. Denote the partial derivative of $y$ with respect to $x_i$ by $\frac{\partial y}{\partial x_i}$, i.e.,

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}.$$

Also we can use $f_i$ to denote $\frac{\partial y}{\partial x_i}$. If the function happens to be written in terms of unsubscripted variables, such as $y = f(u, v, w)$, one also uses, $f_u, f_v, f_w$ to denote the partial derivatives.
Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold the other independent variables constant while allowing one variable to vary.

**Example 7.5.1** Suppose \( y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2 \). Find \( \partial y/\partial x_1 \) and \( \partial y/\partial x_2 \).

\[
\frac{\partial y}{\partial x_1} = \frac{\partial f}{\partial x_1} = 6x_1 + x_2;
\]
\[
\frac{\partial y}{\partial x_2} = \frac{\partial f}{\partial x_2} = x_1 + 8x_2.
\]

**Example 7.5.2** For \( y = f(u, v) = (u + 4)(3u + 2v) \), we have

\[
\frac{\partial y}{\partial u} = f_u = (3u + 2v) + (u + 4) \cdot 3
\]
\[
= 6u + 2v + 12;
\]

\[
\frac{\partial y}{\partial v} = f_v = 2(u + 4).
\]

When \( u = 2 \) and \( v = 1 \), then \( f_u(2, 1) = 26 \) and \( f_v(2, 1) = 12 \).

**Example 7.5.3** Given \( y = (3u - 2v)/(u^2 + 3v) \),

\[
\frac{\partial y}{\partial u} = \frac{3(u^2 + 3v) - (3u - 2v)(2u)}{(u^2 + 3v)^2} = \frac{-3u^2 + 4uv + 9v}{(u^2 + 3v)^2};
\]

and

\[
\frac{\partial y}{\partial v} = \frac{-2(u^2 + 3v) - (3u - 2v) \cdot 3}{(u^2 + 3v)^2} = \frac{-u(2u + 9)}{(u^2 + 3v)^2}.
\]

### 7.6 Applications to Comparative-Static Analysis

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: name-
ly, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

**Market Model**

For the one-commodity market model:

\[
Q_d = a - bp \quad (a, b > 0);
\]
\[
Q_s = -c + dp \quad (c, d > 0),
\]

the equilibrium price and quantity are given by

\[
\bar{p} = \frac{a + c}{b + d};
\]
\[
\bar{Q} = \frac{ad - bc}{b + d}.
\]

These solutions will be referred to as being in the reduced form: the two endogenous variables have been reduced to explicit expressions of the four independent variables, \(a, b, c\), and \(d\).

To find how an infinitesimal change in one of the parameters will affect the value of \(\bar{p}\) or \(\bar{Q}\), one has only to find out its partial derivatives. If the sign of a partial derivative can be determined, we will know the direction in which \(\bar{p}\) will move when a parameter changes; this constitutes a qualitative conclusion. If the magnitude of the partial derivative can be ascertained, it will constitute a quantitative conclusion. To avoid misunderstanding, a clear distinction should be made between the two derivatives, say, \(\partial\bar{Q}/\partial a\) and \(\partial Q_d/\partial a\). The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative \(\partial\bar{Q}/\partial a\), on the other hand, to the equilibrium quantity which takes into account interaction of demand and supply together. To emphasize this distinction, we refer to the partial derivatives of \(\bar{p}\) and \(\bar{Q}\).
with respect to the parameters as comparative-static derivatives.

For instance, for $\bar{p}$, we have

\[
\frac{\partial \bar{p}}{\partial a} = \frac{1}{b + d};
\]

\[
\frac{\partial \bar{p}}{\partial b} = \frac{a + c}{(b + d)^2};
\]

\[
\frac{\partial \bar{p}}{\partial c} = \frac{1}{b + d};
\]

\[
\frac{\partial \bar{p}}{\partial b} = \frac{a + c}{(b + d)^2}.
\]

Thus $\frac{\partial \bar{p}}{\partial a} > 0$ and $\frac{\partial \bar{p}}{\partial b} < 0$.

Figure 7.2:
National-Income Model

\[ Y = C + I_0 + G_0 \quad \text{(equilibrium condition)}; \]
\[ C = \alpha + \beta(Y - T) \quad (\alpha > 0; 0 < \beta < 1); \]
\[ T = \gamma + \delta Y \quad (\gamma > 0; 0 < \delta < 1), \]

where the endogenous variables are the national income \( Y \), consumption \( C \), and taxes \( T \). The equilibrium income (in reduced form) is

\[ \bar{Y} = \frac{\alpha - \beta \gamma + I_0 + G_0}{1 - \beta + \beta \delta}. \]

Thus,

\[ \frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1 - \beta + \beta \delta} > 0 \quad \text{(the government-expenditure multiplier)}; \]
\[ \frac{\partial \bar{Y}}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta \delta} < 0; \]
\[ \frac{\partial \bar{Y}}{\partial \delta} = \frac{-\beta(\alpha - \beta \gamma + I_0 + G_0)}{(1 - \beta + \beta \delta)^2} = \frac{-\beta \bar{Y}}{(1 - \beta + \beta \delta)} < 0. \]

### 7.7 Note on Jacobian Determinants

Partial derivatives can also provide a means of testing whether there exists functional (linear or nonlinear) dependence among a set of \( n \) variables. This is related to the notion of Jacobian determinants.

Consider \( n \) differentiable functions in \( n \) variables not necessary linear.

\[ y_1 = f^1(x_1, x_2, \cdots, x_n); \]
\[ y_2 = f^2(x_1, x_2, \cdots, x_n); \]
7.7. NOTE ON JACOBIAN DETERMINANTS

\[ \cdots; \]

\[ y_n = f^n(x_1, x_2, \cdots, x_n), \]

where the symbol \( f^i \) denotes the \( i \)th function, we can derive a total of \( n^2 \) partial derivatives.

\[ \frac{\partial y_i}{\partial x_j} (i = 1, 2, \cdots, n; j = 1, 2, \cdots, n). \]

We can arrange them into a square matrix, called a Jacobian matrix and denoted by \( J \), and then take its determinant, the result will be what is known as a Jacobian determinant (or a Jacobian, for short), denoted by \(|J|\):

\[ |J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}. \]

**Example 7.7.1** Consider two functions:

\[ y_1 = 2x_1 + 3x_2; \]

\[ y_2 = 4x_1^2 + 12x_1x_2 + 9x_2^2. \]

Then the Jacobian is

\[ |J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix}. \]

A Jacobian test for the existence of functional dependence among a set of \( n \) functions is provided by the following theorem:

**Theorem 7.7.1** The \( n \) functions \( f^1, f^2, \cdots, f^n \) are functionally (linear or non-linear) dependent if and only if the Jacobian \( |J| \) defined above will be identically
zero for all values of \( x_1, x_2, \ldots, x_n \).

For the above example, since

\[
|J| = \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right| = (24x_1 + 36x_2) - (24x_1 + 36x_2) = 0
\]

for all \( x_1 \) and \( x_2 \), \( y^1 \) and \( y^2 \) are functionally dependent. In fact, \( y_2 \) is simply \( y_1 \) squared.

**Example 7.7.2** Consider the linear-equation system: \( Ax = d \), i.e.,

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1; \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2; \\
\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n. 
\end{align*}
\]

We know that the rows of the coefficient matrix \( A \) are linearly dependent if and only if \( |A| = 0 \). This result can now be interpreted as a special application of the Jacobian criterion of functional dependence.

Take the left side of each equation \( Ax = d \) as a separate function of the \( n \) variables \( x_1, x_2, \ldots, x_n \), and denote these functions by \( y_1, y_2, \ldots, y_n \). Then we have \( \frac{\partial y_i}{\partial x_j} = a_{ij} \). In view of this, the elements of \( |J| \) will be precisely the elements of \( A \), i.e., \( |J| = |A| \) and thus the Jacobian criterion of functional dependence among \( y_1, y_2, \ldots, y_n \) is equivalent to the criterion \( |A| = 0 \) in the present linear case.
Chapter 8

Comparative-Static Analysis of General-Functions

The study of partial derivatives has enables us, in the preceding chapter, to handle the simple type of comparative-static problems, in which the equilibrium solution of the model can be explicitly stated in the reduced form. We note that the definition of the partial derivative requires the absence of any functional relationship among the independent variables. As applied to comparative-static analysis, this means that parameters and/or exogenous variables which appear in the reduced-form solution must be mutually independent.

However, no such expediency should be expected when, owing to the inclusion of general functions in a model, no explicit reduced-form solution can be obtained. In such a case, we will have to find the comparative-static derivatives directly from the originally given equations in the model. Take, for instance, a simple national-income model with two endogenous variables $Y$ and $C$:

$$Y = C + I_0 + G_0 \ \text{(equilibrium condition);}$$
CHAPTER 8. COMPARATIVE-STATIC ANALYSIS OF GENERAL-FUNCTIONS

\[ C = C(Y, T_0) \ (T_0: \text{exogenous taxes}) , \]

which reduces to a single equation

\[ Y = C(Y, T_0) + I_0 + G_0 \]

to be solved for \( \bar{Y} \). We must, therefore, find the comparative-static derivatives directly from this equation. How might we approach the problem?

Let us suppose that an equilibrium solution \( \bar{Y} \) does exists. We may write the equation

\[ \bar{Y} = \bar{Y}(I_0, G_0, T_0) , \]

even though we are unable to determine explicitly the form which this function takes. Furthermore, in some neighborhood of \( \bar{Y} \), the following identical equality will hold:

\[ \bar{Y} \equiv C(\bar{Y}, T_0) + I_0 + G_0 . \]

Since \( \bar{Y} \) is a function of \( T_0 \), the two arguments of the \( C \) function are not independent. \( T_0 \) can in this case affect \( C \) not only directly, but also indirectly via \( \bar{Y} \). Consequently, partial differentiation is no longer appropriate for our purposes. In this case, we must resort to total differentiation (as against partial differentiation). The process of total differentiation can lead us to the related concept of total derivative. Once we become familiar with these concepts, we shall be able to deal with functions whose arguments are not all independent so that we can study the comparative-static of a general-function model.
8.1 Differentials

The symbol $dy/dx$ has been regarded as a single entity. We shall now reinterpret as a ratio of two quantities, $dy$ and $dx$.

**Differentials and Derivatives**

Given a function $y = f(x)$, we can use the difference quotient $\Delta y/\Delta x$ to represent the ratio of change of $y$ with respect to $x$. Since

$$\Delta y \equiv \left[ \frac{\Delta y}{\Delta x} \right] \Delta x,$$

(8.1.1)

the magnitude of $\Delta y$ can be found, once the $\Delta y/\Delta x$ and the variation $\Delta x$ are known. If we denote the infinitesimal changes in $x$ and $y$, respectively, by $dx$ and $dy$, the identity (7.1) becomes

$$dy \equiv \left[ \frac{dy}{dx} \right] dx \text{ or } dy = f'(x)dx.$$

(8.1.2)

The symbols $dy$ and $dx$ are called the differentials of $y$ and $x$, respectively.

Dividing the two identities in (8.1.2) throughout by $dx$, we have

$$\frac{(dy)}{(dx)} \equiv \left( \frac{dy}{dx} \right) \text{ or } \frac{(dy)}{(dx)} \equiv f'(x).$$

This result show that the derivative $dy/dx = f'(x)$ may be interpreted as the quotient of two separate differentials $dy$ and $dx$.

On the basis of (8.1.2), once we are given $f'(x)$, $dy$ can immediately be written as $f'(x)dx$. The derivative $f'(x)$ may thus be viewed as a “converter” that serves to convert an infinitesimal change $dx$ into a corresponding change $dy$.

**Example 8.1.1** Given $y = 3x^2 + 7x - 5$, find $dy$. Since $f'(x) = 6x + 7$, the
desired differential is
\[ dy = (6x + 7)dx. \]

**Remark 8.1.1** The purpose of finding the differential \( dy \) is called differentiation. Recall that we have used this term as a synonym for derivation. To avoid confusion, the word “differentiation” with the phrase “with respect to \( x \)” when we take derivative \( dy/dx \).

The following diagram shows the relationship between “\( \Delta y \)” and “\( dy \)”: 

\[ \Delta y \equiv \left[ \frac{\Delta y}{\Delta x} \right] \Delta x = \frac{CB}{AC} AC = CB; \]
\[ dy = \left[ \frac{dy}{dx} \right] \Delta x = \frac{CD}{AC} AC = CD, \]

which differs from \( \Delta y \) by an error of \( DB \).
Differentials and Point Elasticity

As an illustration of the application of differentials in economics, let us consider the elasticity of a function. For a demand function $Q = f(P)$, for instance, the elasticity is defined as $(\Delta Q/Q)/(\Delta P/P)$, the ratio of percentage change in quantity and percentage change in price. Now if $\Delta P \to 0$, the $\Delta P$ and $\Delta Q$ will reduce to the differential $dP$ and $dQ$, and the elasticity becomes

$$\epsilon_d = \frac{dQ/Q}{dP/P} = \frac{dQ/dP}{Q/P} = \text{marginal demand function} / \text{average demand function}.$$

In general, for a given function $y = f(x)$, the point elasticity of $y$ with respect to $x$ as

$$\epsilon_{yx} = \frac{dy/dx}{y/x} = \text{marginal function} / \text{average function}.$$

**Example 8.1.2** Find $\epsilon_d$ if the demand function is $Q = 100 - 2P$. Since $dQ/dP = -2$ and $Q/P = (100 - 2P)/P$, so $\epsilon_d = (-P)/(50 - P)$. Thus the demand is inelastic ($|\epsilon_d| < 1$) for $0 < P < 25$, unit elastic ($|\epsilon_d| = 1$) for $P = 25$, and elastic for $25 < P < 50$.

### 8.2 Total Differentials

The concept of differentials can easily be extended to a function of two or more independent variables. Consider a saving function

$$S = S(Y, i),$$

where $S$ is savings, $Y$ is national income, and $i$ is interest rate. If the function is continuous and possesses continuous partial derivatives, *the total*
differential is defined by
\[ dS = \frac{\partial S}{\partial Y} dY + \frac{\partial S}{\partial i} di.\]

That is, the infinitesimal change in \( S \) is the sum of the infinitesimal change in \( Y \) and the infinitesimal change in \( i \).

**Remark 8.2.1** If \( i \) remains constant, the total differential will reduce to a partial differential:
\[ \frac{\partial S}{\partial Y} = \left( \frac{dS}{dY} \right)_{i \text{ constant}}.\]

Furthermore, general case of a function of \( n \) independent variables \( y = f(x_1, x_2, \cdots, x_n) \), the total differential of this function is given by
\[ df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} f_i dx_i,\]
in which each term on the right side indicates the amount of change in \( y \) resulting from an infinitesimal change in one of the independent variables.

Similar to the case of one variable, the \( n \) partial elasticities can be written as
\[ \epsilon_{fx_i} = \frac{\partial f x_i}{\partial x_i} f \ (i = 1, 2, \cdots, n).\]

### 8.3 Rule of Differentials

Let \( c \) be constant and \( u \) and \( v \) be two functions of the variables \( x_1, x_2, \cdots, x_n \).

The following rules are valid:

- **Rule I** \( dc = 0; \)
- **Rule II** \( d(cu^a) = cau^{a-1} du; \)
- **Rule III** \( d(u \pm v) = du \pm dv; \)
Rule IV \( d(uv) = vdu + udv \);
Rule V \( d(u/v) = 1/v^2(vdu - udv) \).

Example 8.3.1 Find \( dy \) of the function \( y = 5x^2 + 3x_2 \). There are two ways to find \( dy \). One is the straightforward method by finding \( \partial f/\partial x_1 \) and \( \partial f/\partial x_2 \):
\[
\partial f/\partial x_1 = 10x_1 \quad \text{and} \quad \partial f/\partial x_2 = 3,
\]
which will then enable us to write
\[
dy = f_1dx_1 + f_2dx_2 = 10x_1dx_1 + 3dx_2.
\]
The other way is to use the rules given above by letting \( u = 5x^2 \) and \( v = 3x_2 \);
\[
dy = d(5x^2) + d(3x_2) \quad \text{(by rule III)}
\]
\[
= 10x_1dx_1 + 3dx_2 \quad \text{(by rule II)}.
\]

Example 8.3.2 Find \( dy \) of the function \( y = 3x^2_1 + x_1x_2^2 \). Since \( f_1 = 6x_1 + x_2^2 \) and \( f_2 = 2x_1x_2 \), the desired differential is
\[
dy = (6x_1 + x_2^2)dx + 2x_1x_2dx_2.
\]

By applying the given rules, the same result can be arrived at
\[
dy = d(3x^2_1) + d(x_1x_2^2)
\]
\[
= 6x_1dx_1 + x_2^2dx_1 + 2x_1x_2dx_2
\]
\[
= (6x_1 + x_2^2)dx + 2x_1x_2dx_2.
\]

Example 8.3.3 For the function
\[
y = \frac{x_1 + x_2}{2x_1^2}
\]
\[ f_1 = \frac{-(x_1 + 2x_2)}{2x_1^3} \quad \text{and} \quad f_2 = \frac{1}{2x_1^2}, \]

then

\[ dy = \frac{-(x_1 + 2x_2)}{2x_1^3} \, dx_1 + \frac{1}{2x_1^2} \, dx_2. \]

The same result can also be obtained by applying the given rules:

\[
dy = \frac{1}{4x_1} \left[ (2x_1^2 dx_1 + dx_2) - (x_1 + x_2) 4x_1 dx_1 \right] \quad \text{[by rule V]}
\]
\[
= \frac{1}{4x_1} \left[ 2x_1^2 dx_1 + dx_2 - (x_1 + x_2) 4x_1 dx_1 \right]
\]
\[
= \frac{1}{4x_1} \left[ -2x_1(x_1 + 2x_2) dx_1 + 2x_1^2 dx_2 \right]
\]
\[
= -\frac{x_1 + 2x_2}{2x_1^3} \, dx_1 + \frac{1}{2x_1^2} \, dx_2
\]

For the case of more than two functions, we have

Rule VI \( d(u \pm v \pm w) = du \pm dv \pm dw; \)

Rule VII \( d(uvw) = vwdv + uwdv + uvwdw. \)

### 8.4 Total Derivatives

Consider any function

\[ y = f(x, w) \quad \text{where} \quad x = g(w). \]

Unlike a partial derivative, a total derivative does not require the argument \( x \) to remain constant as \( w \) varies, and can thus allow for the postulated relationship between the two variables. The variable \( w \) can affect \( y \) through two channels: (1) indirectly, via the function \( g \) and then \( f \), and (2) directly, via the function. Whereas the partial derivative \( f_w \) is adequate for expressing the direct effect alone, a total derivative is needed to express both effects jointly.
8.4. TOTAL DERIVATIVES

To get the total derivative, we first get the total differential \( dy = f_x \, dx + f_w \, dw \). Dividing both sides of this equation by \( dw \), we have the total derivative:

\[
\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw} = \frac{\partial y}{\partial x} \frac{dx}{dw} + \frac{\partial y}{\partial w}.
\]

Example 8.4.1 Find the \( dy/dw \), given the function

\[
y = f(x, w) = 3x - w^2 \quad \text{where} \quad g(w) = 2w^2 + w + 4.
\]

\[
\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = 3(4w + 1) - 2w = 10w + 3.
\]

As a check, we may substitute the function \( g \) into \( f \), to get

\[
y = 3(2w^2 + w + 4) - w^2 = 5w^2 + 3w + 12,
\]

which is now a function of \( w \) alone. Then, we also have

\[
\frac{dy}{dw} = 10w + 3,
\]

and thus we have the identical answer.

A Variation on the Theorem

For the function

\[
y = f(x_1, x_2, w)
\]

with \( x_1 = g(w) \) and \( x_2 = h(w) \), the total derivative of \( y \) is given by

\[
\frac{dy}{dw} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dw} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dw} + \frac{\partial f}{\partial w}.
\]
Example 8.4.2 Let the production function be

\[ Q = Q(K, L, t), \]

where \( K \) is the capital input, \( L \) is the labor input, and \( t \) is the time which indicates that the production can shift over time in reflection of technological change. Since capital and labor can also change over time, we may write

\[ K = K(t) \text{ and } L = L(t). \]

Thus the rate of output with respect to time can be denoted as

\[ \frac{dQ}{dt} = \frac{\partial Q}{\partial K} \frac{dK}{dt} + \frac{\partial Q}{\partial L} \frac{dL}{dt} + \frac{\partial Q}{\partial t}. \]

Another Variation on the Theme

Now if a function is given,

\[ y = f(x_1, x_2, u, v) \]

with \( x_1 = g(u, v) \) and \( x_2 = h(u, v) \), we can find the total derivative of \( y \) with respect to \( u \) (while \( v \) is held constant). Since

\[ dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv, \]

dividing both sides of the above equation by \( du \), we have

\[ \frac{dy}{du} = \frac{\partial y}{\partial x_1} \frac{dx_1}{du} + \frac{\partial y}{\partial x_2} \frac{dx_2}{du} + \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} \frac{dv}{du} \]

\[ = \frac{\partial y}{\partial x_1} \frac{dx_1}{du} + \frac{\partial y}{\partial x_2} \frac{dx_2}{du} + \frac{\partial y}{\partial u} \left[ \frac{dv}{du} = 0 \text{ since } v \text{ is constant} \right]. \]

Since \( v \) is held constant, the above is the partial total derivative, we
redenote the above equation by the following notation:
\[
\frac{\partial y}{\partial u} = \frac{\partial y}{\partial x_1} \frac{\partial x_1}{\partial u} + \frac{\partial y}{\partial x_2} \frac{\partial x_2}{\partial u} + \frac{\partial y}{\partial u}.
\]

Remark 8.4.1 In the cases we have discussed, the total derivative formulas can be regarded as expressions of the chain rule, or the composite-function rule. Also the chain of derivatives does not have to be limited to only two “links”; the concept of the total derivative should be extendible to cases where there are three or more links in the composite function.

8.5 Implicit Function Theorem

The concept of total differentials can also enable us to find the derivatives of the so-called “implicit functions.”

Implicit Functions

A function given in the form of \( y = f(x_1, x_2, \ldots, x_n) \) is called an explicit function, because the variable \( y \) is explicitly expressed as a function of \( x \). But in many cases \( y \) is not an explicit function \( x_1, x_2, \ldots, x_n \), instead, the relationship between \( y \) and \( x_1, \ldots, x_n \) is given with the form of

\[
F(y, x_1, x_2, \ldots, x_n) = 0.
\]

Such an equation may also be defined as implicit function \( y = f(x_1, x_2, \ldots, x_n) \).

Note that an explicit function \( y = f(x_1, x_2, \ldots, x_n) \) can always be transformed into an equation \( F(y, x_1, x_2, \ldots, x_n) \equiv y - f(x_1, x_2, \ldots, x_n) = 0 \). The reverse transformation is not always possible.

In view of this uncertainty, we have to impose certain conditions under which we can be sure that a given equation \( F(y, x_1, x_2, \ldots, x_n) = 0 \) does
indeed define an implicit function \( y = f(x_1, x_2, \cdots, x_n) \). Such a result is given by the so-called “implicit-function theorem.”

**Theorem 8.5.1 (Implicit-Function Theorem)** Given \( F(y, x_1, x_2, \cdots, x_n) = 0 \), suppose that the following conditions are satisfied:

1. The function \( F \) has continuous partial derivatives \( F_y, F_{x_1}, F_{x_2}, \cdots, F_{x_n} \);
2. At point \( (y_0, x_{10}, x_{20}, \cdots, x_{n0}) \) satisfying \( F(y_0, x_{10}, x_{20}, \cdots, x_{n0}) = 0 \), \( F_y \) is nonzero.

Then there exists an \( n \)-dimensional neighborhood of \( (x_{10}, x_{20}, \cdots, x_{n0}) \), \( N \) such that \( y \) is an implicitly defined function of variables \( x_1, x_2, \cdots, x_n \), in the form of \( y = f(x_1, x_2, \cdots, x_n) \), and \( F(y, x_1, x_2, \cdots, x_n) = 0 \) for all points in \( N \). Moreover, the implicit function \( f \) is continuous, and has continuous partial derivatives \( f_1, \cdots, f_n \).

**Derivatives of Implicit Functions**

Differentiating \( F \), we have \( dF = 0 \), or

\[
F_y dy + F_{x_1} dx_1 + \cdots + F_{x_n} dx_n = 0.
\]

Suppose that only \( y \) and \( x_1 \) are allowed to vary. Then the above equation reduce to \( F_y dy + F_{x_1} dx_1 = 0 \). Thus

\[
\left. \frac{dy}{dx_1} \right|_{\text{other variable constant}} = \frac{\partial y}{\partial x_1} = -\frac{F_{x_1}}{F_y}.
\]

In the simple case where the given equation is \( F(y, x) = 0 \), the rule gives

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}.
\]

**Example 8.5.1** Suppose \( y - 3x^4 = 0 \). Then \( \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-12x^3}{1} = 12x^3 \).
In this particular case, we can easily solve the given equation for \( y \), get \( y = 3x^4 \) so that \( \frac{dy}{dx} = 12x^3 \).

**Example 8.5.2** \( F(x, y) = x^2 + y^2 - 9 = 0 \). Thus,

\[
\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y} = -\frac{x}{y}, \quad (y \neq 0).
\]

**Example 8.5.3** \( F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0 \), we have

\[
\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}.
\]

In particular, at point \((1,1,1)\), \( \frac{\partial y}{\partial x} = -3/4 \).

**Example 8.5.4** Suppose that the equation \( F(Q, K, L) = 0 \) implicitly defines a production function \( Q = f(K, L) \). Then we can find \( MP_K \) and \( MP_L \) as follows:

\[
MP_K \equiv \frac{\partial Q}{\partial K} = -\frac{F_K}{F_Q};
\]

\[
MP_L \equiv \frac{\partial Q}{\partial L} = -\frac{F_L}{F_Q}.
\]

In particular, we can also find the \( MRTS_{LK} \) which is given by

\[
MRTS_{LK} \equiv \left| \frac{\partial K}{\partial L} \right| = \frac{F_L}{F_K}.
\]

**Extension to the Simultaneous-Equation Case**

Consider a set of simultaneous equations.

\[
F^1(y_1, \cdots, y_n; x_1, \cdots, x_m) = 0;
\]

\[
F^2(y_1, \cdots, y_n; x_1, \cdots, x_m) = 0;
\]

\[
\cdots
\]
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\[ F^n(y_1, \ldots, y_n; x_1, \ldots, x_m) = 0. \]

Suppose that \( F^1, F^2, \ldots, F^n \) are differentiable. Then we have

\[
\frac{\partial F^1}{\partial y_1} dy_1 + \frac{\partial F^1}{\partial y_2} dy_2 + \cdots + \frac{\partial F^1}{\partial y_n} dy_n = - \left[ \frac{\partial F^1}{\partial x_1} dx_1 + \frac{\partial F^1}{\partial x_2} dx_2 + \cdots + \frac{\partial F^1}{\partial x_m} dx_m \right];
\]

\[
\frac{\partial F^2}{\partial y_1} dy_1 + \frac{\partial F^2}{\partial y_2} dy_2 + \cdots + \frac{\partial F^2}{\partial y_n} dy_n = - \left[ \frac{\partial F^2}{\partial x_1} dx_1 + \frac{\partial F^2}{\partial x_2} dx_2 + \cdots + \frac{\partial F^2}{\partial x_m} dx_m \right];
\]

\[ \vdots \]

\[
\frac{\partial F^n}{\partial y_1} dy_1 + \frac{\partial F^n}{\partial y_2} dy_2 + \cdots + \frac{\partial F^n}{\partial y_n} dy_n = - \left[ \frac{\partial F^n}{\partial x_1} dx_1 + \frac{\partial F^n}{\partial x_2} dx_2 + \cdots + \frac{\partial F^n}{\partial x_m} dx_m \right].
\]

Or in matrix form,

\[
\begin{bmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n}
\end{bmatrix}
\begin{bmatrix}
dy_1 \\
dy_2 \\
\vdots \\
dy_n
\end{bmatrix}
= - \begin{bmatrix}
\frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_m} \\
\frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_m}
\end{bmatrix}
\begin{bmatrix}
dx_1 \\
dx_2 \\
\vdots \\
dx_m
\end{bmatrix}.
\]

Now suppose the following Jacobian determinant is nonzero:

\[
|J| = \left| \frac{\partial(F^1, F^2, \ldots, F^n)}{\partial(y_1, y_2, \ldots, y_n)} \right| = \begin{vmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n}
\end{vmatrix} \neq 0.
\]

Then, we can obtain differentials \( dy = (dy_1, dy_2, \ldots, dy_n)' \) by inverting \( J \).

\[
dy = J^{-1} F_x dx,
\]
where

\[
F_x = \begin{bmatrix}
\frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} & \cdots & \frac{\partial F^1}{\partial x_3} \\
\frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} & \cdots & \frac{\partial F^2}{\partial x_3} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial x_1} & \frac{\partial F^n}{\partial x_2} & \cdots & \frac{\partial F^n}{\partial x_3}
\end{bmatrix}
\]

If we want to obtain partial derivatives with respect to \(x_i (i = 1, 2, \ldots, m)\), we can let \(dx_t = 0\) for \(t \neq i\). Thus, we have the following equation:

\[
\begin{bmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_3} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_3} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_1}{\partial x_i} \\
\frac{\partial y_2}{\partial x_i} \\
\vdots \\
\frac{\partial y_n}{\partial x_i}
\end{bmatrix}
= -\begin{bmatrix}
\frac{\partial F^1}{\partial x_i} \\
\frac{\partial F^2}{\partial x_i} \\
\vdots \\
\frac{\partial F^n}{\partial x_i}
\end{bmatrix}
\]

Then, by Cramer’s rule, we have

\[
\frac{\partial y_j}{\partial x_i} = \frac{|J^j_i|}{|J|} \quad (j = 1, 2, \ldots, n; i = 1, 2, \ldots, m),
\]

where \(|J^j_i|\) is obtained by replacing the jth column of \(|J|\) with

\[
F_{x_i} = \begin{bmatrix}
\frac{\partial F^1}{\partial x_i} & \frac{\partial F^2}{\partial x_i} & \cdots & \frac{\partial F^n}{\partial x_i}
\end{bmatrix}^T.
\]

Of course, we can find these derivatives by inverting the Jacobian matrix \(J\):

\[
\begin{bmatrix}
\frac{\partial y_1}{\partial x_i} \\
\frac{\partial y_2}{\partial x_i} \\
\vdots \\
\frac{\partial y_n}{\partial x_i}
\end{bmatrix}
= -\begin{bmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_3} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_3} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_3}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\partial F^1}{\partial x_i} \\
\frac{\partial F^2}{\partial x_i} \\
\vdots \\
\frac{\partial F^n}{\partial x_i}
\end{bmatrix}.
\]

In the compact notation,

\[
\frac{\partial y}{\partial x_i} = -J^{-1}F_{x_i}.
\]
Example 8.5.5 Let the national-income model be rewritten in the form:

\[
Y - C - I_0 - G_0 = 0;
\]

\[
C - \alpha - \beta (Y - T) = 0;
\]

\[
T - \gamma - \delta Y = 0.
\]

Then

\[
|J| = \begin{vmatrix}
\frac{\partial F}{\partial Y} & \frac{\partial F}{\partial C} & \frac{\partial F}{\partial T} \\
\frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial C} & \frac{\partial F^2}{\partial T} \\
\frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial C} & \frac{\partial F^3}{\partial T}
\end{vmatrix} = \begin{vmatrix}
1 & -1 & 0 \\
-\beta & 1 & \beta \\
-\delta & 0 & 1
\end{vmatrix} = 1 - \beta + \beta \delta.
\]

Suppose all exogenous variables and parameters are fixed except \( G_0 \).

Then we have

\[
\begin{bmatrix}
1 & -1 & 0 \\
-\beta & 1 & \beta \\
-\delta & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial Y}{\partial G_0} \\
\frac{\partial C}{\partial G_0} \\
\frac{\partial T}{\partial G_0}
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

We can solve the above equation for, say, \( \frac{\partial Y}{\partial G_0} \) which comes out to be

\[
\frac{\partial Y}{\partial G_0} = \frac{1}{1 - \beta + \beta \delta}.
\]
8.6 Comparative Statics of General-Function Models

Consider a single-commodity market model:

\[ Q_d = Q_s, \quad [\text{equilibrium condition}] \]
\[ Q_d = D(P, Y_0), \quad [\partial D/\partial P < 0; \partial D/\partial Y_0 > 0] \]
\[ Q_s = S(P), \quad [dS/dP > 0] \]

where \( Y_0 \) is an exogenously determined income. From this model, we can obtain a single equation:

\[ D(P, Y_0) - S(P) = 0. \]

Even though this equation cannot be solved explicitly for the equilibrium price \( \bar{P} \), by the implicit-function theorem, we know that there exists the equilibrium price \( \bar{P} \) that is the function of \( Y_0 \):

\[ \bar{P} = \bar{P}(Y_0), \]

such that

\[ D(\bar{P}, Y_0) - S(\bar{P}) \equiv F(\bar{P}, Y_0) = 0. \]

It then requires only a straight application of the implicit-function rule to produce the comparative-static derivative, \( d\bar{P}/dY_0 \):

\[ \frac{d\bar{P}}{dY_0} = -\frac{\partial F/\partial Y_0}{\partial F/\partial P} = -\frac{\partial D/\partial Y_0}{\partial D/\partial P - dS/dP} > 0. \]
CHAPTER 8. COMPARATIVE-STATIC ANALYSIS OF GENERAL-FUNCTIONS

Since $\bar{Q} = S(\bar{P})$, thus we have

$$\frac{d\bar{Q}}{dY_0} = \frac{dS}{dP} \frac{d\bar{P}}{dY_0} > 0.$$  

### 8.7 Matrix Derivatives

Matrix derivatives play an important role in economic analysis, especially in econometrics. If $A$ is $n \times n$ non-singular matrix, the derivative of its determinant with respect to $A$ is given by

$$\frac{\partial}{\partial A} |A| = [C_{ij}]$$

where $[C_{ij}]$ is the matrix of cofactors of $A$.

#### Some Useful Formulas

Let $a, b$ be $k \times 1$ vectors and $M$ be a $k \times k$ matrix. Then we have:

$$\frac{da'b}{db} = a;$$
$$\frac{db'a}{db} = a;$$
$$\frac{dMb}{db} = M';$$
$$\frac{dl'Mb}{db} = (M + M')b.$$ 

**Example 8.7.1** How to Find the Least Square Estimator in a Multiple Regression Model. Consider the multiple regression model:

$$y = X\beta + \epsilon,$$

where $n \times 1$ vector $y$ is the dependent variable, $X$ is a $n \times k$ matrix of $k$ explanatory variables with $\text{rank}(X) = k$, $\beta$ is a $k \times 1$ vector of coefficients
which are to be estimated and \( \epsilon \) is a \( n \times 1 \) vector of disturbances. We assume that the matrices of observations \( X \) and \( y \) are given. Our goal is to find an estimator \( b \) for \( \beta \) using the least squares method. The least squares estimator of \( \beta \) is a vector \( b \), which minimizes the expression

\[
E(b) = (y - Xb)'(y - Xb) = y'y - y'Xb - b'X'y + b'X'Xb.
\]

The first-order condition for extremum is:

\[
\frac{dE(b)}{db} = 0 \Rightarrow -2X'y + 2X'Xb = 0 \Rightarrow b = (X'X)^{-1}X'y.
\]

On the other hand, by the third derivation rule above, we have:

\[
\frac{dE^2(b)}{db^2} = (2X'X)' = 2X'X.
\]

It will be seen in Chapter 11 that, to check whether the solution \( b \) is indeed a minimum, we need to prove the positive definiteness of the matrix \( X'X \). First, notice that \( X'X \) is a symmetric matrix. To prove positive definite-ness, we take an arbitrary \( k \times 1 \) vector \( z, z \neq 0 \) and check the following quadratic form:

\[
z'(X'X)z = (Xz)'(Xz)
\]

The assumptions \( \text{rank}(X) = k \) and \( z \neq 0 \) imply \( Xz \neq 0 \). Thus \( X'X \) is positive definite.
Chapter 9

Optimization: Maxima and Minima of a Function of One Variable

The optimization problem is the core issue in economics. Rationality is the most basic assumption about individual decision-makers in economics. Individuals pursue maximized personal interests, and the basis of their analysis is solving optimization problems.

From now on, our attention will thus be turned to the study of goal equilibrium, in which the equilibrium state is defined as the optimal position for a given economic unit and in which the said economic unit will be deliberately striving for attainment of that equilibrium. Our primary focus will be on the classical techniques for locating optimal positions - those using differential calculus.
9.1 Optimal Values and Extreme Values

Economics is by large a science of choice. When an economic project is to be carried out, there are normally a number of alternative ways of accomplishing it. One (or more) of these alternatives will, however, be more desirable than others from the stand-point of some criterion, and it is the essence of the optimization problem to choose.

The most common criterion of choice among alternatives in economics is the goal of maximizing something or of minimizing something. Economically, we may categorize such maximization and minimization problems under general heading of optimization. From a purely mathematical point of view, the collective term for maximum and minimum is the more matter-of-fact designation extremum, meaning an extreme value.

In formulating an optimization problem, the first order of business is to delineate an objective function in which the dependent variable represents the objects whose magnitudes the economic unit in question can pick and choose. We shall therefore refer to the independent variables as choice variables.

Consider a general-form objective function \( y = f(x) \). Three specific cases of functions are depicted in the following Figures:

Figure 9.1:
Remark 9.1.1 The point E and F in (c) are relative (or local) extremum, in the sense that each of these points represents an extremum in some neighborhood of the point only. We shall continue our discussion mainly with reference to the search for relative extreme. Since an absolute (or global) maximum must be either a relative maxima or one of the ends of the function. Thus, if we know all the relative maxima, it is necessary only to select the largest of these and compare it with the end points in order to determine the absolute maximum. Hereafter, the extreme values considered will be relative or local ones, unless indicated otherwise.

9.2 Existence of Extremum for Continuous Function

Let $X$ be a domain of a function $f$. First, we give the following concepts:

Definition 9.2.1 (Local Optimum) If $f(x)$ is continuous in a neighborhood $U$ of a point $x_0$, it is said to have a local or relative maximum (minimum) at $x_0$ if for all $x \in U, x \neq x_0, f(x) \leq f(x_0)$ ($f(x) \geq f(x_0)$).

Definition 9.2.2 (Global Optimum) If $f(x^*) \geq f(x)(f(x^*) > f(x))$ for all $x$ in the domain $X$ of the function, then the function is said to have global (unique) maximum at $x^*$; if $f(x^*) \leq f(x)$ ($f(x^*) < f(x)$) for all $x$ in the domain of the function, then the function is said to have global (unique) minimum at $x^*$.

A classical conclusion about global optimization is the so-called Weierstrass theorem.

Proposition 9.2.1 (Weierstrass’s Theorem) If $f$ is continuous on a closed and bounded subset $X$ of $R^1$ (or, in the general case, of $R^n$), then $f$ reaches its maximum and minimum in $X$, i.e. there exist points $m, M \in X$ such that

$$f(x) \leq f(m)$$
$$f(x) \geq f(M)$$
CHAPTER 9. OPTIMIZATION: MAXIMA AND MINIMA OF A FUNCTION OF ONE VARIABLE

\[ f(m) \leq f(x) \leq f(M), \text{ for all } x \in X. \] Moreover, the set of maximal (minimal) points is compact.

In order to easily determine whether a function has an extreme point, the following gives the method of finding extreme values by differential method. Generally, there are two necessary conditions for the interior extreme point, that is, the first and second order necessary conditions.

### 9.3 First-Derivative Test for Relative Maximum and Minimum

Given a function \( y = f(x) \), the first derivative \( f'(x) \) plays a major role in our search for its extreme values. For smooth functions, relative extreme values can only occur where \( f'(x) = 0 \), which is a necessary (but not sufficient) condition for a relative extremum (either maximum or minimum).

![Figure 9.2](image-url)

We summarize this as in the following proposition.
9.3. FIRST-DERIVATIVE TEST FOR RELATIVE MAXIMUM AND MINIMUM

Proposition 9.3.1 (Fermat’s Theorem on the Necessary Condition for Extremum)
If \( f(x) \) is differentiable on \( X \) and has a local extremum (minimum or maximum) at an interior point \( x_0 \) of \( X \), then \( f'(x_0) = 0 \).

Note that if the first derivative vanishes at some point, it does not imply that at this point \( f \) possesses an extremum. We can only state that \( f \) has a stationary point. We have some useful results about stationary points.

Proposition 9.3.2 (Rolle Theorem) If \( f \) is continuous in \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) \), then there exists at least one point \( c \in (a, b) \) such that \( f'(c) = 0 \).

From Roll Theorem, we can prove the well-known Mean-Value Theorem, also called Lagrange’s Theorem.

![Figure 9.3: The Mean-Value Theorem implies that there exists some \( c \) in the interval \((a, b)\) such that the secant joining the endpoints of the interval \([a, b]\) is parallel to the tangent at \( c \).](image)

Proposition 9.3.3 (the Mean-Value Theorem or Lagrange’s Theorem) If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists at least one point \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).
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Proof. Let \( g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x \). Then, \( g \) is continuous in \([a, b]\), differentiable on \((a, b)\) and \( g(a) = g(b) \). Thus, by Rolle or Fermat’s Theorem, there exists one point \( c \in (a, b) \) such that \( g'(c) = 0 \), and therefore \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

An variation of the above mean-value theorem is in form of integral calculus:

**Theorem 9.3.1 (Mean-Value Theorem of Integral Calculus)** If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), then there exists a number \( c \in (a, b) \) such that

\[
\int_a^b f(x)\,dx = f'(c)(b - a).
\]

The second variation of the mean-value theorem is the generalized mean-value theorem:

**Proposition 9.3.4 (Cauchy’s Theorem or the Generalized Mean Value Theorem)** If \( f \) and \( g \) are continuous in \([a, b]\) and differentiable in \((a, b)\), then there exists at least one point \( c \in (a, b) \) such that

\[
(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).
\]

To verify if it has a maximum or minimum, we can use the following first-derivative test relative extremum:

**First-Derivative Test Relative Extremum**

If \( f'(x_0) = 0 \), then the value of the function at \( x_0 \), \( f'(x_0) \), will be

(a) A relative maximum if \( f''(x) \) changes its sign from positive to negative from the immediate left of the point \( x_0 \) to its immediate right.

(b) A relative minimum if \( f''(x) \) changes its sign from negative to positive from the immediate left of the point \( x_0 \) to its immediate right.

(c) No extreme points if \( f''(x) \) has the same sign on some neighborhood.
Example 9.3.1 \( y = (x - 1)^3 \).

\( x = 1 \) is not extreme point even \( f'(1) = 0 \).

Example 9.3.2 \( y = f(x) = x^3 - 12x^2 + 36x + 8 \).

Since \( f'(x) = 3x^2 - 24x + 36 \), to get the critical values, i.e., the values of \( x \) satisfying the condition \( f'(x) = 0 \), we set \( f'(x) = 0 \), and thus

\[
3x^2 - 24x + 36 = 0.
\]

Its roots are \( \bar{x}_1 = 2 \) and \( \bar{x}_2 = 6 \). It is easy to verify that \( f'(x) > 0 \) for \( x < 2 \) and \( f'(x) < 0 \) for \( x > 2 \). Thus \( x = 2 \) is a maximal point and the corresponding maximum value of the function \( f(2) = 40 \). Similarly, we can verify that \( x = 6 \) is a minimal point and \( f(6) = 8 \).

Example 9.3.3 Find the relative extremum of the average-cost function

\[
AC = f(Q) = Q^2 - 5Q + 8.
\]

Since \( f'(2.5) = 0 \), \( f'(Q) < 0 \) for \( Q < 2.5 \), and \( f'(Q) > 0 \) for \( Q > 2.5 \), so \( Q = 2.5 \) is a minimal point.

9.4 Second and Higher Derivatives

Since the first derivative \( f'(x) \) of a function \( y = f(x) \) is also a function of \( x \), we can consider the derivative of \( f'(x) \), which is called second derivative. Similarly, we can find derivatives of even higher orders. These will enable us to develop alternative criteria for locating the relative extremum of a function.

The second derivative of the function \( f \) is denoted by \( f''(x) \) or \( d^2y/dx^2 \). If the second derivative \( f''(x) \) exists for all \( x \) values, \( f(x) \) is said to be twice
differentiable; if, in addition, \( f''(x) \) is continuous, \( f(x) \) is said to be twice continuously differentiable.

The higher-order derivatives of \( f(x) \) can be similarly obtained and symbolized along the same line as the second derivative:

\[
f''(x), f^{(4)}(x), \ldots, f^{(n)}(x),
\]

or

\[
\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}.
\]

**Remark 9.4.1** \( d^n y/dx^n \) can be also written as \((d^n /dx^n)y\), where the \( d^n /dx^n \) part serves as an operator symbol instructing us to take the nth derivative with respect to \( x \).

**Example 9.4.1** \( y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1 \).

Then

\[
f'(x) = 16x^3 - 3x^2 + 34x + 3;
\]

\[
f''(x) = 48x^2 - 6x + 34;
\]

\[
f'''(x) = 96x - 6;
\]

\[
f^{(4)}(x) = 96;
\]

\[
f^{(5)}(x) = 0.
\]

**Example 9.4.2** Find the first four derivatives of the function

\[
y = g(x) = \frac{x}{1 + x} \quad (x \neq -1).
\]
9.5. SECOND-DERIVATIVE TEST

\[ g'(x) = (1 + x)^{-2}; \]
\[ g''(x) = -2(1 + x)^{-3}; \]
\[ g'''(x) = 6(1 + x)^{-4}; \]
\[ g^{(4)}(x) = -24(1 + x)^{-5}. \]

**Remark 9.4.2** A negative second derivative is consistently reflected in an inverse U-shaped curve; a positive second derivative is reflected in an U-shaped curve.

### 9.5 Second-Derivative Test

Recall the meaning of the first and the second derivatives of a function \( f \).

The sign of the first derivative tells us whether the value of the function increases \((f' > 0)\) or decreases \((f' < 0)\), whereas the sign of the second derivative tells us whether the slope of the function increases \((f'' > 0)\) or decreases \((f'' < 0)\). This gives us an insight how to verify that at a stationary point there exists a maximum or minimum. Thus, we have the following **second-derivative test for relative extremum**.

**Second-Derivative Test for Relative Extremum:**

If \( f'(x_0) = 0 \), then the value of the function at \( x_0 \), \( f(x_0) \) will be

- **(a)** a relative maximum if \( f''(x_0) < 0 \);
- **(b)** a relative minimum if \( f''(x_0) > 0 \).

This test is in general more convenient to use than the first-derivative test, since it does not require us to check the derivative sign to both the left and right of \( x \).
Example 9.5.1 \( y = f(x) = 4x^2 - x \).

Since \( f'(x) = 8x - 1 \) and \( f''(x) = 8 \), we know \( f(x) \) reaches its minimum at \( \bar{x} = 1/8 \). Indeed, since the function plots as a U-shaped curve, the relative minimum is also the absolute minimum.

Example 9.5.2 \( y = g(x) = x^3 - 3x^2 + 2 \).

The \( y' = g'(x) = 3x^2 - 6x \) and \( y'' = 6x - 6 \). Setting \( g'(x) = 0 \), we obtain the critical values \( \bar{x}_1 = 0 \) and \( \bar{x}_2 = 2 \), which in turn yield the two stationary values \( g(0) = 2 \) (a maximum because \( g'(0) < 0 \)) and \( g(2) = -2 \) (a minimum because \( g'(2) > 0 \)).

Remark 9.5.1 Note that when \( f'(x_0) = 0 \), \( f''(x_0) < 0 \) \( f''(x_0) > 0 \) is a sufficient condition for a relative maximum (minimum) but not a necessary condition. However, the condition \( f''(x_0) \leq 0 \) \( f''(x_0) \geq 0 \) is a necessary (even though not sufficient) for a relative maximum (minimum).

Condition for Profit Maximization

Let \( R = R(Q) \) be the total-revenue function and let \( C = C(Q) \) be the total-cost function, where \( Q \) is the level of output. The profit function is then given by

\[
\pi = \pi(Q) = R(Q) - C(Q).
\]

To find the profit-maximizing output level we need to find \( \bar{Q} \) such that

\[
\pi'(\bar{Q}) = R'(\bar{Q}) - C'(\bar{Q}),
\]

or

\[
R'(\bar{Q}) = C'(\bar{Q}), \text{ or } MR(\bar{Q}) = MC(\bar{Q}).
\]

To be sure the first-order condition leads to a maximum, we require

\[
\frac{d^2 \pi}{dQ^2} = \pi''(\bar{Q}) - C''(\bar{Q}) < 0.
\]
Economically, this would mean that, if the rate of change of $MR$ is less than the rate of change of $MC$ at $\bar{Q}$, then that output $Q$ will maximize profit.

**Example 9.5.3** Let $R(Q) = 1200Q - 2Q^2$ and $C(Q) = Q^3 - 61.25Q^2 + 152.5Q + 2000$. Then the profit function is

$$
\pi(Q) = -Q^3 + 59.2Q^2 - 328.5Q - 2000.
$$

Setting $\pi'(Q) = -3Q^2 + 118.5Q - 328.5 = 0$, we have $\bar{Q}_1 = 3$ and $\bar{Q}_2 = 36.5$. Since $\pi''(3) = -18 + 118.5 = 100.5 > 0$ and $\pi''(36.5) = -219 + 118.5 = -100.5 < 0$, so the profit-maximizing output is $\bar{Q} = 36.5$.

### 9.6 Taylor Series

This section considers the so-called “expansion” of a function $y = f(x)$ into what is known as *Taylor series* (expansion around any point $x = x_0$). To expand a function $y = f(x)$ around a point $x_0$ means to transform that function into a *polynomial* form, in which the coefficients of the various terms are expressed in terms of the derivative values $f'(x_0), f''(x_0)$, etc. - all evaluated at the point of expansion $x_0$.

**Proposition 9.6.1 (Taylor’s Theorem)** Given an arbitrary function $\phi(x)$, if we know the values of $\phi(x_0), \phi'(x_0), \phi''(x_0)$, etc., then this function can be expanded around the point $x_0$ as follows:

$$
\phi(x) = \phi(x_0) + \phi'(x_0)(x - x_0) + \frac{1}{2!}\phi''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}\phi^{(n)}(x_0)(x - x_0)^n + R_n
$$

$$
\equiv P_n + R_n,
$$

where $P_n$ represents the $n$th-degree polynomial and $R_n$ denotes a remainder which
can be denoted by the so-called Lagrange form of the remainder:

\[ R_n = \frac{\phi^{(n+1)}(P)}{(n+1)!} (x-x_0)^{n+1} \]

with \( P \) being a point between \( x \) and \( x_0 \). Here \( n! \) is the “n factorial”, defined as

\[ n! \equiv n(n - 1)(n - 2) \cdots (3)(2)(1). \]

**Remark 9.6.1** When \( n = 0 \), the Taylor series reduce to the mean-value theorem we discussed in Section 9.3:

\[ \phi(x) = P_0 + R_0 = \phi(x_0) + \phi'(P)(x-x_0), \]

or

\[ \phi(x) - \phi(x_0) = \phi'(P)(x-x_0), \]

which states that the difference between the value of the function \( \phi \) at \( x_0 \) and at any other \( x \) value can be expressed as the product of the difference \( (x - x_0) \) and \( \phi'(P) \) with \( P \) being some point between \( x \) and \( x_0 \).

![Figure 9.4:](image-url)
Remark 9.6.2 If \(x_0 = 0\), then Taylor series reduce to the so-called Maclaurin series:

\[
\phi(x) = \phi(0) + \phi'(0)x + \frac{1}{2!}\phi''(0)x^2 + \cdots + \frac{1}{n!}\phi^{(n)}(0)x^n + \frac{1}{(n+1)!}\phi^{(n+1)}(P)x^{n+1},
\]

where \(P\) is a point between 0 and \(x\).

Example 9.6.1 Expand the function

\[
\phi(x) = \frac{1}{1 + x}
\]

around the point \(x_0 = 1\), with \(n = 4\). Since \(\phi(1) = \frac{1}{2}\) and

\[
\begin{align*}
\phi'(x) &= -(1 + x)^{-2}, \quad \phi'(1) = -1/4; \\
\phi''(x) &= 2(1 + x)^{-3}, \quad \phi''(1) = 1/4; \\
\phi^{(3)}(x) &= -6(1 + x)^{-4}, \quad \phi^{(3)}(1) = -3/8; \\
\phi^{(4)}(x) &= 24(1 + x)^{-5}, \quad \phi^{(4)}(1) = 3/4,
\end{align*}
\]

we obtain the following Taylor series:

\[
\phi(x) = 1/2 - 1/4(x - 1) + 1/8(x - 1)^2 - 1/16(x - 1)^3 + 1/32(x - 1)^4 + R_n.
\]

9.7 Nth-Derivative Test

A relative extremum of the function \(f\) can be equivalently defined as follows:

A function \(f(x)\) attains a relative maximum (minimum) value at \(x_0\) if \(f(x) - f(x_0)\) is nonpositive (nonnegative) for values of \(x\) in some neighborhood of \(x_0\).

Assume \(f(x)\) has finite, continuous derivatives up to the desired order at \(x = x_0\), then the function can be expanded around \(x = x_0\) as a Taylor
CHAPTER 9. OPTIMIZATION: MAXIMA AND MINIMA OF A FUNCTION OF ONE VARIABLE

series:

\[ f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + \frac{1}{(n + 1)!}f^{(n+1)}(x_0)(x - x_0)^{n+1}. \]

From the above expansion, we have the following test:

**Nth-Derivative Test:**

If \( f'(x_0) = 0 \), and if the first nonzero derivative value at \( x_0 \) encountered in successive derivation is that of the Nth derivative, \( f^{(N)}(x_0) \neq 0 \), then the stationary value \( f(x_0) \) will be

(a) a relative maximum if \( N \) is an even number and \( f^{(N)}(x_0) < 0 \);

(b) a relative minimum if \( N \) is an even number and \( f^{(N)}(x_0) > 0 \);

(c) a inflection point if \( N \) is odd.

**Example 9.7.1** \( y = (7 - x)^4 \).

Since \( f'(7) = 4(7-7)^3 = 0, f''(7) = 12(7-7)^2 = 0, f'''(7) = 24(7-7) = 0, f^{(n)}(7) = 24 > 0 \), so \( x = 7 \) is a minimal point such that \( f(7) = 0 \).
Chapter 10

Exponential and Logarithmic Functions

Exponential functions, as well as the closely related logarithmic functions, have important applications in economics, especially in connection with growth problems, and in economic dynamics in general. This chapter gives some basic properties and derivatives of exponential and logarithmic functions.

10.1 The Nature of Exponential Functions

In this simple version, the exponential function may be represented in the form:

\[ y = f(t) = b^t \quad (b > 1), \]

where \( b \) denotes a fixed base of the exponent. Its generalized version has the form:

\[ y = ab^{ct}. \]

Remark 10.1.1 \( y = ab^{ct} = a(b^{c})^{t} \). Thus we can consider \( b^{c} \) as a base of exponent \( t \). It changes exponent from \( ct \) to \( t \) and changes base \( b \) to \( b^{c} \).
If the base is an irrational number \( e = 2.71828 \ldots \), the function:

\[ y = ae^{rt} \]

is referred to the natural exponential function, which can alternatively denote

\[ y = a \exp(rt) \].

**Remark 10.1.2** It can be proved that \( e \) may be defined as the limit:

\[ e \equiv \lim_{n \to \infty} f(n) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n. \]

### 10.2 Logarithmic Functions

For the exponential function \( y = b^t \) and the natural exponential function \( y = e^t \), taking the log of \( y \) to the base \( b \) (denote by \( \log_b y \)) and the base \( e \) (denoted by \( \log_e y \)) respectively, we have

\[ t = \log_b y, \]

and

\[ t = \log_e y \equiv \ln y. \]

The following rules are familiar to us:

**Rules:**

(a) \( \ln(uv) = \ln u + \ln v \) [log of product];

(b) \( \ln(u/v) = \ln u - \ln v \) [log of quotient];

(c) \( \ln u^a = a \ln u \) [log of power];
10.3. DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

(d) \( \log_b u = (\log_b e)(\log_e u) = (\log_b e)(\ln u) \) [conversion of log base];

(e) \( \log_b e = 1/(\log_e b) = 1/\ln b \) [inversion of log base].

Properties of Log:

(a) \( \log y_1 = \log y_2 \) iff \( y_1 = y_2 \);

(b) \( \log y_1 > \log y_2 \) iff \( y_1 > y_2 \);

(c) \( 0 < y < 1 \) iff \( \log y < 0 \);

(d) \( y = 1 \) iff \( \log y = 0 \);

(e) \( \log y \to \infty \) as \( y \to \infty \);

(f) \( \log y \to -\infty \) as \( y \to 0 \).

Remark 10.2.1 \( t = \log_b y \) and \( t = \ln y \) are the respective inverse functions of the exponential functions \( y = b^t \) and \( y = e^t \).

10.3 Derivatives of Exponential and Logarithmic Functions

(a) \( \frac{d}{dt} \ln t = \frac{1}{t} \);

(b) \( \frac{de^t}{dt} = e^t \);

(c) \( \frac{de^{f(t)}}{dt} = f'(t)e^{f(t)} \);

(d) \( \frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)} \).

Example 10.3.1 The following are examples to find derivatives:
CHAPTER 10. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

(a) Let \( y = e^{rt} \). Then \( \frac{dy}{dt} = re^{rt} \);
(b) Let \( y = e^{-t} \). Then \( \frac{dy}{dt} = -e^{-t} \);
(c) Let \( y = \ln at \). Then \( \frac{dy}{dt} = \frac{a}{at} = \frac{1}{t} \);
(d) Let \( y = \ln t^c \). Since \( y = \ln t^c = c \ln t \), so \( \frac{dy}{dt} = c(1/t) \);
(e) Let \( y = t^3 \ln t^2 \). Then \( \frac{dy}{dt} = 3t^2 \ln t + 2t^3 / t = 2t^2(1 + 3 \ln t) \).

The Case of Base \( b \)

(a) \( \frac{db}{dt} = b'(\ln b) \);
(b) \( \frac{d}{dt} \log_b t = \frac{1}{\ln b} \);
(c) \( \frac{d}{dt} b^{f(t)} = f'(t)b^{f(1)} \ln b \);
(d) \( \frac{d}{dt} \log_b f(t) = \frac{f'(t)}{f(t)} \frac{1}{\ln b} \).

Proof of (a). Since \( b^t = e^{\ln b^t} = e^{t \ln b} \), then \( (d/dt)b^t = (d/dt)e^{t \ln b} = (\ln b)(e^{t \ln b}) = b't \ln b \).

Proof of (b). Since

\[
\log_b t = (\log_b e)(\log_e t) = (1/\ln b) \ln t,
\]

\[
(d/dt)(\log_b t) = (d/dt)[(1/\ln b) \ln t] = (1/\ln b)(1/t)
\]

Example 10.3.2 (a) Let \( y = 12^{1-t} \). Then \( \frac{dy}{dt} = \frac{d(1-t)}{dt}12^{1-t} \ln 12 = -12^{1-t} \ln 12 \)

An Application

Example 10.3.3 Find \( dy/dx \) from \( y = x^a e^{kx-c} \). Taking the natural log of both sides, we have

\[
\ln y = a \ln x + kx - c.
\]

Differentiating both with respect to \( x \), we get

\[
\frac{1}{y} \frac{dy}{dx} = \frac{a}{x} + k.
\]
Thus \( \frac{dy}{dx} = \left( \frac{a}{x} + k \right)y = (a/x + k)xe^{kx - c} \).

Use the above technical method, we can similarly find the derivative of \( y = \phi^\psi \).
Chapter 11

Optimization: Maxima and Minima of a Function of Two or More Variables

This chapter develops a way of finding the extreme values of an objective function that involves two or more choice variables. As before, our attention will be focused heavily on relative extrema, and for this reason we should often drop the adjective “relative,” with the understanding that, unless otherwise specified, the extrema referred to are relative.

11.1 The Differential Version of Optimization Condition

This section shows the possibility of equivalently expressing the derivative version of first and second conditions in terms of differentials.

Consider the function $z = f(x)$. Recall that the differential of $z = f(x)$ is

$$dz = f'(x)dx.$$
Since \( f'(x) = 0 \), which implies \( dz = 0 \), is the necessary condition for extreme values, so \( dz = 0 \) is also the necessary condition for extreme values. This first-order condition requires that \( dz = 0 \) as \( x \) is varied. In such a context, with \( dx \neq 0 \), \( dz = 0 \) if and only if \( f'(x) = 0 \).

What about the sufficient conditions in terms of second-order differentials?

Differentiating \( dz = f'(x)dx \), we have

\[
d^2z \equiv d(dz) = d[f'(x)dx]
\]

\[
= d[f'(x)]dx
\]

\[
= f''(x)dx^2.
\]

Note that the symbols \( d^2z \) and \( dx^2 \) are fundamentally different. \( d^2z \) means the second-order differential of \( z \); but \( dx^2 \) means the squaring of the first-order differential \( dx \).

Thus, from the above equation, we have \( d^2z < 0 \) (\( d^2z > 0 \)) if and only if \( f''x < 0 \) (\( f''x > 0 \)). Therefore, the second-order sufficient condition for maximum (minimum) of \( z = f(x) \) is \( d^2z < 0 \) (\( d^2z > 0 \)).

### 11.2 Extreme Values of a Function of Two Variables

For a function of one variable, an extreme value is represented graphically by the peak of a hill or the bottom of a valley in a two-dimensional graph. With two choice variables, the graph of the function \( z = f(x, y) \) becomes a surface in a 3-space, and while the extreme values are still to be associated with peaks and bottoms.
First-Order Condition

For the function \( z = f(x, y) \), the first-order necessary condition for an extremum again involves \( dz = 0 \) for arbitrary values of \( dx \) and \( dy \): an extremum must be a stationary point, at which \( z \) must be constant for arbitrary infinitesimal changes of two variables \( x \) and \( y \).

In the present two-variable case, the total differential is

\[
dz = f_x dx + f_y dy.
\]

Thus, the equivalent derivative version of the first-order condition \( dz = 0 \) is

\[
f_x = f_y = 0 \quad \text{or} \quad \partial f / \partial x = \partial f / \partial y = 0.
\]

As in the earlier discussion, the first-order condition is necessary, but not sufficient. To develop a sufficient condition, we must look to the second-order total, which is related to second-order partial derivatives.

Second-Order Partial Derivatives

From the function \( z = f(x, y) \), we can have two first-order partial derivatives, \( f_x \) and \( f_y \). Since \( f_x \) and \( f_y \) are themselves functions of \( x \), we can find
second-order partial derivatives:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} f_x \equiv \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right); \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} f_y \equiv \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right); \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \frac{\partial z}{\partial y} \equiv \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right); \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \frac{\partial z}{\partial x} \equiv \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right).
\end{align*}
\]

The last two are called cross (or mixed) partial derivatives.

**Theorem 11.2.1 (Schwarz’s Theorem or Young’s Theorem)** If at least one of the two partials is continuous, then

\[
\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \ldots, n.
\]

**Remark 11.2.1** Even though \(f_{xy}\) and \(f_{yx}\) have been separately defined, they will – according to Young’s theorem, be identical with each other, as long as the two cross partial derivatives are both continuous. In fact, this theorem applies also to functions of three or more variables. Given \(z = g(u, v, w)\), for instance, the mixed partial derivatives will be characterized by \(g_{uv} = g_{vu}, g_{uw} = g_{wu}\), etc. provided these partial derivatives are continuous.

**Example 11.2.1** Find all second-order partial derivatives of \(z = x^3 + 5xy - y^2\). The first partial derivatives of this function are

\[
\begin{align*}
f_x &= 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y.
\end{align*}
\]

Thus, \(f_{xx} = 6x, f_{yx} = 5,\) and \(f_{yy} = -2\). As expected, \(f_{yx} = f_{xy}\).
11.2. EXTREME VALUES OF A FUNCTION OF TWO VARIABLES

Example 11.2.2 For \( z = x^2 e^{-y} \), its first partial derivatives are

\[
\begin{align*}
  f_x &= 2x e^{-y} \\
  f_y &= -x^2 e^{-y}.
\end{align*}
\]

Again, \( f_{yx} = f_{xy} \).

Second-Order Total Differentials

From the first total differential

\[
dz = f_x dx + f_y dy,
\]

we can obtain the second-order total differential \( d^2 z \):

\[
d^2 z = d(dz) = \frac{\partial(dz)}{\partial x} dx + \frac{\partial(dz)}{\partial y} dy
\]

= \frac{\partial}{\partial x} (f_x dx + f_y dy) dx + \frac{\partial}{\partial y} (f_x dx + f_y dy) dy

= (f_{xx} dx^2 + f_{xy} dy dx + f_{yx} dx dy + f_{yy} dy^2)

= f_{xx} dx^2 + 2f_{xy} dx dy + f_{yy} dy^2 \quad \text{[if } f_{xy} = f_{yx}] .
\]

We know that if \( f(x, y) \) satisfy the conditions of Schwarz’s theorem, we have \( f_{xy} = f_{yx} \).

Example 11.2.3 Given \( z = x^3 + 5xy - y^2 \), find \( dz \) and \( d^2 z \).

\[
dz = f_x dx + f_y dy
\]

= \((3x^2 + 5y)dx + (5x - 2y)dy.
\]
\[ d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \]
\[ = 6dx^2 + 10dxdy - 2dy^2. \]

Note that the second-order total differential can be written in matrix form

\[ d^2z = \begin{bmatrix} dx, dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \]

for the function \( z = f(x, y) \), where the matrix

\[ \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \]

is the Hessian matrix (or simply a Hessian).

Then, by the discussion on quadratic forms in Chapter 5, we have

(a) \( d^2z \) is positive definite iff \( f_{xx} > 0 \) and \( f_{xx}f_{yy} - (f_{xy})^2 > 0 \);

(b) \( d^2z \) is negative definite iff \( f_{xx} < 0 \) and \( f_{xx}f_{yy} - (f_{xy})^2 > 0 \).

From the inequality \( f_{xx}f_{yy} - (f_{xy})^2 > 0 \), it implies that \( f_{xx} \) and \( f_{yy} \) are required to take the same sign.

**Example 11.2.4** Give \( f_{xx} = -2, f_{xy} = 1, \) and \( f_{yy} = -1 \) at a certain point on a function \( z = f(x, y) \), does \( d^2z \) have a definite sign at that point regardless of the values of \( dx \) and \( dy \)? The Hessian determinant is in this case

\[
\begin{vmatrix}
-2 & 1 \\
1 & -1
\end{vmatrix},
\]
with principal minors $|H_1| = -2 < 0$ and

$$|H_2| = \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 2 - 1 = 1 > 0.$$  

Thus $d^2z$ is negative definite.

**Second-Order Condition for Extremum**

Using the concept of $d^2z$, we can state the second-order sufficient condition for extremum:

(a) A maximum of $z = f(x, y)$ if $d^2z < 0$ for any values of $dx$ and $dy$, not both zero;

(b) A minimum of $z = f(x, y)$ if $d^2z > 0$ for any values of $dx$ and $dy$, not both zero.

**Example 11.2.5** Give $f_{xx} = -2$, $f_{xy} = 1$, and $f_{yy} = -1$ at a certain point on a function $z = f(x, y)$, does $d^2z$ have a definite sign at that point regardless of the values of $dx$ and $dy$? The Hessian determinant is in this case

$$\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix},$$  

with principal minors $|H_1| = -2 < 0$ and

$$|H_2| = \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 2 - 1 = 1 > 0.$$  

Thus $d^2z$ is negative definite.

For operational convenience, second-order differential conditions can be translated into equivalent conditions on second-order derivatives. The
Main Result

For any values of $dx$ and $dy$, not both zero,

$$d^2z < 0 \iff f_{xx} < 0; f_{yy} < 0; \text{ and } f_{xx}f_{yy} > (f_{xy})^2;$$

$$d^2z > 0 \iff f_{xx} > 0; f_{yy} > 0; \text{ and } f_{xx}f_{yy} > (f_{xy})^2.$$ 

From the first- and second-order conditions, we obtain conditions for relative extremum:

Conditions for Maximum:

1. $f_x = f_y = 0$ (necessary condition);
2. $f_{xx} < 0; f_{yy} < 0; \text{ and } f_{xx}f_{yy} > (f_{xy})^2.$

Conditions for Minimum:

1. $f_x = f_y = 0$ (necessary condition);
2. $f_{xx} > 0; f_{yy} > 0; \text{ and } f_{xx}f_{yy} > (f_{xy})^2.$

Example 11.2.6 Find the extreme values of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$.

$$f_x = 24x^2 + 2y - 6x, \quad f_y = 2x + 2y;$$

$$f_{xx} = 48x - 6, \quad f_{yy} = 2, \quad f_{xy} = 2.$$ 

Setting $f_x = 0$ and $f_y = 0$, we have

$$24x^2 + 2y - 6x = 0;$$
11.3. OBJECTIVE FUNCTIONS WITH MORE THAN TWO VARIABLES

\[ 2y + 2x = 0. \]

Then \( y = -x \) and thus from \( 24x^2 + 2y - 6y \), we have \( 24x^2 - 8x = 0 \) which yields two solutions for \( x \): \( \bar{x}_1 = 0 \) and \( \bar{x}_2 = 1/3 \).

Since \( f_{xx}(0,0) = -6 \) and \( f_{yy}(0,0) = 2 \), it is impossible \( f_{xx}f_{yy} \geq (f_{xy})^2 = 4 \), so the point \( (\bar{x}_1, \bar{y}_1) = (0,0) \) is not extreme point. For the solution \( (\bar{x}_2, \bar{y}_2) = (1/3, -1/3) \), we find that \( f_{xx} = 10 > 0 \), \( f_{yy} = f_{xy} = 2 > 0 \), and \( f_{xx}f_{yy} - (f_{xy})^2 = 20 - 4 > 0 \), so \( (\bar{x}, \bar{y}, \bar{z}) = (1/3, -1/3, 23/27) \) is a relative minimum point.

Example 11.2.7 \( z = x + 2ey - e^x - e^{2y} \). Letting \( f_x = 1 - e^x \) and \( f_y = 2e - 2e^{2y} \), we have \( \bar{x} = 0 \) and \( \bar{y} = 1/2 \). Since \( f_{xx} = -e^x \), \( f_{yy} = -4e^{2y} \), and \( f_{xy} = 0 \), we have \( f_{xx}(0, 1/2) = -1 < 0 \), \( f_{yy}(0, 1/2) = -4e < 0 \), and \( f_{xx}f_{yy} - (f_{xy})^2 > 0 \), so \( (\bar{x}, \bar{y}, \bar{z}) = (0, 1/2, -1) \) is the maximum point.

11.3 Objective Functions with More than Two Variables

When there are \( n \) choice variables, the objective function may be expressed as

\[ z = f(x_1, x_2, \ldots, x_n). \]

The total differential will then be

\[ dz = f_1dx_1 + f_2dx_2 + \cdots + f_n dx_n, \]

so that the necessary condition for extremum is \( dz = 0 \) for arbitrary \( dx_i \), which in turn means that all the \( n \) first-order partial derivatives are required to be zero:

\[ f_1 = f_2 = \cdots = f_n = 0. \]
It can be verified that the second-order differential \(d^2z\) can be written as

\[
d^2z = \begin{bmatrix}
    dx_1, dx_2, \ldots, dx_n
\end{bmatrix}
\begin{bmatrix}
    f_{11} & f_{12} & \cdots & f_{1n} \\
    f_{21} & f_{22} & \cdots & f_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{n1} & f_{n2} & \cdots & f_{nn}
\end{bmatrix}
\begin{bmatrix}
    dx_1 \\
    dx_2 \\
    \vdots \\
    dx_n
\end{bmatrix}
\equiv (dx)'Hdx.
\]

Thus the Hessian determinant is

\[
|H| = \begin{vmatrix}
    f_{11} & f_{12} & \cdots & f_{1n} \\
    f_{21} & f_{22} & \cdots & f_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{n1} & f_{n2} & \cdots & f_{nn}
\end{vmatrix}
\]

and the second-order sufficient condition for extremum is, as before, that all the \(n\) principal minors be positive for a minimum in \(z\) and that they duly alternate in sign for a maximum in \(z\), the first one being negative.

In summary, we have

**Conditions for Maximum:**

1. \(f_1 = f_2 = \cdots = f_n = 0\) (necessary condition);
2. \(|H_1| < 0, |H_2| > 0, |H_3| < 0, \ldots, (-1)^n|H_n| > 0\). [\(d^2z\) is negative definite].

**Conditions for Minimum:**

1. \(f_1 = f_2 = \cdots = f_n = 0\) (necessary condition);
2. \(|H_1| > 0, |H_2| > 0, |H_3| > 0, \ldots, |H_n| > 0\). [\(d^2z\) is positive]
Example 11.3.1 Find the extreme values of

\[ z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2. \]

From the first-order conditions:

\[
\begin{align*}
    f_1 &= 0 : 4x_1 + x_2 + x_3 = 0; \\
    f_2 &= 0 : x_1 + 8x_2 + 0 = 0; \\
    f_3 &= 0 : x_1 + 0 + 2x_3 = 0,
\end{align*}
\]

we can find a unique solution \( \bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0 \). This means that there is only one stationary value, \( \bar{z} = 2 \). The Hessian determinant of this function is

\[
|H| = \begin{vmatrix}
    f_{11} & f_{12} & f_{13} \\
    f_{21} & f_{22} & f_{23} \\
    f_{31} & f_{32} & f_{33}
\end{vmatrix} = \begin{vmatrix}
    4 & 1 & 1 \\
    1 & 8 & 0 \\
    1 & 0 & 2
\end{vmatrix}.
\]

Since the principal minors of which are all positive: \( |H_1| = 4, |H_2| = 31, \) and \( |H_3| = 54 \), we can conclude that \( \bar{z} = 2 \) is a minimum.

11.4 Second-Order Conditions in Relation to Concavity and Convexity

Second-order conditions which are always concerned with whether a stationary point is the peak of a hill or the bottom of a valley are closely related to the so-called (strictly) concave or convex functions.

A function that gives rise to a hill (valley) over the entire domain is said to be a concave (convex) function. If the hill (valley) pertains only to a subset \( S \) of the domain, the function is said to be concave (convex) on \( S \).
Mathematically, a function is concave (or convex) if and only if, for any pair of distinct points \( u \) and \( v \) in the domain of \( f \), and for any \( 0 < \theta < 1 \),

\[
\theta f(u) + (1 - \theta) f(v) \leq f(\theta u + (1 - \theta)v)
\]

(or \( \theta f(u) + (1 - \theta) f(v) \geq f(\theta u + (1 - \theta)v) \))

Further, if the weak inequality “\( \leq \)” (“\( \geq \)”)

inequality “\( < \)” (“\( > \)”), the function is said to be strictly concave (strictly convex).

**Remark 11.4.1** \( \theta u + (1 - \theta)v \) consists of line segments between points \( u \) and \( v \) when \( \theta \) takes values of \( 0 \leq \theta \leq 1 \). Thus, in the sense of geometry, the function \( f \) is concave (convex) if and only if the line segment of any two points \( u \) and \( v \) lies on or below (above) the surface. The function is strictly concave (strictly convex) if and only if the line segment lies entirely below (above) the surface, except at \( M \) and \( N \).

![Figure 11.2:](image)

From the definition of concavity and convexity, we have the following three theorems:
11.4. SECOND-ORDER CONDITIONS IN RELATION TO CONCAVITY AND CONVEXITY

**Theorem I (Linear functions).** If \( f(x) \) is a linear function, then it is a concave function as well as a convex function, but not strictly so.

**Theorem II (Negative of a function).** If \( f(x) \) is a (strictly) concave function, then \(-f(x)\) is a (strictly) convex function, and vice versa.

**Theorem III (Sum of functions).** If \( f(x) \) and \( g(x) \) are both concave (convex) functions, then \( f(x) + g(x) \) is a concave (convex) function. Further, in addition, either one or both of them are strictly concave (strictly convex), then \( f(x) + g(x) \) is strictly concave (convex).

In view of the association of concavity (convexity) with a global hill (valley) configuration, an extremum of a concave (convex) function must be a peak - a maximum (a bottom - a minimum). Moreover, the maximum (minimum) must be a peak an absolute maximum (minimum). Further, the maximum (minimum) is unique if the function is strictly concave (strictly convex).

In the preceding paragraph, the properties of concavity and convexity are taken to be global in scope. If they are valid only for a portion of surface (only in a subset \( S \) of domain), the associated maximum and minimum are relative to that subset of the domain.

We know that when \( z = f(x_1, \cdots, x_n) \) is twice continuous differentiable, \( z = f(x_1, \cdots, x_n) \) reaches its maximum (minimum) if \( d^2z \) is negative (positive) definite.

The following proposition shows the relationship between concavity (convexity) and negative definiteness.

**Proposition 11.4.1** A twice continuous differentiable function \( z = f(x_1, x_2, \cdots, x_n) \) is concave (convex) if and only if \( d^2z \) is everywhere negative (positive) semidefinite. The said function is strictly concave (convex) if (but not only if) \( d^2z \) is everywhere negative (positive) definite.

This proposition is useful. It can easily verify whether a function is
strictly concave (strictly convex) by checking whether its Hessian matrix is negative (positive) definite.

**Example 11.4.1** Check \( z = -x^4 \) for concavity or convexity by the derivative condition.

Since \( d^2 z = -12x^2 \leq 0 \) for all \( x \), it is concave. This function, in fact, is strictly concave.

**Example 11.4.2** Check \( z = x_1^2 + x_2^2 \) for concavity or convexity.

Since

\[
|H| = \begin{vmatrix}
 f_{11} & f_{12} \\
 f_{21} & f_{22}
\end{vmatrix} = \begin{vmatrix}
 2 & 0 \\
 0 & 2
\end{vmatrix},
\]

\( |H_1| = 2 > 0 \), \( |H_2| = 4 > 0 \). Thus, by the proposition, the function is strictly convex.

If we only require a function be differentiable, but not twice differentiable, the following position fully characterizes the concavity of the function.

**Proposition 11.4.2** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable. Then \( f \) is concave if and only if for any \( x, y \in \mathbb{R} \), we have

\[
f(y) \leq f(x) + f'(x)(y - x). \tag{11.4.1}
\]

Indeed, for a concave function, the from Figure ??, we have

\[
\frac{u(x) - u(x^*)}{x - x^*} \leq u'(x^*),
\]

which means (??).
When there are two or more independent variables, the above proposition becomes:

**Proposition 11.4.3** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then $f$ is concave if and only if for any $x, y \in \mathbb{R}$, we have

$$f(y) \leq f(x) + \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} (y_j - x_j).$$  \hfill (11.4.2)

### 11.5 Economic Applications

**Problem of a Multiproduct Firm**

**Example 11.5.1** Suppose a competitive firm produces two products. Let $Q_i$ represents the output level of the $i$-th product and let the prices of the products are denoted by $P_1$ and $P_2$. Since the firm is a competitive firm, it takes the prices as given. Accordingly, the firm’s revenue function will be

$$TR = P_1 Q_1 + P_2 Q_2$$
The firm’s cost function is assumed to be
\[ C = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2. \]

Then, the profit function of this hypothetical firm is given by
\[ \pi = TR - C = P_1 Q_1 + P_2 Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2. \]

The firm wants to maximize the profit by choosing the levels of \( Q_1 \) and \( Q_2 \). For this purpose, setting
\[ \frac{\partial \pi}{\partial Q_1} = 0 : 4Q_1 + Q_2 = P_1; \]
\[ \frac{\partial \pi}{\partial Q_2} = 0 : Q_1 + 4Q_2 = P_2, \]
we have
\[ \bar{Q}_1 = \frac{4P_1 - P_2}{15} \quad \text{and} \quad \bar{Q}_2 = \frac{4P_2 - P_1}{15}. \]

Also the Hessian matrix is
\[ H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}. \]

Since \( |H_1| = -4 < 0 \) and \( |H_2| = 16 - 1 > 0 \), the Hessian matrix (or \( d^2 z \)) is negative definite, and the solution does maximize. If fact, since \( H \) is everywhere negative definite, the maximum profit found above is actually a unique absolute maximum.

**Example 11.5.2** Let us now transplant the problem in the above example into the setting of a monopolistic market.
Suppose that the demands facing the monopolist firm are as follows:

\[ Q_1 = 40 - 2P_1 + P_2; \]
\[ Q_2 = 15 + P_1 - P_2. \]

Again, the cost function is given by

\[ C = Q_1^2 + Q_1Q_2 + Q_2^2. \]

From the monopolistic’s demand function, we can express prices \( P_1 \) and \( P_2 \) as functions of \( Q_1 \) and \( Q_2 \). That is, solving

\[
\begin{align*}
-2P_1 + P_2 &= Q_1 - 40; \\
P_1 - P_2 &= Q_2 - 15,
\end{align*}
\]

we have

\[
\begin{align*}
P_1 &= 55 - Q_1 - Q_2; \\
P_2 &= 70 - Q_1 - 2Q_2.
\end{align*}
\]

Consequently, the firm’s total revenue function \( TR \) can be written as

\[
TR = P_1Q_1 + P_2Q_2 \\
= (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2; \\
= 55Q_1 + 70Q_2 - 2Q_1Q_2 - Q_1^2 - 2Q_2^2.
\]

Thus the profit function is

\[
\pi = TR - C \\
= 55Q_1 + 70Q_2 - 3Q_1Q_2 - 2Q_1^2 - 3Q_2^2,
\]
which is an object function with two choice variable. Setting

$$\frac{\partial \pi}{\partial Q_1} = 0 : \quad 4Q_1 + 3Q_2 = 55;$$
$$\frac{\partial \pi}{\partial Q_2} = 0 : \quad 3Q_1 + 6Q_2 = 70,$$

we can find the solution output level are

$$(\bar{Q}_1, \bar{Q}_2) = (8, \frac{2}{3}).$$

The prices and profit are

$$\bar{P}_1 = 39 \frac{1}{3}, \quad \bar{P}_2 = 46 \frac{2}{3}, \text{ and } \bar{\pi} = 488 \frac{1}{3}.$$ 

Inasmuch as the Hessian determinant is

$$\begin{vmatrix} -4 & -3 \\ -3 & -6 \end{vmatrix},$$

we have $|H_1| = -4 < 0$ and $|H_2| = 15 > 0$ so that the value of $\bar{\pi}$ does represent the maximum. Also, since Hessian matrix is everywhere negative definite, it is a unique absolute maximum.
Chapter 12

Optimization with Equality Constraints

The last chapter presented a general method for finding the relative extrema of an objective function of two or more variables. One important feature of that discussion is that all the choice variables are independent of one another, in the sense that the decision made regarding one variable does not impinge upon the choice of the remaining variables. However, in many cases, optimization problems are the constrained optimization problem. Say, every consumer maximizes his utility subject to his budget constraint. A firm minimizes the cost of production with the constraint of production technique.

In the present chapter, we shall consider the problem of optimization with equality constraints. Our primary concern will be with relative constrained extrema.
12.1 Effects of a Constraint

In general, for a function, say \( z = f(x, y) \), the difference between a constrained extremum and a free extremum may be illustrated in Figure ??.

The free extremum in this particular graph is the peak point of entire domain, but the constrained extremum is at the peak of the inverse U-shaped curve situated on top of the constraint line. In general, a constraint (less freedom) maximum can be expected to have a lower value than the free (more freedom) maximum, although by coincidence, the two maxima may happen to have the same value. But the constrained maximum can never exceed the free maximum. In general, the number of constraints should be less than the number of choice variables.
12.2 Finding the Stationary Values

For illustration, let us consider a consumer choice problem: maximizes his utility:

\[ u(x_1, x_2) = x_1 x_2 + 2x_1 \]

subject to the budget constraint:

\[ 4x_1 + 4x_2 = 60. \]

Even without any new technique of solution, the constrained maximum in this problem can easily be found. Since the budget line implies

\[ x_2 = \frac{60 - 4x_1}{2} = 30 - 2x_1 \]

we can combine the constraint with the objective function by substitution. The result is an objective function in one variable only:

\[ u = x_1(30 - 2x_1) + 2x_1 = 32x_1 - 2x_1^2, \]

which can be handled with the method already learned. By setting

\[ \frac{\partial u}{\partial x_1} = 32 - 4x_1 = 0, \]

we get the solution \( \bar{x}_1 = 8 \) and thus, by the budget constraint, \( \bar{x}_2 = 30 - 2\bar{x}_1 = 30 - 16 = 14 \) since \( d^2u/dx_1^2 = -4 < 0 \), that stationary value constitutes a (constrained) maximum.

However, when the constraint is itself a complicated function, or when the constraint cannot be solved to express one variable as an explicit function of the other variables, the technique of substitution and elimination of variables could become a burdensome task or would in fact be of no
The essence of the Lagrange-multiplier method is to condition of the free-extremum problem can still be applied.

In general, given an objective function

\[ z = f(x, y) \]

subject to the constraint

\[ g(x, y) = c, \]

where \( c \) is a constant, we can write the Lagrange function as

\[ Z = f(x, y) + \lambda[c - g(x, y)]. \]

The symbol \( \lambda \), representing some as yet undermined number, is called a Lagrange multiplier. If we can somehow be assured that \( g(x, y) = c \), so that the constraint will be satisfied, then the last term of \( Z \) will vanish regardless of the value of \( \lambda \). In that event, \( Z \) will be identical with \( u \). Moreover, with the constraint out of the way, we only have to seek the free maximum. The question is: How can we make the parenthetical expression in \( Z \) vanish?

The tactic that will accomplish this is simply to treat \( \lambda \) as an additional variable, i.e., to consider \( Z = Z(\lambda, x, y) \). For stationary values of \( Z \), then
12.2. FINDING THE STATIONARY VALUES

the first-order condition for free extremum is

\[
Z_\lambda \equiv \frac{\partial Z}{\partial \lambda} = c - g(x, y) = 0; \\
Z_x \equiv \frac{\partial Z}{\partial x} = f_x - \lambda g_x(x, y) = 0; \\
Z_y \equiv \frac{\partial Z}{\partial y} = f_y - \lambda g_y(x, y) = 0.
\]

**Example 12.2.1** Let us again consider the consumer’s choice problem above. The Lagrange function is

\[
Z = x_1 x_2 + 2x_1 + \lambda [60 - 4x_1 - 2x_2,]
\]

for which the necessary condition for a stationary value is

\[
Z_\lambda = 60 - 4x_1 - 2x_2 = 0; \\
Z_x = x_2 + 2 - 4\lambda = 0; \\
Z_y = x_1 - 2\lambda = 0.
\]

Solving the critical values of the variables, we find that \( \bar{x}_1 = 8, \bar{x}_2 = 14, \) and \( \lambda = 4. \) As expected, \( \bar{x}_1 = 8 \) and \( \bar{x}_2 = 14 \) are the same obtained by the substitution method.

**Example 12.2.2** Find the extremum of \( z = xy \) subject to \( x + y = 6. \) The Lagrange function is

\[
Z = xy + \lambda (6 - x - y).
\]

The first-order condition is

\[
Z_\lambda = 6 - x - y = 0; \\
Z_x = y - \lambda = 0; \\
Z_y = x - \lambda = 0.
\]
Thus, we find $\bar{\lambda} = 3$, $\bar{x} = 3$, $\bar{y} = 3$.

**Example 12.2.3** Find the extremum of $z = x_1^2 + x_2^2$ subject to $x_1 + 4x_2 = 2$.

The Lagrange function is

$$Z = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2).$$

The first-order condition (FOC) is

$$Z_\lambda = 2 - x_1 - 4x_2 = 0;$$
$$Z_{x_1} = 2x_1 - \lambda = 0;$$
$$Z_{x_2} = 2x_2 - 4\lambda = 0.$$

The stationary value of $Z$, defined by the solution

$$\bar{\lambda} = \frac{4}{17}, \bar{x}_1 = \frac{2}{17}, \bar{x}_2 = \frac{8}{17}, \bar{z} = \frac{4}{17}.$$

To tell whether $\bar{z}$ is a maximum, we need to consider the second-order condition.

**An Interpretation of the Lagrange Multiplier**

The Lagrange multiplier $\bar{\lambda}$ measures the sensitivity of $Z$ to change in the constraint. If we can express the solution $\bar{\lambda}$, $\bar{x}$, and $\bar{y}$ all as implicit functions of the parameter $c$:

$$\bar{\lambda} = \bar{\lambda}(c), \quad \bar{x} = \bar{x}(c), \quad \text{and} \quad \bar{y} = \bar{y}(c),$$
all of which will have continuous derivative, we have the identities
\[ c - g(\bar{x}, \bar{y}) \equiv 0; \]
\[ f_x(\bar{x}, \bar{y}) - \bar{\lambda}g_x(\bar{x}, \bar{y}) \equiv 0; \]
\[ f_y(\bar{x}, \bar{y}) - \bar{\lambda}g_y(\bar{x}, \bar{y}) \equiv 0. \]

Thus, we can consider \( Z \) as a function of \( c \):
\[ Z = f(\bar{x}, \bar{y}) + \bar{\lambda}[c - g(\bar{x}, \bar{y})]. \]

Then
\[ \frac{dZ}{dc} = f_x \frac{d\bar{x}}{dc} + f_y \frac{d\bar{y}}{dc} + [c - g(\bar{x}, \bar{y})] \frac{d\bar{\lambda}}{dc} + \lambda \left[ 1 - g_x \frac{d\bar{x}}{dc} - g_y \frac{d\bar{y}}{dc} \right] = \lambda. \]

**n-Variable and Multiconstraint Cases**

The generalization of the Lagrange-multiplier method to \( n \) variable can be easily carried. The objective function is
\[ z = f(x_1, x_2, \cdots, x_n) \]
subject to
\[ g(x_1, x_2, \cdots, x_n) = c. \]

It follows that the Lagrange function will be
\[ Z = f(x_1, x_2, \cdots, x_n) + \lambda[c - g(x_1, x_2, \cdots, x_n)], \]
for which the first-order condition will be given by

\[ Z_\lambda = c - g(x_1, x_2, \cdots, x_n); \]
\[ Z_i = f_i(x_1, x_2, \cdots, x_n) - \lambda g_i(x_1, x_2, \cdots, x_n) \quad [i = 1, 2, \cdots, n]. \]

If the objective function has more than one constraint, say, two constraints

\[ g(x_1, x_2, \cdots, x_n) = c \quad \text{and} \quad h(x_1, x_2, \cdots, x_n) = d. \]

The Lagrange function is

\[ Z = f(x_1, x_2, \cdots, x_n) + \lambda[c - g(x_1, x_2, \cdots, x_n)] + \mu[d - h(x_1, x_2, \cdots, x_n)], \]

for which the first-order condition consists of \((n + 2)\) equations:

\[ Z_\lambda = c - g(x_1, x_2, \cdots, x_n) = 0; \]
\[ Z_\mu = d - h(x_1, x_2, \cdots, x_n) = 0; \]
\[ Z_i = f_i(x_1, x_2, \cdots, x_n) - \lambda g_i(x_1, x_2, \cdots, x_n) - \mu h_i(x_1, x_2, \cdots, x_n) = 0. \]

### 12.3 Second-Order Condition

From the last section, we know that finding the constrained extremum is equivalent to find the free extremum of the Lagrange function \(Z\) and give the first-order condition. This section gives the second-order condition for the constrained extremum of \(f\).

We only consider the case where the objective functions take form

\[ z = f(x_1, x_2, \cdots, x_n) \]
subject to

\[ g(x_1, x_2, \cdots, x_n) = c. \]

The Lagrange function is then

\[ Z = f(x_1, x_2, \cdots, x_n) + \lambda [c - g(x_1, x_2, \cdots, x_n)]. \]

Define the bordered Hessian determinant \( |\bar{H}| \) by

\[
|\bar{H}| = \begin{vmatrix}
0 & g_1 & g_2 & \cdots & g_n \\
g_1 & Z_{11} & Z_{12} & \cdots & Z_{1n} \\
g_2 & Z_{21} & Z_{22} & \cdots & Z_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_n & Z_{n1} & Z_{n2} & \cdots & Z_{nn}
\end{vmatrix}
\]

where \( Z_{ij} = f_{ij} - \lambda g_{ij} \). Note that by the first-order condition,

\[ \lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_n}{g_n}. \]

The bordered principal minors can be defined as

\[
|\bar{H}_2| = \begin{vmatrix}
0 & g_1 \\
g_1 & Z_{11} \\
g_2 & Z_{21} \\
\end{vmatrix}, \quad |\bar{H}_3| = \begin{vmatrix}
0 & g_1 & g_2 & g_3 \\
g_1 & Z_{11} & Z_{12} & Z_{13} \\
g_2 & Z_{21} & Z_{22} & Z_{23} \\
g_3 & Z_{31} & Z_{32} & Z_{33}
\end{vmatrix} \quad (\text{etc.})
\]

The Conditions for Maximum:

\begin{enumerate}
\item \( Z_\lambda = Z_1 = Z_2 = \cdots = Z_n = 0 \) [necessary condition];
\item \( |\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0, \cdots, (-1)^n |\bar{H}_n| > 0. \)
\end{enumerate}
The Conditions for Minimum:

(1) $Z_\lambda = Z_1 = Z_2 = \cdots = Z_n = 0$ [necessary condition];

(2) $|\bar{H}_2| < 0, |\bar{H}_3| < 0, |\bar{H}_4| < 0, \ldots, (|\bar{H}_n| < 0$.

**Example 12.3.1** For the objective function $z = xy$ subject to $x + y = 6$, we have shown that $(\bar{x}, \bar{y}, \bar{z}) = (3, 6, 9)$ is an extremum point. Since $Z_x = y - \lambda$ and $Z_y = x - \lambda$, then $Z_{xx} = 0$, $Z_{xy} = 1$, and $Z_{yy} = 0$, $g_x = g_y = 1$. Thus, we find that

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 > 0,$$

which establishes the value of $\bar{z} = 9$ as an maximum.

**Example 12.3.2** For the objective function of $z = x_1^2 + x_2^2$ subject to $x_1 + 4x_2 = 2$, we have shown that $(\bar{x}, \bar{y}, \bar{z}) = (2/17, 8/17, 4/7)$ is the extremum point. To tell whether it is maximum or minimum, we check the second-order sufficient condition. Since $Z_1 = 2x_1 - \lambda$ and $Z_2 = 2x_2 - \lambda$ as well as $g_1 = 1$ and $g_2 = 4$, we have $Z_{11} = 2$, $Z_{22} = 2$, and $Z_{12} = Z_{21} = 0$. It thus follows that the bordered Hessian is

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -34 < 0,$$

and the value $\bar{z} = 4/17$ is a minimum.

### 12.4 General Setup of the Problem

Now consider the most general setting of the problem with $n$ variables and $m$ equality constraints ("Extremize" means to find either the minimum or the maximum of the objective function $f$):
12.4. GENERAL SETUP OF THE PROBLEM

Extremize \( f(x_1, \ldots, x_n) \) (12.4.1)

s.t. \( g^j(x_1, \ldots, x_n) = b_j, j = 1, 2, \ldots, m < n. \)

Again, \( f \) is called the objective function, \( g^1, g^2, \ldots, g^m \) are the constraint functions, \( b^1, b^2, \ldots, b^m \) are the constraint constants. The difference \( n - m \) is the number of degrees of freedom of the problem.

Note that \( n \) must be strictly greater than \( m \), otherwise there is no freedom to choose.

If it is possible to explicitly express (from the constraint functions) \( m \) independent variables as functions of the other \( n - m \) independent variables, we can eliminate \( m \) variables in the objective function, thus the initial problem will be reduced to the unconstrained optimization problem with respect to \( n - m \) variables. However, in many cases it is not technically feasible to explicitly express one variable as function of the others.

Instead of the substitution and elimination method, we may resort to the easy-to-use and well-defined method of Lagrange multipliers.

Let \( f \) and \( g^1, \ldots, g^m \) be continuous functions and the Jacobian \( J = (\frac{\partial g^j}{\partial x_i}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m) \) have the full rank, i.e. \( \text{rank}(J) = m. \)

We introduce the Lagrangian function as

\[
L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j (b_j - g^j(x_1, \ldots, x_n)),
\]

where \( \lambda_1, \ldots, \lambda_m \) are constant (Lagrange multipliers).

What are the Necessary Conditions for the Solution of the problem (??)?
Equate all partials of $L$ with respect to $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$ to zero:

$$\frac{\partial L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)}{\partial x_i} = \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g^j(x_1, \ldots, x_n)}{\partial x_i} = 0, \ i = 1, 2, \ldots, n,$$

$$\frac{\partial L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)}{\partial \lambda_j} = b_j - g^j(x_1, \ldots, x_n) = 0, \ j = 1, 2, \ldots, m.$$

Solve these equations for $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_m$. In the end we will get a set of stationary points of the Lagrangian. If $x^* = (x^*_1, \ldots, x^*_n)$ is a solution of the problem (??), it should be a stationary point of $L$.

It is important to assume that $\text{rank}(J) = m$, and the functions are continuously differentiable.

If we need to check whether a stationary point is a maximum or minimum, the following local sufficient conditions can be applied:

**Proposition 12.4.1** Let us introduce a bordered Hessian $|\bar{H}_r|$ as

$$|\bar{H}_r| = \text{det} \left( \begin{array}{cccccccc} 0 & \cdots & 0 & \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^1}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g^m}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_r} \\ \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^1}{\partial x_r} & \cdots & \frac{\partial g^m}{\partial x_r} & \frac{\partial^2 L}{\partial x_r \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_r \partial x_r} \end{array} \right), \ r = 1, 2, \ldots, n.$$

Let $f$ and $g^1, \ldots, g^m$ are twice continuous functions and let $x^*$ satisfy the necessary conditions for the problem (??).

Let $|\bar{H}_r(x^*)|$ be the bordered Hessian determinant evaluated at $x^*$. Then

(1) if $(-1)^m |\bar{H}_r(x^*)| > 0, r = m + 1, \ldots, n$, then $x^*$ is a local minimum point for the problem (??);
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(2) if $(-1)^r |\bar{H}_r(x^*)| > 0$, $r = m + 1, \ldots, n$, then $x^*$ is a local maximum point for the problem (??).

12.5 Quasiconcavity and Quasiconvexity

For a problem of free extremum, we know that the concavity (convexity) of the objective function guarantees the existence of absolute maximum (absolute minimum). For a problem of constrained optimization, we will demonstrate the quasiconcavity (quasiconvexity) of the objective function guarantees the existence of absolute maximum (absolute minimum).

Algebraic Characterization

Quasiconcavity and quasiconvexity, like concavity and convexity, can be either strict or nonstrict:

**Definition 12.5.1** A function is quasiconcave (quasiconvex) if and only if for any pair of distinct points $u$ and $v$ in the convex domain of $f$, and for $0 < \theta < 1$, we have

$$f(\theta u + (1 - \theta)v) \geq \min\{f(u), f(v)\}$$

$$[f(\theta u + (1 - \theta)v) \leq \max\{f(u), f(v)\}].$$

Note that when $f(v) \geq f(u)$, the above inequalities imply respectively

$$f(\theta u + (1 - \theta)v) \geq f(u)$$

$$[f(\theta u + (1 - \theta)v) \leq f(v)].$$

Furthermore, if the weak inequality “$\geq$” (“$\leq$”) is replaced by the strict inequality “$>$” (“$<$”), $f$ is said to be strictly quasiconcave (strictly quasiconvex).
 Remark 12.5.1 From the definition of quasiconcavity (quasiconvexity), we can know that quasiconvexity (quasiconvexity) is a weaker condition than concavity (convexity).

**Theorem I (Negative of a function).** If \( f(x) \) is quasiconcave (strictly quasiconcave), then \(-f(x)\) is quasiconvex (strictly quasiconvex).

**Theorem II (concavity versus quasiconcavity).** Any (strictly) concave (convex) function is (strictly) quasiconcave (quasiconvex), but the converse may not be true.

**Theorem III (linear function).** If \( f(x) \) is linear, then it is quasiconcave as well as quasiconvex.

**Theorem IV (monotone function with one variable).** If \( f \) is a function of one variable, then it is quasiconcave as well as quasiconvex.

**Remark 12.5.2** Note that, unlike concave (convex) functions, a sum of two quasiconcave (quasiconvex) functions is not necessarily quasiconcave (quasiconvex).

Sometimes it may prove easier to check quasiconcavity and quasiconvexity by the following alternative (equivalent) definitions. We state it as a proposition.
Proposition 12.5.1 A function $f(x)$, where $x$ is a vector of variables in the domain $X$, is quasiconcave (quasiconvex) if and only if, for any constant $k$, the set $S^\geq \equiv \{ x \in X : f(x) \geq k \} \ (S^\leq \equiv \{ x \in X : f(x) \leq k \})$ is convex.

Example 12.5.1 (1) $Z = x^2$ is quasiconvex since $S^\leq$ is convex.
(2) $Z = f(x, y) = xy$ is quasiconcave since $S^\geq$ is convex.
(3) $Z = f(x, y) = (x - a)^2 + (y - b)^2$ is quasiconvex since $S^\leq$ is convex.

The above facts can be seen by looking at graphs of these functions.

Differentiable Functions

Similar to the concavity, when a function is differentiable, we have the following result.

Proposition 12.5.2 Suppose $f : \mathcal{R} \to \mathcal{R}$ is differentiable. Then $f$ is concave if and only if for any $x, y \in \mathcal{R}$, we have

$$f(y) \geq f(x) \Rightarrow f'(x)(y - x) \geq 0.$$ (12.5.2)
When there are two or more independent variables, the above proposition becomes:

**Proposition 12.5.3** Suppose \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is differentiable. Then \( f \) is concave if and only if for any \( x, y \in \mathbb{R} \), we have

\[
f(y) \geq f(x) \Rightarrow \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} (y_j - x_j) \geq 0.
\] (12.5.3)

If a function \( z = f(x_1, x_2, \cdots, x_n) \) is twice continuously differentiable, quasiconcavity and quasiconvexity can be checked by means of the first and second order partial derivatives of the function.

Define a bordered determinant as follows:

\[
|B| = \begin{vmatrix}
0 & f_1 & f_2 & \cdots & f_n \\
f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\
f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_n & f_{n1} & f_{n2} & \cdots & f_{nn}
\end{vmatrix}
\]

**Remark 12.5.3** The determinant \(|B|\) is different from the bordered Hessian \(|H|\). Unlike \(|H|\), the border in \(|B|\) is composed of the first derivatives of the function \( f \) rather than an extraneous constraint function \( g \).

We can define successive principal minors of \( B \) as follows:

\[
|B_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix}, \quad |B_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}, \cdots, \quad |B_n| = |B|.
\]
A necessary condition for a function \( z = f(x_1, \ldots, x_n) \) defined the non-negative orthant to be quasiconcave is that

\[
|B_1| \leq 0, \quad |B_2| \geq 0, \quad |B_3| \leq 0, \quad \cdots, \quad (-1)^n |B_n| \geq 0.
\]

A sufficient condition for \( f \) to be strictly quasiconcave on the nonnegative orthant is that

\[
|B_1| < 0, \quad |B_2| > 0, \quad |B_3| < 0, \quad \cdots, \quad (-1)^n |B_n| > 0.
\]

For strict quasiconvexity, the corresponding sufficient condition is that

\[
|B_1| < 0, \quad |B_2| < 0, \quad \cdots, \quad |B_n| < 0.
\]

**Example 12.5.2** \( z = f(x_1, x_2) = x_1x_2 \) for \( x_1 > 0 \) and \( x_2 > 0 \). Since \( f_1 = x_2, f_2 = x_1, f_{11} = f_{22} = 0, \) and \( f_{12} = f_{21} = 1 \), the relevant principal minors turn out to be

\[
|B_1| = \begin{vmatrix} 0 & x_2 \\ x_2 & 0 \end{vmatrix} = -x_2^2 < 0, \quad |B_2| = \begin{vmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{vmatrix} = 2x_1x_2 > 0.
\]

Thus \( z = x_1x_2 \) is strictly quasiconcave on the positive orthant.

**Example 12.5.3** Show that \( z = f(x, y) = x^ay^b \) \( (x, y > 0; a, b > 0) \) is quasiconcave.

Since

\[
f_x = ax^{a-1}y^b, \quad f_y = bx^ay^{b-1},
\]

\[
f_{xx} = a(a-1)x^{a-2}y^b, \quad f_{xy} = abx^{a-1}y^{b-1}, \quad f_{yy} = b(b-1)x^ay^{b-2},
\]
thus

\[ |B_1| = \begin{vmatrix} 0 & f_x \\ f_x & f_{xx} \end{vmatrix} = -(ax^{a-1}y^b)^2 < 0; \]

\[ |B_2| = \begin{vmatrix} 0 & f_x & f_y \\ f_x & f_{xx} & f_{xy} \\ f_y & f_{yx} & f_{yy} \end{vmatrix} = \left[2a^2b^2 - a(a - 1)b^2 - a^2b(b - 1)\right]x^{3a-2}y^{3b-2} \]

\[ = ab(a + b)x^{3a-2}y^{3b-2} > 0. \]

Hence it is strictly quasiconcave.

**Remark 12.5.4** When the constraint \( g \) is linear: \( g(x) = a_1x_1 + \cdots + a_nx_n = c \), the second-order partial derivatives of \( g \) disappears, and thus, from the first order condition \( f_i = \lambda g_i \), the bordered determinant \( |B| \) and the bordered Hessian determinant have the following relationship:

\[ |B| = \lambda^2|\bar{H}|. \]

Consequently, in the linear-constraint case, the two bordered determinants always have the same sign at the stationary of \( z \). The same is true for principal minors. Thus, the first-order necessary condition is also a sufficient condition for the maximization problem.

**Absolute versus Relative Extrema**

If a function is quasiconcave (quansiconvex), by the similar reasons for concave (convex) functions, its relative maximum (relative minimum) is an absolute maximum (absolute minimum).
12.6 Utility Maximization and Consumer Demand

Let us now re-examine the consumer choice problem – utility maximization problem. For simplicity, only consider the two-commodity case. The consumer wants to maximize his utility

\[ u = u(x, y) \quad (u_x > 0, u_y > 0) \]

subject to his budget constraint

\[ P_x x + P_y y = B \]

by taking prices \( P_x \) and \( P_y \) as well as his income as given.

**First-Order Condition**

The Lagrange function is

\[ Z = u(x, y) + \lambda (B - P_x x - P_y y) \].

At the first-order condition, we have the following equations:

\[
Z_\lambda = B - P_x x - P_y y = 0; \\
Z_x = u_x - \lambda P_x = 0; \\
Z_y = u_y - \lambda P_y = 0.
\]

From the last two equations, we have

\[
\frac{u_x}{P_x} = \frac{u_y}{P_y} = \lambda,
\]
or

\[
\frac{u_x}{u_y} = \frac{P_x}{P_y}
\]

The term \( \frac{u_x}{u_y} \equiv MRS_{xy} \) is the so-called marginal rate of substitution of \( x \) for \( y \). Thus, we obtain the well-known equality: \( MRS_{xy} = \frac{P_x}{P_y} \) which is the necessary condition for the interior solution.

![Figure 12.4: Indifference curves and budget line](image)

**Second-Order Condition**

If the bordered Hessian in the present problem is positive, i.e., if

\[
|\bar{H}| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & u_{xx} & u_{xy} \\ P_y & u_{yx} & u_{yy} \end{vmatrix} = 2P_xP_yu_{xy} - P_y^2u_{xx} - P_x^2u_{yy} > 0,
\]

(with all the derivatives evaluated at the critical value of \( \bar{x} \) and \( \bar{y} \)), then the stationary value of \( u \) will assuredly be maximum.

Since the budget constraint is linear, from the result in the last section,
we have

$$|B| = \lambda^2 |H|.$$  

Thus, as long as $|B| > 0$, we know the second-order condition holds.

Recall that $|B| > 0$ means that utility function is strictly quasi-concave.

Also, quasi-concavity of a utility function means the indifference curves represented by the utility function is convex, i.e., the upper contour set

$$\{ y : u(y) \geq u(x) \}$$

is convex, in which case we call the preferences represented by the utility function is convex.

Remark 12.6.1 The convexity of preferences implies that people want to diversify their consumptions, and thus, convexity can be viewed as the formal expression of basic measure of economic markets for diversification. Also, the strict quasi-concavity implies that the strict convexity of $\succ_i$, which in turn implies the conventional diminishing marginal rates of substitution (MRS), and weak convexity of $\succeq_i$ is equivalent to the quasi-concavity of utility function $u_i$.  

Chapter 13

Optimization with Inequality Constraints

Classical methods of optimization (the method of Lagrange multipliers) deal with optimization problems with equality constraints in the form of \( g(x_1, \ldots, x_n) = c \). Nonclassical optimization, also known as mathematical programming, tackles problems with inequality constraints like \( g(x_1, \ldots, x_n) \leq c \).

Mathematical programming includes linear programming and nonlinear programming. In linear programming, the objective function and all inequality constraints are linear. When either the objective function or an inequality constraint is nonlinear, we face a problem of nonlinear programming.

13.1 Non-Linear Programming

The nonlinear programming problem is that of choosing nonnegative values of certain variables so as to maximize or minimize a given (non-linear) function subject to a given set of (non-linear) inequality constraints.
The nonlinear programming maximum problem is

\[
\max \ f(x_1, \ldots, x_n) \\
\text{s.t.} \ g^i(x_1, \ldots, x_n) \leq b_i, \ i = 1, 2, \ldots, m; \\
x_1 \geq 0, \ldots, x_n \geq 0.
\]

Similarly, the minimization problem is

\[
\min \ f(x_1, \ldots, x_n) \\
\text{s.t.} \ g^i(x_1, \ldots, x_n) \geq b_i, \ i = 1, 2, \ldots, m; \\
x_1 \geq 0, \ldots, x_n \geq 0.
\]

First, note that there are no restrictions on the relative size of \(m\) and \(n\), unlike the case of equality constraints. Second, note that the direction of the inequalities (\(\leq\) or \(\geq\)) at the constraints is only a convention, because the inequality \(g^i \leq b_i\) can be easily converted to the \(\geq\) inequality by multiplying it by -1, yielding \(-g^i \geq -b_i\). Third, note that an equality constraint, say \(g^k = b_k\), can be replaced by the two inequality constraints, \(g^k \leq b_k\) and \(-g^k \leq -b_k\).

**Definition 13.1.1 (Binding Constraint)** A constraint \(g^i \leq b_j\) is called binding (or active) at \(x^0 = (x^0_1, \ldots, x^0_n)\) if \(g^i(x^0) = b_j\).

**Example 13.1.1** The following is an example of a nonlinear program:

\[
\max \ \pi = x_1(10 - x_1) + x_2(20 - x_2)
\]
Example 13.1.2 (How to Solve a Nonlinear Program Graphically) An intuitive way to understand a low-dimensional non-linear program is to represent the feasible set and the objective function graphically.

For instance, we may try to represent graphically the optimization problem from Example (??), which can be re-written as

\[
\begin{align*}
\text{max} \quad & \pi = 125 - (x_1 - 5)^2 - (x_2 - 10)^2 \\
\text{s.t.} \quad & 5x_1 + 3x_2 \leq 40; \\
& x_1 \leq 5; \\
& x_2 \leq 10; \\
& x_1 \geq 0, x_2 \geq 0.
\end{align*}
\]

Or

\[
\begin{align*}
\text{min} \quad & C = (x_1 - 5)^2 + (x_2 - 10)^2 \\
\text{s.t.} \quad & 5x_1 + 3x_2 \leq 40; \\
& x_1 \leq 5; \\
& x_2 \leq 10; \\
& x_1 \geq 0, x_2 \geq 0.
\end{align*}
\]

Region $OABCD$ below shows the feasible set of the nonlinear program.

The value of the objective function can be interpreted as the square of the distance from the point $(5, 10)$ to the point $(x_1, x_2)$. Thus, the nonlinear program requires us to find the point in the feasible region which minimizes the distance to the point $(5, 10)$.

As can be seen in the figure above, the optimal point $(x_1^*, x_2^*)$ lies on the
line $5x_1 + 3x_2 = 40$ and it is also the tangency point with a circle centered at $(5, 10)$. These two conditions are sufficient to determine the optimal solution.

Consider $(x_1 - 5)^2 + (x_2 - 10)^2 - C = 0$. From the implicit-function theorem, we have

$$\frac{dx_2}{dx_1} = -\frac{x_1 - 5}{x_2 - 10}.$$ 

Thus, the tangent to the circle centered at $(5, 10)$ which has the slope $-\frac{5}{3}$ is given by the equation

$$-\frac{5}{3} = -\frac{x_1 - 5}{x_2 - 10},$$

or

$$3x_1 - 5x_2 + 35 = 0.$$
Solving the system of equations

\[
\begin{align*}
5x_1 + 3x_2 &= 40 \\
3x_1 - 5x_2 &= -35
\end{align*}
\]

we find the optimal solution \((x_1^*, x_2^*) = \left(\frac{95}{34}, \frac{295}{34}\right)\).

### 13.2 Kuhn-Tucker Conditions

For the purpose of ruling out certain irregularities on the boundary of the feasible set, a restriction on the constrained functions is imposed. This restriction is called the constraint qualification.

**Definition 13.2.1** Denote by \(C\) the constraint set. We say that the constraint qualification condition is satisfied at \(x^* \in C\) if the constraints that hold at \(x^*\) with equality are independent; that is, if the gradients (the vectors of partial derivatives) of \(g^j\)-constraints that are valuated and binding at \(x^*\) are linearly independent.

Define the Lagrangian function for optimization with inequality constraints:

\[
L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j (b_j - g^j(x_1, \ldots, x_n)).
\]

The following results is the necessity theorem about Kuhn-Tucker conditions for a local optimum if the constraint qualification is satisfied.

**Proposition 13.2.1 (Necessity Theorem of Kuhn-Tucker conditions)** Suppose that that the constraint qualification condition is satisfied and that the objective functions and constraint functions are differentiable. Then, we have
(1) the Kuhn-Tucker (in short, \(KT\)) necessary conditions for maximization are:

\[
\frac{\partial L}{\partial x_i} \leq 0, \quad x_i \geq 0 \quad \text{and} \quad x_i \frac{\partial L}{\partial x_i} = 0, \ i = 1, \ldots, n; \\
\frac{\partial L}{\partial \lambda_j} \geq 0, \quad \lambda_j \geq 0 \quad \text{and} \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0, \ j = 1, \ldots, m. 
\]

(2) the Kuhn-Tucker necessary conditions for minimization are:

\[
\frac{\partial L}{\partial x_i} \geq 0, \quad x_i \geq 0 \quad \text{and} \quad x_i \frac{\partial L}{\partial x_i} = 0, \ i = 1, \ldots, n; \\
\frac{\partial L}{\partial \lambda_j} \leq 0, \quad \lambda_j \geq 0 \quad \text{and} \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0, \ j = 1, \ldots, m. 
\]

In general, the \(KT\) conditions are neither necessary nor sufficient for a local optimum without other conditions such as constraint qualification condition. However, if certain assumptions are satisfied, the \(KT\) conditions become necessary and even sufficient.

**Example 13.2.1** The Lagrangian function of the nonlinear program in Example (??) is:

\[
L = x_1(10 - x_1) + x_2(20 - x_2) - \lambda_1(5x_1 + 3x_2 - 40) - \lambda_2(x_1 - 5) - \lambda_3(x_2 - 10). 
\]

The KT conditions are:

\[
\frac{\partial L}{\partial x_1} = 10 - 2x_1 - 5\lambda_1 - \lambda_2 \leq 0; \\
\frac{\partial L}{\partial x_2} = 20 - 2x_2 - 3\lambda_1 - \lambda_2 \leq 0; \\
\frac{\partial L}{\partial \lambda_1} = -(5x_1 + 3x_2 - 40) \geq 0; \\
\frac{\partial L}{\partial \lambda_2} = x_1 - 5 \geq 0; \\
\frac{\partial L}{\partial \lambda_3} = x_2 - 10 \geq 0.
\]
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\[ \frac{\partial L}{\partial \lambda_2} = -(x_1 - 5) \geq 0; \]
\[ \frac{\partial L}{\partial \lambda_3} = -(x_2 - 10) \geq 0; \]
\[ x_1 \geq 0, x_2 \geq 0; \]
\[ \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0; \]
\[ x_1 \frac{\partial L}{\partial x_1} = 0, \quad x_2 \frac{\partial L}{\partial x_2} = 0; \]
\[ \lambda_i \frac{\partial L}{\partial \lambda_i} = 0, \quad i = 1, 2, 3. \]

Notice that the failure of the constraint qualification signals certain irregularities at the boundary kinks of the feasible set. Only if the optimal solution occurs in such a kink may the KT conditions not be satisfied.

If all constraints are linear, the constraint qualification is always satisfied.

**Example 13.2.2** The constraint qualification for the nonlinear program in (??) is satisfied since all constraints are linear. Therefore, the optimal solution \((\frac{95}{34}, \frac{205}{34})\) must satisfy the KT conditions in example (??).

**Example 13.2.3** The following example illustrates a case where the Kuhn-Tucker conditions are not satisfied in the solution of an optimization problem. Consider the problem:

\[
\begin{align*}
\text{max} & \quad y \\
\text{s.t.} & \quad x + (y - 1)^3 \leq 0; \\
& \quad x \geq 0, y \geq 0.
\end{align*}
\]

The solution to this problem is \((0, 1)\). (If \(y > 1\), then the restriction \(x + (y + 1)^3 \leq 0\) implies \(x < 0\).) The Lagrangian function is:

\[ L = y + \lambda[-x - (y - 1)^3]. \]
One of the Kuhn-Tucker conditions requires
\[ \frac{\partial L}{\partial y} \leq 0, \]
or
\[ 1 - 3\lambda(y - 1)^2 \leq 0. \]

As can be observed, this condition is not verified at the point \((0, 1)\).

**Proposition 13.2.2 (Kuhn-Tucker Sufficiency Theorem- 1)** Suppose the following conditions are satisfied:

(a) \( f \) is differentiable and concave in the nonnegative orthant;
(b) each constraint function is differentiable and convex in the nonnegative orthant;
(c) the point \( x^* \) satisfies the Kuhn-Tucker necessary maximum conditions.

Then \( x^* \) gives a global maximum of \( f \).

The concavity of the objective function and convexity of constraints may be weakened to quasiconcavity and quasiconvexity, respectively. To adapt this theorem for minimization problems, we need to interchange the two words “concave” and “convex” in a) and b) and use the Kuhn-Tucker necessary minimum condition in c).

**Proposition 13.2.3 (Kuhn-Tucker Sufficiency Theorem- 2)** The KT conditions are sufficient conditions for \( x^* \) to be a local optimum of a maximization (minimization) program if the following assumptions are satisfied:
(a) the objective function $f$ is differentiable and quasiconcave (or quasiconvex);

(b) each constraint $g^i$ is differentiable and quasiconvex (or quasiconcave);

(c) any one of the following is satisfied:

(c.i) there exists $j$ such that $\frac{\partial f(x^*)}{\partial x_j} < 0 \ (>)$;

(c.ii) there exists $j$ such that $\frac{\partial f(x^*)}{\partial x_j} > 0 \ (< 0)$ and $x_j$ can take a positive value without violating the constraints;

(c.iii) $f$ is concave (or convex).

The problem of finding the nonnegative vector $(x^*, \lambda^*)$, $x^* = (x_1^*, \ldots, x_n^*)$, $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ which satisfies Kuhn-Tucker necessary conditions and for which

$L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda) \ \forall \ x = (x_1, \ldots, x_n) \geq 0, \lambda = (\lambda_1, \ldots, \lambda_m) \geq 0$

is known as the saddle point problem.

**Proposition 13.2.4** If $(x^*, \lambda^*)$ solves the saddle point problem then $(x^*, \lambda^*)$ solves the problem (??).

### 13.3 Economic Applications

#### Corner Solution for Linear Utility Maximization

Suppose the preference ordering is represented by the linear utility function:

$$u(x, y) = ax + by.$$
Since the marginal rate of substitution is \( a/b \) and the economic rate of substitution is \( p_x/p_y \) are both constant, they cannot be in general equal. So the first-order condition cannot hold with equality as long as \( \frac{a}{b} \neq \frac{p_x}{p_y} \). In this case the answer to the utility-maximization problem typically involves a boundary solution: only one of the two goods will be consumed. It is worthwhile presenting a more formal solution since it serves as a nice example of the Kuhn-Tucker theorem in action. The Kuhn-Tucker theorem is the appropriate tool to use here, since we will almost never have an interior solution.

The Lagrange function is

\[
L(x, y, \lambda) = ax + by + \lambda(m - p_x x - p_y y),
\]

and thus

\[
\begin{align*}
\frac{\partial L}{\partial x} &= a - \lambda p_x; \\
\frac{\partial L}{\partial y} &= b - \lambda p_y; \\
\frac{\partial L}{\partial \lambda} &= m - p_x - p_y.
\end{align*}
\]

There are four cases to be considered:

Case 1. \( x > 0 \) and \( y > 0 \). Then we have \( \frac{\partial L}{\partial x} = 0 \) and \( \frac{\partial L}{\partial y} = 0 \). Thus, \( \frac{a}{b} = \frac{p_x}{p_y} \). Since \( \lambda = \frac{a}{p_x} > 0 \), we have \( p_x x + p_y y = m \) and thus all \( x \) and \( y \) that satisfy \( p_x x + p_y y = m \) are the optimal consumptions.

Case 2. \( x > 0 \) and \( y = 0 \). Then we have \( \frac{\partial L}{\partial x} = 0 \) and \( \frac{\partial L}{\partial y} \leq 0 \). Thus, \( \frac{a}{b} \geq \frac{p_x}{p_y} \). Since \( \lambda = \frac{a}{p_x} > 0 \), we have \( p_x x + p_y y = m \) and thus \( x = \frac{m}{p_x} \) is the optimal consumption.

Case 3. \( x = 0 \) and \( y > 0 \). Then we have \( \frac{\partial L}{\partial x} \leq 0 \) and \( \frac{\partial L}{\partial y} = 0 \). Thus, \( \frac{a}{b} \leq \frac{p_x}{p_y} \). Since \( \lambda = \frac{b}{p_y} > 0 \), we have \( p_x x + p_y y = m \) and thus \( y = \frac{m}{p_y} \) is the optimal consumption.
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Case 4. \( x = 0 \) and \( y = 0 \). Then we have \( \frac{\partial L}{\partial x} \leq 0 \) and \( \frac{\partial L}{\partial y} \leq 0 \). Since \( \lambda \geq \frac{1}{p_y} > 0 \), we have \( p_x x + p_y y = m \) and thus \( m = 0 \) because \( x = 0 \) and \( y = 0 \).

In summary, the demand functions are given by

\[
x(p_x, p_y, m), y(p_x, p_y, m) = \begin{cases} 
(m/p_x, 0) & \text{if } a/b > p_x/p_y \\
(0, m/p_y) & \text{if } a/b < p_x/p_y \\
(x, m/p_x - p_y/p_x x) & \text{if } a/b = p_x/p_y
\end{cases}
\]

for all \( x \in [0, m/p_x] \).

Remark 13.3.1 In fact, it is easily found out the optimal solutions by comparing relatives steepness of the indifference curves and the budget line. For instance, as shown in Figure 13.2 below, when \( \frac{a}{b} > \frac{p_x}{p_y} \), the indifference curves become steeper, and thus the optimal solution is the one the consumer spends his all income on good \( x \). When \( \frac{a}{b} < \frac{p_x}{p_y} \), the indifference curves become flatter, and thus the optimal solution is the one the consumer spends his all income on good \( y \). When \( \frac{a}{b} \neq \frac{p_x}{p_y} \), the indifference...
curves and the budget line are parallel and coincide at the optimal solutions, and thus the optimal solutions are given by all the points on the budget line.

**Economic Interpretation of a Nonlinear Program and Kuhn-Tucker Conditions**

A maximization program in the general form, for example, is the production problem facing a firm which has to produce $n$ goods such that it maximizes its revenue subject to $m$ resource (factor) constraints.

The variables have the following economic interpretations:

- $x_j$ is the amount produced of the $j$th product;
- $r_i$ is the amount of the $i$th resource available;
- $f$ is the profit (revenue) function;
- $g^i$ is a function which shows how the $i$th resource is used in producing the $n$ goods.

The optimal solution to the maximization program indicates the optimal quantities of each good the firm should produce.

In order to interpret the Kuhn-Tucker conditions, we first have to note the meanings of the following variables:

- $f_j = \frac{\partial f}{\partial x_j}$ is the marginal profit (revenue) of product $j$;
- $\lambda_i$ is the shadow price of resource $i$;
- $g^i_j = \frac{\partial g^i}{\partial x_j}$ is the amount of resource $i$ used in producing a marginal unit of product $j$;
- $\lambda_i g^i_j$ is the imputed cost of resource $i$ incurred in the production of a marginal unit of product $j$. 


The $KT$ condition $\frac{\partial L}{\partial x_j} \leq 0$ can be written as $f_j \leq \sum_{i=1}^{m} \lambda_i g_j$ and it says that the marginal profit of the $j$th product cannot exceed the aggregate marginal imputed cost of the $j$th product.

The condition $x_j \frac{\partial L}{\partial x_j} = 0$ implies that, in order to produce good $j$ ($x_j > 0$), the marginal profit of good $j$ must be equal to the aggregate marginal imputed cost ($\frac{\partial L}{\partial x_j} = 0$). The same condition shows that good $j$ is not produced ($x_j = 0$) if there is an excess imputation $x_j \frac{\partial L}{\partial x_j} < 0$.

The $KT$ condition $\frac{\partial L}{\partial \lambda_i} \geq 0$ is simply a restatement of constraint $i$, which states that the total amount of resource $i$ used in producing all the $n$ goods should not exceed total amount available $r_i$.

The condition $\frac{\partial L}{\partial \lambda_i} = 0$ indicates that if a resource is not fully used in the optimal solution ($\frac{\partial L}{\partial \lambda_i} > 0$), then its shadow price will be 0 ($\lambda_i = 0$). On the other hand, a fully used resource ($\frac{\partial L}{\partial \lambda_i} = 0$) has a strictly positive price ($\lambda_i > 0$).

**Example 13.3.1** Let us find an economic interpretation for the maximization program given in Example (??):

\[
\begin{align*}
\text{max} & \quad R = x_1(10 - x_1) + x_2(20 - x_2) \\
\text{s.t.} & \quad 5x_1 + 3x_2 \leq 40; \\
& \quad x_1 \leq 5; \\
& \quad x_2 \leq 10; \\
& \quad x_1 \geq 0, x_2 \geq 0. 
\end{align*}
\]

A firm has to produce two goods using three kinds of resources available in the amounts 40, 5, 10 respectively. The first resource is used in the production of both goods: five units are necessary to produce one unit of good 1, and three units to produce one unit of good 2. The second resource is used only in producing good 1 and the third resource is used
only in producing good 2.

The sale prices of the two goods are given by the linear inverse demand equations \( p_1 = 10 - x_1 \) and \( p_2 = 20 - x_2 \). The problem the firm faces is how much to produce of each good in order to maximize revenue \( R = x_1p_1 + x_2p_2 \). The solution \((2, 10)\) gives the optimal amounts the firm should produce.
Chapter 14

Differential Equations

We first introduce the general concept of ordinary differential equations defined on Euclidean spaces.

Definition 14.0.1  An equation consists of independent variable $x$, unknown function $y = y(x)$ of the independent variable, its first derivative $y' = y'(x)$ to the $n$th order derivative $y^{(n)} = y^{(n)}(x)$,

$$F(x, y, y', \cdots, y^{(n)}) = 0,$$  \hfill (14.0.1)

is called an ordinary differential equation. If the highest order derivative in the equation is $n$, the equation is also called $n$th-order ordinary differential equation.

If for all $x \in I$, the function $y = \psi(x)$ satisfies

$$F(x, \psi(x), \psi'(x), \cdots, \psi^{(n)}(x)) = 0,$$

then $y = \psi(x)$ is called a solution to the ordinary differential equation (14.0.1).

Sometimes the solutions of the ordinary differential equations are not unique, and there may even exist infinite solutions. For example, $y = \frac{C}{x} + \frac{1}{5}x^4$ is the solution of the ordinary differential equation $\frac{dy}{dx} + \frac{y}{x} = x^3$,
where $C$ is an arbitrary constant. Next we introduce the concept of general solutions and particular solutions of ordinary differential equations.

**Definition 14.0.2** The solution of the $n$th-order ordinary differential equation (14.0.2)

$$y = \psi(x, C_1, \cdots, C_n)$$

(14.0.2)

containing $n$ independent arbitrary constants, $C_1, \cdots, C_n$, is called the *general solution* to ordinary differential equation (14.0.2). Here, independence means that the Jacobi determinant

$$D[\psi, \psi^{(1)}, \cdots, \psi^{(n-1)}] = \frac{D[C_1, \cdots, C_n]}{D[C_1', \cdots, C_n']}$$

is not equal to 0.

If a solution of an ordinary differential equation, denoted $y = \psi(x)$, does not contain any constant, it is called a particular solution. Obviously, a general solution becomes a particular solution when the arbitrary constants are determined. In general, the restrictions of some initial conditions determine the value of any constants. For example, for ordinary differential equation (14.0.3), if there are some given initial conditions:

$$y(x_0) = y_0, y^{(1)}(x_0) = y^{(1)}_0, \cdots, y^{(n-1)}(x_0) = y^{(n-1)}_0,$$  

(14.0.3)

then the ordinary differential equation (14.0.3) and the initial value conditions (14.0.3) are said to be the Cauchy problem or initial value problem for the $n$th-order ordinary differential equations. Then the question is what con-
14.1. EXISTENCE AND UNIQUENESS THEOREM OF SOLUTIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

ditions the function $F$ should satisfy so that the above ordinary differential equations are uniquely solvable. This problem is the existence and uniqueness of solutions for ordinary differential equations.

14.1 Existence and Uniqueness Theorem of Solutions for Ordinary Differential Equations

First, we consider an ordinary differential equation of first-order $y' = f(x, y)$ that satisfies initial condition $(x_0, y_0)$, that is, $y(x_0) = y_0$. Let $y(x)$ be a solution to the differential equation.

First, we introduce the concept of Lipschitz conditions.

**Definition 14.1.1** Considering a function $f(x, y)$ defined on $D \subseteq \mathbb{R}^2$, if there exists a neighborhood $U \subseteq D$ of $(x_0, y_0)$, and a positive number $L$ such that

$$|f(x, y) - f(x, z)| \leq L|y - z|, \forall (x, y), (x, z) \in U,$$

then we say $f$ satisfies the **local Lipschitz condition** with respect to $y$ at the point $(x_0, y_0) \in D$.

If there is a positive number $L$ such that

$$|f(x, y) - f(x, z)| \leq L|y - z|, \forall (x, y), (x, z) \in D,$$

we call $f(x, y)$ satisfies **global Lipschitz condition** in $D \subseteq \mathbb{R}^2$.

The following lemma characterizes the properties of the function satisfying Lipschitz condition.

**Lemma 14.1.1** Suppose that $f(x, y)$ defined on $D \subseteq \mathbb{R}^2$ is continuously differentiable. If there is an $\epsilon > 0$ such that $f_y(x, y)$ is bounded on $U = \{(x, y) : |x - x_0| < \epsilon, |y - y_0| < \epsilon\}$, then $f(x, y)$ satisfies the local Lipschitz condition. If $f_y(x, y)$ is bounded on $D$, then $f(x, y)$ satisfies the global Lipschitz condition.
Theorem 14.1.1 If \( f \) is continuous on an open set \( D \), then for any \((x_0, y_0) \in D\), there always exists a solution \( y(x) \) of the differential equation, and it satisfies \( y' = f(x, y) \) and \( y(x_0) = y_0 \).

The following is the theorem on the uniqueness of the solution for differential equations.

Theorem 14.1.2 If \( f \) is continuous on an open set \( D \), and \( f \) satisfies the global Lipschitz condition with respect to \( y \), then for any \((x_0, y_0) \in D\), there always exists a unique solution \( y(x) \) satisfying \( y' = f(x, y) \) and \( y(x_0) = y_0 \).

For \( n \)th order ordinary differential equations, \( y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \), if the Lipschitz condition is changed to for \( y, y', \ldots, y^{(n-1)} \) instead of for \( y \), we have similar conclusions about existence and uniqueness. See Ahmad and Ambrosetti (2014) for the specific proof of existence and uniqueness.

### 14.2 Some Common Ordinary Differential Equations with Explicit Solutions

Generally, we hope to obtain the concrete form of solutions, namely explicit solutions, for differential equations. However, in many cases, there is no explicit solution. Here we give some common cases in which differential equations can be solved explicitly.

#### Case of Separable Equations

Consider a separable differential equation \( y' = f(x)g(y) \), and \( y(x_0) = y_0 \). It can be rewritten as:

\[
\frac{dy}{g(y)} = f(x)dx.
\]

Integrating both sides, then we get the solution to the differential equation.
14.2. SOME COMMON ORDINARY DIFFERENTIAL EQUATIONS WITH EXPLICIT SOLUTIONS

For example, for \((x^2 + 1)y' + 2xy^2 = 0, y(0) = 1\), using the above solving procedure, we get the solution as

\[ y(x) = \frac{1}{\ln(x^2 + 1) + 1}. \]

In addition, the differential equation with the form \(y' = f(y)\) is called autonomous system, since \(y'\) is only determined by \(y\).

Homogeneous Type of Differential Equation

Some differential equations with constant coefficients have explicit solutions. We first give the definition of homogeneous functions.

**Definition 14.2.1** We call the function \(f(x, y)\) a **homogeneous function of degree \(n\)** if for any \(\lambda\), \(f(\lambda x, \lambda y) = \lambda^n f(x, y)\).

Differential equations have the form of homogeneous functions if \(M(x, y)dx + N(x, y)dy = 0\), where \(M(x, y)\) and \(N(x, y)\) are homogeneous functions with the same order.

By variable transformation \(z = \frac{y}{x}\), the above differential equations can be transformed into the separable form. Suppose \(M(x, y)\) and \(N(x, y)\) are homogeneous functions of degree \(n\), \(M(x, y)dx + N(x, y)dy = 0\) is transformed to \(z + x \frac{dz}{dx} = -\frac{M(1, z)}{N(1, z)}\), and then the final form is \(\frac{dz}{dx} = \frac{z + \frac{M(1, z)}{N(1, z)}}{x}\), where \(z + \frac{M(1, z)}{N(1, z)}\) is a function of \(z\).

Exact Differential Equation

Given a simply connected and open subset \(D \subseteq \mathbb{R}^2\) and two functions \(M\) and \(N\) which are continuous on \(D\), then an implicit first-order ordinary
differential equation of the form

\[ M(x, y)dx + N(x, y)dy = 0 \]

is called an exact differential equation, or a total differential equation. The nomenclature of “exact differential equation” refers to the exact derivative of a function. Indeed, if \( \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \), then the solution is \( F(x, y) = C \), where the constant \( C \) is determined by the initial value, and \( F(x, y) \) satisfies \( \frac{\partial F}{\partial x} = M(x, y) \) or \( \frac{\partial F}{\partial y} = N(x, y) \).

It is clear that a separable differential equation is a special case of an exact differential equation \( y' = f(x)g(y) \) or \( \frac{1}{g(y)}dy - f(x)dx = 0 \), and then we have \( M(x, y) = -f(x), N(x, y) = \frac{1}{g(y)}, \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} = 0 \).

For example, \( 2xy^3dx + 3x^2y^2dy = 0 \) is an exact differential equation, of which the general solution is \( x^2y^3 = C \), and \( C \) is a constant.

When solving differential equations with explicit solutions, we usually convert differential equations into the form of exact differential equations.

First-Order Linear Equation

Consider the first order linear equation of the following form:

\[ \frac{dy}{dx} + p(x)y = q(x). \]  \hspace{1cm} (14.2.4)

When \( q(x) = 0 \), the above differential equation (??) is a separable differential equation , and its solution is assumed to be \( y = \psi(x) \).

Suppose that \( \psi_1(x) \) is a particular solution of the differential equation (??), then \( y = \psi(x) + \psi_1(x) \) is clearly the solution of the equations (??).

It is easy to show that the solution to \( \frac{dy}{dx} + p(x)y = 0 \) is \( y = Ce^{- \int p(x)dx} \). Next we find a particular solution to the differential equation (??).
14.2. SOME COMMON ORDINARY DIFFERENTIAL EQUATIONS WITH EXPLICIT SOLUTIONS

Suppose that
\[ y = c(x)e^{-\int p(x)\,dx}, \]
and differentiating this gives
\[ y' = c'(x)e^{-\int p(x)\,dx} + c(x)p(x)e^{-\int p(x)\,dx}, \]
then substituting this back into the original differential equation, we have
\[ c'(x)e^{-\int p(x)\,dx} + c(x)p(x)e^{-\int p(x)\,dx} = p(x)c(x)e^{-\int p(x)\,dx} + q(x), \]
and thus
\[ c'(x) = q(x)e^{\int p(x)\,dx}. \]

We have
\[ c(x) = \int q(x)e^{\int p(x)\,dx} \, dx + C. \]
Thus, the solution is
\[ y(x) = e^{-\int p(x)\,dx} \left( \int q(x)e^{\int p(x)\,dx} \, dx + C \right). \]

Bernoulli Equation

The following differential equation is called the Bernoulli equation:
\[ \frac{dy}{dx} + p(x)y = q(x)y^n \quad (14.2.5) \]
where \( n \neq 0, 1 \) is a natural number.

Multiplying both sides by \((1 - n)y^{(-n)}\) gives:
\[ (1 - n)y^{(-n)} \frac{dy}{dx} + (1 - n)y^{(1-n)}p(x) = (1 - n)q(x). \]
Let $z = y^{(1-n)}$, and get:

$$\frac{dz}{dx} + (1 - n)zp(x) = (1 - n)q(x),$$

which becomes a first-order linear equation whose explicit solution can be obtained.

The differential equations with explicit solutions have other forms, such as some special forms of Ricatti equations, and the equations similar to $M(x, y)dx + N(x, y)dy = 0$, but not satisfying

$$\frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial N(x, y)}{\partial x}.$$

### 14.3 Higher Order Linear Equations with Constant Coefficients

Consider a differential equation of degree $n$ with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(x). \quad (14.3.6)$$

If $f(x) \equiv 0$, then the differential equation (??) is called **homogeneous differential equation of degree** $n$, otherwise it is called **nonhomogeneous differential equation**.

There is a method for finding the general solution $y_g(x)$ of a homogeneous differential equation of degree $n$. The general solution is the sum of $n$ bases of solutions $y_1, \ldots, y_n$, that is, $y_g(x) = C_1y_1(x) + \cdots + C_ny_n(x)$, where $C_1, \ldots, C_n$ are arbitrary constants. These arbitrary constants are uniquely determined by initial-value conditions. Find a function $y(x)$ sat-
14.3. Higher Order Linear Equations with Constant Coefficients

isfying

\[ y(x) = y_0, \ y'(x) = y_0, \ldots, y^{(n-1)}(x) = y_{0_{n-1}}, \ \text{when } x = x_0, \]

where \( x_0, y_0, y_0_1, \ldots, y_{0_{n-1}} \) are given initial values.

The procedures for solving the fundamental solution of homogeneous differential equations are given below:

1. Solve the characteristic equation with respect to \( \lambda \):

\[ \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n = 0. \]

Suppose that the roots of the characteristic equation are \( \lambda_1, \ldots, \lambda_n \). Some roots may be complex and some are multiple.

2. If \( \lambda_i \) is the non-multiple real characteristic root, then the fundamental solution corresponding to this root is \( y_i(x) = e^{\lambda_i x} \).

3. If \( \lambda_i \) is the real characteristic root of multiplicity \( k \), then there are \( k \) fundamental solutions:

\[ y_{i1}(x) = e^{\lambda_i x}, y_{i2}(x) = xe^{\lambda_i x}, \ldots, y_{ik}(x) = x^{k-1}e^{\lambda_i x}. \]

4. If \( \lambda_j \) is the non-multiple complex characteristic root, \( \lambda_j = \alpha_j + i\beta_j, i = \sqrt{-1} \), its complex conjugate denoted by \( \lambda_{j+1} = \alpha_j - i\beta_j \) is also the characteristic root, thus there are two fundamental solutions generated by these complex conjugate roots \( \lambda_j, \lambda_{j+1} \):

\[ y_{j1} = e^{\alpha_j x} \cos \beta_j x, \quad y_{j2} = e^{\alpha_j x} \sin \beta_j x. \]

5. If \( \lambda_j \) is the complex characteristic root of multiplicity \( l \), \( \lambda_j = \alpha_j + i\beta_j \), its complex conjugate is also the complex characteristic root of multiplicity
Thus these $2l$ complex roots generate $2l$ fundamental solutions:

$$y_{j_1} = e^{\alpha_j x} \cos \beta_j x, \ y_{j_2} = x e^{\alpha_j x} \cos \beta_j x, \ldots, y_{j_l} = x^{l-1} e^{\alpha_j x} \cos \beta_j x;$$

$$y_{j_1+1} = e^{\alpha_j x} \sin \beta_j x, \ y_{j_2+2} = x e^{\alpha_j x} \sin \beta_j x, \ldots, y_{j_{2l}} = x^{l-1} e^{\alpha_j x} \sin \beta_j x.$$

The following is a general method for solving nonhomogeneous differential equations.

The general form of solution to nonhomogeneous differential equations is $y_{nh}(x) = y_g(x) + y_p(x)$, where $y_g(x)$ is the corresponding general solution of the homogeneous equation, and $y_p(x)$ is the particular solution of the nonhomogeneous equation.

Next are some procedures for solving for particular solutions of nonhomogeneous equations.

(1) If $f(x) = P_k(x)e^{bx}$, and $P_k(x)$ is the polynomial of degree $k$, then the form of particular solutions is:

$$y_p(x) = x^s Q_k(x)e^{bx},$$

where $Q_k(x)$ is also a polynomial of degree $k$. If $b$ is not a characteristic root corresponding to the characteristic equation, then $s = 0$; if $b$ is a characteristic root of multiplicity $m$, then $s = m$.

(2) If $f(x) = P_k(x)e^{px} \cos qx + Q_k(x)e^{px} \sin qx$, and $P_k(x)$ and $Q_k(x)$ are all polynomials of degree $k$, then the form of particular solutions is:

$$y_p(x) = x^s R_k(x)e^{px} \cos qx + x^s T_k(x)e^{px} \sin qx,$$

where $R_k(x), T_k(x)$ are also polynomials of degree $k$. If $p + iq$ is not a root of the characteristic equation, then $s = 0$; if $p + iq$ is a characteristic root of multiplicity $m$, then $s = m$.

(3) A general method for solving nonhomogeneous differential equa-
14.3. HIGHER ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The variation of parameters or the method of undetermined coefficients.

Suppose that the general solution of a homogeneous equation is given as follows:

\[ y_g = C_1 y_1(x) + \cdots + C_n y_n(x), \]

where \( y_i(x) \) is the fundamental solution. Regard constants \( C_1, \cdots, C_n \) as the functions with respect to \( x \), such as \( u_1(x), \cdots, u_n(x) \), so the form of particular solutions to the nonhomogeneous equation can be expressed as

\[ y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x), \]

where \( u_1(x), \cdots, u_n(x) \) are the solutions of the following equations

\[ u'_1(x)y_1(x) + \cdots + u'_n(x)y_n(x) = 0, \]
\[ u'_1(x)y'_1(x) + \cdots + u'_n(x)y'_n(x) = 0, \]
\[ \vdots \]
\[ u'_1(x)y^{(n-2)}_1(x) + \cdots + u'_n(x)y^{(n-2)}_n(x) = 0, \]
\[ u'_1(x)y^{(n-1)}_1(x) + \cdots + u'_n(x)y^{(n-1)}_n(x) = f(x). \]

(4) If \( f(x) = f_1(x) + f_2(x) + \cdots + f_r(x) \), and \( y_{p1}(x), \cdots, y_{pr}(x) \) are the particular solutions corresponding to \( f_1(x), \cdots, f_r(x) \), then

\[ y_p(x) = y_{p1}(x) + \cdots + y_{pr}(x). \]

Here is an example to familiarize the application of this method.

Example 14.3.1 Solve \( y'' - 5y' + 6y = t^2 + e^t - 5 \).

The characteristic roots are \( \lambda_1 = 2 \) and \( \lambda_2 = 3 \). Thus, the general solu-
tion of the homogeneous equation is:

\[ y(t) = C_1e^{2t} + C_2e^{3t}. \]

Next, to find a particular solution of the nonhomogeneous equation, its form is written as:

\[ y_p(t) = at^2 + bt + c + de^t. \]

We first substitute this particular solution in the initial equation to determine the coefficients \( a, b, c, d \):

\[ 2a + de^t - 5(2at + b + de^t) + 6(at^2 + bt + c + de^t) = t^2 - 5 + e^t. \]

The coefficients of both sides should be consistent, thus we get:

\[ 6a = 1, \quad -5 \times 2a + 6b = 0, \quad 2a - 5b + 6c = -5, \quad d - 5d + 6d = 1, \]

Therefore, \( d = 1/2, a = 1/6, b = 5/18 \) and \( c = -71/108 \).

Finally, the general solution of the nonhomogeneous differential equation is:

\[ y(t) = C_1e^{2t} + C_2e^{3t} + \frac{t^2}{6} + \frac{5t}{18} - \frac{71}{108} + \frac{e^t}{2}. \]

### 14.4 System of Ordinary Differential Equations

The general form is:

\[ \dot{x}(t) = A(t)x(t) + b(t), \quad x(0) = x_0, \]

where \( t \) (time) is an independent variable, \( x(t) = (x_1(t), \ldots, x_n(t))^t \) is a vector of dependent variables, \( A(t) = (a_{ij}(t))_{n \times n} \) is an \( n \times n \) matrix of
real varying coefficients, and \( b(t) = (b_1(t), \cdots, b_n(t))' \) is an \( n \)-dimensional varying vector.

Consider the case that \( A \) is a constant coefficient matrix and \( b \) is a constant vector, also called the \textit{system of differential equations with constant coefficients}:

\[
\dot{x}(t) = A(t)x(t) + b, \quad x(0) = x_0, \tag{14.4.7}
\]

where \( A \) is assumed to be nonsingular.

The system of differential equations (14.4.7) can be solved by the following two steps.

Step 1: we consider the system of homogeneous equations (i.e. \( b = 0 \)):

\[
\dot{x}(t) = A(t)x(t), \quad x(0) = x_0. \tag{14.4.8}
\]

And its solution is denoted by \( x_c(t) \).

Step 2: find a particular solution \( x_p \) to the nonhomogeneous equation (14.4.7). The constant vector \( x_p \) is a particular solution so that \( Ax_p = -b \), namely \( x_p = -A^{-1}b \).

Given the general solution of the homogeneous equation and the particular solution to the nonhomogeneous equation, the general solution of the system of differential equations (14.4.7) is:

\[
x(t) = x_c(t) + x_p.
\]

There are two methods for solving the system of homogeneous differential equations (14.4.7).

The first one is that we can eliminate \( n - 1 \) dependent variables so that the system of differential equations becomes the differential equation of order \( n \), such as the following example.
Example 14.4.1 The system of differential equation is:

\[
\begin{cases}
    \dot{x} = 2x + y, \\
    \dot{y} = 3x + 4y.
\end{cases}
\]

We differentiate the first equation to eliminate \(y\) and \(\dot{y}\). Since \(\dot{y} = 3x + 4y = 3x + 4\dot{x} - 4 \cdot 2x\), we obtain the corresponding quadratic homogeneous differential equation:

\[
\ddot{x} - 6\dot{x} + 5x = 0,
\]

thus the general solution is \(x(t) = C_1e^t + C_2e^{5t}\). Since \(y(t) = \dot{x} - 2x\), \(y(t) = -C_1e^t + 3C_2e^{5t}\).

The second method is to rewrite the homogeneous differential equation (??) as:

\[
x(t) = e^{At}x_0,
\]

where

\[
e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots.
\]

Now we solve \(e^{At}\) in three different cases.

**Case 1: \(A\) has different real eigenvalues**

Matrix \(A\) has different real eigenvalues, which means that its eigenvectors are linearly independent. Thus \(A\) can be diagonalized, that is,

\[
A = P\Lambda P^{-1},
\]

where \(P = [v_1, v_2, \cdots, v_n]\) consists of the eigenvectors of \(A\), and moreover \(\Lambda\) is a diagonal matrix whose diagonal elements are the eigenvalues of \(A\), thus we have

\[
e^A = Pe^\Lambda P^{-1}.
\]
Therefore, the solution to the system of differential equation (??) is:

\[ x(t) = Pe^{\Lambda}P^{-1}x_0 \]
\[ = Pe^{\Lambda}c \]
\[ = c_1v_1e^{\lambda_1t} + \cdots + c_nv_ne^{\lambda_nt}, \]

where \( c = (c_1, c_2, \ldots, c_n) \) is a vector of arbitrary constants, and it is determined by the initial value, namely \( c = P^{-1}x_0 \).

**Case 2: A has multiple real eigenvalues, but no complex eigenvalues**

First, consider a simple case that \( A \) has only one eigenvalue of multiplicity \( m \). In this case, there are at most \( m \) linearly independent eigenvectors, which means that the matrix \( P \) can not be constructed as a matrix consisting of linearly independent eigenvectors, so \( A \) can not be diagonalized.

Thus, the solution has the following form:

\[ x(t) = \sum_{i=1}^{m} c_i h_i(t), \]

where \( h_i(t), \forall i, \) are quasi-polynomials, and \( c_i, \forall i, \) are determined by initial conditions. For example, when \( m = 3 \), we have:

\[ h_1(t) = e^{\lambda t}v_1, \]
\[ h_2(t) = e^{\lambda t}(tv_1 + v_2), \]
\[ h_3(t) = e^{\lambda t}(t^2v_1 + 2tv_2 + 3v_3), \]

where \( v_1, v_2, v_3 \) are determined by following conditions:

\[ (A - \lambda I)v_i = v_{i-1}, v_0 = 0. \]

If \( A \) has more than one multiple real eigenvalues, then the solution of the
differential equation (??) can be obtained by summing up the solutions corresponding to each eigenvalue.

Case 3: \( A \) has complex eigenvalues

Since \( A \) is a real matrix, complex eigenvalues will be generated in the form of conjugate pairs.

If an eigenvalue of \( A \) is \( \alpha + \beta i \), then its conjugate complex \( \alpha - \beta i \) is also an eigenvalue.

Now consider a simple case: \( A \) has only one pair of complex eigenvalues, \( \lambda_1 = \alpha + \beta i \) and \( \lambda_2 = \alpha - \beta i \).

Let \( v_1 \) and \( v_2 \) be the eigenvectors corresponding to \( \lambda_1 \) and \( \lambda_2 \); then we have \( v_2 = \bar{v}_1 \), where \( \bar{v}_1 \) refers to the conjugation of \( v_1 \). The solution of the differential equation (??) can be expressed as:

\[
x(t) = e^{At}x_0 = Pe^{A}P^{-1}x_0 = Pe^{A}c = c_1v_1e^{(\alpha+\beta)t} + c_2v_2e^{(\alpha-\beta)t} = c_1v_1e^{\alpha t}(\cos \beta t + i \sin \beta t) + c_2v_2e^{\alpha t}(\cos \beta t - i \sin \beta t) = (c_1v_1 + c_2v_2)e^{\alpha t} \cos \beta t + i(c_1v_1 - c_2v_2)e^{\alpha t} \sin \beta t = h_1e^{\alpha t} \cos \beta t + h_2e^{\alpha t} \sin \beta t,
\]

where \( h_1 = c_1v_1 + c_2v_2 \) and \( h_2 = i(c_1v_1 - c_2v_2) \).

If \( A \) has many pairs of conjugate complex eigenvalues, then the solution of the differential equation (??) is obtained by summing up the solutions corresponding to all eigenvalues.
14.5 Simultaneous Differential Equations and Stability of Equilibrium

Consider the case of simultaneous differential equations with two variables \( x = x(t) \) and \( y = y(t) \), where \( t \) (time) is an independent variable.

The two dimensional autonomous differential equations have the following form:

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y), \\
\frac{dy}{dt} &= g(x, y).
\end{align*}
\]

Such simultaneous differential equations are called planar dynamical systems, and multidimensional dynamical systems can be similarly defined.

If \( f(x^*, y^*) = g(x^*, y^*) = 0 \) is established, the point \( (x^*, y^*) \) is called the equilibrium of this dynamical system.

Let \( J \) be the Jacobian

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\]

evaluated at \( (x^*, y^*) \), and \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of this Jacobian.

Then the stability of equilibrium point is characterized as follows:

1. It is a stable (or unstable) node if \( \lambda_1 \) and \( \lambda_2 \) are different real numbers and are negative (or positive);

2. It is a saddle point if eigenvalues are real numbers but with opposite signs, namely \( \lambda_1 \lambda_2 < 0 \);

3. It is a stable (or unstable) focus if \( \lambda_1 \) and \( \lambda_2 \) are complex numbers and \( Re(\lambda_1) < 0 \) (or \( Re(\lambda_1) > 0 \));

4. It is a center if \( \lambda_1 \) and \( \lambda_2 \) are complex and \( Re(\lambda_1) = 0 \);
(5) It is a **stable (or unstable) improper node** if $\lambda_1$ and $\lambda_2$ are real, $\lambda_1 = \lambda_2 < 0$ (or $\lambda_1 = \lambda_2 > 0$) and the Jacobian is not a diagonal matrix;

(6) It is a **stable (or unstable) star node** if $\lambda_1$ and $\lambda_2$ are real, $\lambda_1 = \lambda_2 < 0$ (or $\lambda_1 = \lambda_2 > 0$) and the Jacobian is a diagonal matrix.

Figure 2.5 below depicts six types of equilibrium point.

![Figure 14.1: The Types of Equilibrium Points](image)

**14.6 Stability of Dynamical System**

In a dynamical system, Lyapunov method studies the global stability of equilibrium points.
14.6. STABILITY OF DYNAMICAL SYSTEM

Consider the following dynamical system:

\[ \dot{x} = f(t, x), \]  

(14.6.9)

where \( x = (x_1, \cdots, x_n) \), \( f(t, x) \) is continuously differentiable with respect to \( x \in \mathbb{R}^n \) and meanwhile satisfies the initial condition \( x(0) = x_0 \). If \( x^* \) is the equilibrium point of the dynamical system, then \( f(t, x^*) = 0 \).

Let \( \bar{x}(t, x_0) \) be the unique solution of the dynamical system (14.6.9), and \( B_r(x) = \{ x' \in D : |x' - x| < r \} \) be an open ball of radius \( r \) centered at \( x \).

The following is the definition of stability of equilibrium point.

**Definition 14.6.1** The equilibrium point \( x^* \) of the dynamical system (14.6.9)

(1) is **globally stable** if for any \( r > 0 \), there is a neighbourhood \( U \) of \( x^* \) such that

\[ \bar{x}(t, x_0) \in B_r(x^*), \forall x_0 \in U. \]

(2) is **globally asymptotically stable** if for any \( r > 0 \), there is a neighbourhood \( U' \) of \( x^* \) such that

\[ \lim_{t \to \infty} \bar{x}(t, x_0) = x^*, \forall x_0 \in U. \]

(3) is **globally unstable** if it is neither globally stable nor asymptotically globally stable.

**Definition 14.6.2** Let \( x^* \) be the equilibrium point of the dynamical system (14.6.9), \( Q \subseteq \mathbb{R}^n \) be an open set containing \( x^* \), and \( V(x) : Q \to \mathbb{R} \) be a continuously differentiable function. If it satisfies:

(1) \( V(x) > V(x^*), \forall x \in Q, x \neq x^*; \)

(2) \( \dot{V}(x) \) is defined as:

\[ \dot{V}(x) \overset{\text{def}}{=} \nabla V(x) f(t, x) \leq 0, \forall x \in Q, \]  

(14.6.10)
where $\nabla V(x)$ is the gradient of $V$ with respect to $x$,

thus it is called Lyapunov function.

The following is the Lyapunov theorem about the equilibrium points of dynamical systems.

**Theorem 14.6.1** If there exists a Lyapunov function $V$ for the dynamical system (??), then the equilibrium point $x^*$ is globally stable.

If the Lyapunov function (??) of the dynamical system satisfies $\dot{V}(x) < 0, \forall x \in Q, x \neq x^*$, then the equilibrium point $x^*$ is asymptotically globally stable.
Chapter 15

Difference Equations

Difference equations can be regarded as discretized differential equations, and many of their properties are similar to those of differential equations.

Let $y$ be a real-valued function defined on natural numbers. $y_t$ means the value of $y$ at $t$, where $t = 0, 1, 2, \cdots$, which can be regarded as time points.

**Definition 15.0.3** The first-order difference of $y$ at $t$ is:

$$\Delta y(t) = y(t + 1) - y(t).$$

The second-order difference of $y$ at $t$ is:

$$\Delta^2 y(t) = \Delta(\Delta y(t)) = y(t + 2) - 2y(t + 1) + y(t).$$

Generally, the $n$th-order difference of $y$ at $t$ is:

$$\Delta^n y(t) = \Delta(\Delta^{n-1} y(t)), \ n > 1.$$ 

**Definition 15.0.4** The difference equation is a function of $y$ and its differ-
ences $\Delta y, \Delta^2 y, \cdots, \Delta^{n-1} y$,

$$F(y, \Delta y, \Delta^2 y, \cdots, \Delta^n y, t) = 0, t = 0, 1, 2, \cdots. \quad (15.0.1)$$

If $n$ is the highest order of nonzero coefficient in the formula (15.0.1), the above equation is called an \textit{n\textsuperscript{th}-order difference equation}.

If $F(\psi(t), \Delta \psi(t), \Delta^2 \psi(t), \cdots, \Delta^n \psi(t), t) = 0$ holds for $\forall t$, then we call $y = \psi(k)$ a solution of the difference equation. Similar to differential equations, the solutions of difference equations also have general solutions and particular solutions. The general solutions usually contain some arbitrary constants that can be determined by initial conditions.

The difference equations can also be expressed in the following form by variable conversion:

$$F(y(t), y(t + 1), \cdots, y(t + n), t) = 0, t = 0, 1, 2, \cdots. \quad (15.0.2)$$

If the coefficients of $y_0(k), y_n(k)$ are not zero, and the highest corresponding order is $n$, then it is called an \textit{n\textsuperscript{th}-order difference equation}.

The followings are mainly focused on the difference equations with constant coefficients. A common expression is written as:

$$f_0 y(t + n) + f_1 y(t + n - 1) + \cdots + f_{n-1} y(t + 1) + f_n y(t) = g(t), t = 0, 1, 2, \cdots, \quad (15.0.3)$$

where $f_0, f_1, \cdots, f_n$ are real numbers, and $f_0 \neq 0, f_n \neq 0$.

Dividing both sides of the equation by $f_0$, and making $a_i = \frac{f_i}{f_0}$ for $i = 0, \cdots, n$, $r(t) = \frac{g(t)}{f_0}$, the $n$th order difference equation can be written as the simpler form:
Here are three procedures that are usually used to solve \(n\)th order linear difference equations:

Step 1: find the general solution of the homogeneous difference equation
\[
y(t + n) + a_1y(t + n - 1) + \cdots + a_{n-1}y(t + 1) + a_ny(t) = 0,
\]
and let the general solution be \(Y\).

Step 2: find a particular solution \(y^*\) of the difference equation (2.9.53).

Step 3: the solution of the difference equation (2.9.53) is
\[
y(t) = Y + y^*.
\]

The followings are the solutions of the first-order, second-order and \(n\)th-order difference equations, respectively.

## 15.1 First-order Difference Equations

The first-order difference equation is defined as:
\[
y(t + 1) + ay(t) = r(t), t = 0, 1, 2, \ldots.
\] (15.1.5)

The corresponding homogeneous difference equation is:
\[
y(t + 1) + ay(t) = 0,
\]
and the general solution is \(y(t) = c(-a)^t\), where \(c\) is an arbitrary constant.
Next, we discuss how to get a particular solution for a nonhomogeneous difference equation.

First, consider \( r(t) = r \), that is, the case that does not change over time.

Obviously, a particular solution is as follows:

\[
y^* = \frac{r}{1 + a}, \quad a \neq -1,
\]

\[
y^* = rt, \quad a = -1.
\]

Hence, the solution of the nonhomogeneous difference equation \((\text{?})\) is:

\[
y(t) = \begin{cases} 
  c(-a)^t + \frac{r}{1 + a}, & \text{if } a \neq -1, \\
  c + rt, & \text{if } a = -1.
\end{cases} \tag{15.1.6}
\]

If the initial condition \( y(0) = y_0 \) is known, the solution of the difference equation \((\text{?})\) is:

\[
y(t) = \begin{cases} 
  (y_0 - \frac{r}{1 + a}) \times (-a)^t + \frac{r}{1 + a}, & \text{if } a \neq -1, \\
  y_0 + rt, & \text{if } a = -1.
\end{cases} \tag{15.1.7}
\]

If \( r \) depends on \( t \), a particular solution is:

\[
y^* = \sum_{i=0}^{t-1} (-a)^{-i} r(i),
\]

thus the solution of the difference equation \((\text{?})\) is:

\[
y(t) = (-a)^ty_0 + \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i), \quad t = 1, 2, \cdots.
\]

For a general function \( r(t) = f(t) \), the coefficients of \( A_0, \cdots, A_m \) can be determined by using the method of undetermined-coefficients, namely considering \( y^* = f(A_0, A_1, \cdots, A_m; t) \). The following is to solve for
a particular solution in a case that \( r(t) \) is a polynomial.

**Example 15.1.1** Solve the following difference equation:

\[
y(t + 1) - 3y(t) = t^2 + t + 2.
\]

The homogeneous equation is:

\[
y(t + 1) - 3y(t) = 0,
\]

The general solution is:

\[
Y = C3^t.
\]

Using the method of undetermined-coefficients to get the particular solution of the nonhomogeneous equation, suppose that the particular solution has the form:

\[
y^* = At^2 + Bt + D.
\]

Substitute \( y^* \) into the nonhomogeneous difference equation, and get:

\[
A(t + 1)^2 + B(t + 1) + D - 3At^2 - 3Bt - 3D = t^2 + t + 2,
\]

or

\[
-2At^2 + 2(A - B)t + A + B - 2D = t^2 + t + 2.
\]

Since equality holds for each \( t \), we must have:

\[
\begin{cases}
-2A = 1 \\
2(A - B) = 1 \\
A + B - 2D = 2,
\end{cases}
\]

which gives \( A = \frac{1}{2} \), \( B = -1 \) and \( D = \frac{3}{4} \), thus we have a particular solution: \( y^* = -\frac{1}{2}t^2 - t - \frac{3}{4} \). Therefore, a particular solution of the nonho-
mogogeneous equation is \( y(t) = Y + y^* = C3^t - \frac{1}{2}t^2 - t - \frac{3}{4} \).

We can also solve the case with an exponential function by using the method of undetermined-coefficients.

**Example 15.1.2** Consider the first-order difference equation:

\[
y(t + 1) - 3y(t) = 4e^t.
\]

Suppose that the form of particular solution is \( y^* = Ae^t \), then substituting it into the nonhomogeneous difference equation gives: \( A = \frac{4}{e - 3} \). Therefore, the general solution of the first-order difference equation is:

\[
y(t) = Y + y^* = C3^t + \frac{4e^t}{e - 3}.
\]

Here are some of the common ways for finding particular solutions:

1. when \( r(t) = r \), a usual form of particular solution is: \( y^* = A \);
2. when \( r(t) = r + ct \), a usual form of particular solution is:
    \[
y^* = A_1 t + A_2;
    \]
3. when \( r(t) = t^n \), a usual form of particular solution is: \( y^* = A_0 + A_1 t + \cdots + A_n t^n \);
4. when \( r(t) = c^t \), a usual form of particular solution is: \( y^* = Ac^t \);
5. when \( r(t) = \alpha \sin(ct) + \beta \cos(ct) \), a usual form of particular solution is: \( y^* = A_1 \sin(ct) + A_2 \cos(ct) \).

**15.2 Second-order Difference Equation**

The second-order difference equation is defined as:

\[
y(t + 2) + a_1 y(t + 1) + a_2 y(t) = r(t).
\]
The corresponding homogeneous differential equation is:

\[ y(t + 2) + a_1 y(t + 1) + a_2 y(t) = 0. \]

Then, its general solution depends on the roots of the following linear equation:

\[ m^2 + a_1 m + a_2 = 0, \]

which is called the **auxiliary equation or characteristic equation** of second order difference equations. Let \( m_1 \) and \( m_2 \) be the roots of this equation. Since \( a_2 \neq 0 \), both \( m_1 \) and \( m_2 \) are not 0.

**Case 1:** \( m_1 \) and \( m_2 \) are different real roots.

Now the general solution of the homogeneous equation is

\[ Y = C_1 m_1 t + C_2 m_2, \]

where \( C_1, C_2 \) are arbitrary constants.

**Case 2:** \( m_1 \) and \( m_2 \) are the same real roots.

Now the general solution of the homogeneous equation is

\[ Y = (C_1 + C_2 t) m_1. \]

**Case 3:** \( m_1 \) and \( m_2 \) are two complex roots, namely \( r(\cos \theta \pm i \sin \theta) \) with \( r > 0, \theta \in (-\pi, \pi] \). Now the general solution of the homogeneous equation is

\[ Y = C_1 r^t \cos(t \theta + C_2). \]

For a general function \( r(t) \), it can be solved by the method of undetermined-coefficients.

### 15.3 Difference Equations of Order \( n \)

The general \( n \)th-order difference equation is defined as:

\[ y(t+n)+a_1y(t+n-1)+\cdots+a_{n-1}y(t+1)+a_ny(t)=r(t), \quad t=0, 1, 2, \ldots \quad (15.3.8) \]
CHAPTER 15. DIFFERENCE EQUATIONS

The corresponding homogeneous equation is:

\[ y(t + n) + a_1 y(t + n - 1) + \cdots + a_{n-1} y(t + 1) + a_n y(t) = 0, \]

and its characteristic equation is:

\[ m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \]

Let its \( n \) characteristic roots be \( m_1, \ldots, m_n \).

The general solutions of the homogeneous equations are the sum of the bases generated by these eigenvalues, and its concrete forms are as follows:

Case 1: The formula generated by a single real root \( m \) is \( C_1 m^k \).

Case 2: The formula generated by the real root \( m \) of multiplicity \( p \) is:

\[ (C_1 + C_2 t + C_3 t^2 + \cdots + C_p t^{p-1}) m^t. \]

Case 3: The formula generated by a pair of nonrepeated conjugate complex roots \( r (\cos \theta \pm i \sin \theta) \) is:

\[ C_1 r^t \cos(t \theta + C_2). \]

Case 4: The formula generated by a pair of conjugate complex roots \( r (\cos \theta \pm i \sin \theta) \) of multiplicity \( p \) is:

\[ r^t [C_{1,1} \cos(t \theta + C_{1,2}) + C_{2,1} t \cos(t \theta + C_{2,2}) + \cdots + C_{p,1} t^{p-1} \cos(t \theta + C_{p,2})]. \]

The general solution of the homogeneous difference equation is obtained by summing up all formulas generated by eigenvalues.
A particular solution $y^*$ of a nonhomogeneous difference equation can be generated by the method of undetermined-coefficients.

A particular solution is:

$$y^* = \sum_{s=1}^{n} \theta_s \sum_{i=0}^{\infty} m_s^i r(t - i),$$

where

$$\theta_s = \frac{m_s}{\Pi_{j \neq s} (m_s - m_j)}.$$

### 15.4 Stability of $n$th-Order Difference Equations

Consider an $n$th-order difference equation

$$y(t + n) + a_1 y(t + n - 1) + \cdots + a_{n - 1} y(t + 1) + a_n y(t) = r(t), \quad t = 0, 1, 2, \cdots. \quad (15.4.9)$$

The corresponding homogeneous equation is:

$$y(t + n) + a_1 y(t + n - 1) + \cdots + a_{n - 1} y(t + 1) + a_n y(t) = 0, \quad t = 0, 1, 2, \cdots. \quad (15.4.10)$$

**Definition 15.4.1** If an arbitrary solution $Y(t)$ of the homogeneous equation (15.4.10) satisfies $Y(t)|_{t \to \infty} = 0$, then the difference equation (15.4.9) is asymptotically stable.

Let $m_1, \cdots, m_n$ be the solution of their characteristic equation:

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \quad (15.4.11)$$

**Theorem 15.4.1** If the modulus of all eigenvalues of the characteristic equation are less than 1, the difference equation (15.4.9) is asymptotically stable.
When the following inequality conditions are satisfied, the modulus of all eigenvalues of the characteristic equation are less than 1.

\[
\begin{vmatrix}
1 & a_n \\
a_n & 1
\end{vmatrix} > 0, \\
\begin{vmatrix}
1 & 0 & a_n & a_{n-1} \\
a_1 & 1 & 0 & a_n \\
a_n & 0 & 1 & a_1 \\
a_{n-1} & a_n & 0 & 1
\end{vmatrix} > 0, \\
\begin{vmatrix}
1 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\
a_1 & 1 & \cdots & 0 & 0 & a_n & a_{n-1} & \cdots & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & a_{n-2} & \cdots & 1 & 0 & 0 & \cdots & a_n \\
a_n & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_{n-1} \\
a_{n-1} & a_n & \cdots & 0 & 0 & 1 & \cdots & a_{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & 1
\end{vmatrix} > 0.
\]

15.5 Difference Equations with Constant Coefficients

The difference equation with constant coefficients is defined as:

\[
x(t) = Ax(t - 1) + b, \quad (15.5.12)
\]

where \( x = (x_1, \cdots, x_n)' \), \( b = (b_1, \cdots, b_n)' \). Suppose the matrix \( A \) is diagonalizable, the corresponding eigenvalues are \( \lambda_1, \cdots, \lambda_n \) and the matrix \( P \)
formed by linearly independent eigenvectors such that

$$A = P^{-1} \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n \\
\end{pmatrix} P.$$

A necessary and sufficient condition for the differential equation (??) to be (asymptotically) stable is that the modulus of all eigenvalues $\lambda_i$ are less than 1. When the modulus of all eigenvalues $\lambda_i$ are less than 1, the equilibrium point $x^* = \lim_{t \to \infty} x(t) = (I - A)^{-1}b$.