Matching with Couples: Semi-Stability and Algorithm

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Abstract

This paper introduces the notion of semi-stability for matching problem with couples, which is a natural generalization of, and identical to, the conventional stability for matching without couples. It is shown that there always exists a semi-stable matching for matching markets with strict preferences, and further the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice. We also provide sufficient conditions for stability, which enable us to provide a new explanation to the puzzle of NRMP raised in Kojima *et al.* (2013). Moreover, we introduce the notion of asymptotical stability and present sufficient conditions for a matching sequence to be asymptotically stable. Another important contribution of the paper is to develop a new algorithm, called the Persistent Improvement Algorithm (PI-Algorithm in short), for finding semi-stable matchings, which not only generalizes but is also more efficient than the Gale-Shapley algorithm that only fits the matching market with singles. Lastly, this paper investigates the welfare property and incentive issues of semi-stable mechanisms.

Keywords: Matching with couples; stability; semi-stability; asymptotic stability; algorithm.

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1 Introduction

This paper studies the matching problem with couples. One typical feature of the problem is that the stable matchings with couples may not exist. To avoid this defect, we introduce the notion of semi-stability as a generalized solution for matching problem with couples, which is a relaxation and natural generalization of the conventional stability, and further identical to, the conventional stability for matching without couples.

Matching is one of the most important natures of market. Many problems, such as trade problem in consumption goods markets, employment problem in labor markets and auction problem of indivisible/public goods, etc., can be regarded as matching problems. Gale and Shapley (1962) were the first to introduce the notion of stable matching and regarded it as the solution of matching problem. The *deferred acceptance algorithm* proposed by them reveals that stable matching always exists in one-to-one matching markets with strict preferences. Since then, a lot of important theoretical results and their practice on matching have been developed (cf. Roth (2008) for detailed literature review).

Stable matching mechanism has wide applications such as the National Resident Matching Program (NRMP) that has a long history. The original algorithm for NRMP proposed by Mullen and Stalnaker (1952) was an unstable mechanism, and later, it was revised repeatedly as discussed in detail in Roth and Peranson (1999). One of the reasons is that, since the 1970s, more and more female medical students had entered the job market, which made NRMP's algorithm run into difficulties in finding stable outcomes. For instance, couples would often decline the job offers assigned by the clearinghouse and search positions themselves in order to stay together, say, they would prefer to have jobs in the same city, although the choice may not be the best for their professional development. This implies that couple students' preferences are complements. In order to make the NRMP's algorithm also work for couple medical students, Roth and Peranson (1999) designed the NRMP's present algorithm, but it may result in an empty set of stable matchings since stable matching does not exist at all.

Sun and Yang (2006, 2009) studied the auction problem in economies where agents of the same type are substitutes for one another, but agents of different types are complements. They showed that equilibrium always exists in economies with quasi-linear preferences. Ostrovsky (2008) studied a more generalized problem of supply chain networks in which there are similar restrictions—same-side substitutability and cross-side complementarity, and showed that the set of chain-stable networks is non-empty. The problem of matching with couples, however, is different from that of cross-side complementarity in that the agents of same-couple are complements. As shown in Roth (1984), there may not exist any stable matching in matching markets with couples. Ronn (1990) demonstrated that when couple factors in preferences are taken into account, it is an NP-complete problem to show whether there exist stable matching by computational methods.[1]

As Kojima *et al.* (2013) pointed out, although there may not exist any stable matching in couple markets and the NRMP's present algorithm proposed by Roth and Peranson (1999) is an adjustment of the instability-chaining algorithm for markets with singles, which was proposed by in Roth and Vande Vate (1990), so that the revised matching system can accept couples' preferences, the market practice in the past two decades indicated that the algorithm rarely failed for the clearinghouse to find stable matchings. Why can the clearinghouse find stable outcomes while the theories indicate that there may not exist any stable outcomes for couple markets? Kojima *et al.* (2013) regarded it as a puzzle and thought the reason may be that the market size is very large while couples take a very small proportion. They also showed that under some regularity conditions, as the size of market tends to infinity whereas the number of couples relative to the size of market does not grow rapidly, the probability that there exists a stable matching tends to 1.

In their attempt to find a generalized solution for the matching problem with couples, Klijn and Masso (2003) introduced the notion of weakly stable matching[2] in order to extend the existence result to a larger class of preferences. For markets with singles only, they showed that the set of weakly stable and weakly efficient matchings is identical to Zhou's (1994) bargaining set. However, as Klaus and Klijn (2005) indicated, the set of weakly stable matchings may still be empty in matching markets with couples. When do there exist stable matchings in a matching market with couples? Klaus and Klijn (2005) showed that there exists a stable matching when all couples' preferences are (weakly) responsive. But, (weakly) responsive preferences actually imply that couples' preferences would fail to be complements. When couples' preferences are not (weakly) responsive, that is, when they are complements, Klaus and Klijn[3] showed that stable outcomes may not exist even if in the system containing only one couple. Thus, their result is limited in application.

Moreover, Aldershof and Carducci (1996) showed that, even when the set of stable matchings is nonempty, there may not be a lattice structure, the set of unmatched objects may not be

[1]The abbreviation NP refers to nondeterministic polynomial time, which is a common term in computational complexity theory.

[2]A matching is weakly stable if it is individually rational and all blocking coalitions are dominated. The detailed definition can be seen in Klijn and Masso (2003), Klaus and Klijn (2005).

[3]The counterexample is seen in Roth (2008), but it belongs to Klaus and Klijn.

the same at every stable matching, and further there may not be any strategy-proof stable mechanisms. Klaus and Klijn (2005) demonstrated that there are not any ready parallels to any of the standard results in marriage matching markets, even if preferences are responsive.

All in all, there has been no satisfactory result so far to the problem of matching with couples. There is neither any concept of outcome that is generally applicable, nor any generalized algorithm applicable to matching markets with couples for general settings.

In this paper we introduce the notion of semi-stability that can be seen as a generalized solution of the matching problem with couples, and shows that there always exists a semi-stable matching for couples markets with strict preferences. A semi-stable matching means that it is individually rational and there does not exist any blocking coalition of the matching that contains singleton, that is, any blocking coalition of a semi-stable matching contains a real couple. As such, the set of stable matchings is clearly a subset of semi-stable matchings. For a special matching market containing only singletons, a semi-stable matching is identical to a stable matching. As a result, the notion of semi-stable matching is a natural generalization of the conventional stable matching without couples.

We then provide sufficient conditions for the existence of stable matchings with couples even in the presence of complementary preferences. It is shown that there exists a stable matching with couples provided every real couple plays reservation strategies, i.e., some reservation preferences, which can secure a pair of jobs if they want, are placed on top of its rank list of preferences. The reason why couples play reservation strategies is that their preferences have couple-complementarity, that is, although popular jobs are personally desirable, the pair of popular jobs may not be the most preferred for couples, as the pair of jobs may not be at the same hospital or in the same city. In order to stay together, the most preferred pair of jobs may not be popular jobs, which is consistent with the practice of NRMP. As a by-product of the results, it provides another explanation for the puzzle of NRMP raised in Kojima *et al.* (2013).

This paper also introduces the notion of asymptotic stability. In a large matching market with couples, if the number of couples is sufficiently small relative to that of singletons, a semi-stable matching can be deemed as an approaching stable matching. The number of blocking coalitions of any semi-stable matching must be very small, relative to the size of the market, if the length of rank list of couples' preferences is bounded and the market goes very large. We then introduce the notion of the degree of instable matching to indicate the unorderly degree of a matching. The degree of instability is 0 when stable matching exists, and 1 for the null matching in which all players are unmatched. It is shown that under some simple regularity conditions, matching markets with couples is asymptotically stable, i.e., there exists a matching sequence whose unstable degree tends to zero when the size of markets tends to infinity. This conclusion is similar to the result in Kojima *et al.* (2013), who demonstrated that the probability that a stable matching exists converges to 1 as the market size approaches infinity under some regularity conditions. The simple regularity conditions defined in this paper are weaker than their regularity conditions. As such, their result can be regarded as a special case of our result.

Another important contribution of the paper is to provide an algorithm, called Persistent Improvement Algorithm (PI-Algorithm in short), for finding a semi-stable matching, which not only generalizes but is also more efficient than the Gale-Shapley algorithm that only fits the matching market with singles. Crawford and Knoer (1981) and Kelso and Crawford (1982) studied the employment problem in labor markets, and generalized the Gale-Shapley algorithm by introducing the salary adjustment process. Hatfield and Milgrom (2005) extended the Gale-Shapley algorithm into a generalized algorithm for matching with contracts, which is in turn a generalization of the salary adjustment process of Kelso and Crawford (1982). Ostrovsky (2008) studied a more generalized problem about supply chain networks in which there are restrictions—same-side substitutability and cross-side complementarity, and presented the T-Algorithm which generalizes the algorithms in Kelso and Crawford (1982), Hatfield and Milgrom (2005), as well as the Gale-Shapley algorithm for marriage matching. However, the problem of matching with couples is different from the problem of cross-side complementarity in that the agents of same-couple are complements. Those algorithms mentioned above cannot fit into the matching markets with couples. Roth and Vande Vate (1990) presented the instability-chaining algorithm for one-to-one matching markets. The NRMP's present algorithm proposed in Roth and Peranson (1999) is an improvement of the instability-chaining algorithm for single markets such that the clearinghouse can accept the preferences of couples, but the algorithm may not converge. This paper presents the PI-Algorithm which fits the matching markets with couples. Moreover, it is a strict generalization of the Gale-Shapley algorithm and further is more efficient, by which we can find a semi-stable matching quickly.

We also show that the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice, and in each of which there exist optimal semi-stable matchings for one-side, truth telling is a dominant strategy for one-side, and the set of unmatched objects is the same at every semi-stable matching. We study the problems of welfare property and incentive issues of semi-stable matching mechanisms from the perspective of market design, and generalizes the respective results in marriage matching markets.

The remainder of this paper is organized as follows. Section II describes the setup and introduces the notions of semi-stability in matchings with couples. Section III provides the main results on the existence of semi-stable matchings, stable matchings, asymptotically stable matching sequence, and a generalized lattice theorem. We also provides a new explanation for the NRMP's puzzle. Section IV presents the PI-Algorithm and its properties. Section V discusses the welfare and incentive properties of semi-stable matching mechanisms from the perspective of market design. Section VI concludes and the appendix gives proofs of theorems.

2 The Model of Matching with Couples

A matching market consists of jobs of hospitals, job-seeking medical students and their preferences. Although a hospital may provide many jobs, yet as Gale and Shapley (1962) and Roth and Sotomayor (1990) pointed out, when medical students' preferences are on specific jobs, it is equivalent to the one-to-one marriage matching market. In fact, a hospital may provide some jobs of special profession, such as physician jobs, surgeon jobs or gynecologist jobs, etc., and the requirements for the jobs are generally different. As such, in this paper, matching objects of medical students are jobs rather than hospitals. Let H denote the set of jobs of hospitals, S the set of medical students and C the set of student couples. Their elements are written as h, s, and $c = (s, s')$, respectively, where $s \in S$, $s' \in S \cup \{\phi\}$. When $s' = \phi$, $c = (s, \phi)$ denotes a special couple—single student. By this method, the model in this paper can actually be applied to three markets, i.e., markets with singles only, markets with couples only, and the markets containing both singles and couples.

We assume that all preferences of jobs and couples are strict. Let $\succ_h$ and $\succ_c$ denote h's and c's preference, and P^h and P^c denote h's and c's preferences rank list, respectively. It is said that $s \in S$ is acceptable (resp. unacceptable) to h if $s \succ_h \phi$ (resp $\phi \succ_h s$), and $(h, h') \in [H \cup \{\phi\}]^2$ is acceptable (resp. unacceptable) pair of jobs to c if $(h, h') \succ_c (\phi, \phi)$ (resp. $(\phi, \phi) \succ_c (h, h')$). For convenience of discussion, we assume that ϕ and (ϕ, ϕ) are at the last in P^h and P^c, respectively. Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ denote a market of matching with couples.

A matching μ is a one-to-one idempotent function from the set $H \cup S \cup \{\phi\}$ onto itself (i.e., $\mu^2(x) = x$ for all x) such that $\mu(s) \in H \cup \{\phi\}$ and $\mu(h) \in S \cup \{\phi\}$, where $\mu(s)$ and $\mu(h)$ are the matched objects of s and h in μ. When a medical student or a job is not matched in μ, we denote by ϕ its matched object. For convenience, we assume $\mu(\phi) = \phi$. Let $\mu(c) = (\mu(s), \mu(s'))$ with $\mu(s) \in H \cup \{\phi\}$ and $\mu(s') \in H \cup \{\phi\}$. For any μ, $\mu(s) = h$ if and only if $\mu(h) = s$; $\mu(c) = (h, h')$ if and only if $\mu(h) = s$ and $\mu(h') = s'$.

If a job or couple cannot be improved by voluntarily abandoning its matched object, the matching is individually rational. Formally,

Definition 2.1 A matching μ is *individually rational*, if (i) for all $h \in H$ with $\mu(h) \neq \phi$, $\mu(h) \succ_h \phi$; and (ii) for all $c = (s, s') \in C$, $\mu(c) \succ_c (\phi, \mu(s'))$ when $\mu(s) \neq \phi$, $\mu(c) \succ_c (\mu(s), \phi)$ when $\mu(s') \neq \phi$, and $\mu(c) \succ_c (\phi, \phi)$ when $\mu(c) \neq (\phi, \phi)$.

A couple and a pair of hospital jobs constitute a coalition. We then have the following definitions.

Definition 2.2 $\{(s, s'), (h, h')\}$ is called a *blocking coalition of matching* μ if (i) $(h, h') \succ_c \mu(c)$; and (ii) [$h \neq \phi$ and $\mu(h) \neq s$ imply $s \succ_h \mu(h)$] and [$h' \neq \phi$ and $\mu(h') \neq s'$ imply $s' \succ_{h'} \mu(h')$].

Thus, a blocking coalition means that agents can be improved upon by matching with each other.

Definition 2.3 A matching said to be *stable* if it is individually rational and there exist no blocking coalitions.

Definition 2.4 A matching is said to be *semi-stable* if it is individually rational and there are no blocking coalitions containing a single.

It is obvious that a stable matching is semi-stable, but the reverse in general is not true. However, a semi-stable matching is a stable matching for any matching market containing only singletons. Indeed, when all couples are (s, ϕ), it is identical to the definition of stable matching for singles markets. Thus, the notion of semi-stability for couples markets is a natural generalization of the conventional stability for singles markets. As Gale and Shapley (1962) showed, there always exists a stable matching for singles markets with strict preferences. However, for matching markets with couples, Roth (1984) showed that there may not exist any stable matchings. In the next section, departing from Roths example, we show that there always exists a semi-stable matching for any matching market containing couples with strict preferences and also provide sufficient conditions for the existence of stable matchings.

3 Main Existence Results and NRMP Puzzle Revisit

In this section, we first investigate the existence of semi-stable matching for couples markets with strict preferences. We then show that the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice. We also provide sufficient conditions for the existence of stability. The results can enable us to provide a new explanation for the puzzle of NRMP raised in Kojima *et al.* (2013). Moreover, we introduce the notion of asymptotical stability and provide sufficient conditions for a matching sequence to be asymptotically stable.

3.1 Existence of Semi-Stable Matchings and Distributive Lattice

The example in Klaus and Klijn (2005) shows that, even if there is only one couple in a matching market, there may not exist any stable matching. As such, if one focuses only on stable matchings, the set of outcomes may be empty. This makes us to introduce the notion of semi-stable matching, which means that there does not exist any blocking coalitions of the matching that contains singletons. A question then is whether there exists a semi-stable matching for a matching market with couples. The following theorem gives an affirmative answer.

Theorem 3.1 (Existence of Semi-Stable Matching) *For any matching market with couples and strict preferences* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$*, there exists a semi-stable matching* μ*.*

The theorem indicates there always exists a semi-stable matching for strict preferences. Since the theorem is proved by a constructive way, we actually obtain an algorithm to find a semi-stable outcome. In addition, the algorithm also provides an approach to find a stable matching, if any, in matching markets with couples. Indeed, we first find a semi-stable matching, and then see if the semi-stable matching to be stable by verifying that each real couple does not form a blocking coalition. Of course, this is only a sufficient condition for stable matching, that is, if the semi-stable matching is not stable, we cannot assert that there does not exist any stable matchings.

The Conway lattice theorem in the literature shows that the set of all stable matchings forms a distributive lattice for a matching market of singletons with strict preferences.[4] Can we have a similar result of this nice properties for semi-stable matchings with couples? The answer is in the affirmative in some sense. In any matching market containing couples with strict preferences, there is a partition of the set of semi-stable matchings, each of which forms distributive lattices. To see this, define a partial ordering relation $\geq_C$ on matchings as follows. For any $c \in C$ and two matchings μ_1 and μ_2, $\mu_1 \geq_c \mu_2$ if and only if $\mu_1(c) \succ_c \mu_2(c)$ or $\mu_1(c) = \mu_2(c)$. It is easily seen that $\geq_C$ is a partial ordering relation, i.e., it is irreflexive, anti-symmetric and transitive.

Consider a matching market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ with couples. Let F be the set of all semi-stable matchings. Define operators $\vee_C$ and $\wedge_C$ as follows: for any $c \in C$ and

[4]The theorem is seen in Knuth (1976) and Roth and Sotomayor (1990), but it belongs to John Conway.

$\mu_1, \mu_2 \in F$, let $\lambda = \mu_1 \vee_C \mu_2$ and $\nu = \mu_1 \wedge_C \mu_2$ where $\lambda(c) = \max_{\succ c}\{\mu_1(c), \mu_2(c)\}$, $\nu(c) = \min_{\succ c}\{\mu_1(c), \mu_2(c)\}$. We then have the following generalized lattice theorem.

Theorem 3.2 (Generalized Lattice Theorem) *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a matching market with couples and strict preferences. Then the set of all semi-stable matchings F can be partitioned into subsets, each of them forms distributive lattices for operators $\vee_C$ and $\wedge_C$.*

For a matching market without couples, the set of all semi-stable matchings is identical to the set of all stable matchings. As a corollary, the above theorem generalizes the Conway lattice theorem for marriage matching markets.

Corollary 3.1 (Conway Lattice Theorem) *If all preferences are strict, then the set of all stable matchings in marriage matching markets forms a lattice for partial ordering relation.*

3.2 Sufficient Condition for Existence of Stable Matchings

For a market of matching with couples $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, under what conditions, does there exist a stable matching? Klaus and Klijn (2005) provided an answer by introducing the notion of (weakly) responsive preference[5], and showed that relative personal preferences can be induced from the preference of couple when it is (weakly) responsive, and these personal preferences induced must be unique. In such situations, the stable matchings found by the Gale-Shapley algorithm are also stable in couple context.

However, (weakly) responsive preference implies that there is no complementarity for preferences of couple, but in real world, they are generally complementary. For example, although for an individual s, $h_p \succ_s h_r$, yet for couple $c = (s, s')$, $(h_r, h_q) \succ_c (h_p, h_q)$, as h_q and h_r are in Boston whereas h_p is in New York. Thus, the preference of the couple c is not (weakly) responsive. If so, there may not exist any stable outcomes even in markets containing only one couple.

In this subsection, we provide a sufficient condition for the existence of stable matching even in the presence of complementary preferences of couples. To do so, we first introduce the following notions.

[5]The preference of couple $c = (s, s') \in C$ is (weakly) responsive if there exist single preferences $\succ_s$ and $\succ_{s'}$, such that: 1) for all $h \in H \cup \{\phi\}$, $(h, \phi) \succ_c (\phi, \phi)$ if and only if $h \succ_s \phi$; $(\phi, h) \succ_c (\phi, \phi)$ if and only if $h \succ_{s'} \phi$; and 2) for all $h_p, h_q, h_r \in H \cup \{\phi\}$, if $h_q \succ_{s'} \phi$, $h_p \succ_s h_r \succ_s \phi$, then $(h_p, h_q) \succ_c (h_r, h_q)$; if $h_q \succ_s \phi$, $h_p \succ_{s'} h_r \succ_{s'} \phi$, then $(h_q, h_p) \succ_c (h_q, h_r)$.

Definition 3.1 (Reservation Preference) Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples. A pair of jobs (h, h') is said to be a couple $c = (s, s')$'s *reservation preference* if (i) $(h, h') \succ_c (\phi, \phi)$, (ii) whenever $h \neq \phi$, $s \succ_h \widetilde{s}$ for all $\widetilde{s} \in P^h \backslash \{s\}$, and (iii) whenever $h' \neq \phi$, $s' \succ_{h'} \widetilde{s'}$ for all $\widetilde{s'} \in P^{h'} \backslash \{s'\}$.

A reservation preference of a couple means that the couple can get a pair of jobs if they want, as the members of the couple are respectively the most preferred medical student for the relevant jobs of hospitals.

Definition 3.2 (Effective Preference) Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market and $c = (s, s')$ be a couple. A pair of jobs (h, h') is said to be c's *effective preference* if (i) $(h, h') \succ_c (\phi, \phi)$, (ii) $s \succ_h \phi$ whenever $h \neq \phi$, and (iii) $s' \succ_{h'} \phi$ whenever $h' \neq \phi$. Student s is said to be an *effective preference of* h if (i) $s \succ_h \phi$ and (ii) there exists $\overline{h} \in H \cup \{\phi\}$ such that $(h, \overline{h}) \succ_c (\phi, \phi)$. Student s' is h''s effective preference can be similarly defined.

If jobs in a couple c's preference can accept its corresponding members, then the couple's preference is an effective preference to the couple. If (h, h') is not an effective preference of c, then there does not exist any individually rational matching μ such that $\mu(c) = (h, h')$. Similarly, if s is not an effective preference of h, there does not exist any individually rational matching μ such that $\mu(h) = s$.

We then have the following theorem which shows that there must exist a stable matching if the first preference of each couple is one of its reservation preference.

Theorem 3.3 (Sufficient Condition I for the Existence of Stable Matchings) *Let* $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ *be a matching market with couples and strict preferences. Suppose that for all* $c \in C$ *with* $s' \neq \phi$*, the first preference in* P^c (h, h') *is a reservation preference of* c*. Then, there exists a stable matching* μ*.*

The reason why a reservation preference may be the first priority of a real couple is that their preferences have couple-complementarity, that is, although one wants some popular jobs, the pair of popular jobs may not be the most preferred for couples. Since the pair of popular jobs may not be at the same hospital or in the same city, the most preferred pair for a couple may not be popular jobs, but is its reservation preference. In the later subsection, we will give a generalized version of the theorem with more slack condition that reservation preference of real couples may not be their first preference.

3.3 Asymptotical Stability

If we regard stable matchings as orderly matchings whereas unstable matchings as unorderly matchings, then the degree of a unstable matching may be used to measure the unorderly degree of a matching. For a matching market $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, the most unorderly matching is the null matching μ^0, i.e., for any $c \in C$, $\mu^0(c) = (\phi, \phi)$, whose unstable degree is denoted by 1. The most orderly matchings are stable matchings. For any stable matching μ^1, its unstable degree is denoted by 0. For any unstable matching μ, its unstable degree is a real number between 0 and 1. The intuition is that the higher the unstable degree of a matching is, the more unorderly the matching is. Formally, we have

Definition 3.3 Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples. If the set C contains n elements, the rank list of preference P^{c_i} contains l_i effective preferences, the number of blocking coalitions of matching μ is m, then the unstable degree of μ is denoted by $\theta(\mu) = m/N$, where $N = \sum_{i=1}^{n} l_i$.

Definition 3.3 implies that unstable degree is a function from the set of all the matchings in matching market Γ onto the unit interval. The null matching μ^0 has $N = \sum_{i=1}^{n} l_i$ blocking coalitions[6] and each stable matching μ^1 has no blocking coalition, so $\theta(\mu^0) = 1$ and $\theta(\mu^1) = 0$. Thus, for any matching μ, if it has m blocking coalitions, obviously $0 \leq m \leq N$, and thus the degree of instability $\theta(\mu) \in [0, 1]$. Intuitively, the more blocking coalitions a matching has, the more unorderly it is. Thus, the lower the unstable degree of a matching is, the more orderly and stable it is.

Definition 3.4 Let $\{\Gamma^k\}_{k=1}^{\infty}$ be a sequence of matching markets with couples where $\Gamma^k = (H^k, S^k, C^k, (\succ_h)_{h\in H^k}, (\succ_c)_{c\in C^k})$, and let μ^k be a matching of Γ^k, $k = 1, 2, \cdots$. The matching sequence $\{\mu^k\}_{k=1}^{\infty}$ is said to be *asymptotically stable* if $\lim_{k\to\infty}\{\theta(\mu^k)\}_{k=1}^{\infty} = 0$. $\{\Gamma^k\}_{k=1}^{\infty}$ is said to be asymptotically stable if there exists *a matchings sequence* $\{\mu^k\}_{k=1}^{\infty}$ that is asymptotically stable.

Based on some common features of large matching markets in reality, Kojima and Pathak (2009) first presented the notion of regular markets. Kojima *et al.* (2013) then defined the regular markets for matchings with couples, and demonstrated that for a regular sequence of markets with couples, the probability that a stable matching exists converges to 1 as the market

[6]Each blocking coalition $\{c, (h, h')\}$ of the null matching μ^0 contains a couple and one of its effective preferences; conversely, for any $c \in C$ and any of its effective preferences (h, h'), $\{c, (h, h')\}$ must be a blocking coalition of the null matching μ^0. Thus, the null matching μ^0 has exactly $N = \sum_{i=1}^{n} l_i$ blocking coalitions.

size approaches infinity whereas the number of couples relative to the market size does not grow rapidly.

Here we introduce the notion of simple regular markets with couples. Consider a sequence of markets of different sizes. For a sequence of matching markets with couples, $\{\Gamma^k\}_{k=1}^{\infty}$ with $\Gamma^k = (H^k, S^k, C^k, (\succ_h)_{h \in H^k}, (\succ_c)_{c \in C^k})$, there are $|C^k| = n_k$, m_k real couples, and l_c effective preference in P^c.

Definition 3.5 A sequence of markets $\{\Gamma^k\}_{k=1}^{\infty}$ is said to be *simple regular* if it satisfies the following conditions:

(1) $m_k = o(n_k)$;

(2) There exists a number q such that for any $c \in C$ with $s' \neq \phi$, $l_c \leq q$;

(3) (Participation Restriction) For any $c \in C$, $l_c > 0$.

Condition (1) implies the fact that the number of real couples is small relative to the number of singletons. Condition (2) requires that the numbers of effective preferences of real couples is bounded by q. Condition (3) is actually a participation restriction. For any couple c, if $l_c = 0$, then for any individually rational matching μ, $\mu(c) = (\phi, \phi)$, so it will not participate in the matching market. In fact, provided the number of real couples that satisfy the participation restriction is small relative to that of singletons satisfying the participation restriction, the first and third conditions can be omitted. We then have the following theorem.

Theorem 3.4 (Asymptotic Stability) *Suppose that* $\{\Gamma^k\}_{k=1}^{\infty}$ *is a sequence of simple regular markets with couples and strict preferences where* $\Gamma^k = (H^k, S^k, C^k, (\succ_h)_{h \in H^k}, (\succ_c)_{c \in C^k})$. *Then there exists a matching sequence* $\{\mu^k\}_{k=1}^{\infty}$ *that is asymptotically stable when* n_k *tends to infinity, that is,* $\{\Gamma^k\}_{k=1}^{\infty}$ *is asymptotically stable.*

The theorem indicates that there almost always exist some stable matchings when the size of a simple regular market tends to infinity. As the simple regularity conditions are weaker than the regularity conditions proposed in Kojima *et al.* (2013), their result that the probability that a stable matching exists converges to 1 as the market size approaches infinity can be regarded as a special case of our asymptotic stability theorem.

3.4 NRMP Puzzle Revisit

In the past two decades, NRMP's practice has shown that the clearinghouse seldom fails to find a stable matching. Kojima *et al.* (2013) pointed out that it is a puzzle. In fact, the reason is that the NRMP's market has many special features, which are described as eight stylized facts by them. Here are the first four stylized facts.

Fact 1: Applicants who participate as couples constitute a small fraction of all participating applicants.

Fact 2: The length of single applicants' rank lists is small relative to the number of possible programs.

Fact 3: Applicants who participate as couples rank more programs than single applicants. However, the number of distinct programs ranked by a couple member is small relative to the number of possible programs.

Fact 4: The most popular programs are ranked as a top choice by a small number of applicants.

Kojima *et al.* (2013) pointed out that, in the data of NRMP during 1992-2009, applicants who participated as couples are on average 4.4% of all applicants, the length of single applicants' preference lists is on average about 7-9 programs, which is about 0.3% of the number of all possible programs, and the length of couple applicants' rank lists is about 81 on average.

Since a matching of which the unstable degree is zero must be stable, the asymptotic stability theorem indicates that there almost always exist a stable matching when the size of simple regular markets approaches infinity. Facts 1, 2 and 3 imply that the NRMP's practice satisfies the simple regularity conditions, so the asymptotic stability theorem is a good interpretation for the puzzle of NRMP.

Considering the stylized fact 4 of NRMP's market, most of popular jobs are placed at the top of their preference rank lists by only a small number of medical students, which contradicts the intuition that popular jobs ought to be preferred by most medical students. This indicates that the first preferences of most students are not their most preferred choices. If all real couples play reservation strategies, i.e., they place one of their reservation preferences at the top of their rank order list (ROL) of preferences, then there exists a stable matching by Theorem 3.3. In fact, real couples have more incentives to play reservation strategies than singletons do due to couple-complementarity, that is, although one wants some popular jobs, the pair of popular jobs may not be the most preferred for couples as they may not be at the same hospital or in the same city.

From the statistical data in Table 1, we can see that the fraction of rank order list where both members rank the same region, i.e., the preference having couple-complementarity, is 72.7%. It shows that the pair of jobs having couple-complementarity surely provides the couple with extra welfare, so it gives couples more incentives than singletons to play reservation strategies. Real couples play reservation strategies, which coincides with Fact 4 that the most popular programs are ranked as a top choice by a small number of applicants. This actually shows that not only most real couples but also most singletons play reservation strategies.

	Total	Mean length for rank-order list (ROLL)	Geographic similarity for preference
single doctors	3010	7.6	
♯ Regions ranked			2.5
couples			
♯ Regions ranked	19	81.2	4%
Fraction of ROLL where with members Rank same Region			72.7%

Table 1: Summary statics of Psychology Labor Market

Notes: The data are from Kojima *et al.* (2013). This table reports descriptive information from the Association of Psychology Postdoctoral and Internship Centers match, averaged over 1999-2007. Single doctors' rank order lists consist of a ranking over hospital jobs, while couples' indicate rankings over pairs of hospital jobs.

Mathcing Market	Doctor Type	Doctor's Choice Received					
		1st	2nd	3rd	4th	5th	unassigned
without couples	single	36.8%	16.9%	10.1%	6.4%	11.2%	18.9%
with couples	single	36.0%	16.6%	10.1%	6.2%	11.6%	19.5%
couple		18.0%	10.6%	8.7%	5.1%	52.5%	5.2%

Table 2: Summary statics of Psychology Labor Market

Notes: The data are from Kojima *et al.* (2013). This table reports the choice received in the doctor-optimal stable matching in a market with single doctors and without couples versus a stable matching in the market with couples in the Association of Psychology Postdoctoral and Internship Centers match, averaged over 1999-2007. A doctor is counted as unassigned even if being unassigned is among her top five choices.

Intuitively, if the first preference of a couple or singleton is successfully matched, then the preference may be seen as one of its reservation preferences. The data in Table 2 show that the fractions of the first preference of singletons that is successfully matched are respectively 36.8% and 36.0% in markets without couples and with couples. If the second preference of singletons that is successfully matched is also seen as their reservation preference, then the fractions of singletons that play reservation strategies are respectively 53.7% and 52.6%. Although the fraction of the first and second preferences[7] of couples that are successfully matched is only 28.6% and it is 42.4% plus the third and fourth preferences, as couples have more incentives than singletons to play reservation strategies, we can conclude that there is a larger fraction of couples than 53.7% to place their reservation preferences at the top of their rank order list. It can be partly explained by the fractions of couples and singletons unassigned. On average, about 19% of all singletons are unassigned but it is 5.2% for couples, that is, about 81% of all singletons and 94.8% of all couples are assigned. It may be interpreted as a larger fraction of couples than singletons play reservation strategies.[8]

Theorem 3.3 indicates that if all real couples play reservation strategies, then there must exist a stable matching. In fact, the condition of Theorem 3.3 can be weakened. Provided the preferences in front of real couples' first reservation preference are not pairs of popular jobs, then there exists a stable matching. Since the number of couples is very small relative to the number of singletons, we may consider that all popular jobs are assigned to singletons. The data in Table 1 show that the number of singletons is 3010 whereas the number of couples is 19, and the fraction of singletons whose first preferences are successfully assigned is over 36% (which means more than 1000 jobs, and we may consider that almost all popular jobs are among the 1000 jobs). The following theorem shows that there exist a stable matching in such markets. It is another strong interpretation for the puzzle of NRMP.

Theorem 3.5 (Sufficient Condition II for the Existence of Stable Matching) *Let* $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ *be a matching market with strict preferences. Suppose that*

(1) for any $c \in C$ *with* $s' \neq \phi$*, the first reservation preference in* P^c *is* (h, h')*;*

(2) for any preference $(\overline{h}, \overline{h'})$ *before* (h, h')*,* $(\overline{h}, \phi) \neq (\phi, \phi)$ *or* $(\overline{h'}, \phi) \neq (\phi, \phi)$ *implies that it is not only a reservation preference item but also the first preference of a singleton.*

[7] Since the preference of couples is a pair of jobs, two jobs can constitute two preferences, such as (a,b) and (b,a).

[8] Of course, it also has another interpretation that, originally, jobs a and b are not to be accepted by each member of a couple, yet both jobs are at the same hospital or in the same city, so as a couple they may accept the pairs of jobs (a,b) or (b,a). Thus, the probability that the couple is assigned is increased.

Then, there exists a stable matching.

It can be easily seen that, when the conditions of theorem 3.3 are met, the conditions of theorem 3.5 must also be satisfied. Thus, theorem 3.5 is a generalization of theorem 3.3.

4 Persistent Improvement Algorithm (PI-Algorithm)

Hatfield and Milgrom (2005) presented the generalized Gale-Shapley algorithm for matching with contracts. Ostrovsky (2008) studied the more generalized problem about supply chain networks with same-side substitutability and cross-side complementarity. He presented the T-Algorithm that generalizes the result of Hatfield and Milgrom (2005) and also the Gale-Shapley algorithm for one-to-one matching. However, the problem of matching with couples is different from that of cross-side complementarity where the agents of same-couple are complements. As such, these algorithms cannot be applied to matching markets with couples. This section provides a new algorithm, called PI-Algorithm, which fits the matching markets with couples. The PI-Algorithm not only generalizes but is also more efficient than the Gale-Shapley algorithm, by which we can find a semi-stable matching according to the steps described in the proof of Theorem 3.1.

4.1 PI-Algorithm

Given a matching market with couples $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, let P^h and P^c be h's and c's preferences rank list. Following Hatfield and Milgrom (2005), we denote the space of contracts $X = S \times H$. A contract $x = (x_S, x_H) \in X$ denotes a pair of matching between medical student x_S and job x_H, and a medical student can sign only one contract with any given jobs of hospitals. Thus, each contract is a binary relation so that it is associated with one medical student and one job. The running steps of PI-Algorithm are a sequential process in which contracts are chosen by jobs of hospitals and medical students.

Given a set of contracts $X' \subsetneqq X$, if there does not exist any acceptable contract for h, h's choice set is empty; otherwise, it just contains h's most preferred contract. For students' choices, it is more complicated. The couple $c = (s, s')$ makes an optimal choice by its preferences rank list in which it may choose two contracts simultaneously, one contract or no contract. If c does not choose any contract, the choice sets of s and s' are empty; if c only chooses one contract (s, h), s's choice set is $\{(s, h)\}$ and s''s choice set is empty; if c chooses two contracts (s, h) and (s', h'), s's choice set is $\{(s, h)\}$ and s''s choice set is $\{(s', h')\}$. Formally, the choice mappings

$Ch_h(\cdot)$ and $Ch_s(\cdot)$ of jobs in hospitals and medical students are defined as follows:

$$Ch_h(X') = \begin{cases} \emptyset & \text{if } \{(x'_S, h) \in X' : x'_S \succ_h \phi\} = \emptyset \\ \{(x_S, h)\} & \text{otherwise} \end{cases}$$

where $x_S = \max \succ_h \{x'_S : (x'_S, h) \in X'\}$.

For $c = (s, \phi), s \in S$,

$$Ch_s(X') = \begin{cases} \emptyset & \text{if } \{(s, x'_H) \in X' : (x'_H, \phi) \succ_c (\phi, \phi)\} = \emptyset \\ \{(s, x_H)\} & \text{otherwise} \end{cases}$$

where $(x_H, \phi) = \max \succ_c \{(x'_H, \phi) : (s, x'_H) \in X'\}$.

For $c = (s, s')$ with $s, s' \in S$,

$$Ch_s(X') = \begin{cases} \emptyset & \text{if } \overline{h} = \phi \\ \{(s, \overline{h})\} & \text{otherwise} \end{cases}$$

and

$$Ch_{s'}(X') = \begin{cases} \emptyset & \text{if } \overline{h'} = \phi \\ \{(s', \overline{h'})\} & \text{otherwise} \end{cases}$$

where $(\overline{h}, \overline{h'}) = \max \succ_c \{(h, h') : h = \phi \textit{ or } (s, h) \in X', h' = \phi \textit{ or } (s', h') \in X'\}$.

Denote by $Ch_S(X') = \cup_{s \in S} Ch_s(X')$ the choice set for all medical students and $Ch_H(X') = \cup_{h \in H} Ch_h(X')$ the choice set for all hospital jobs .

PI-Algorithm starts from the initial matching μ_0 at which matched objects of all the medical students are ϕ. After running each round, a new matching μ_t is created, which is a Pareto improvement on μ_{t-1} for all couples, i.e., all couples c weakly prefer $\mu_t(c)$ to $\mu_{t-1}(c)$ with at least one strictly preferring $\mu_t(c)$. PI-Algorithm ends if there is no further Pareto improvement for all couples.

In round 0 of PI-Algorithm, it produces preferences rank list $P^c(0)$ of each couple c through the method that all such items (h, h') of P^c will be removed whenever s or s' is unacceptable to job h or job h' respectively. PI-Algorithm consists of repeated rounds of calculation. There are four steps in each round except round 0. Step 1 determines preferences rank list $P^c(t)$ of each couple c where all the items of $P^c(t-1)$ behind $\mu_{t-1}(c)$ will be removed. Step 2 determines the set of contracts $X(t)$ submitted by medical students. Step 3 determines the choice set $Ch_H(X(t))$ for all of the hospital jobs. Step 4 determines the choice set $Ch_S(Ch_H(X(t)))$ for all medical students. All of the contracts in the choice set $Ch_S(Ch_H(X(t)))$ form a matching μ_t in round t. Running round-by-round calculation, when $Ch_S(Ch_H(X(t))) = Ch_S(Ch_H(X(t-1)))$, PI-Algorithm ends and all of the contracts in the choice set $Ch_S(Ch_H(X(t)))$ form the last matching μ_E. Formally, we have

Round 0, for all $c \in C$, $P^c(0) = P^c \setminus \{(h, h') \in P^c : s \notin P^h \ or \ s' \notin P^{h'}\}$, $t = 1$.

Round t, for all $c \in C$,

Step 1: $P^c(t) = P^c(t-1) \setminus \{(h, h') \in P^c(t-1) : \mu_{t-1}(c) \succ_c (h, h')\}$;

Step 2: $X(t) = X_1(t) \cup X_2(t)$ where

$X_1(t) = \cup_{c \in C}\{(s, h) \in X : h \in H, \ \text{exist} \ \ h' \in (H \cup \{\phi\}) \ \text{such that} \ \ (h, h') \in P^c(t)\}$,

$X_2(t) = \cup_{c \in C}\{(s', h') \in X : h' \in H, \ \text{exist} \ \ h \in (H \cup \{\phi\}) \ \text{such that} \ \ (h, h') \in P^c(t)\}$;

Step 3: $Ch_H(X(t))$;

Step 4: $Ch_S(Ch_H(X(t)))$, if $Ch_S(Ch_H(X(t))) = Ch_S(Ch_H(X(t-1)))$, then end, else $t = t + 1$ and goto Round t.

We will illustrate these steps of PI-Algorithm by the following example.

4.2 PI-Algorithm: An Example

The running procedures of PI-Algorithm for Example 1 are specified as in Table 3.

Example 1: $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$ are couples, $c_3 = (s_5, \phi)$ and $c_4 = (s_6, \phi)$ are singletons. There are five hospital jobs h_1, h_2, h_3, h_4 and h_5. Their preferences rank lists are as follows:

$c_1 : \{(h_1, h_2), (h_3, h_4), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_4), (\phi, \phi)\}$;

$c_2 : \{(h_1, h_2), (h_3, h_5), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_5), (\phi, \phi)\}$;

$c_3 : \{(h_1, \phi), (h_2, \phi), (h_3, \phi), (h_5, \phi), (\phi, \phi)\}$;

$c_4 : \{(h_1, \phi), (h_2, \phi), (h_3, \phi), (h_4, \phi), (\phi, \phi)\}$;

$h_1 : \{s_1, s_3, s_5, \phi\}$; $h_2 : \{s_2, s_4, s_6, \phi\}$; $h_3 : \{s_1, s_3, s_5, \phi\}$;

$h_4 : \{s_2, s_4, s_6, \phi\}$; $h_5 : \{s_2, s_4, s_5, \phi\}$.

In round 0, it removes all the items of preferences rank lists of all couples that cannot be acceptable to hospital jobs. After round 1 and round 2, it has actually produced the last matching. Round 3 repeats round 2 and thus PI-Algorithm ends. We obtain the following matching.

$\mu(c_1) = (h_1, h_2)$, $\mu(c_2) = (h_3, h_5)$, $\mu(c_3) = (\phi, \phi)$, $\mu(c_4) = (h_4, \phi)$.

We can easily verify the matching obtained is stable. The matching object of c_1 is the most preferred and therefore there do not exist any blocking coalition containing c_1. As h_1 and h_2 also obtain their most preferred objects, there do not exist any blocking coalitions containing h_1 or h_2. Thus, possible blocking coalitions must contain $\{s_1, h_3\}$, $\{s_2, h_4\}$, $\{s_4, h_4\}$ or $\{s_2, h_5\}$. However, such blocking coalition does not exist because there is no participation incentive for c_1 and c_2. Hence there does not exist any blocking coalition.

Round 0		$P^{c_1}(0)=\{(h_1,h_2),(h_3,h_4),(h_1,\phi),(h_3,\phi),(\phi,h_2),(\phi,h_4),(\phi,\phi)\}$; $P^{c_2}(0)=\{(h_1,h_2),(h_3,h_5),(h_1,\phi),(h_3,\phi),(\phi,h_2),(\phi,h_5),(\phi,\phi)\}$; $P^{c_3}(0)=\{(h_1,\phi),(h_3,\phi),(h_5,\phi),(\phi,\phi)\}$; $P^{c_4}(0)=\{(h_2,\phi),(h_4,\phi),(\phi,\phi)\}$.
Round 1	step 1	$P^{c_1}(1)=P^{c_1}(0),P^{c_2}(1)=P^{c_2}(0),P^{c_3}(1)=P^{c_3}(0),P^{c_4}(1)=P^{c_4}(0)$
	step 2	$X(1)=\{(s_1,h_1),(s_1,h_3),(s_2,h_2),(s_2,h_4),(s_3,h_1),(s_3,h_3),$ $(s_4,h_2),(s_4,h_5),(s_5,h_1),(s_5,h_3),(s_5,h_5),(s_6,h_2),(s_6,h_4)\}$
	step 3	$Ch_H(X(1))=\{(s_1,h_1),(s_2,h_2),(s_1,h_3),(s_2,h_4),(s_4,h_5)\}$
	step 4	$Ch_s(Ch_H(X(1)))=\{(s_1,h_1),(s_2,h_2),(s_4,h_5)\}$
Round 2	step 1	$P^{c_1}(2)=\{(h_1,h_2)\},P^{c_2}(2)=P^{c_2}(1)\backslash\{\phi,\phi\},P^{c_3}(2)=P^{c_3}(1),P^{c_4}(2)=P^{c_4}(1)$
	step 2	$X(2)=\{(s_1,h_1),(s_2,h_2),(s_3,h_1),(s_3,h_3),(s_4,h_2),(s_4,h_5),(s_5,h_1),$ $(s_5,h_3),(s_5,h_5),(s_6,h_2),(s_6,h_4)\}$
	step 3	$Ch_H(X(2))=\{(s_1,h_1),(s_2,h_2),(s_3,h_3),(s_6,h_4),(s_4,h_5)\}$
	step 4	$Ch_S(Ch_H(X(2)))=\{(s_1,h_1),(s_2,h_2),(s_3,h_3),(s_6,h_4),(s_4,h_5)\}$
Round 3	step 1	$P^{c_1}(3)=\{(h_1,h_2)\}$, $P^{c_2}(3)=\{(h_1,h_2),(h_3,h_5)\}$, $P^{c_3}(3)=P^{c_3}(2)$, $P^{c_4}(3)=\{(h_2,\phi),(h_4,\phi)\}$;
	step 2	$X(3)=\{(s_1,h_1),(s_2,h_2),(s_3,h_1),(s_3,h_3),(s_4,h_2),(s_4,h_5),(s_5,h_1),$ $(s_5,h_3),(s_5,h_5),(s_6,h_2),(s_6,h_4)\}$
	step 3	$Ch_H(X(3))=\{(s_1,h_1),(s_2,h_2),(s_3,h_3),(s_6,h_4),(s_4,h_5)\}$
	step 4	$Ch_S(Ch_H(X(3)))=\{(s_1,h_1),(s_2,h_2),(s_3,h_3),(s_6,h_4),(s_4,h_5)\}$ Since $Ch_S(Ch_H(X(3)))=Ch_S(Ch_H(X(2)))$, END.

Table 3: PI-Algorithm Running Procedures

4.3 Properties of PI-Algorithm

Suppose PI-Algorithm ends in round T. Denote the matching produced in round k by μ_k and the last matching by μ_E. Then PI-Algorithm implies the following two lemmas.

Lemma 4.1 $\mu_{T-1}=\mu_T=\mu_E$*;* $P^c(t+1)\subsetneqq P^c(t)$*,* $X(t+1)\subsetneqq X(t)$ *for any* $c\in C$*,* $0<t<T$*.*

Lemma 4.2 $\mu_t(c)\succ_c\mu_{t-1}(c)$ *or* $\mu_t(c)=\mu_{t-1}(c)$ *for all* $c\in C$*, and* $\mu_t(c)\succ_c\mu_{t-1}(c)$ *for some* $c\in C$*,* $0<t<T$*.*

Lemma 4.2 implies that PI-Algorithm brings a Pareto improvement in each round except round 0 and the last round. In addition, Lemma 4.2 implies that PI-Algorithm must end in finite rounds. Suppose there exist n couples and l_i+1 preferences in couple c_i's preferences list

P^{c_i}. Since at least one couple gets strict improvement in each round except round 0 and the last round, PI-Algorithm ends at most $T = \sum_{i=1}^{n} l_i + 1$ rounds. We then have the following theorem.

Theorem 4.1 *For any matching market with couples and strict preferences* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$*, the matching* μ_E *obtained by running PI-Algorithm is a stable matching for all* $c \in C$ *and* $h \in H$ *provided* $\mu_E(h) \neq \phi$.

The theorem implies that PI-Algorithm converges to a matching μ_E that is a stable matching for a subset $(C \cup \overline{H})$ of $(C \cup H)$ with $\mu_E(\overline{h}) \neq \phi$ for all $\overline{h} \in \overline{H}$. In fact, the matching μ_N found by the NRMP's present algorithm[9] is also a stable matching for a subset $(\overline{C} \cup H)$ of $(C \cup H)$ with $\mu_N(\overline{c}) \neq (\phi, \phi)$ for all $\overline{c} \in \overline{C}$.[10] However, the NRMP's present algorithm does not necessarily converge, which will encounter an infinite loop when no stable matching exists. Even so, we cannot assert that there does not exist any stable matchings by this argument because infinite loop may also occur when there exist some stable matchings. Compared with the NRMP's present algorithm, PI-Algorithm must end after finite rounds.

Theorem 4.2 *For any matching market with couples and strict preferences* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$*, the matching* μ_E *obtained by running PI-Algorithm is a stable matching when the market contains no real couples, i.e., for any* $c \in C$*,* $s' = \phi$.

The theorem indicates that PI-Algorithm also finds a stable matching when a matching market only contains single medical students. It simplifies the matching process of the Gale-Shapley algorithm and thus is more efficient. To compare the two algorithms, we first briefly state the running process of the Gale-Shapley algorithm in Gale and Shapley (1962).

The running procedures of the Gale-Shapley algorithm that jobs first propose to medical students are as follows: each job h proposes to medical students of its preference list starting from its first choice (if it has some acceptable choices); each medical student "holds" the most preferred job offer and rejects all others; any job rejected at some steps makes a new proposal

[9]See Roth and Peranson (1999) for detailed description of the algorithm. In order to avoid the infinite loop that may occur, Kojima *et al.* (2013) presented the sequential couples algorithm similar to the Roth-Peranson algorithm, which are slightly different in two aspects. Firstly, where the sequential couples algorithm fails, the Roth-Peranson algorithm proceeds and tries to find a stable matching. Secondly, in the Roth-Peranson algorithm, when a couple is added to the market with single doctors, any single doctor who is displaced by the couple is placed before another couple is added. By contrast, the sequential couples algorithm holds any displaced single doctor without letting her apply, until it processes applications by all couples.

[10]When the NRMP's present algorithm stops at an infinite loop, or when the sequential couples algorithm ends, the matching obtained is stable if not considering the unmatched medical students. See Roth and Peranson (1999) and Kojima *et al.* (2013) for details.

by sequential order to the next preferred medical student who has not yet rejected it; when no further proposals are made, the job finally accepted by medical students (if any) forms the last matching.

As for PI-algorithm, in round 1, each medical student proposes to all of his or her acceptable choices, and each job chooses its most preferred contract and sends back to the student. The result is identical to that each job selects its most preferred student from its preference rank list because its most preferred student must have proposed to it.[11] As such, each medical student's choice by the two algorithms in round 1 is perfectly identical. After round 1, PI-Algorithm is varied from the Gale-Shapley algorithm, as medical students do not propose to those jobs to which they prefer their current matched objects. The procedure is that each job proposes to medical students from more preferred to less preferred ones, but not in strict sequential order. In other words, it will skip those medical students who will reject its proposals. Compared with the Gale-Shapley algorithm, PI-Algorithm obviously accelerates the matching process and improves the efficiency of algorithm.

For a matching market without couples, similar to the Gale-Shapley algorithm, PI-Algorithm may not only begin with proposals by medical students, but also begin with proposals by jobs. That is, PI-Algorithm can begin from the set S to the set H, or similarly from the set H to the set S, and obtain stable matchings μ_S^E and μ_H^E. The following theorem shows that they are respectively identical to μ_H and μ_S that are obtained by Gale-Shapley algorithm. Thus, PI-Algorithm can be seen as the generalization of Gale-Shapley algorithm which also fits the matching market with couples.

Theorem 4.3 *For any matching market containing only singletons with strict preferences* $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, *the matchings* μ_S^E *and* μ_H^E *obtained by running PI-Algorithm are respectively identical to* μ_H *and* μ_S *obtained by Gale-Shapley algorithm.*

5 Market Design

In the practice of matching markets, stable matching mechanisms play an important role. However, in some special markets, theoretically there may not exist any stable matching mechanisms, such as roommate allocation problem (Gale and Shapley,1962) and matching with couples, etc. For matching markets with couples, if we employ semi-stable matching mechanisms, The-

[11] Assume that each preference of hospital jobs is acceptable to couples; otherwise, this preference item cannot be considered effective, which does not affect any individually rational matching and can be deleted from the preference lists of hospital jobs.

orem 3.1 guarantees the existence of semi-stable matching mechanisms, and PI-Algorithm also ensures that semi-stable matching mechanisms are computationally feasible.

Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples, and let $Q = \{P^c_1, P^c_2, ..., P^c_m, P^h_1, P^h_2, ..., P^h_n\}$ be the set of stated preference lists, one for each couple and hospital job, where each P^c and P^h are couple's and job's preferences.

Definition 5.1 A matching mechanism induced by the matching market Γ is a function h whose range is the set of all possible inputs (C, H, Q), and whose output $h(Q)$ is a matching between C and H. If $h(Q)$ is always stable with respect to Q, it will be called a stable matching mechanism; If $h(Q)$ is always semi-stable with respect to Q, it will be called a semi-stable matching mechanism.[12]

For any matching market without couples, stable mechanisms have the following properties: 1) at every stable matching, the set of unassigned agents is the same (cf. McVitie and Wilson (1970)); 2) there exist weakly Pareto efficient stable matchings for one side of agents (cf. Roth (1982a)); 3) stable mechanisms in general are not strategy-proof (cf. Dubins and Freedman (1981), Roth (1982, 1982a, 1985), Sonmez (1997), Martinez *et al.* (2004), Abdulkadiroglu (2005), Hatfield and Milgrom (2005), Klaus and Klijn (2005)). For matching markets with couples, in general, there does not exist a stable matching mechanism, but Theorem 3.1 guarantees the existence of a semi-stable matching mechanism.

Let μ^E denote a semi-stable matching obtained by PI-Algorithm following the steps described in the proof of Theorem 3.1, then when $h(Q) = \mu^E$, Theorem 3.1 ensures that the mechanism is a semi-stable matching revelation mechanism, which is called PI-Algorithm mechanism. For a matching market $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, let F be the set of all semi-stable matchings. Define a correspondence $K : F \to 2^F$ by $K(\mu) = \{\nu : \nu(c) = \mu(c)$ for all $c = (s, s') \in C$ with $s' \neq \phi\}$, i.e., the matched objects to real couples are the same in every matching of $K(\mu)$.

For marriage matching markets with strict preferences, McVitie and Wilson (1970) shows that the set of unmatched men and women is the same at every stable matching. The following theorem generalizes the result of McVitie and Wilson (1970), which shows that for any matching market with couples, the subset of semi-stable matchings $K(\mu^E)$ coincides with the set of all stable matchings.

Theorem 5.1 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences. Suppose that μ is a semi-stable matching of Γ. Then, the set of unmatched medical*

[12] This definition follows Roth and Sotomayor (1990). Mechanisms in which players must state their preferences are called revelation mechanisms, and the mechanism h is called a revelation mechanism in the literature.

students and hospital jobs is the same in every semi-stable matching of $K(\mu)$.

Theorem 5.2 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences. Then, the semi-stable matching μ^E obtained by PI-Algorithm mechanism is weakly Pareto efficient on $K(\mu^E)$ for the side of hospital jobs.*

Theorem 5.2 implies that we have an optimal result for the side of hospital jobs in matching markets with couples. It then can be regarded as a generalization of the optimal theorem on marriage matching markets in Roth (1982a). Also, by Theorem 4.3, for a matching market without couples, $K(\mu^E)$ coincides with the set of stable matchings, and thus we have the following corollary.

Corollary 5.1 *For matching markets containing singles only with strict preferences $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, the stable matching μ^E obtained by running PI-Algorithm is weakly Pareto efficient for the side of jobs of hospitals.*

While Theorem 5.2 shows that the PI-Algorithm mechanism results in weak Pareto optimality with respect to hospital jobs, the following theorem, however, shows that it is not strategy-proof.

Theorem 5.3 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences. Then PI-Algorithm mechanism is not strategy-proof.*

For any matching market with strict preferences $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, Theorem 3.2 shows that the set of semi-stable matchings forms a partition[13], that is, $F = \cup_{r=1}^{m} K(\mu_r)$, where F is the set of all semi-stable matchings and $K(\mu_i) \cap K(\mu_j) = \Phi$, $1 \leq i \leq j \leq m$. Without loss of generality, assume that semi-stable matching μ_r is the optimal matching in $K(\mu_r)$ for the side of hospital jobs. Although Theorem 5.3 is a negative result on strategy-proof of the PI-Algorithm mechanism in the whole domain, the following theorem is relatively positive, which shows that the PI-Algorithm mechanism is strategy-proof on every distributive lattice.

Theorem 5.4 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences. Suppose that the PI-Algorithm mechanism is restricted in $K(\mu_r)$. Then it is a dominant strategy for every job of hospitals to state its ture preferences on $K(\mu_r)$.*

Since the matching market without couples is a special case of the matching market with couples, Theorem 5.4 generalizes the result in (Dubins and Freedman,1981; Roth,1982a).

[13]For details, see the proof of Theorem 3.2 in Appendix.

In PI-Algorithm mechanism, the outcome will be a random one among $\mu_1, \mu_2, \cdots, \mu_m$. Example 5.1 below is an instance that the outcome is completely random. For hospital jobs, though their welfare may be different among $\mu_1, \mu_2, \cdots, \mu_m$, they cannot anticipate which μ_r will be obtained by the algorithm. Therefore, every hospital job still has incentive to tell the truth in PI-Algorithm mechanism. The reason is that every μ_r is the optimal semi-stable matching in $K(\mu_r)$ for hospital jobs, and thus, when the outcomes of mechanism are restricted in $K(\mu_r)$, truth-telling is a dominant strategy for every hospital job.

Example 5.1 Consider a matching market with two couples $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$, one singleton $c_3 = (s_5, \phi)$, and four jobs of hospitals. Their rank lists of preferences are as follows:

$P^{c_1} : \{(h_1, h_2), (h_3, h_1), (h_3, \phi), (\phi, \phi)\}$; $P^{c_2} : \{(h_1, h_2), (\phi, h_4), (\phi, \phi)\}$; $P^{c_3} : \{(h_1, \phi), (h_2, \phi), (\phi, \phi)\}$;

$P^{h_1} : \{s_3, s_2, s_5, \phi\}$; $P^{h_2} : \{s_2, s_4, s_5, \phi\}$; $P^{h_3} : \{s_1, \phi\}$; $P^{h_4} : \{s_4, \phi\}$.

By PI-Algorithm mechanism, after processing PI-A1, we get a matching μ is as follows:

$\mu(c_1) = (h_3, \phi), \mu(c_2) = (\phi, h_4), \mu(c_3) = (\phi, \phi)$.

In the process of PI-A2, if the blocking coalition $\{c_3, (h_1, \phi)\}$ is stochastically obtained first, we finally get the semi-stable matching μ_1 with $\mu_1(c_1) = (h_3, \phi), \mu_1(c_2) = (\phi, h_4), \mu_1(c_3) = (h_1, \phi)$.

In the process of PI-A2, if the blocking coalition $\{c_3, (h_2, \phi)\}$ is stochastically obtained first, we finally get the semi-stable matching μ_2 with $\mu_2(c_1) = (h_3, h_1), \mu_2(c_2) = (\phi, h_4), \mu_2(c_3) = (h_2, \phi)$.

In the two semi-stable matching μ_1 and μ_2, for couple c_1 , singleton c_2 , job h_1 and job h_2, their welfare is changed.

6 Conclusion

This paper studies the problem of matching with couples, which can be seen as an instance of problems with same-side complementarity. One of the typical characteristics of such problems is that stable matching may not exist. To overcome this defect, we introduce the notion of semi-stable matching and considers it as a generalized solution, which is a natural generalization of, and identical to, the conventional stability for matching without couples. It is shown that there always exists a semi-stable matchings for matching markets with couples and strict preferences, and futher the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice. When the matching market with couples is specialized as the single market, semi-stable matchings of the market become stable matchings. We also provide sufficient conditions for the existence of stable outcomes. If the first preference of all real couples is exactly one of their reservation preferences, then there exist some stable matchings.

We also discuss the puzzle of NRMP introduced by Kojima *et al.* (2013). For a matching market with couples, if all couples play reservation strategies, i.e., they place their reservation preferences at the top of their rank order list of preferences, which is consistent with the stylized fact 4 of NRMP's market described by Kojima *et al.* (2013), then the semi-stable matching obtained by PI-Algorithm is a stable matching. We introduce the notion of simple regularity market, which simplifies the regularity market presented by Kojima *et al.* (2013). The stylized facts imply that NRMP's market satisfies the conditions of simple regularity market. For a sequence of simple regularity market with couples, when the size of market tends to infinity, the sequence of semi-stable matchings found by PI-Algorithm is asymptotically stable. This gives a new interpretation for the puzzle of NRMP from another perspective.

Another remarkable contribuation of this paper is that we have provided a uniform algorithm, called PI-Algorithm for matching with couples, which ensures to find a semi-stable matching and a stable matching on a subset of the domain that exists. Moreover, if a matching is not a semi-stable matching during the process, PI-Algorithm goes on processing by canceling some items from some couples' rank list of preferences until it converges to a semi-stable matching. This approach ensures to find a semi-stable matching in matching markets with couples for strict preferences.

Moreover, this paper studies the welfare property and incentive issues of PI-Algorithm mechanism from the perspective of market design. For a matching market without couples, the result obtained by PI-Algorithm mechanism is the same with that of Gale-Shapley algorithm mechanism, while PI-Algorithm is more efficient. For a matching market with couples, the semi-stable matching μ^E obtained by PI-Algorithm mechanism is the optimal semi-stable matching in a subset $K(\mu^E)$ of the set of all semi-stable matchings for the side of hospital jobs.

This paper also motivates further topics for research. For instance, what are the necessary and sufficient conditions for the semi-stable matching obtained by PI-Algorithm mechanism to be stable? In what matching mechanisms will players have incentives to play reservation strategies? Another important future research is to study similar issues for matching markets with couples and weak preferences.

Appendix: proofs

Proof of Theorem 3.1. We prove the theorem by finding a semi-stable matching through PI-Algorithm. To do so, we first prove the following two lemmas.

Lemma 6.1 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences, and let the matching obtained by PI-Algorithm be μ_E. Suppose that PI-Algorithm ends in round T and $\{(s, s'), (h, h')\}$ is a blocking coalition of μ_E. Then, we have (1) whenever $h \neq \phi$ and $\mu_E(h) \neq s$, $\mu_E(h) = \phi$; and (2) whenever $h' \neq \phi$ and $\mu_E(h') \neq s'$, $\mu_E(h') = \phi$.*

Proof: We only show statement (1), the proof of (2) is similar. Since PI-Algorithm ends in round T, we have $\mu_{T-1} = \mu_T = \mu_E$ by Lemma 4.1. Suppose, by way of contradiction, that $\mu_E(h) \neq \phi$ for $h \neq \phi$ and $\mu_E(h) \neq s$. There are two cases to be considered.

Case 1: $h' = \phi$. Since $\{(s, s'), (h, \phi)\}$ is a blocking coalition of μ_E, we have $s \succ_h \mu_E(h)$ and $(h, \phi) \succ_c \mu_E(c)$. Also, since $\mu_{T-1} = \mu_T = \mu_E$, we have $s \succ_h \mu_{T-1}(h)$ and $(h, \phi) \succ_c \mu_{T-1}(c)$. Thus $(h, \phi) \in P^c(T)$, and so $(s, h) \in X(T)$. We then have $(s, h) \notin Ch_H(X(T))$. Indeed, if $(s, h) \in Ch_H(X(T))$, then $\mu_T(c) \succ_c (h, \phi)$ or $\mu_T(c) = (h, \phi)$, which contradicts the fact that $(h, \phi) \succ_c \mu_E(c) = \mu_T(c)$. Therefore, there exists $\overline{s} \neq \phi$ such that $(\overline{s}, h) \in X(T)$ and $(\overline{s}, h) \in Ch_H(X(T))$. Thus $Ch_h(X(T)) = \{(\overline{s}, h)\}$. Now, if $(\overline{s}, h) \notin Ch_S(Ch_H(X(T)))$, then $\mu_T(h) = \phi$, contradicting $\mu_T(h) = \mu_E(h) \neq \phi$. Thus, we must have $(\overline{s}, h) \in Ch_S(Ch_H(X(T)))$, and therefore $\mu_T(h) = \overline{s}$. But, $(s, h) \notin Ch_H(X(T))$, and $(\overline{s}, h) \in Ch_H(X(T))$ implies $\mu_E(h) = \mu_T(h) = \overline{s} \succ_h s$, which contradicts to $s \succ_h \mu_E(h)$. Hence, we must have $\mu_E(h) = \phi$.

Case 2: $h' \neq \phi$. Obviously, it implies $s' \neq \phi$. Since $\{(s, s'), (h, h')\}$ is a blocking coalition of μ_E, $(h, h') \succ_c \mu_E(c)$, $s \succ_h \mu_E(h)$ or $s = \mu_E(h)$, $s' \succ_{h'} \mu_E(h')$ or $s' = \mu_E(h')$. As $(h, h') \succ_c \mu_E(c) = \mu_{T-1}(c)$, we have $(h, h') \in P^c(T)$. Thus, $(s, h) \in X(T)$ and $(s', h') \in X(T)$. If $(s, h) \in Ch_H(X(T))$ and $(s', h') \in Ch_H(X(T))$, then $\mu_T(c) \succ_c (h, h')$ or $\mu_T(c) = (h, h')$, which contradicts to $(h, h') \succ_c \mu_E(c) = \mu_T(c)$. Thus, either $(s, h) \notin Ch_H(X(T))$ or $(s', h') \notin Ch_H(X(T))$. Without loss of generality, suppose $(s, h) \notin Ch_H(X(T))$. Then, there exists $\overline{s} \neq \phi$ such that $(\overline{s}, h) \in X(T)$ and $(\overline{s}, h) \in Ch_H(X(T))$. As $\mu_T(h) = \mu_E(h) \neq \phi$, $\mu_T(h) = \overline{s}$. $(s, h) \notin Ch_H(X(T))$ and $(\overline{s}, h) \in Ch_H(X(T))$ imply $\mu_E(h) = \mu_T(h) = \overline{s} \succ_h s$, which contradicts to $s \succ_h \mu_E(h)$ or $s = \mu_E(h)$. Hence, we must also have $\mu_E(h) = \phi$.

Thus, in either case, we have proved that whenever $h \neq \phi$ and $\mu_E(h) \neq s$, we must have $\mu_E(h) = \phi$. Q.E.D.

Lemma 6.2 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences, and let the matching finally obtained by PI-Algorithm be μ_E. Suppose that for*

$c = (s, \phi) \in C$, $s \neq \phi$, $h \in H$, $\{c, (h, \phi)\}$ *is a blocking coalition of the matching* μ_E. *Then there exist* $\bar{c} = (\bar{s}, \bar{s}')$, $\bar{s}' \neq \phi$ *and* $\bar{h} \in H$ *such that* $(h, \bar{h}) \succ_{\bar{c}} \mu_E(\bar{c})$.

Proof: Since PI-Algorithm ends in round T, we have $\mu_{T-1} = \mu_T = \mu_E$ by Lemma 4.1. Also, since $\{c, (h, \phi)\}$ is a blocking coalition of μ_E, we have $s \succ_h \mu_E(h)$ and $(h, \phi) \succ_c \mu_E(c)$. Thus $(h, \phi) \succ_c \mu_{T-1}(c)$ and $(h, \phi) \in P^c(T)$. Therefore, $(s, h) \in X(T)$. Then we must have $(s, h) \notin Ch_H(X(T))$. Indeed, if $(s, h) \in Ch_H(X(T))$, we have $\mu_T(c) \succ_c (h, \phi)$ or $\mu_T(c) = (h, \phi)$, contradicting to $(h, \phi) \succ_c \mu_E(c) = \mu_T(c)$. Thus, there is $\bar{s} \neq \phi$ such that $(\bar{s}, h) \in X(T)$ and $(\bar{s}, h) \in Ch_H(X(T))$.

We show $\bar{s}$ must be a member of real couple. Suppose not. $\bar{s}$ is then a singleton. Let $c_1 = (\bar{s}, \phi)$. $(\bar{s}, h) \in Ch_H(X(T))$ implies $\mu_T(c_1) = (h, \phi)$ or $\mu_T(c_1) \succ_{c_1} (h, \phi)$. Then we have $\mu_{T-1}(c_1) = \mu_T(c_1) \succ_{c_1} (h, \phi)$, which implies $(h, \phi) \notin P^{c_1}(T)$. (Indeed, if $\mu_T(c_1) = (h, \phi)$, $\mu_T(h) = \bar{s}$, which contradicts to $\mu_T(h) = \phi$ by lemma 6.1.) Therefore, $(\bar{s}, h) \notin X(T)$, but this is impossible by noting that $(\bar{s}, h) \in X(T)$. Hence, we must have $\bar{s}$ is a member of real couple.

Let $\bar{c} = (\bar{s}, \bar{s}')$ with $\bar{s}' \neq \phi$. We now show $(h, \phi) \notin P^{\bar{c}}(T)$. Note that $(\bar{s}, h) \in Ch_H(X(T))$ implies $\mu_T(\bar{c}) \succ_{\bar{c}} (h, \phi)$ or $\mu_T(\bar{c}) = (h, \phi)$. We then have $\mu_{T-1}(\bar{c}) = \mu_T(\bar{c}) \succ_{\bar{c}} (h, \phi)$, which implies $(h, \phi) \notin P^{\bar{c}}(T)$. To see this, suppose $\mu_T(\bar{c}) = (h, \phi)$. Then $\mu_T(h) = \bar{s}$, which is impossible by noting that $\mu_T(h) = \phi$. Thus $(\bar{s}, h) \in X(T)$ implies the existence of $\bar{h} \in H$ such that $(h, \bar{h}) \in P^{\bar{c}}(T)$, which in turn implies $(h, \bar{h}) \succ_{\bar{c}} \mu_{T-1}(\bar{c}) = \mu_T(\bar{c})$ or $\mu_T(\bar{c}) = \mu_{T-1}(\bar{c}) = (h, \bar{h})$. If $\mu_T(\bar{c}) = (h, \bar{h})$, $\mu_T(h) = \bar{s}$, which contradicts to $\mu_T(h) = \phi$. Therefore, $(h, \bar{h}) \succ_{\bar{c}} \mu_T(\bar{c}) = \mu_E(\bar{c})$. Q.E.D.

With these two lemmas, we are now ready to prove the theorem. The process of PI-Algorithm can be divided into two stages as follows:

Process PI-A1: For any matching market with couples and strict preferences $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, operating the procedures described in section IV, and supposing PI-Algorithm ends at round T, the matching finally obtained is μ_E. If μ_E is a semi-stable matching, we complete the searching, and otherwise go on to the second stage.

Process PI-A2: There is a blocking coalition $\{(s, \phi), (h, \phi)\}$ of μ_E. By Lemma 6.2, there exist $\bar{c} = (\bar{s}, \bar{s}')$, $\bar{s}' \neq \phi$ and $\bar{h} \in H$ such that $(h, \bar{h}) \succ_{\bar{c}} \mu_E(\bar{c})$ or $(\bar{h}, h) \succ_{\bar{c}} \mu_E(\bar{c})$. For each $\bar{c}$, deleting all of preferences before $\mu_E(\bar{c})$ containing h from the preference list of $\bar{c}$, and letting $P^{\bar{c}}(T+1) = P^{\bar{c}}(T) \setminus \{(h_1, h_2) \in H \times H : h_1 = h, or, h_2 = h\}$, PI-Algorithm continues on the basis of the round T. Letting the new process start from round $T+1$ to round T_1, the matching obtained at the new ending is μ_E^1.

We first prove for any $c \in C$, $\mu_{T+1}(c) \succ_c \mu_T(c)$ or $\mu_{T+1}(c) = \mu_T(c)$. In round $T+1$, obviously $X(T+1) \subset X(T)$. Then, for any $\widetilde{h} \in H$, if $\mu_T(\widetilde{h}) = \widetilde{s} \neq \phi$, we have $(\widetilde{s}, \widetilde{h}) \in X(T+1)$, and $Ch_{\widetilde{h}}(X(T+1)) = Ch_{\widetilde{h}}(X(T))$. This implies for any $\widetilde{c} \in C$, if $\mu_T(\widetilde{c}) \neq (\phi, \phi)$, $\mu_T(\widetilde{c})$ can be selected by $\widetilde{c}$ at round $T+1$. Therefore, $\mu_{T+1}(\widetilde{c}) \succ_{\widetilde{c}} \mu_T(\widetilde{c})$ or $\mu_{T+1}(\widetilde{c}) = \mu_T(\widetilde{c})$.

We then prove there is at least one $c \in C$, such that $\mu_{T+1}(c) \succ_c \mu_T(c)$. Let $c_1 = (s_1, s_1')$ with $s_1' = \phi$. Since $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_E, $(h, \phi) \succ_{c_1} \mu_E(c_1) = \mu_T(c_1)$, and $(h, \phi) \in P^{c_1}(T) = P^{c_1}(T+1)$, and $(s_1, h) \in X(T+1)$. Suppose $Ch_h(X(T+1)) = \{(s, h)\}$. Then $s \succ_h s_1$ or $s = s_1$. If $s = s_1$, then $\mu_{T+1}(c_1) \succ_{c_1} (h, \phi)$ or $\mu_{T+1}(c_1) = (h, \phi)$, and thus $\mu_{T+1}(c_1) \succ_{c_1} \mu_T(c_1)$; if $s \succ_h s_1$, $\mu_T(h) = \mu_E(h) \neq s_1$ because $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_E. Thus we have $\mu_T(h) = \mu_E(h) = \phi$ by Lemma 6.1, which implies $\mu_T(s) \neq h$. $Ch_h(X(T+1)) = \{(s, h)\}$ implies $(s, h) \in X(T+1)$. And for any real couple $\bar{c}$, we have already deleted all the preferences before $\mu_E(\bar{c})$ containing h from the preference list of $\bar{c}$, so s must be a singleton. Let $c = (s, \phi)$, then $(s, h) \in X(T+1)$, which means $(h, \phi) \in P^c(T+1)$. $\mu_T(s) \neq h$ implies $\mu_T(c) \neq (h, \phi)$, and $(h, \phi) \succ_c \mu_T(c)$. As $Ch_h(X(T+1)) = \{(s, h)\}$, $\mu_{T+1}(c) \succ_c (h, \phi)$ or $\mu_{T+1}(c) = (h, \phi)$, and $\mu_{T+1}(c) \succ_c \mu_T(c)$. Thus we have shown there is a $c \in C$ such that $\mu_{T+1}(c) \succ_c \mu_T(c)$.

By Lemma 4.2, for any $c \in C$, $\mu_E^1(c) \succ_c \mu_{T+1}(c)$ or $\mu_E^1(c) = \mu_{T+1}(c)$. Thus $\mu_E^1(c) \succ_c \mu_T(c)$ or $\mu_E^1(c) = \mu_T(c)$ and least one $c \in C$ makes $\mu_E^1(c) \succ_c \mu_T(c)$. If μ_E^1 is a semi-stable matching, the process ends, otherwise goes on with PI-A2. Repeating this process, we can get matchings μ_E^1, μ_E^2, $\cdots$, μ_E^m until there is no blocking coalition of μ_E^m, $\{(s, \phi), (h, \phi)\}$. The repetition is to be terminated, because the terms of the preference list of all $c \in C$ is finite, and for all of $c \in C$, the new matching is a Pareto improvement after each repetition. As this repetition is always in process, each single medical students will obtain his/her most favorite jobs, that is to say, the first item in his/her preference list. In this case, there is no exist blocking coalitions $\{(s, \phi), (h, \phi)\}$ of μ_E^m.

Obviously, PI-Algorithm indicates that the matchings μ_E and μ_E^1, μ_E^2, $\cdots$, μ_E^m are also individually rational. Thus we have shown that the matching finally obtained μ_E^m is a semi-stable matching of the matching market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$. Q.E.D.

Proof of Theorem 3.2: Define a correspondence $K : F \to 2^F$ so that every matching μ in F corresponds with $K(\mu)$, a subset of F. All the matchings in $K(\mu)$ respectively match the same objects to real couples, i.e., for any $\mu \in F$, $K(\mu) = \{\nu : \forall c = (s, s') \in C, s' \neq \phi, \neg \nu(c) = \mu(c)\}$. All of $K(\mu)$ constitute a partition of F because $F = \cup_{\mu \in E} K(\mu)$, and $K(\mu_1) \cap K(\mu_2) = \emptyset$ for $K(\mu_1) \neq K(\mu_2)$. If we can show that operators $\vee_C$ and $\wedge_C$ are closed in $K(\mu)$, it is easy to verify that they meet the requirements of idempotent law, commutative law, associative law, absorption law and distributive law, and then, by the definition of distributive lattice,[14] $K(\mu)$ constitutes a distributive lattice for operators $\vee_C$ and $\wedge$. Consequently, the theorem is proved.

To show that operator $\vee_C$and $\wedge_C$ is indeed closed in $K(\mu)$, i.e., $\lambda = \mu_1 \vee_C \mu_2 \in K(\mu)$ and $\upsilon = \mu_1 \wedge_C \mu_2 \in K(\mu)$, consider any $c = (s, s') \in C$. If $s' \neq \phi$, by the definition of $K(\mu)$, $\mu_1(c) = \mu_2(c)$, thus $\lambda(c) = \upsilon(c) = \mu_1(c) = \mu_2(c)$; if $s' = \phi$, when $\mu_1(c) = \mu_2(c)$, $\lambda(c) = \upsilon(c) = \mu_1(c) = \mu_2(c)$; when $\mu_1(c) \succ_c \mu_2(c)$, $\lambda(c) = \mu_1(c)$ and $\upsilon(c) = \mu_2(c)$; when $\mu_2(c) \succ_c \mu_1(c)$, $\lambda(c) = \mu_2(c)$ and $\upsilon(c) = \mu_1(c)$.

We now show that $\lambda \in K(\mu)$. We first show that λ is a matching. To do so, we need to show that for any $c_1 \in C$, $c_2 \in C$, $c_1 \neq c_2$, (1) $\lambda(s_1) \neq \lambda(s_2)$; (2) $\lambda(s'_1) \neq \lambda(s'_2)$; (3) $\lambda(s'_1) \neq \lambda(s_2)$, and (4) $\lambda(s_1) \neq \lambda(s'_2)$:

(1) $\lambda(s_1) \neq \lambda(s_2)$. Suppose not. There are two cases to be considered.

Case 1: $s'_1 \neq \phi$ or $s'_2 \neq \phi$, without loss of generality, suppose $s'_1 \neq \phi$.

Since $\lambda(c_1) = \mu_1(c_1) = \mu_2(c_1)$, $\lambda(s_1) = \mu_1(s_1) = \mu_2(s_1)$. If $\lambda(c_2) = \mu_1(c_2)$, $\lambda(s_2) = \mu_1(s_2)$, then $\mu_1(s_1) = \mu_1(s_2)$, which contradicts that μ_1 is a matching; if $\lambda(c_2) = \mu_2(c_2)$, $\lambda(s_2) = \mu_2(s_2)$, then $\mu_2(s_1) = \mu_2(s_2)$, which contradicts μ_2 being a matching.

Case 2: $s'_1 = \phi$ and $s'_2 = \phi$.

Case A: $\lambda(c_1) = \mu_1(c_1)$ and $\lambda(c_2) = \mu_1(c_2)$. It implies $\lambda(s_1) = \mu_1(s_1)$ and $\lambda(s_2) = \mu_1(s_2)$. Then, $\mu_1(s_1) = \mu_1(s_2)$, which contradicts that μ_1 is a matching.

Case B: $\lambda(c_1) = \mu_2(c_1)$ and $\lambda(c_2) = \mu_2(c_2)$. It implies $\lambda(s_1) = \mu_2(s_1)$ and $\lambda(s_2) = \mu_2(s_2)$. Then, $\mu_2(s_1) = \mu_2(s_2)$, which contradicts that μ_2 is a matching;

Case C: $\lambda(c_1) = \mu_1(c_1)$ and $\lambda(c_2) = \mu_2(c_2)$. It implies $\lambda(s_1) = \mu_1(s_1)$ and $\lambda(s_2) = \mu_2(s_2)$. Then, $\mu_1(s_1) = \mu_2(s_2)$. Let $h = \mu_1(s_1) = \mu_2(s_2)$, $\mu_1(h) = s_1$ and $\mu_2(h) = s_2$. Since $s'_1 = \phi$ and $s'_2 = \phi$, $\mu_1(c_1) = (h, \phi)$ and $\mu_2(c_2) = (h, \phi)$. $\lambda(c_1) = \mu_1(c_1)$ and $\lambda(c_2) = \mu_2(c_2)$ respectively implies that $(h, \phi) = \mu_1(c_1) \succ_{c_1} \mu_2(c_1)$ and $(h, \phi) = \mu_2(c_2) \succ_{c_2} \mu_1(c_2)$. $c_1 \neq c_2$ implies $s_1 \neq s_2$. Therefore, if $s_1 \succ_h s_2$, $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_2, which contradicts that μ_2 is a semi-stable matching; if $s_2 \succ_h s_1$, $\{c_2, (h, \phi)\}$ is a blocking coalition of μ_1, which contradicts that μ_1 is a semi-stable matching.

Case D: $\lambda(c_1) = \mu_2(c_1)$ and $\lambda(c_2) = \mu_1(c_2)$. It can also be proved as the same way of Case C.

Thus, in the light of case 1 and case 2, if $c_1 \neq c_2$, we must have $\lambda(s_1) \neq \lambda(s_2)$.

[14]From Birkhoff and Mac Lane (2007), a lattice is a set L of elements, with two binary operations $\wedge$ and $\vee$ which are idempotent, commutative, and associative and which satisfy the absorption law. If in addition the distributive laws hold, L is called a distributive lattice.

(2) $\lambda(s_1') \neq \lambda(s_2')$. As $\lambda(s_1') \neq \phi$ and $\lambda(s_2') \neq \phi$, we have $s_1' \neq \phi$ and $s_2' \neq \phi$, and thus $\lambda(c_1) = \mu_1(c_1) = \mu_2(c_1)$ and $\lambda(c_2) = \mu_1(c_2) = \mu_2(c_2)$. Since $c_1 \neq c_2$ implies $\mu_1(s_1') \neq \mu_1(s_2')$, $\lambda(s_1') \neq \lambda(s_2')$.

(3) $\lambda(s_1') \neq \lambda(s_2)$. Since $\lambda(s_1') \neq \phi$ implies $s_1' \neq \phi$, $\lambda(c_1) = \mu_1(c_1) = \mu_2(c_1)$. Also, $c_1 \neq c_2$ implies $\mu_1(s_1') \neq \mu_1(s_2)$ and $\mu_2(s_1') \neq \mu_2(s_2)$, but $\lambda(s_1') = \mu_1(s_1') = \mu_2(s_1')$ and $\lambda(s_2) = \mu_1(s_2)$ or $\mu_2(s_2)$, thus $\lambda(s_1') \neq \lambda(s_2)$.

(4) $\lambda(s_1) \neq \lambda(s_2')$. The proof is similar to that of (3).

Therefore, for all $c_1 \in C$, $c_2 \in C$, if $c_1 \neq c_2$, then $\lambda(s_1) \neq \lambda(s_2)$, $\lambda(s_1') \neq \lambda(s_2')$, $\lambda(s_1) \neq \lambda(s_2')$ and $\lambda(s_1') \neq \lambda(s_2)$. Hence $\lambda = \mu_1 \vee_C \mu_2$ is a matching.

Next, we prove $\lambda = \mu_1 \vee_C \mu_2$ is semi-stable matching.

Let $C = C_1 \cup C_2$, where $\forall c_1 \in C_1$, then $s_1' = \phi$, and $\forall c_2 \in C_2$, then $s_2' \neq \phi$. Obviously, $C_1 \cap C_2 = \emptyset$. Let $H_C(\mu) = \{h \in H, \exists c_2 \in C_2, \neg\mu(h) = s_2, or, \mu(h) = s_2'\}$. For any $\alpha \in K(\mu)$, let $\widetilde{\alpha}$ denote a restricted matching which is denoted in $C_1 \cup [H \setminus H_C(\mu)]$, where $\forall c_1 \in C_1$ then $\widetilde{\alpha}(c_1) = \alpha(c_1)$. Since α is a semi-stable matching, $\widetilde{\alpha}$ is a stable matching. Otherwise, there exists a blocking coalition $\{c_1, (h_1, \phi)\}$ of $\widetilde{\alpha}$, and obviously $\{c_1, (h_1, \phi)\}$ is also a blocking coalition of α, which contradicts that α is a semi-stable matching.

For any $\mu_1 \in K(\mu)$ and $\mu_2 \in K(\mu)$, $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are stable matchings denoted in $C_1 \cup [H \setminus H_C(\mu)]$. By Conway's lattice theorem for singleton matching markets, $\widetilde{\lambda} = \widetilde{\mu}_1 \vee_C \widetilde{\mu}_2$ is also a stable matching denoted in $C_1 \cup [H \setminus H_C(\mu)]$. Therefore, $\lambda = \mu_1 \vee_C \mu_2$ must be a semi-stable matching. Otherwise, there exists a blocking coalition $\{c_1, (h, \phi)\}$ of λ, where $c_1 \in C_1$. If $h \in H \setminus H_C(\mu)$, $\{c_1, (h, \phi)\}$ must be a blocking coalition of $\widetilde{\lambda}$, which contradicts that $\widetilde{\lambda}$ is a stable matching; if $h \in H_C(\mu)$, $\lambda(h) = \mu_1(h) = \mu_2(h)$. Since $\lambda(c_1) = \mu_1(c_1)$ or $\lambda(c_1) = \mu_2(c_1)$. And $\{c_1, (h, \phi)\}$ is a blocking coalition of λ implies $s_1 \succ_h \lambda(h)$ and $(h, \phi) \succ_{c_1} \lambda(c_1)$. Therefore, $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_1 or μ_2, which contradicts that μ_1 and μ_2 are semi-stable matchings. So, $\lambda = \mu_1 \vee_C \mu_2$ must be a semi-stable matching. Thus $\lambda = \mu_1 \vee_C \mu_2 \in K(\mu)$. Similarly, we can prove that $\upsilon = \mu_1 \wedge_C \mu_2 \in K(\mu)$. Thus, the theorem is proved. Q.E.D.

Proof of Theorem 3.3: Let the matching obtained by PI-Algorithm be μ_E. We show it must be a stable matching under the imposed conditions. Obviously, PI-Algorithm implies that μ_E must be an individually rational matching. We first show $\mu_E(c) = (h, h')$ for any $c \in C$ with $s' \neq \phi$. Indeed, at round 1 of PI-Algorithm, the first preference item in P^c, (h, h') is c's a reservation preference item. If $h \neq \phi$, then $(s, h) \in X(T)$ and $(s, h) \in Ch_H(X(T))$; if $h' \neq \phi$, then $(s', h') \in X(T)$ and $(s', h') \in Ch_H(X(T))$. Thus, (h, h') is selectable for c, and it is the first item of P^c. Therefore, $\mu_1(c) = (h, h')$. By Lemma 4.2, $\mu_E(c) \succ_c \mu_1(c)$ or $\mu_E(c) = \mu_1(c)$. As (h, h') is the first item of P^c, $(h, h') \succ_c \mu_E(c)$ or $(h, h') = \mu_E(c)$, and consequently we have $\mu_E(c) = (h, h')$.

We then show that μ_E is stable. If not, there exists at least a blocking coalition $\{c_1, (h_1, h'_1)\}$ with $c_1 = (s_1, s'_1)$. If $s'_1 \neq \phi$, $(h_1, h'_1) \succ_c \mu_E(c_1)$, which contradicts that $\mu_E(c_1)$ equals to the first preference of c_1. If: $s'_1 = \phi$, then by Lemma 6.2, there is $\overline{c} = (\overline{s}, \overline{s}')$ with $\overline{s}' \neq \phi$ and $\overline{h} \in H$ such that $(h_1, \overline{h}) \succ_{\overline{c}} \mu_E(\overline{c})$ or $(\overline{h}, h_1) \succ_{\overline{c}} \mu_E(\overline{c})$, which also contradicts that $\mu_E(\overline{c})$ equals to the first preference of $\overline{c}$. Thus, μ_E must be stable. Q.E.D.

Proof of Theorem 3.4: By theorem 3.1, for any $k = 1, 2, \ldots$, there is a semi-stable matching μ_E^k in market Γ^k. By the definition of the semi-stable matching, any blocking coalition $\{c, (h, h')\}$ of μ_E^k contains a real couple $c = (s, s')$, and (h, h') must be its an effective preference. By Condition 2 for simple regularity, the number of blocking coalitions of μ_E^k must be less than $q \cdot m_k$. For the null matching μ^0, the number of its blocking coalitions is $N = \sum_{c \in C^k} l_c$. By Condition 3 for simple regularity, we have $N \geq n_k$. Thus, the unstable degree is $\theta(\mu_E^k) = m/N \leq q \cdot m_k/n_k$. By Condition 1 for simple regularity, $m_k = o(n_k)$. Thus $\theta(\mu_E^k) = o(1)$, that is, as n_k tend to infinity, the sequence of unstable degree $\{\theta(\mu_E^k)\}_{k=1}^{\infty}$ tends to zero. Therefore, the matching sequence $\{\mu_E^k\}_{k=1}^{\infty}$ is asymptotically stable, that is, $\{\Gamma^k\}_{k=1}^{\infty}$ is asymptotically stable. Q.E.D.

Proof of Theorem 3.5: Let the semi-stable matching obtained by the algorithm described in Section 4 be μ_E. We want to show μ_E must be a stable matching. It is clear that PI-Algorithm indicates that μ_E must be individually rational.

We first show that for any $c \in C$ with $s' \neq \phi$, $\mu_E(c) = (h, h')$. Indeed, at round 1 of PI-Algorithm, (h, h') is c's a reservation preference, which implies that if $h \neq \phi$, then $(s, h) \in X(T)$ and $(s, h) \in Ch_H(X(T))$; if $h' \neq \phi$, then $(s', h') \in X(T)$ and $(s', h') \in Ch_H(X(T))$. Thus, (h, h') is selectable for c at round 1, and $\mu_1(c) \succ_c (h, h')$ or $\mu_1(c) = (h, h')$.

We now show $\mu_1(c) = (h, h')$. Suppose, by way of contradiction, that $\mu_1(c) \neq (h, h')$. Then we must have $\mu_1(c) = (\overline{h}, \overline{h}') \succ_c (h, h')$. Thus, $(\overline{h}, \phi) \neq (\phi, \phi)$ or $(\overline{h'}, \phi) \neq (\phi, \phi)$ implies that it is not only a reservation preference but also the first preference of a singleton. Therefore, if $\overline{h} \neq \phi$, there exists $\overline{c} = (\overline{s}, \phi)$ such that $(\overline{s}, \overline{h}) \in X(T)$ and $(\overline{s}, \overline{h}) \in Ch_H(X(T))$, and $\mu_1(\overline{c}) = (\overline{h}, \phi)$ because $(\overline{h}, \phi)$ is the first preference of $\overline{c}$; if $\overline{h}' \neq \phi$, there exists $\widetilde{c} = (\widetilde{s}, \phi)$ such that $(\widetilde{s}, \overline{h}') \in X(T)$ and $(\widetilde{s}, \overline{h}') \in Ch_H(X(T))$, and $\mu_1(\widetilde{c}) = (\overline{h'}, \phi)$ because $(\overline{h}', \phi)$ is the first preference of $\widetilde{c}$. This contradicts $\mu_1(c) = (\overline{h}, \overline{h}')$. Hence $\mu_1(c) = (h, h')$.

For any preference $(\overline{h}, \overline{h}')$ of c before (h, h'), if $\overline{h} \neq \phi$, there exists $\overline{c} = (\overline{s}, \phi)$ such that $\mu_1(\overline{c}) = (\overline{h}, \phi)$. The process of the PI-algorithm for finding semi-stable matching μ_E implies that $\mu_E(c) \succ_c \mu_1(c)$ for any $c \in C$. But, since $(\overline{h}, \phi)$ is the first preference of $\overline{c}$, $\mu_E(\overline{c}) = (\overline{h}, \phi)$. Likewise, if $\overline{h}' \neq \phi$, there exists $\widetilde{c} = (\widetilde{s}, \phi)$ such that $\mu_E(\widetilde{c}) = (\overline{h'}, \phi)$. Thus $\mu_E(c) = \mu_1(c) = (h, h')$ by noting that it cannot be matched a object which is better than (h, h') for c.

Now, if the semi-stable matching μ_E is not a stable matching, there exists at least a blocking coalition $\{c_1, (h_1, h_1')\}$ with $s_1' \neq \phi$. As such, we have $(h_1, h_1') \succ_c \mu_E(c_1)$. Moreover, $s_1 \succ_{h_1} \mu_E(h_1)$ if $h_1 \neq \phi$, and $s_1' \succ_{h_1'} \mu_E(h_1')$ if $h_1' \neq \phi$. However, as shown above, if $h_1 \neq \phi$, then there exists $\overline{c}_1 = (\overline{s}_1, \phi)$ such that $\mu_E(\overline{c}_1) = (h_1, \phi)$, which contradicts $s_1 \succ_{h_1} \mu_E(h_1) = \overline{s}_1$. Indeed, since (h_1, ϕ) is a reservation preference of $\overline{c}_1$ implies that $\overline{s}_1 \succ_{h_1} s_1$, if $h_1' \neq \phi$, then there exists $\widetilde{c}_1 = (\widetilde{s}_1, \phi)$ such that $\mu_1(\widetilde{c}_1) = (h_1', \phi)$, which contradicts $s_1' \succ_{h_1'} \mu_E(h_1') = \widetilde{s}_1$ by noting that (h_1', ϕ) is a reservation preference of $\widetilde{c}_1$ implies that $\widetilde{s}_1 \succ_{h_1'} s_1'$. Thus μ_E must be a stable matching. Q.E.D.

Proof of Lemma 4.1: PI-Algorithm ends when the matching of the current round repeats the one of the previous round. Obviously, $\mu_{T-1} = \mu_T = \mu_E$. The process of PI-Algorithm indicates that the preference list of each round is derived from the previous one by deleting a part of elements. Deleting the items after $\mu_t(c)$ from $P^c(t+1)$ results in $P^c(t+1) \subsetneqq P^c(t)$. And $X(t)$ and $X(t+1)$ are respectively derived from $P^c(t)$ and $P^c(t+1)$. As such, $X(t+1) \subsetneqq X(t)$. Q.E.D.

Proof of Lemma 4.2: In the light of $X(t) \subsetneqq X(t-1)$ by Lemma 4.1, for any $h \in H$, provided $(s, h) \in X(t)$, we have $(s, h) \in X(t-1)$. Thus, if $Ch_h(X(t-1)) = \{(s, h)\}$ and $(s, h) \in X(t)$, then we must have $Ch_h(X(t)) = \{(s, h)\}$.

Consider two cases: (1) $\mu_{t-1}(c) = (\phi, \phi)$. Then $\mu_t(c) \succ_c \mu_{t-1}(c)$ or $\mu_t(c) = \mu_{t-1}(c)$. (2) $\mu_{t-1}(c) = (h, h') \neq (\phi, \phi)$. Since $(h, h') \in P^c(t)$, in the round t, we have $(s, h) \in X(t)$ and $(s, h) \in Ch_h(X(t))$ if $h \neq \phi$, also $(s', h') \in X(t)$ and $(s', h') \in Ch_{h'}(X(t))$ if $h' \neq \phi$. Hence, (h, h') is selectable for c. Thus, we also have $\mu_t(c) \succ_c \mu_{t-1}(c)$ or $\mu_t(c) = \mu_{t-1}(c)$. When $0 < t < T$, there exists some $c \in C$ such that $\mu_t(c) \succ_c \mu_{t-1}(c)$, otherwise, PI-Algorithm ends before round T, which contradicts the fact that it ends in round T. Q.E.D.

Proof of Theorem 4.1: It is clear that PI-Algorithm implies the matching μ_E is an individually rational matching. Suppose, by way of contradiction, that μ_E is not a stable matching. Then there exists a blocking coalition $\{(s, s'), (h, h')\}$ with $\mu_E(h) \neq \phi$ and $\mu_E(h') \neq \phi$. But, by Lemma 6.1, $\mu_E(h) = \phi = \mu_E(h')$ when $h \neq \phi$, $\mu_E(h) \neq s$, $h' \neq \phi$, $\mu_E(h') \neq s'$. As such, $\mu_E(h) \neq \phi$ and $\mu_E(h') \neq \phi$ imply $\mu_E(h) = s$ and $\mu_E(h') = s'$, which contradicts $\{(s, s'), (h, h')\}$ being a blocking coalition of μ_E. Q.E.D.

Proof of Theorem 4.2: Again, PI-Algorithm clearly implies that the matching μ_E must be an individually rational matching. Suppose that μ_E is not a stable matching, there exists at least a blocking coalition $\{(s, s'), (h, \phi)\}$ of μ_E. By Lemma 6.2, there exist $\overline{c} = (\overline{s}, \overline{s}')$ with $\overline{s}' \neq \phi$ and $\overline{h} \in H$ such that $(h, \overline{h}) \succ_{\overline{c}} \mu_E(\overline{c})$ or $(\overline{h}, h) \succ_{\overline{c}} \mu_E(\overline{c})$, which contradicts that Γ has no real couples. Q.E.D.

Proof of Theorem 4.3: For any matching market containing only singletons $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$, the subset $K(\mu_S^E)$ of the set of semi-stable matchings is just the set of stable matchings. By Theorem 5.2, the matching μ_S^E is the optimal stable matching for the side of jobs of hospitals. The optimal theorem of marriage matching markets (Roth,1982a) implies that μ_H is the optimal stable matching for the side of jobs of hospitals. Since the optimal matching is unique, we have $\mu_S^E = \mu_H$. For any matching market containing only singletons, μ_H^E and μ_S^E are logically symmetrical, likewise, we have $\mu_H^E = \mu_S$. QED.

Proof of Theorem 5.1: Since matched object of each real couple is the same for all semi-stable matchings in $K(\mu)$, the set of unmatched real couples is the same at every semi-stable matching of in $K(\mu)$. After excluding all real couples and their matched objects, the semi-stable matching is actually a stable matching for all singletons and all remaining jobs of hospitals. By McVitie and Wilson theorem, the set of unmatched single medical students and remaining jobs of hospitals is the same for every stable matching. Q.E.D.

Proof of Theorem 5.2: To prove the theorem, we need the following lemma:

Lemma 6.3 *Let $\Gamma = (H, S, C, (\succ_h)_{h\in H}, (\succ_c)_{c\in C})$ be a matching market with couples and strict preferences, and let the outcome obtained by PI-Algorithm mechanism be μ^E. Suppose that for every $\mu \in K(\mu^E)$, there exists $h \in H$ such that $s = \mu(h) \succ_h \mu^E(h)$. Then s must be a singleton and $h_1 = \mu^E(s) \neq \phi$, $\mu(h_1) \succ_{h_1} \mu^E(h_1)$.*

Proof: Since μ and μ^E are semi-stable matchings, they are both individually rational, and $\mu^E(h) \succ_h \phi$ or $\mu^E(h) = \phi$. Since $s = \mu(h) \succ_h \mu^E(h)$, $s = \mu(h) \succ_h \phi$, which implies $\mu(s) = h \neq \phi$. By Theorem 5.1, the set of unmatched medical students is the same for every semi-stable matching in $K(\mu^E)$, which implies that the set of matched medical students is also the same for every semi-stable matching in $K(\mu^E)$. Thus, $h_1 = \mu^E(\mu(h)) = \mu^E(s) \neq \phi$, $s = \mu^E(h_1)$. Let $s_1 = \mu(h_1)$, $\mu(h) \succ_h \mu^E(h)$ implies $s = \mu(h) \neq \mu^E(h)$. Thus $h_1 = \mu^E(s) \neq h$, and $s_1 = \mu(h_1) \neq \mu(h) = s$. We show s must be a singleton. If not, s is then a member of a real couple c, then $\mu(c) = \mu^E(c)$ because $\mu \in K(\mu^E)$. Thus $h = \mu(s) = \mu^E(s) = h_1$, contradicting to $h_1 \neq h$. Next, we show that $h_1 \succ_s h = \mu(s)$.

Since μ and μ^E are individually rational and $h_1 \neq \phi \neq h$, we have $h_1 = \mu^E(s) \succ_s \phi$ and $h = \mu(s) \succ_s \phi$. Thus, at step1 of round 1 in the process of PI-Algorithm, we have $(s, h) \in X(1)$ and $(s, h_1) \in X(1)$. Noting PI-Algorithm ends at round T and letting $\overline{s} = \mu^E(h)$, we have $(\overline{s}, h) \in X(T)$ and $(\overline{s}, h) \in Ch_h(X(T))$. Since $s = \mu(h) \succ_h \mu^E(h) = \overline{s}$, we have $(s, h) \notin X(T)$, otherwise, it contradicts $(\overline{s}, h) \in Ch_h(X(T))$. Since $(s, h) \notin X(T)$, $h_1 = \mu^E(s) = \mu^E_{T-1}(s) \succ_s h$.

If $s \succ_{h_1} s_1 = \mu(h_1)$, as $h_1 \succ_s h = \mu(s)$, $\{(s, \phi), (h_1, \phi)\}$ constitutes a blocking coalition of matching μ, which contradicts that μ is a semi-stable matching. Hence $\mu(h_1) = s_1 \succ_{h_1} s = \mu^E(h_1)$. Q.E.D.

Now we begin to prove the theorem. Suppose, by way of contradiction, that μ^E is not an optimal semi-stable matching in $K(\mu^E)$ for the side of jobs of hospitals, there exists a matching $\mu \in K(\mu^E)$ such that for all $h \in H$, $\mu(h) \succ_h \mu^E(h)$ or $\mu(h) = \mu^E(h)$; and there exists a $h \in H$ such that $\mu(h) \succ_h \mu^E(h)$. Let $s_1 = \mu(h)$ and $h_1 = \mu^E(s_1)$. By Lemma 6.3, s_1 must be a singleton, and $h_1 \neq \phi$ and $\mu(h_1) \succ_{h_1} \mu^E(h_1)$. Repeatedly applying Lemma 6.3, we can obtain two sequences, one is the sequence of medical students $\{s_1, s_2, \cdots\}$, and the other is the sequence of jobs of hospitals $\{h_1, h_2, \ldots,\}$, where $s_{k+1} = \mu(h_k)$ and $h_k = \mu^E(s_k)$ for any $k > 0$. Due to the limited number of medical students, there exists the least k and r, such that $s_{k+r} = s_k$. Thus $k = 1$, otherwise $h_{k+r-1} = \mu(s_k) = h_{k-1}$, and $s_{k+r-1} = \mu^E(h_{k-1}) = s_{k-1}$, which contradicts that k is the least. Therefore, $k = 1$, that is, $s_1 = s_{r+1}$, and $h_0 = h_r$. It is easy to deduce that $s_l = s_{l+r}$ and $h_l = h_{l+r}$ for any $l > 0$. Thus, we obtain twin circulating sequences, the sequence of medical students $\{s_1, s_2, \cdots, s_r\}$, and the sequence of jobs of hospitals $\{h_1, h_2, \cdots, h_r\}$, such that $s_{k+1} = \mu(h_k)$ and $h_k = \mu^E(s_k)$ and $s_{k+1} \succ_{h_k} s_k$ for any $k > 0$. We now show that there are no such twin circulating sequences. As a result, the theorem is proved.

Denote by $\overline{H} \subseteq H$ the set of all jobs meeting the condition $\mu(h) \succ_h \mu^E(h)$. We consider two cases.

Case 1: $\overline{H} = \{h_1, h_2, \cdots, h_r\}$. For any $2 \leq k \leq r + 1$, μ and μ^E are both semi-stable matchings, which implies that they are both individually rational matchings, and thus $h_k =$

$\mu^E(s_k) \succ_{s_k} \phi$ and $h_{k-1} = \mu^E(s_{k-1}) \succ_{s_{k-1}} \phi$. At the step 1 of round 1 in the process of PI-Algorithm, $(s_k, h_k) \in X(1)$ and $(s_{k-1}, h_{k-1}) \in X(1)$. Since PI-Algorithm ends at round T and gives the temporary matching μ_t at round t, we have $(s_{k-1}, h_{k-1}) \in X(T)$ and $(s_{k-1}, h_{k-1}) \in Ch_H(X(T))$. Also, since $s_k = \mu(h_{k-1}) \succ_{h_{k-1}} \mu^E(h_{k-1}) = s_{k-1}$, $(s_k, h_{k-1}) \notin X(T)$ (otherwise, $(s_k, h_{k-1}) \in X(T)$, contradicting to $(s_{k-1}, h_{k-1}) \in Ch_H(X(T))$). Thus, $(s_k, h_{k-1}) \in X(1)$ and $(s_k, h_{k-1}) \notin X(T)$ imply that there exists $1 < t_k < T$ such that $(s_k, h_{k-1}) \in X(t_k)$ and $(s_k, h_{k-1}) \notin X(t_k + 1)$. Without loss of generality, suppose t_k is the least one. Then $\overline{h} \equiv \mu_{t_k}(s_k) \succ_{s_k} h_{k-1} = \mu(s_k)$. Thus $\overline{h} \notin \overline{H}$ (otherwise, $\mu_{t_k}(s_k) = \overline{h} = h_j = \mu^E(s_j)$. Then, $(s_j, h_j) \in Ch_H(X(t_k))$, and $(s_{j+1}, h_j) \notin X(t_k)$ by noting that $s_{j+1} = \mu(h_j) \succ_{h_j} \mu^E(h_j) = s_j$. Thus we have $t_j < t_k$, which contradicts the hypothesis that t_k is the least). So $\overline{h} \notin \overline{H}$.

Let $\overline{s} = \mu(\overline{h})$. Then $\overline{s} \neq \phi$. Suppose not. Since $\overline{h} = \mu_{t_k}(s_k) \succ_{s_k} h_{k-1} = \mu(s_k)$, $s_k \succ_{\overline{h}} \phi$, and s_k is a singleton, $\{(s_k, \phi), (\overline{h}, \phi)\}$ forms a blocking coalition of matching μ, which contradicts the fact that μ is a semi-stable matching. So we must have $\overline{s} \neq \phi$. Then, by Theorem 5.1, we have $\widetilde{s} \equiv \mu^E(\overline{h}) \neq \phi$. Therefore, $\overline{s}$, $\widetilde{s}$ and s_k are all acceptable medical students for $\overline{h}$. Since $\overline{h}$ has chosen s_k before the final matching $\widetilde{s}$, we have $s_k \succ_{\overline{h}} \widetilde{s}$. If $\widetilde{s} \succ_{\overline{h}} \overline{s}$ or $\widetilde{s} = \overline{s}$, then $s_k \succ_{\overline{h}} \overline{s} = \mu(\overline{h})$. Consequently, $\{(s_k, \phi), (\overline{h}, \phi)\}$ is a blocking coalition of the matching μ, which contradicts that μ is a semi-stable matching. Thus $\overline{s} \succ_{\overline{h}} \widetilde{s}$, that is, $\mu(\overline{h}) \succ_{\overline{h}} \mu^E(\overline{h})$, and $\overline{h} \in \overline{H}$. But, this contradicts $\overline{h} \notin \overline{H}$. Hence μ^E must be an optimal semi-stable matching in $K(\mu^E)$ for the side of jobs of hospitals when $\overline{H} = \{h_1, h_2, \cdots, h_r\}$.

Case 2: $\overline{H} \neq \{h_1, h_2, \cdots, h_r\}$. By repeating the above proof, we can obtain the circulating sequence of medical students $\{s_1, s_2, \cdots, s_r\}$ and the circulating sequence of jobs of hospitals $\{h_1, h_2, \cdots, h_r\}$ such that $s_{k+1} = \mu(h_k)$ and $h_k = \mu^E(s_k)$ and $s_{k+1} \succ_{h_k} s_k$ for any $k > 0$. The prerequisite of twin circulating sequences is that there exists at least one $\overline{h} \notin \{h_1, h_2, \cdots, h_r\}$ such that $\mu(\overline{h}) \succ_{\overline{h}} \mu^E(\overline{h})$. Applying Lemma 6.3 again, we obtain another twin circulating sequences satisfying the same conditions. By repeating the proving process in Case 1, the prerequisite of this new twin circulating sequences is another new twin circulating sequences. It similarly show that these different twin circulating sequences do not intersect with each other, and so on. Since H is a finite set, all these twin circulating sequences are bound to constitute a causal circle, that is, the prerequisite of the first twin circulating sequences is the existence of the second twin circulating sequences, and the prerequisite of the second twin circulating sequences is the existence of the third twin circulating sequences, and so on, which leads to a contradiction. Thus, μ^E also must be an optimal semi-stable matching in $K(\mu^E)$ for the side of jobs of hospitals when $\overline{H} \neq \{h_1, h_2, \cdots, h_r\}$. Q.E.D.

Proof of Theorem 5.3: To prove the theorem, it is sufficient to domonstate some matching markets such that truthtell is not the best response for some agent even though all others state their preferences truthfully.

Consider a matching market with one couple $c_1 = (s_1, s_2)$ and one single $c_2 = (s_3, \phi)$, and four jobs of hospitals. Their rank lists of preferences are as follows:

$P^{c_1} : \{(h_1, h_2), (h_2, h_1), (h_3, \phi), (\phi, \phi)\}$; $P^{c_2} : \{(h_1, \phi), (h_4, \phi), (h_2, \phi), (\phi, \phi)\}$;

$P^{h_1} : \{s_2, s_1, s_3, \phi\}$; $P^{h_2} : \{s_3, s_1, s_2, \phi\}$; $P^{h_3} : \{s_1, \phi\}$; $P^{h_4} : \{s_3, \phi\}$.

After processing PI-A1 in PI-Algorithm mechanism, we get a matching μ with $\mu(c_1) = (h_3, \phi)$ and $\mu(c_2) = (h_4, \phi)$. $\{c_2, (h_1, \phi)\}$ is the only blocking coalition containing singleton. Thus, after deleting the preference containing h_1 in P^{c_1}, in the process of PI-A2, we get a semi-stable matching μ_1 with $\mu_1(c_1) = (h_3, \phi)$ and $\mu_1(c_2) = (h_1, \phi)$ by continuing PI-Algorithm.

Even all others state their ture preferences, c_1 can be better off by manipulating their own preferences by reporting $P'^{c_1} : \{(h_1, h_2), (\phi, \phi)\}$. As such, we obtain another semi-stable matching μ_2 with $\mu_2(c_1) = (h_1, h_2)$ and $\mu_2(c_2) = (h_4, \phi)$ by PI-Algorithm mechanism, resulting in $\mu_2(c_1) = (h_1, h_2) \succ_{c_1} (h_3, \phi) = \mu_1(c_1)$. Therefore, PI-Algorithm mechanism is not strategy-proof.

We can also provide an exmple of market in which a job has incentive not to state its true preferences. To see this, consider another matching marekt with two couples $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$, and four jobs of hospitals. Their rank lists of preferences are as follows:

$P^{c_1} : \{(h_1, h_2), (h_3, \phi), (\phi, \phi)\}$; $P^{c_2} : \{(h_1, h_2), (h_4, \phi), (\phi, \phi)\}$;

$P^{h_1} : \{s_1, s_3, \phi\}$; $P^{h_2} : \{s_4, s_2, \phi\}$; $P^{h_3} : \{s_1, \phi\}$; $P^{h_4} : \{s_3, \phi\}$.

After processing PI-A1 in PI-Algorithm mechanism, we get a semi-stable matching μ_1 with $\mu_1(c_1) = (h_3, \phi)$ and $\mu_1(c_2) = (h_4, \phi)$.

Supposing that all others state their true preferences, h_1 can be better off by manipulating its own preferences by reporting $P'^{h_1} : \{s_3, \phi\}$. After processing PI-A1 in PI-Algorithm mechanism, we get another semi-stable matching μ_2 with $\mu_2(c_1) = (h_3, \phi)$ and $\mu_2(c_2) = (h_1, h_2)$. $\mu_2(h_1) = s_3 \succ_{h_1} \phi = \mu_1(h_1)$. Again, this shows PI-Algorithm mechanism is not strategy-proof. Q.E.D.

Proof of Theorem 5.4: Since matched objects of real couples are the same for every semi-stable matching in $K(\mu_r)$, the welfare is unchanged for all real couples and their matched jobs. Thus, truthtelling is a dominant strategy for all of real couples and their matched jobs.

After excluding all real couples and their matched objects, the semi-stable matching is actually a stable matching for all singletons and remaining hospital jobs. By Theorem 4.3, for any matching market containing only singletons and with strict preferences, the matching obtained by PI-Algorithm is identical to μ_H which are obtained by jobs optimal Gale-Shapley algorithm. By the dominant strategy theorem on marriage matching markets in Dubins and Freedman (1981) and Roth (1982a), telling the truth is a dominant strategy for each remaining job of hospitals. Q.E.D.

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