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On the existence of equilibria in games with arbitrary strategy spaces and preferences[☆]

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ABSTRACT

This paper provides necessary and sufficient conditions for the existence of pure strategy Nash equilibria by replacing the assumptions concerning continuity and quasiconcavity with a unique condition, passing strategy space from topological vector spaces to arbitrary topological spaces. Preferences may also be nontotal/nontransitive, discontinuous, nonconvex, or nonmonotonic. We define a single condition, *recursive diagonal transfer continuity (RDTC)* for aggregator payoff function and *recursive weak transfer quasi-continuity (RWTQC)* for individuals' preferences, respectively, which establishes the existence of pure strategy Nash equilibria in games with arbitrary (topological) strategy spaces and preferences without imposing any kind of quasiconcavity-related conditions.

1. Introduction

The notion of Nash equilibrium is probably one of the most important solution concepts in economics in general and game theory in particular, which has wide applications in almost all areas of economics as well as in business and other social sciences. The classical existence theorems on Nash equilibrium (e.g. in Nash, 1950, 1951; Debreu, 1952; Glicksberg, 1952; Nikaido and Isoda, 1955) typically assume *continuity* and *quasiconcavity* for the payoff functions, in addition to convexity and compactness of strategy spaces, which require strategy spaces be topological vector spaces. However, in many important economic models, such as those in Bertrand, 1883, Hotelling (1929), Dasgupta and Maskin (1986), and Jackson (2009), payoffs are discontinuous and/or non-quasiconcave, and strategy spaces are nonconvex and/or noncompact.

Accordingly, economists continually strive to seek weaker conditions that can guarantee the existence of equilibrium. Some seek to weaken the quasiconcavity of payoffs or substitute it with some types of transitivity/monotonicity of payoffs (cf. McManus, 1964; Roberts and Sonnenschein, 1977; Topkis, 1979; Nishimura and Friedman, 1981; Milgrom and Roberts, 1990; Vives, 1990), some seek to weaken the continuity of payoff functions (cf. Dasgupta and Maskin, 1986; Simon, 1987; Simon and Zame, 1990; Tian, 1992a,b,c, 1994; Reny, 1999, 2009; Bagh and Jofre, 2006; Monteiro and Page, 2007; Morgan and Scalzo, 2007; Nessah and Tian, 2008; Carmona, 2009, 2011; Scalzo, 2010; Balder, 2011; Prokopovych, 2011, 2013), while others seek to weaken both quasiconcavity and continuity (cf. Baye et al., 1993, de Castro, 2011, McLennan et al., 2011; Barelli and Meneghel, 2013).

However, all the existing results are under the assumption of topological vector spaces, impose linear (convex) or lattice structures and only provide sufficient conditions for the existence of equilibrium.[1] In order to apply a fixed-point theorem (say, Brouwer, Browder, Kakutani, Michael, or Knaster, Kuratowski, and Mazurkiewicz, etc.), they all need to assume some forms of quasiconcavity[2] (or transitivity/monotonicity) and continuity of payoffs,

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1 McLennan et al. (2011) and Barelli and Meneghel (2013) recently provide necessary and sufficient conditions for the existence of Nash equilibrium. However, they obtain their existence results under the linear structures.

2 For mixed strategy Nash equilibrium, quasiconcavity is automatically satisfied since the mixed extension has linear payoff functions. Thus only some form of continuity matters for the existence of mixed strategy Nash equilibrium.

in addition to compactness and convexity of strategy space. While it may be the convex structure that easily connects economics to mathematics, in many important situations where commodities or alternatives are invisible so that the choice spaces are discrete, there are no convex/lattice structures.

Thus, convexity assumption excludes the possibility of considering discrete games, and consequently seriously limits the applicability of economic theory. As such, the intrinsic nature of equilibrium has not been fully understood. Why does or does not a game have an equilibrium? Are continuity and quasiconcavity both essential to the existence of equilibrium? If so, can continuity and quasiconcavity be combined into one single condition? One can easily find simple examples of economic games that have or do not have an equilibrium (see Examples 3.1 and 3.2), but none of them can be used to reveal the existence/non-existence of equilibria in these games. This paper sheds some light on these questions.

We fully characterize the existence of pure strategy Nash equilibrium in general games with arbitrary topological strategy spaces[3] that may be discrete or non-convex and payoffs (resp. preferences) that may be discontinuous (resp. discontinuous or nontotal/nontransitive) or do not have any form of quasi-concavity (resp. convexity) or monotonicity. We introduce the notions of recursive transfer continuities, specifically *recursive diagonal transfer continuity (RDTC)* for aggregator payoff function and *recursive weak transfer quasi-continuity (RWTQC)* for individuals' preferences.

It is shown that the single condition, RDTC (resp. RWTQC) is necessary, and further, under compactness of strategy space, sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs (resp. preferences).[4] We also provide an existence theorem for a strategy space that may not be compact. We show that RDTC (resp. RWTQC) with respect to a compact set is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary (topological) strategy spaces and general payoffs (resp. preferences). RDTC (resp. RWTQC) also permits the existence of symmetric pure strategy Nash equilibria in games with general strategy spaces and payoffs (resp. preferences).

RDTC (resp. RWTQC) strengthens diagonal transfer continuity introduced in Baye et al. (1993) (resp. weak transfer quasi-continuity introduced in Nessah and Tian, 2008) to allow recursive (sequential) transfers in order to get rid of the diagonal transfer quasiconcavity assumption (resp. the strong diagonal transfer quasiconcavity assumption) so that these conditions turn out to be necessary and sufficient for the existence of equilibria in compact games. As such, no quasiconcavity/monotonicity-related conditions are assumed. These results may be used to argue the existence of equilibrium in general games with no linear (convex) structures such as equilibrium issues in market design theory and matching theory. In the paper, we also provide sufficient conditions for the existence of equilibrium without imposing any form of quasiconcavity.

The remainder of the paper is organized as follows. Section 2 provides basic notation and definitions, and analyzes the essence of Nash equilibrium. Section 3 investigates the existence of pure strategy Nash equilibrium by using aggregate payoffs and individuals' preferences respectively. We also provide sufficient conditions for recursive transfer continuities. Section 4 extends the results to symmetric pure strategy Nash equilibrium. Concluding remarks are offered in Section 5.

[3] In particular, the strategy spaces may not be metrizable, locally convex, Hausdorff, or even not regular.

[4] As such, one *cannot* say that RDTC (resp. RWTQC) is equivalent to Nash equilibrium.

2. Preliminaries: Nash equilibrium and its intrinsic nature

2.1. Notions and definitions

Let I be the set of players that is either finite or countably infinite. Each player i's strategy space X_i is a general topological space that may not be metrizable, locally convex, Hausdorff, or even not regular. Denote by $X = \prod_{i\in I} X_i$ the Cartesian product of the sets of strategy profiles, equipped with the product topology. For each player $i \in I$, denote by $-i$ all other players rather than player i. Also denote by $X_{-i} = \prod_{j\neq i} X_j$ the Cartesian product of the sets of strategies of players $-i$. Without loss of generality, assume that player i's preference relation is given by the weak preference $\succcurlyeq_i$ defined on X, which may be nontotal or nontransitive.[5] Let $\succ_i$ denote the asymmetric part of $\succcurlyeq_i$, i.e., $y \succ_i x$ if and only if $y \succcurlyeq_i x$ but not $x \succcurlyeq_i y$.

A *game* $G = (X_i, \succcurlyeq_i)_{i\in I}$ is simply a family of ordered tuples $(X_i, \succcurlyeq_i)$.

When $\succcurlyeq_i$ can be represented by a payoff function $u_i : X \to \mathbb{R}$, the game $G = (X_i, u_i)_{i\in I}$ is a special case of $G = (X_i, \succcurlyeq_i)_{i\in I}$.

A strategy profile $x^* \in X$ is a *pure strategy Nash equilibrium* of a game G iff,

$$x^* \succcurlyeq_i (y_i, x^*_{-i}) \quad \forall i \in I,\ \forall y_i \in X_i.$$

2.2. The essence of equilibrium and why the existing results are only sufficient

Before proceeding to the notions of recursive transfer continuities, we first analyze the intrinsic nature of Nash equilibrium, and why the conventional continuity is unnecessarily strong and most of the existing results provide only sufficient but not necessary conditions.

In doing so, we define an "upsetting" (irreflexive) binary relation on X, denoted by $\succ$ as follows:

$$y \succ x \quad \text{iff} \quad \exists i \in I \quad s.t.\ (y_i, x_{-i}) \succ_i x. \tag{1}$$

In this case, we say strategy profile y *upsets* strategy profile x. It is clear that "$y \succ x$ for $x, y \in X$" is equivalent to "$x \in X$ is not an equilibrium". We will use these terms interchangeably. Then, one can easily see that a strategy profile $x^* \in X$ is a pure strategy Nash equilibrium if and only if there does not exist any strategy y in X that upsets x^*.

When $x \in X$ is not a pure strategy Nash equilibrium, then there exists a strategy profile $y \in X$ such that $y \succ x$. To establish the existence of an equilibrium, it usually requires *all* strategies in a neighborhood $\mathcal{V}_x$ of x be upset by some strategy profile $z \in X$, denoted by $z \succ \mathcal{V}_x$, i.e., $z \succ x'$ for all $x' \in \mathcal{V}_x$. The topological structure of the conventional continuity surely secures this upsetting relation locally at x by y, i.e., there always exists a neighborhood $\mathcal{V}_x$ of x such that $y \succ \mathcal{V}_x$. As such, no transfers (say, from y to z) or switchings (from player i to j) are needed for securing this upsetting relation locally at x. However, when u_i is not continuous, such a topological relation between the upsetting point y and the neighborhood $\mathcal{V}_x$ may no longer be true, i.e., we may not have $y \succ \mathcal{V}_x$. But, if y can be transferred to z so that $z \succ \mathcal{V}_x$, then the upsetting relation $\succ$ can be secured locally at x. This naturally leads to the following notion of transfer continuity, which is a weak notion of continuity

[5] The results obtained for weak preferences $\succcurlyeq_i$ can also be used to get the results for strict preferences $\succ_i$. Indeed, from $\succ_i$, we can define a weak preference $\succcurlyeq_i$ on $X \times X$ as follows: $y \succcurlyeq_i x$ if and only if $\neg x \succ_i y$. The preference $\succcurlyeq_i$ defined in such a way is called the completion of $\succ_i$. A preference $\succcurlyeq_i$ is said to be complete iff, for any $x, y \in X$, either $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$. A preference $\succcurlyeq_i$ is said to be total iff, for any $x, y \in X$, $x \neq y$ implies $x \succcurlyeq_i y_i$ or $y \succcurlyeq_i x$.

first introduced in Tian (1992a), Tian and Zhou (1992, 1995) and Baye et al. (1993) in order to study preference maximization and the existence of equilibrium in discontinuous games. Since then, this approach has become a main tool that was adopted by Reny (1999) and many others in studying the existence of equilibrium in discontinuous games.

Definition 2.1. The upsetting relation $\succ$ is *transfer continuous* iff, whenever $y \succ x$ for $x, y \in X$, there exists a deviation strategy profile $z \in X$ and a neighborhood $\mathcal{V}_x \subset X$ of x such that $z \succ x'$ for all $x' \in \mathcal{V}_x$, i.e., the upsetting relation $\succ$ can be secured locally at x.

When individuals' preferences $\succcurlyeq_i$ can be represented by numerical payoff functions u_i and the number of players is finite,[6] define the aggregator function $U : X \times X \to \Re$ by

$$U(y, x) = \sum_{i \in I} u_i(y_i, x_{-i}), \quad \forall (x, y) \in X \times X, \tag{2}$$

which refers to the aggregate payoff across individuals where for every player i assuming she or he deviates to y_i given that all other players follow the strategy profile x. The "upsetting" relation $\succ$ is then defined as:

$$y \succ x \quad \text{iff} \quad U(y, x) > U(x, x). \tag{3}$$

The above definition of transfer continuity in turn immediately reduces to the notion of diagonal transfer continuity introduced by Baye et al. (1993) for aggregator function, which together with diagonal transfer quasiconcavity, guarantees the existence of pure strategy Nash equilibrium.[7]

Definition 2.2. A game $G = (X_i, u_i)_{i \in I}$ is *diagonally transfer continuous (DTC)* iff, whenever $U(y, x) > U(x, x)$ for $x, y \in X$ (i.e., whenever x is not an equilibrium), there exists another deviation strategy profile $z \in X$ and a neighborhood $\mathcal{V}_x \subset X$ of x such that $U(z, x') > U(x', x')$ for all $x' \in \mathcal{V}_x$.

Note that, to secure "upsetting" relation locally at x by z: $z \succ \mathcal{V}_x$, it is unnecessary for all players i to have $u_i(z_i, x'_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}_x$, but it is enough for just one player. Diagonal transfer continuity in Baye et al. (1993), better-reply security in Reny (1999), weak transfer continuity in Nessah and Tian (2008), for instance, weaken the conventional continuity along this line.

It may be remarked that, in contrast to the better-reply secure mixed extension, the concept of diagonal transfer continuity makes the problem on the existence of a mixed strategy equilibrium considerably more tractable in games where the sum of the payoff functions is not necessarily upper semicontinuous. Indeed, as Prokopovych and Yannelis (2014) point out, verifying whether a game that is not upper semicontinuous-sum has a better-reply secure mixed extension is quite challenging.

Moreover, to secure "upsetting" relation locally at x by z, it is unnecessary to just fix one player, but it can be done by switching among players. If for every $x' \in \mathcal{V}_x$, there exists a player i such that $(z_i, x'_{-i}) \succ_i x'$, all is done here. In other words, we can secure this "upsetting" relation locally by possibly switching players for every strategy in a neighborhood. This exactly comes up with the notion of weak transfer quasi-continuity introduced by Nessah and Tian (2008), which together with strong diagonal transfer quasiconcavity,[8] guarantees the existence of pure strategy Nash equilibrium.

Definition 2.3. A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer quasi-continuous (WTQC)* iff, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y \in X$ and a neighborhood $\mathcal{V}_x$ of x so that for every $x' \in \mathcal{V}_x$, there exists a player i such that $u_i(y_i, x'_{-i}) > u_i(x')$.

Thus, weak transfer quasi-continuity is a weak form of continuity where a profitable deviation exists not for every point of the neighborhood of x but only for each of the nonequilibrium points belonging to the neighborhood.

Weak transfer quasi-continuity is also called *single-deviation property* in Reny (2009), which is weaker than the better-reply security. It may be remarked that strong diagonal transfer quasiconcavity cannot be replaced by conventional quasiconcavity for weak transfer quasi-continuity to guarantee the existence of equilibrium. Indeed, Reny (2009) shows with the aid of a three-person game that a game that is compact, quasiconcave, and weakly transfer quasi-continuous need not have a pure strategy Nash equilibrium.[9]

Note that, while the weak transfer quasi-continuity (or lower single-deviation property) explicitly exhibits switchings among agents, the diagonal transfer continuity for aggregator function U internalizes (implicitly allows) the switchings. Such implicit switchings have an advantage that it may become easy to check "upsetting" relations, especially for "complementary discontinuities", by which a downward jump in one player's payoff can always be accompanied by an upward jump in another player's payoff (cf. Dasgupta and Maskin, 1986; Simon, 1987), in addition to the advantage that the diagonal transfer continuity can be used to study the existence of mixed strategy equilibrium as shown in Prokopovych and Yannelis (2014).

Although this "upsetting" relation is ensured, it may still not be sufficient for the existence of equilibrium unless some forms of quasiconcavity, transitivity, or monotonicity are imposed. This is why, to make this upsetting relation sufficient for the existence of equilibrium, most of the existing results impose additional assumptions concerning convexity of strategy spaces and quasiconcavity of payoffs in order to use a fixed point theorem. However, such kind of conditions require a preassumption that the strategy space of a game must be a topological vector space, and then can only apply to economic models with linear (convex) structures.

As such, a single upsetting transfer alone may not be enough to guarantee the existence of an equilibrium, some recursive (sequential multiple) upsetting transfers starting from a strategy profile y^0 may be needed for the existence of equilibrium without imposing any quasiconcavity/monotonicity-related conditions. This is similar to extending the weak axiom of revealed preference (WARP) to strong axiom of revealed preference (SARP) in order to fully reveal individuals' preferences. With such recursively upsetting relations, we are able to allow not only sequential transfers, but also the switchings of players in any stage of upsetting transfers and for different strategy profiles in a neighborhood. As a result, such a weak notion of transfer continuity may become both necessary and sufficient for the existence of Nash equilibria for any games. Indeed, we will show that the above intuition and insights turn out

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6 I can be a countably infinite set. In this case, one may define U according to $U(y, x) = \sum_{i \in I} \frac{1}{2^i} u_i(y_i, x_{-i})$. This is a more general formulation.

7 A game $G = (X_i, u_i)_{i \in I}$ is diagonally transfer quasiconcave iff, for any finite subset $Y^m = \{y^1, \ldots, y^m\} \subset X$, there exists a corresponding finite subset $X^m = \{x^1, \ldots, x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\}$, we have $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$, where $co\, D$ denotes the convex hull of the set D.

8 A game $G = (X_i, u_i)_{i \in I}$ is said to be *strongly diagonal transfer quasiconcave* iff for any finite subset $\{y^1, \ldots, y^m\} \subseteq X$, there exists a corresponding finite subset $\{x^1, \ldots, x^m\} \subseteq X$ such that for any subset $\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} \subseteq \{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\}$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\}$, there exists $y \in \{y^{k^1}, \ldots, y^{k^s}\}$ so that $u_i(y_i, x_{-i}) \leq u_i(x)$ for all $i \in I$.

9 However, for two-person games where the strategy sets are subsets of the real line, Prokopovych (2013) shows that the conclusion holds.

to be correct. It is the two requirements—securing upsetting relation and recursive transfers that fully characterize the existence or nonexistence of an equilibrium.

3. Existence of pure strategy Nash equilibria

In this section we investigate the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs/preferences. We will provide two existence theorems. One is based on aggregate payoffs and the other is based on individuals' payoffs/preferences.

3.1. Existence of equilibrium in games with aggregate payoffs

In this subsection we assume that individuals' preferences can be represented by payoff functions. We consider a mapping of individual payoffs into the aggregator function $U : X \times X \to \Re$. The aggregator function approach is pioneered by Nikaido and Isoda (1955), and is also used by Baye et al. (1993). Dasgupta and Maskin (1986) use a similar approach to prove the existence of mixed strategy Nash equilibrium in games with discontinuous payoff functions. An advantage of this approach is that it internalizes the switchings among players so that checking "upsetting" relations are relatively easier due to the complementarity that a game generally has.

Definition 3.1 (*Recursive Upsetting*). A strategy profile $y^0 \in X$ is said to be *recursively upset by* $z \in X$ iff there exists a finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ such that $U(y^1, y^0) > U(y^0, y^0), U(y^2, y^1) > U(y^1, y^1), \ldots, U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$.

For convenience, we say that y^0 is *directly upset by* z iff $z \succ y^0$ and *indirectly upset by* z when $m > 1$. Recursive upsetting says that a strategy profile y^0 can be directly or indirectly upset by a strategy profile z through sequential deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}\}$ in a recursive way that y^0 is upset by y^1, y^1 is upset by $y^2, \ldots,$ and y^{m-1} is upset by z.

Definition 3.2 (*Recursive Diagonal Transfer Continuity*). A game $G = (X_i, u_i)_{i\in I}$ is said to be *recursively diagonal transfer continuous (RDTC)* iff, whenever x is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ (i.e., $U(z, x') > U(x', x')$ for all $x' \in \mathcal{V}_x$) for every z that recursively upsets y^0.

In words, RDTC implies that whenever x is not an equilibrium, there exists a starting point y^0 such that every recursive upsetting chain $\{y^1, y^2, \ldots, y^m\}$ disproves the possibility of an equilibrium in a sufficiently small neighborhood of x, i.e., all points in the neighborhood are upset by all securing strategy profiles that directly or indirectly upset y^0. This implies if the game is not RDTC, then there is a nonequilibrium strategy profile x such that for every $y^0 \in X$ and every neighborhood $\mathcal{V}_x$ of x, some deviation strategy profiles in the neighborhood cannot be upset by a securing strategy profile z that directly or indirectly upsets y^0.

In the definition of RDTC, x is transferred to y^0 that could be any point in X. We can similarly define m-recursive diagonal transfer continuity (m-RDTC).

Definition 3.3 (*m-Recursive Diagonal Transfer Continuity*). A game $G = (X_i, u_i)_{i\in I}$ is said to be *m-recursively diagonal transfer continuous* (*m-RDTC*) iff, whenever x is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for every z that m-recursively upsets y^0.

Thus, a game $G = (X_i, u_i)_{i\in I}$ is RDTC iff it is m-RDTC on X for all $m = 1, 2 \ldots$. Then 1-RDTC implies diagonal transfer continuity, and 1-RWTQC implies weak transfer quasi-continuity (by letting $y^0 = x$), respectively. Since they are in the form of single transfer, (1-)diagonal transfer continuity or (1-)weak transfer quasi-continuity is neither necessary nor sufficient, and thus some form of quasiconcavity such as (strong) diagonal transfer quasiconcavity is needed for the existence of equilibrium as studied in Baye et al. (1993) and Nessah and Tian (2008).

Remark 3.1. Under RDTC, when $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$, we have not only $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$, but also $U(y^{m-1}, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x), \ldots, U(y^1, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. That is, any chain of securing strategy profiles $\{y^1, y^2, \ldots, y^{m-j}\}$ obtained by truncating a recursive upsetting chain $\{y^1, y^2, \ldots, y^{m-1}, z\}$ is also a recursive upsetting chain, including y^1.

Remark 3.2. RDTC neither implies nor is implied by continuity for games with two or more players.[10] This becomes clear when one sees that RDTC is a necessary and sufficient condition for the existence of pure strategy Nash equilibrium while continuity of the aggregator payoff function is neither a necessary nor sufficient condition for the existence of pure strategy Nash equilibrium.

Before formally stating our main result, we first give a brief description and explain why RDTC of an upsetting relation $\succ$ ensures the existence of pure strategy Nash equilibrium in a compact game. Suppose by way of contradiction that a game fails to have a pure strategy Nash equilibrium on a compact strategy space X, by RDTC, for every x, there is a starting transfer strategy profile y^0 such that all points in a neighborhood of x will be upset by any z that recursively upsets y^0. Then there are finite strategy profiles $\{x^1, x^2, \ldots, x^n\}$ whose neighborhoods cover X. Thus, all of the points in a neighborhood, say $\mathcal{V}_{x^1}$, will be upset by a corresponding deviation profile z^1, which means z^1 cannot be in $\mathcal{V}_{x^1}$. If it is in some other neighborhood, say, $\mathcal{V}_{x^2}$, then it can be shown that z^2 will upset all strategy profiles in the union of $\mathcal{V}_{x^1}$ and $\mathcal{V}_{x^2}$ so that z^2 is not in the union of $\mathcal{V}_{x^1}$ and $\mathcal{V}_{x^2}$. Without loss of generality, suppose $z^2 \in \mathcal{V}_{x^3}$. Then we can similarly show that z^3 is not in the union of $\mathcal{V}_{x^1}$, $\mathcal{V}_{x^2}$ and $\mathcal{V}_{x^3}$. With this recursive process going on, we can finally show that $z^n \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots, \cup \mathcal{V}_{x^n}$, which means z^n will not be in the strategy space X, resulting in a contradiction.

Now we are ready to state our main result on the existence of pure strategy Nash equilibrium in games.

Theorem 3.1. *Let X be a compact topological space and $u_i :\to \mathcal{R}$ be arbitrary functions. Then the game $G = (X_i, u_i)_{i\in I}$ possesses a pure strategy Nash equilibrium if and only if it is RDTC on X.*

Proof. Sufficiency. Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium. Then, for each $x \in X$, there exists $y \in X$ such that $U(y, x) > U(x, x)$, and thus, by RDTC, there exists y^0 and a neighborhood $\mathcal{V}_x$ such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ whenever $y^0 \in X$ is recursively upset by z, i.e., for every sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ with $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$, we have $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. Since there is no equilibrium by the contrapositive hypothesis, y^0 is not an equilibrium and thus, by RDTC, such a sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ exists for some $m \geq 1$.

Since X is compact and $X \subseteq \bigcup_{x\in X} \mathcal{V}_x$, there is a finite set $\{x^1, \ldots, x^L\}$ such that $X \subseteq \bigcup_{i=1}^{L} \mathcal{V}_{x^i}$. For each of such x^i, the cor-

[10] In one-player games RDTC is equivalent to the player's utility function possessing a maximum on a compact set, and consequently it implies transfer weak upper continuity introduced in Tian and Zhou (1995), which is weaker than continuity.

responding initial deviation profile is denoted by y^{0i} so that $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ whenever y^{0i} is recursively upset by z^i.

Since there is no equilibrium, for each of such y^{0i}, there exists z^i such that $U(z^i, y^{0i}) > U(y^{0i}, y^{0i})$, and then, by 1-RDTC, we have $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$. Now consider the set of securing strategy profiles $\{z^1, \ldots, z^L\}$. Then, $z^i \notin \mathcal{V}_{x^i}$; otherwise, by $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$, we will have $U(z^i, z^i) > U(z^i, z^i)$, a contradiction. So we must have $z^1 \notin \mathcal{V}_{x^1}$.

Without loss of generality, suppose $z^1 \in \mathcal{V}_{x^2}$. Since $U(z^2, z^1) > U(z^1, z^1)$ by noting that $z^1 \in \mathcal{V}_{x^2}$ and $U(z^1, y^{01}) > U(y^{01}, y^{01})$, then, by 2-RDTC, we have $U(z^2, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1})$. Also, $U(z^2, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$. Thus $U(z^2, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2})$, and consequently $z^2 \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}$.

Again, without loss of generality, suppose $z^2 \in \mathcal{V}_{x^3}$. Since $U(z^3, z^2) > U(z^2, z^2)$ by noting that $z^2 \in \mathcal{V}_{x^3}$, $U(z^2, z^1) > U(z^1, z^1)$, and $U(z^1, y^{01}) > U(y^{01}, y^{01})$, by 3-RDTC, we have $U(z^3, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1})$. Also, since $U(z^3, z^2) > U(z^2, z^2)$ and $U(z^2, y^{02}) > U(y^{02}, y^{02})$, by 2-RDTC, we have $U(z^3, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$. Thus, we have $U(z^3, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3})$, and consequently $z^3 \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}$.

With this recursive process going on, for $k = 3, \ldots, L$, we can show that $z^k \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \ldots, \cup \mathcal{V}_{x^k}$, i.e., z^k is not in the union of $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \ldots, \mathcal{V}_{x^k}$. In particular, for $k = L$, we have $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}$ and thus $z^L \notin X \subseteq \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}$, a contradiction.

Necessity. To see this, first note that, if $x^* \in X$ is a pure strategy Nash equilibrium of a game G, we must have $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$, which is obtained by summing up $u_i(y_i, x^*_{-i}) \leq u_i(x^*) \forall\, y_i \in X_i$ for all players. Let x^* be a pure strategy Nash equilibrium and $U(y, x) > U(x, x)$ for $x, y \in X$. Let $y^0 = x^*$ and $\mathcal{V}_x$ be a neighborhood of x. Then, it is impossible to find any securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$, and thus RDTC holds trivially. ■

Although RDTC is necessary for the existence of a pure strategy Nash equilibrium, it may not be sufficient for the existence of a pure strategy Nash equilibrium when a strategy space is noncompact. As such, RDTC cannot be regarded as being equivalent to the definition of Nash equilibrium without any other restrictions such as the compactness of strategy space. To see this, consider the following counterexample.

Example 3.1. Consider the following two-person game with $X_1 = X_2 = (0, 1)$ and the payoff functions given by

$$u_i(x_1, x_2) = x_i \quad i = 1, 2.$$

The game clearly does not possess a pure strategy Nash equilibrium. However, it is RDTC on X.

Indeed, for any two strategy profiles $x, y \in X$ with $U(y, x) > U(x, x)$, choose $\epsilon > 0$ such that $(x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \subset X$. Let $y^0 = (x_1 + \epsilon, x_2 + \epsilon) \in X$ and $\mathcal{V}_x \subseteq (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon)$. Note that $U(y, x) = y_1 + y_2$. Then, for any finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ with $U(y^1, y^0) > U(y^0, y^0), U(y^2, y^1) > U(y^1, y^1), \ldots, U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$, i.e., $z_1 + z_2 > y_1^{m-1} + y_2^{m-1} > \cdots > y_1^0 + y_2^0$, we have $U(z, x') = z_1 + z_2 > y_1^0 + y_2^0 > x'_1 + x'_2$ for all $x' \in \mathcal{V}_x$. Thus, $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$, which means the game is RDTC on X.

The above theorem assumes that the strategy space of a game is compact. This may still be a restrictive assumption because strategy space of a game may not be closed or bounded when it is finite dimensional. For instance, it is well known that Walrasian mechanism can be regarded as a generalized game. However, when preferences are strictly monotone, excess demand functions are not well defined for zero prices, and then we cannot use Theorem 3.1 to show the existence of competitive equilibrium.

In the following we show that the compactness of strategy space in Theorem 3.1 can also be relaxed.[11] To do so, we first introduce the following stronger version of RDTC.

Definition 3.4. Let B be a subset of X. A game $G = (X_i, u_i)_{i \in I}$ is said to be *RDTC with respect to B* iff, whenever x is not an equilibrium, there exists a strategy profile $y^0 \in B$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that (1) whenever x is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in B and (2) $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for every finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$ for $m \geq 1$.

Condition (1) in the above definition ensures that if a strategy profile x is not an equilibrium for the game $G = (X_i, u_i)_{i \in I}$, it must not be an equilibrium when the strategy space is constrained to be B. Also, when $B = X$, RDTC with respect to B reduces to RDTC on X.

One can easily check the game in Example 3.1 is not RDTC with respect to any compact set B although it is RDTC on X. As such, there is no pure strategy Nash equilibrium by the theorem below.

The following theorem fully characterizes the existence of pure strategy Nash equilibrium in games with an arbitrary (topological) strategy space that may be discrete, continuum, non-convex or non-compact and arbitrary payoff functions that may be discontinuous or nonquasiconcave.

Theorem 3.2. *Let X be an arbitrary topological space and $u_i :\to \mathcal{R}$ be arbitrary functions. Then, the game $G = (X_i, u_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if and only if there exists a compact set $B \subseteq X$ such that the game $G = (X_i, u_i)_{i \in I}$ is RDTC with respect to B.*

Proof. Sufficiency. The proof is essentially the same as that of Theorem 3.1 and we just outline it here. We first show that the game possesses a pure strategy Nash equilibrium x^* in B. Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium in B. Then, since the game G is RDTC with respect to B, for each $x \in B$, there exists $y^0 \in B$ and a neighborhood $\mathcal{V}_x$ such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ for any finite subset of securing strategy profiles $\{y^1, \ldots, y^m\} \subset B$ with $y^m = z$ and $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1}), U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2}), \ldots, U(y^1, y^0) > U(y^0, y^0)$. Since there is no equilibrium in B by the contrapositive hypothesis, y^0 is not an equilibrium in B and thus, by RDTC with respect to B, such a sequence of recursive securing strategy profiles $\{y^1, \ldots, y^{m-1}, z\}$ exists for some $m \geq 1$.

Since B is compact and $B \subseteq \bigcup_{x \in X} \mathcal{V}_x$, there is a finite set $\{x^1, \ldots, x^L\} \subseteq B$ such that $B \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$. For each of such x^i, the corresponding initial deviation profile is denoted by y^{0i} so that $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ whenever y^{0i} is recursively upset by z^i through any finite subset of securing strategy profiles $\{y^{1i}, \ldots, y^{mi}\} \subset B$ with $y^{mi} = z^i$. Then, by the same argument as in the proof of Theorem 3.1, we will obtain that z^k is not in the union of $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \ldots, \mathcal{V}_{x^k}$ for $k = 1, 2, \ldots, L$. For $k = L$, we have $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \ldots \cup \mathcal{V}_{x^L}$ and thus $z^L \notin B \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$, a contradiction. Thus, the game possesses a pure strategy Nash equilibrium x^* in B.

We now show that x^* must be a pure strategy Nash equilibrium in X. Suppose not. x^* will be upset by a strategy profile in $X \setminus B$, and then it is upset by a strategy profile in B, which means x^* is not a Nash equilibrium in B, a contradiction.

Necessity. Suppose x^* is a pure strategy Nash equilibrium. Let $B = \{x^*\}$. Then, the set B is clearly compact. Now, for any $x, y \in X$ such that $U(y, x) > U(x, x)$ for $x, y \in X$, let $y^0 = x^*$ and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$ and $y^0 = x^*$ is a unique element in B, there is no other securing strategy

[11] I thank David Rahman for raising this issue to me. Thanks also to Adam Wong for pointing out a misstatement in an earlier version of Theorem 3.2.

profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$ or x is set by a strategy profile in B. Hence, the game is RDTC with respect to B. ■

The following example illustrates that, although the strategy space is an open unit interval, highly discontinuous and nonquasiconcave, we can use Theorem 3.2 to argue the existence of equilibrium.

Example 3.2. Consider a game with $n = 2, X_1 = X_2 = (0, 1)$ that is an open unit interval set, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2,$$

where $\mathbb{Q} = \{x \in (0, 1) : x \text{ is a rational number}\}$.

Then the game is neither compact nor quasiconcave. It is not weakly transfer quasi-continuous either (thus, as shown in Nessah and Tian, 2008, it is not diagonally transfer continuous, better-reply secure, or weakly transfer continuous). To see this, consider any nonequilibrium x that consists of irrational numbers. Then, for any neighborhood $\mathcal{V}_x$ of x, choosing $x' \in \mathcal{V}_x$ with $x'_1 \in \mathbb{Q}$ and $x'_2 \in \mathbb{Q}$, we have $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$ and $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$ for any $y \in X$. So the game is not weakly transfer quasi-continuous. Thus, there is no existing theorem that can be applied.

However, it is RDTC on X. Indeed, suppose $U(y, x) > U(x, x)$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let y^0 be any vector with rational coordinates, $B = \{y^0\}$, and $\mathcal{V}_x$ be a neighborhood of x. Since $U(y, y^0) \leq U(y^0, y^0)$ for all $y \in X$, it is impossible to find any securing strategy profile y^1 such that $U(y^1, y^0) > U(y^0, y^0)$. Hence, the game is RDTC with respect to B. Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium. In fact, the set of pure strategy Nash equilibria consists of all rational coordinates of $(0, 1) \times (0, 1)$.

In general, the weaker the conditions in an existence theorem, the harder to verify whether the conditions are satisfied in a particular game. The weakness of the concept of RDTC is that it is very hard to verify. For this reason it is useful to provide some sufficient conditions for RDTC.

Definition 3.5 (*Deviation Transitivity*). $G = (X_i, u_i)_{i \in I}$ is said to be *deviational transitive* iff $U(y^2, y^1) > U(y^1, y^1)$ and $U(y^1, y^0) > U(y^0, y^0)$ imply that $U(y^2, y^0) > U(y^0, y^0)$. That is, the upsetting dominance relation is transitive.

We then have the following result without assuming the convexity of strategy space or imposing any form of quasiconcavity.

Proposition 3.1. *Suppose $G = (X_i, u_i)_{i \in I}$ is compact and deviational transitive. If G is 1-RDTC, then there exists a pure strategy Nash equilibrium.*

Proof. We only need to show that, when G is deviational transitive, 1-RDTC implies m-RDTC for $m \geq 1$. Suppose x is not an equilibrium. Then, by 1-RDTC, there exists a strategy profile $y^0 \in X$ and a neighborhood $\mathcal{V}_x$ of x such that $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ whenever $U(z, y^0) > U(y^0, y^0)$ for any $z \in X$.

Now, for any sequence of deviation profiles $\{y^1, \ldots, y^{m-1}, y^m\}$, if $U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1})$, $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$, $\ldots$, $U(y^1, y^0) > U(y^0, y^0)$, we then have $U(y^m, y^0) > U(y^0, y^0)$ by deviation transitivity of U, and thus by 1-RDTC, $U(y^m, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$. Since m is arbitrary, G is RDTC. ■

3.2. Existence of equilibrium in games with individuals' preferences

While the aggregator function approach internalizes the switchings among agents in an upsetting relation so that the proof is relatively simpler, it has some limitations. We need to assume that the preferences of each player can be represented by a payoff function. It is also a cardinal approach, not an ordinal approach. While monotonic transformations keep individuals' upsetting relations unchanged, it may not be true after aggregation, i.e., with a mapping by the aggregator function, a deviation strategy profile y may no longer upset a strategy profile x after some monotonic transformation although y upsets x before the transformation. Besides, the aggregator function approach only reveals the total upsetting relations, but is less clear about individuals' strategic interactions, and thus it lacks a more natural game theoretical analysis.

Nevertheless, the method developed in this paper need not necessarily define the aggregator function U. What matters is the concept of upsetting and recursive securing strategies. We can also show the existence of equilibrium in terms of individuals' payoffs or preferences.

Definition 3.6. A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is said to be *recursively weakly transfer quasi-continuous (RWTQC)* iff, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that for every $x' \in \mathcal{V}_x$ and every finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ with $(y^1_{i_1}, y^0_{-i_1}) \succ_{i_1} y^0$ for some $i_1 \in I$, $(y^2_{i_2}, y^1_{-i_2}) \succ_{i_2} y^1$ for some $i_2 \in I, \ldots, (z_{i_m}, y^{m-1}_{-i_m}) \succ_{i_m} y^{m-1}$ for some $i_m \in I$, there exists player $i \in I$ such that $(z_i, x'_{-i}) \succ_i x'$.

Thus, the notion of RWTQC allows the switchings (transfers) among players in the process of recursive upsetting transfers and at every point in the neighborhood $\mathcal{V}_x$. Similarly, we can define the notions of m-RWTQC and RWTQC with respect to B for $B \subset X$. Notice that, if a game is 1-weakly transfer quasi-continuous, it is WTQC. The converse may not be true.

It is worth noticing the relationship between RWTQC and RDTC, which can help us simplify the proof. By using the upsetting binary relation $\succ$ defined in the previous section, we can define recursive transfer continuity accordingly.

Definition 3.7. The "upsetting" relation $\succ$ is said to be *recursively transfer continuous* iff, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $z \succ \mathcal{V}_x$ for any z that recursively upsets y^0.

Lemma 3.1. *Let $\succ$ be the upsetting relation defined by (1). We then have*

(a) *A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is RWTQC on X if and only if the upsetting relation $\succ$ is recursively transfer continuous on X.*
(b) *A game $G = (X_i, \succcurlyeq_i)_{i \in I}$ is WTQC on X if and only if the upsetting relation $\succ$ is transfer continuous on X.*

The proof is straightforward, and thus it is omitted here.

By Lemma 3.1, we then have the following result that guarantees the existence of pure strategy Nash equilibrium in qualitative games with compact strategy spaces and general preferences.

Theorem 3.3. *Let X be a compact topological space. Then the game $G = (X_i, \succcurlyeq_i)_{i \in I}$ possesses a pure strategy Nash equilibrium if and only if it is RWTQC on X.*

The compactness of strategy space in Theorem 3.3 can also be removed. The following theorem shows the existence of equilibrium in games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and preferences that may not be represented by a payoff function, nontotal/nontransitive, non-convex or discontinuous.

Definition 3.8. Let B be a subset of X. A game $G = (X_i, u_i)_{i \in I}$ is said to be *RWTQC with respect to B* iff, whenever x is not an equilibrium, there exists a strategy profile $y^0 \in B$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that (1) whenever y^0 is upset by a strategy profile in $X \setminus B$, it is upset by a strategy profile in B and (2) for every $x' \in \mathcal{V}_x$ and every finite set of deviation strategy profiles $\{y^1, y^2, \ldots, y^{m-1}, z\}$ with $(y^1_{i_1}, y^0_{-i_1}) \succ_{i_1} y^0$ for some $i_1 \in I$, $(y^2_{i_2}, y^1_{-i_2}) \succ_{i_2} y^1$ for some $i_2 \in I, \ldots, (z_{i_m}, y^{m-1}_{-i_m}) \succ_{i_m} y^{m-1}$ for some $i_m \in I$, there exists player $i \in I$ such that $(z_i, x'_{-i}) \succ_i x'$.

We then have the following full characterization.

Theorem 3.4. *Let X be an arbitrary topological space. Then, the game $G = (X_i, \succcurlyeq_i)_{i \in I}$ has a pure strategy Nash equilibrium if and only if there exists a compact set $B \subseteq X$ such that the game is RWTQC with respect to B.*

Similarly, we can provide some new sufficient conditions by using deviation transitivity.

Definition 3.9 (*Deviation Transfer Transitivity*). $G = (X_i, \succcurlyeq_i)_{i \in I}$ is said to be *deviational transfer transitive* iff for $y^0, y^1, y^2 \in X$, there is some $i \in I$ such that $(y^2_i, y^1_{-i}) \succ_i y^1$ and $(y^1_i, y^0_{-i}) \succ_i y^0$ imply that $(y^2_i, y^0_{-i}) \succ_i y^0$.

Proposition 3.2. *Suppose $G = (X_i, \succcurlyeq_i)_{i \in I}$ is compact and deviational transfer transitive. If G is 1-RWTQC, then there exists a pure strategy Nash equilibrium.*

4. Existence of symmetric pure strategy Nash equilibria

The techniques developed in the previous section can be used to study the existence of symmetric pure strategy Nash equilibrium. Throughout this section, we assume that the strategy spaces for all players are the same. As such, let $X_0 = X_1 = \cdots = X_n$. If in addition, $u_1(y, x, \ldots, x) = u_2(x, y, x, \ldots, x) = \cdots, u_n(x, \ldots, x, y)$ for all $x, y \in X$, we say that $G = (X_i, u_i)_{i \in I}$ is a quasi-symmetric game.

Definition 4.1. A Nash equilibrium $(x^*_1, \ldots, x^*_n)$ of a game G is said to be *symmetric* iff $x^*_1 = \cdots = x^*_n$.

For convenience, we denote, for each player i and for all $x, y \in X_0$, $u_i(x, \ldots, y, \ldots, x)$ the function u_i evaluated at the strategy in which player i chooses y and all others choose x.

Define a quasi-symmetric function $\psi : X_0 \times X_0 \to \mathcal{R}$ by

$$\psi(y, x) = u_i(x, \ldots, y, \ldots, x). \tag{4}$$

Since G is quasi-symmetric, x^* is a symmetric pure strategy Nash equilibrium if and only if $\psi(y, x^*) \leq \psi(x^*, x^*)$ for all $y \in X_i$.

Definition 4.2. $\psi : X_0 \times X_0 \to \mathcal{R}$ is said to be *RDTC* iff, whenever $x \in X$ is not an equilibrium, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $\psi(z, \mathcal{V}_x) > \psi(\mathcal{V}_x, \mathcal{V}_x)$ for any z that recursively upsets y^0.

We then have the following theorem.

Theorem 4.1. *Let X be an arbitrary topological space. Suppose a game $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric and compact. Then it possesses a symmetric pure strategy Nash equilibrium if and only if $\psi : X_0 \times X_0 \to \mathcal{R}$ is RDTC on X.*

Proof. The proof is the same as that of Theorem 3.1 provided U is replaced by ψ, and thus it is omitted here. ■

Similar to Proposition 3.1, we have the following proposition.

Proposition 4.1. *Suppose a game $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, and deviational transitive. If $\psi(x, y)$ defined by 4 is 1-RDTC on X, then it possesses a pure strategy symmetric Nash equilibrium.*

5. Conclusion

This paper fully characterizes the existence of pure strategy Nash equilibrium by replacing two typical forms of conditions — continuity and quasiconcavity with a unique condition, weakening topological vector spaces to arbitrary topological spaces. We have shown that RDTC and RWTQC are not only sufficient but also necessary for the existence of pure strategy Nash equilibrium, respectively for aggregator payoff function and individuals' preferences, in compact games with arbitrary strategy spaces and preferences. Nevertheless, this result cannot be regarded as a definition of Nash equilibrium since a game that satisfies RDTC or RWTQC may not have a Nash equilibrium when the strategy space is not compact.

The basic transfer method we adopt here was systematically developed in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Baye et al. (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. These papers, especially Zhou and Tian (1992), develop three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to study the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian, 1993; Tian and Zhou, 1995). Since an "upsetting" binary relation is defined so that the existence of a Nash equilibrium is equivalent to the existence of a maximal element under this relation, the characterization results on the existence of maximal elements under ordered or nontotal/nontransitive preferences, such as in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Rodríguez-Palmero and Garcíg-Lapresta (2002) can be used to prove or characterize the existence of Nash equilibria. In particular, incorporating recursive transfers into various transfer continuities can be used to obtain full characterization results for many other solution problems. Indeed, the notions of various recursive transfer continuities defined in this paper are equivalent to the notion of transfer irreflexive lower continuity (TILC) introduced in Rodríguez-Palmero and Garcíg-Lapresta (2002) for studying the existence of maximal elements for irreflexive binary relations.

It is worth remarking that extensions in weakening the conventional continuity that use "securing a payoff" have the basic nature of transfer continuity. In fact, payoff security and better-reply security introduced by Reny (1999) and their extensions by many others actually fall in the forms of transfer continuity. Indeed, as Prokopovych (2011) shows in Lemmas 1 and 2, payoff security is equivalent to the transfer lower semicontinuity that was introduced in Tian (1992a), and better-reply security is equivalent to the transfer reciprocal upper semicontinuity in payoff secure games, respectively.

The approach developed in the paper can be similarly used to fully characterize the existence of mixed strategy Nash and Bayesian Nash equilibria in games with general strategy spaces and payoffs. It can also allow us to ascertain the existence of equilibria in important classes of economic games. Tian (2012) shows how it can be employed to characterize the existence of competitive equilibrium for economies with excess demand functions. More importantly, the results obtained can be used to study the existence of equilibrium in various general games with no linear (convex) structures such as in market design theory and matching theory.

As a final remark, we must admit that recursive transfer continuity is very hard to check. As such, the contribution of this paper is not to provide conditions that are easy to check, but to help understand the essence of existence problem in general games. Recursive transfer continuity provides a way to understand what kind of games can or cannot have equilibria. In general, the weaker the conditions in an existence theorem, the harder to verify whether the conditions are satisfied in a particular game. In the paper, we also provide some sufficient conditions for the existence of equilibrium in discontinuous and nonconvex games. A potential future work may be to find more sufficient conditions for recursive diagonal transfer continuity.

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