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RESEARCH ARTICLE

On the existence of Nash equilibrium in discontinuous games

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Abstract This paper offers an equilibrium existence theorem in discontinuous games. We introduce a new notion of very weak continuity, called *quasi-weak transfer continuity* that guarantees the existence of pure strategy Nash equilibrium in compact and quasiconcave games. We also consider possible extensions and improvements of the main result. We present applications to show that our conditions allow for economically meaningful payoff discontinuities.

Keywords Discontinuous games · Quasi-weak transfer continuity · Various notions of transfer continuity · Nash equilibrium

JEL Classification C72 · C62

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1 Introduction

The concept of Nash equilibrium in Nash (1950, 1951) is probably the most important solution concept in game theory. It is immune from unilateral deviations, that is, each player has no incentive to deviate from his/her strategy given that other players do not deviate from theirs. Nash (1951) proved that a finite game has a Nash equilibrium in mixed strategies. Debreu (1952) then showed that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are convex and compact, and (2) players have continuous and quasiconcave payoff functions. However, in many important economic models, such as those in Bertrand (1883), Hotelling (1929), Dasgupta and Maskin (1986), and Jackson (2009), payoffs are particularly discontinuous and/or nonquasiconcave.

Economists then seek weaker conditions that can guarantee the existence of equilibrium. Some seek to weaken the quasiconcavity of payoffs or substitute it with some forms of transitivity/monotonicity of payoffs (cf. McManus 1964; Roberts and Sonnenschein 1977; Nishimura and Friedman 1981; Topkis 1979; Vives 1990; Milgrom and Roberts 1990), some seek to weaken the continuity of payoff functions (cf. Dasgupta and Maskin 1986; Simon 1987; Simon and Zame 1990; Tian 1992a, b, c, 2015; Tian and Zhou 1992, 1995; Reny 1999, 2009, 2013; Bagh and Jofre 2006; Morgan and Scalzo 2007; Carmona 2009, 2011, 2014; Carmona and Podczeck 2015; Prokopovych 2011, 2013; Nessah 2011; He and Yannelis 2015a, b), while others seek to weaken both quasiconcavity and continuity (cf. Yao 1992; Baye et al. 1993; Nessah and Tian 2008; Nessah and Tian 2013; Tian 2015; McLennan et al. 2011; Barelli and Meneghel 2013; Reny 2013).

This paper investigates the existence of pure strategy Nash equilibria in discontinuous games. We introduce a new notion of very weak continuity, called *quasi-weak transfer continuity*, which holds in a large class of discontinuous games. Roughly speaking, a game is quasi-weakly transfer continuous if for every nonequilibrium strategy x^*, there exists a player i, a neighborhood $\mathcal{N}$ and a securing strategy for player i such that for every deviation strategy profile z in $\mathcal{N}$, the lower envelope of agent i's payoff at securing strategy is strictly above the lower envelope of the agent's payoff at the local security level even if the others deviate slightly from z.

We establish that a compact, convex, quasiconcave and quasi-weakly transfer continuous game has a Nash equilibrium. We provide sufficient conditions for quasi-weak transfer continuity such as weak transfer continuity, strong quasi-weak transfer continuity, quasi-weak upper semicontinuity and payoff security, and transfer lower continuity and quasi-upper semicontinuity. These conditions are satisfied in many economic games and are often simple to check. We also provide the existence theorems for symmetric games, and consider further extensions and improvements of our main result. We show that our main results are unrelated to Reny (1999, 2009), Carmona (2009, 2011), Nessah (2011), Prokopovych (2011, 2013, 2015), and Barelli and Meneghel (2013).

The remainder of the paper is organized as follows. In Sect. 2, we first introduce the notion of quasi-weak transfer continuity, and then provide the main existence result on pure strategy Nash equilibrium. We also provide examples illustrating the theorems as well as some sufficient conditions for quasi-weak transfer continuity.

Section 3 considers the equilibrium existence for symmetric games. Section 4 gives some possible extensions and improvements. Section 5 presents some applications of interest to economists that illustrate the usefulness of our results. Section 6 concludes the paper. All the proofs are presented in the "Appendix."

2 Existence of Nash equilibria

Consider a game in normal form: $G = (X_i, u_i)_{i \in I}$, where $I = \{1, \ldots, n\}$ is a finite set of players, X_i is player i's strategy space that is a nonempty subset of a Hausdorff locally convex topological vector space, and u_i is player i's payoff function from the set of strategy profiles $X = \prod_{i \in I} X_i$ to $\mathbb{R}$. For each player $i \in I$, denote by $-i$ all players rather than player i. Also denote by $X_{-i} = \prod_{j \neq i} X_j$ the set of strategies of the players in $-i$. Product sets are endowed with the product topology.

A game $G = (X_i, u_i)_{i \in I}$ is said to be *compact* if for all $i \in I$, u_i is bounded and X_i is compact. A game $G = (X_i, u_i)_{i \in I}$ is said to be *quasiconcave* if for every $i \in I$, X_i is convex and the function u_i is quasiconcave in x_i. A *pure strategy Nash equilibrium* of G is a strategy profile $x^* \in X$ such that $u_i(y_i, x^*_{-i}) \leq u_i(x^*)$ for $y_i \in X_i$ and all $i \in I$.

The following weak notion of continuity, *quasi-weak transfer continuity*, guarantees the existence of equilibrium in compact and quasiconcave games. The function that characterizes quasi-weak transfer continuity is defined as follows: for $x \in X$, let $\Omega(x)$ denote the set of all open neighborhoods of x. For all $i \in I$ and $z \in X$, define the lower envelope of the player's payoff function $u_i(z)$:

$$\underline{u}_i(z) = \sup_{\mathcal{N}_z \in \Omega(z)} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}).$$

Definition 1 A game $G = (X_i, u_i)_{i \in I}$ is said to be *quasi-weakly transfer continuous* if whenever $x \in X$ is not an equilibrium, there exists a player i, $\overline{y}_i \in X_i$ and some neighborhood $\mathcal{N}_x$ of x such that for every $z \in \mathcal{N}_x$, we have $\underline{u}_i(\overline{y}_i, z_{-i}) > \underline{u}_i(z)$.

Quasi-weak transfer continuity means that whenever x is not an equilibrium, some player i has a strategy $\overline{y}_i$ yielding a strictly large lower envelope of the payoff at the local security level even if the others play slightly differently than at x.[1] Our existence result states that a compact quasiconcave game that is quasi-weakly transfer continuous has a Nash equilibrium.

Theorem 1 *If the game $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave, and quasi-weakly transfer continuous, then it possesses a pure strategy Nash equilibrium.*

The proof of the theorem will be presented in the "Appendix." While we can prove this theorem directly, here we choose to prove the theorem by constructing a suitable game with an additional player and then applying Theorem 3.4 in Reny (2013). An

[1] The local security level at z means the value of the least favorable outcome in a neighborhood of z, given by $\underline{u}_i(z)$.

advantage of such a proof is that it nicely connects Theorem 3.4 in Reny (2013) and this approach is insightful. Indeed, when a game fails to have a pure strategy Nash equilibrium, we construct a new game that includes all of the original players I and a new player $i = 0 \notin I$ which is point secure with respect to I. Then, we can show that Theorem 3.4 of Reny (2013) implies that this new game has a Nash equilibrium by quasi-weak transfer continuity and quasiconcavity, and consequently it is also a Nash equilibrium of G by quasi-weak transfer continuity.

Note that, contrary to the results of Reny (1999), Bagh and Jofre (2006), Carmona (2009, 2011), and Prokopovych (2011), which require verifying the closureness of the graph of the vector payoff function, quasi-weak transfer continuity is relatively easier to verify, requiring no analysis of any closures of high-dimensional objects.

Example 1 Consider the following game with two players and the unit square $X_1 = X_2 = [0, 1]$. For player $i = 1, 2$ and $x = (x_1, x_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(x_1, x_2) = \begin{cases} x_i + 1, & \text{if } x_{-i} > \frac{1}{2} \\ 1, & \text{if } x_i > \frac{1}{2} \text{ and } x_{-i} = \frac{1}{2} \\ -1, & \text{if } x_i \leq \frac{1}{2} \text{ and } x_{-i} = \frac{1}{2} \\ x_i - 1, & \text{if } x_{-i} < \frac{1}{2}. \end{cases}$$

It can be verified that the game is not (generalized) better-reply secure so that Proposition 2.4 of Barelli and Meneghel (2013), Theorem 1 of Carmona (2011) and Theorem 3.1 in Reny (1999) cannot be applied. It is not generalized weakly transfer continuous so that Theorem 3.1 of Nessah (2011) cannot be used. It is neither weakly reciprocal upper semicontinuous. As such, Theorem 4 in Prokopovych (2011) and Corollary 2 in Carmona (2009) cannot be applied.

However, it is quasi-weakly transfer continuous. Indeed, let $x = (x_1, x_2)$ be a nonequilibrium strategy profile. If $(x_1, x_2) \neq (\frac{1}{2}, \frac{1}{2})$, then it is easy to show that either player 1 or 2 can secure (x_1, x_2). If $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$, then by nonequilibrium of x and continuity of payoffs at x, there exists a player i, $\overline{y}_i \in X_i$ and some neighborhood $\mathcal{N}$ of x such that for all $z \in \mathcal{N}$, we have

$$\underline{u}_i(\overline{y}_i, z_{-i}) > \underline{u}_i(z).$$

Let $\epsilon > 0$ be sufficiently small, neighborhood $\mathcal{N} \subseteq (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)^2$ of $(\frac{1}{2}, \frac{1}{2})$, and $\overline{y}_i = 1$. Fix any $z \in \mathcal{N}$, then we obtain

(1) If $z_{-i} > \frac{1}{2}$, then $\underline{u}_i(\overline{y}_i, z_{-i}) = 2 > 1 + z_i \geq \underline{u}_i(z)$.
(2) If $z_{-i} \leq \frac{1}{2}$, then $\underline{u}_i(\overline{y}_i, z_{-i}) = 0 > z_i - 1 = \underline{u}_i(z)$.

Since the game is compact and quasiconcave, by Theorem 1, it possesses a pure strategy Nash equilibrium.

Prokopovych (2013) introduced the notion of weak single-deviation property that generalizes better-reply security of Reny (1999), weak transfer quasi-continuity of

Nessah and Tian (2008) and single deviation property of Reny (2009) as follows: A game $G = (X_i, u_i)_{i \in I}$ has the weak single-deviation property if whenever $\overline{x} \in X$ is not an equilibrium, there exists an open neighborhood $\mathcal{N}$ of $\overline{x}$, a set of players $I(\overline{x}) \subseteq I$ and a collection of deviation strategies $\{y_i(\overline{x}) \in X_i : i \in I(\overline{x})\}$ such that for every $z \in \mathcal{N}$ nonequilibrium, there exists a player $j \in I(\overline{x})$ satisfying $u_j(y_j(\overline{x}), z_{-j}) > u_j(z)$. He then provided a theorem (Theorem 2) that shows under the weak single-deviation property and a condition (Condition (ii) in Theorem 2), there is a pure strategy Nash equilibrium in games with compact and convex strategy spaces. Notice that Condition (ii) is unrelated to quasiconcavity. Indeed, Reny (2009) showed with the aid of a three-person game that a game which is compact, quasiconcave, and has the single-deviation property may not have a pure strategy Nash equilibrium[2] (also see Example 2 in Prokopovych (2013)).[3] Moreover, the following example shows that the quasi-weak transfer continuity does not imply the weak single deviation property either. As such, Theorem 1 is unrelated to Theorem 2 in Prokopovych (2013).

Example 2 Consider the following concession game with two players and the unit square $X_1 = X_2 = [0, 1]$. For player $i = 1, 2$ and $x = (x_1, x_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(x_1, x_2) = \begin{cases} 1, & \text{if } x_i = 0 \text{ and } x_{-i} > 0 \\ x_{-i} - x_i + 1, & \text{if } x_i < x_{-i} \text{ and } x_i > 0 \\ 3x_i, & \text{if } x_i = x_{-i} < \frac{1}{2} \\ 0, & \text{if } x_i = x_{-i} \geq \frac{1}{2} \\ x_{-i} - x_i - 1, & \text{if } x_i > x_{-i}. \end{cases}$$

It can be verified that the game does not have the weak single deviation property. Indeed, let $\overline{x} = (\frac{1}{2}, \frac{1}{2})$ be a nonequilibrium. For an open neighborhood $\mathcal{N}$ of $(\frac{1}{2}, \frac{1}{2})$, a set of players $I(\frac{1}{2}, \frac{1}{2}) \subseteq I$ and a collection of deviation strategies $\{y_i(\overline{x}) \in X_i : i \in I(\overline{x})\}$, we can find a nonequilibrium strategy z in $\mathcal{N}$ so that $u_j(y_j(\overline{x}), z_{-j}) \leq u_j(z)$, for each $j \in I(\overline{x})$. To see this, consider two cases:

(1) $I(\frac{1}{2}, \frac{1}{2}) = \{i\}, i = 1, 2$. Let $z \in \mathcal{N}$ so that $z_i = z_{-i} = t \neq y_i$, $t < \frac{1}{2}$ and $t > \frac{1}{2} - \frac{y_i}{2}$ if $y_i > 0$. Then

$$u_i(y_i, z_{-i}) = \begin{cases} 1, & \text{if } y_i = 0 \\ t - y_i + 1, & \text{if } y_i < t \text{ and } y_i > 0 \\ t - y_i - 1, & \text{if } y_i > t. \end{cases}$$
$$\leq 3t = u_i(z).$$

[2] However, for two-person games where the strategy sets are subsets of the real line, Prokopovych (2013) showed that the conclusion holds. Also, the single-deviation property was called weak transfer quasi-continuity in Nessah and Tian (2008) and will be defined below.

[3] Thus, strong transfer quasiconcavity defined below is also unrelated to and cannot be replaced by quasiconcavity. Indeed, as shown in Theorem 4.2, transfer quasi-continuity, together with strong transfer quasiconcavity, guarantees the existence of Nash equilibrium in any compact games, but Reny (2009)'s counterexample shows the nonexistence of Nash equilibrium when strong transfer quasiconcavity is replaced by quasiconcavity.

(2) $I(\frac{1}{2},\frac{1}{2}) = I = \{1,2\}$. Let $z \in \mathcal{N}$ so that $z_i = z_{-i} = t \neq y_i$, for each $i = 1, 2$, $t < \frac{1}{2}$ and $t > \frac{1}{2} - \frac{y_i}{2}$ if $y_i > 0$, $i = 1, 2$. Then $u_j(z) = 3t$ for $j = 1, 2$.
 (i) If $y_1 = y_2 = 0$, then $u_j(y_j, z_{-j}) = 1 < 3t = u_j(z)$, for $j = 1, 2$.
 (ii) If $y_i = 0$ and $y_{-i} > 0$ for $i = 1, 2$, then $u_i(y_i, z_{-i}) = 1 < 3t = u_i(z)$ and

$$u_{-i}(y_{-i}, z_i) = \begin{cases} t - y_{-i} + 1, & \text{if } y_{-i} < t \\ t - y_{-i} - 1, & \text{if } y_{-i} > t. \end{cases}$$
$$\leq 3t = u_{-i}(z).$$

 (iii) If $y_1 > 0$ and $y_2 > 0$, then for $j = 1, 2$ we have

$$u_j(y_j, z_{-j}) = \begin{cases} t - y_j + 1, & \text{if } y_j < t \\ t - y_j - 1, & \text{if } y_j > t. \end{cases}$$
$$\leq 3t = u_j(z).$$

However, it is quasi-weakly transfer continuous. Indeed, $\overline{x} \neq (\frac{1}{2},\frac{1}{2})$ is obviously quasi-weakly transfer continuous. Suppose that $\overline{x} = (\frac{1}{2},\frac{1}{2})$. Let $\epsilon > 0$ be sufficiently small ($\epsilon < \frac{1}{4}$), $\mathcal{N} \subset (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)^2$ and $\overline{y}_i = \epsilon$, for some $i = 1, 2$. Fix any $z \in \mathcal{N}$ and $\mathcal{V}_z \subset \mathcal{N}$. Then $\underline{u}_i(\overline{y}_i, z_{-i}) \geq 1 + z_{-i} - \epsilon > 1 + z_{-i} - z_i + \epsilon \geq \underline{u}_i(z)$.

While it is somewhat simple to verify quasi-weak transfer continuity, it is sometimes even simpler to verify other conditions leading to it.

Definition 2 A game $G = (X_i, u_i)_{i\in I}$ is said to be *strongly quasi-weakly transfer continuous* if whenever $x \in X$ is not an equilibrium, there exists a player i, $\overline{y}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{N}_x$ of x such that for every $z \in \mathcal{N}_x$, we have $u_i(\overline{y}_i, z_{-i}) > \underline{u}_i(z) + \epsilon$.

Definition 3 A game $G = (X_i, u_i)_{i\in I}$ is said to be *quasi upper semicontinuous* (QUSC) if for all $i \in I$, $x \in X$, and $\epsilon > 0$, there exists a neighborhood $\mathcal{N}$ of x such that $u_i(x) \geq \underline{u}_i(z) - \epsilon$, for every $z \in \mathcal{N}$.

Definition 4 A game $G = (X_i, u_i)_{i\in I}$ is said to be *quasi-weakly upper semicontinuous* (QWUSC) if whenever $x \in X$ is not an equilibrium, there exists a player i, $\hat{x}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{N}_x$ of x such that $u_i(\hat{x}_i, x_{-i}) > \underline{u}_i(z) + \epsilon$, for every $z \in \mathcal{N}_x$.

Definition 5 A game $G = (X_i, u_i)_{i\in I}$ is said to be *weakly transfer lower semicontinuous* (WTLSC) if whenever x is not a Nash equilibrium, there exists a player i, $y_i \in X_i$, $\epsilon > 0$ and some neighborhood $\mathcal{N}_x$ of x such that $u_i(y_i, z_{-i}) > u_i(x) + \epsilon$ for all $z \in \mathcal{N}_x$.

It is obvious that (1) upper semicontinuity implies quasi upper semicontinuity, which in turn implies quasi-weak upper semicontinuity, and (2) lower semicontinuity implies payoff security, which in turn implies weak transfer lower semicontinuity. Also, quasi-weak upper semicontinuity and transfer lower semicontinuity, when combined with payoff security[4] and quasi upper semicontinuity, respectively, imply quasi-weak transfer continuity. We then have the following proposition.

Proposition 1 *Suppose that a game G satisfies any of the following conditions:*

(a) it is strongly quasi-weakly transfer continuous;
(b) it is quasi-weakly upper semicontinuous and payoff secure;
(c) it is weakly transfer lower semicontinuous and quasi upper semicontinuous.

Then it is quasi-weakly transfer continuous, and consequently, there exists a pure strategy Nash equilibrium provided that it is also compact and quasiconcave.

Example 3 Consider the two-player game with the following payoff functions defined on $[0, 1] \times [0, 1]$:

$$u_i(x_1, x_2) = \begin{cases} x_i + 1 & \text{if } x_{-i} > \frac{1}{2} \\ x_i - 1 & \text{if } x_{-i} \le \frac{1}{2}. \end{cases}$$

This game is not (generalized) better-reply secure nor (weakly) reciprocal upper semicontinuous. As such, Corollary 3.3, Corollary 3.4 of Reny (1999), Proposition 1 of Bagh and Jofre (2006), and Theorem 4 in Prokopovych (2011) cannot be applied.

However, the game is payoff secure and quasi-weakly upper semicontinuous. To see this, let $i \in I$, $\epsilon > 0$, and $x \in X$. If $x_{-i} \neq \frac{1}{2}$, then it is clear that there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V}$ of x_{-i} such that $u_i(y_i, z_{-i}) \ge u_i(x) - \epsilon$ for each $z_{-i} \in \mathcal{V}$. If $x_{-i} = \frac{1}{2}$, then there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V}$ of x_{-i} such that $u_i(x) - \epsilon = x_i - 1 - \epsilon \le 0 \le u_i(y_i, z_{-i})$ for each $z_{-i} \in \mathcal{V}$. Thus, the game is payoff secure.

Also, let $x = (x_1, x_2)$ be a nonequilibrium strategy profile. Then there exists a player i such that $x_i < 1$. Let $x_i + 2\epsilon < 1$ for some $\epsilon > 0$. If $x_{-i} \neq \frac{1}{2}$, then it is clear that there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V} \subseteq (x_i - \epsilon, x_i + \epsilon) \times [0, 1]$ of x such that for every $z \in \mathcal{V}$ and every neighborhood $\mathcal{V}_z$ of z, $u_i(y_i, x_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{V}_z$. If $x_{-i} = \frac{1}{2}$, then there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V} \subseteq (x_i - \epsilon, x_i + \epsilon) \times [0, 1]$ of x such that for each $z \in \mathcal{V}$ and every neighborhood $\mathcal{V}_z$ of z, there exists $z' \in \mathcal{V}_z$ with $z'_{-i} = \frac{1}{2}$ so that $u_i(z_i, z'_{-i}) + \epsilon = z_i - 1 + \epsilon \le x_i + 2\epsilon - 1 < u_i(y_i, x_{-i}) = 0$. Thus, it is quasi-weakly upper semicontinuous. Since the game is also compact and quasiconcave, then by Proposition 1(b), it possesses a Nash equilibrium.

Example 4 Consider the two-player game with the following payoff functions defined on $[-1, 1] \times [-1, 1]$ by

$$u_i(x_1, x_2) = \begin{cases} x_i + 1 & \text{if } x_{-i} > 0 \\ x_i & \text{if } x_{-i} = 0 \\ x_i - \frac{1}{2} & \text{if } x_{-i} < 0. \end{cases}$$

[4] A game is payoff secure if for every $x \in X$, every $\epsilon > 0$, and every player i, respectively, there exists $\overline{x}_i \in X_i$ such that $u_i(\overline{x}_i, z_{-i}) \ge u_i(x) - \epsilon$ for all z_{-i} in some open neighborhood of x_{-i}.

This game is not generalized better-reply secure. However, it is clearly weakly transfer lower continuous. To see that it is also quasi upper semicontinuous, consider a player i, a strategy x and $\epsilon > 0$. If $x_{-i} \neq 0$, it is obvious that the game is quasi upper semicontinuous. If $x_{-i} = 0$, then there exists a neighborhood $\mathcal{N} \subseteq (x_i - \delta, x_i + \delta] \times (-\delta, \delta)$ of x (with $\delta < \frac{1}{2}$) such that for each $z \in \mathcal{N}$ and each $\mathcal{N}_z$ as a neighborhood of z, there exists $z' \in \mathcal{N}_z$ with $z'_{-i} < 0$ so as $u_i(x) = x_i \geq (x_i + \delta) - \frac{1}{2} - \epsilon \geq u_i(z_i, z'_{-i}) - \epsilon$, which means it is also quasi upper semicontinuous at $x_{-i} = 0$. Since the game is also compact and quasiconcave, then by Proposition 1(c), it possesses a Nash equilibrium.

3 Pure strategy symmetric Nash equilibrium

In this section, it is assumed that $G = (X_i, u_i)_{i \in I}$ is a quasi-symmetric game, i.e., $Z = X_1 = \cdots = X_n$ and $u_1(x, y, \ldots, y) = u_2(y, x, y, \ldots, y) = u_n(y, \ldots, y, x)$ for all $x, y \in Z$. Recall that a Nash equilibrium $(\overline{x}_1, \ldots, \overline{x}_n)$ is symmetric if $\overline{x}_1 = \overline{x}_2 = \cdots = \overline{x}_n$.

Definition 6 A symmetric game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally quasi-weak transfer continuous* if whenever $(x^*, \ldots, x^*) \in X^n$ is not an equilibrium, there exists a player i, a strategy $\overline{y} \in Z$, and a neighborhood $\mathcal{N}$ of x^* such that for every $z \in \mathcal{N}$, we have $\underline{u}_i(z, \ldots, z, \overline{y}, z, \ldots, z) > \underline{u}_i(z, \ldots, z)$.

We then have the following existence theorem for quasi-symmetric games.

Theorem 2 *If the game $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, quasiconcave, and diagonally quasi-weak transfer continuous, then it has a symmetric pure strategy Nash equilibrium.*

The following example illustrates Theorem 2.

Example 5 Consider a timing game between two players on the unit square $X_1 = X_2 = [0, 1]$ studied by Prokopovych (2013). For player $i = 1, 2$ and $x = (x_1, x_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(x_1, x_2) = \begin{cases} 2, & \text{if } x_i < x_{-i} \\ 2, & \text{if } x_i = x_{-i} < \frac{1}{2} \\ 0, & \text{if } x_i = x_{-i} \geq \frac{1}{2} \\ -2, & \text{if } x_i > x_{-i}. \end{cases}$$

It can be verified that the game is not diagonally better-reply secure so that Theorem 4.1 in Reny (1999) cannot be applied. This game is not generalized weakly transfer continuous nor weakly reciprocal upper semicontinuous so that the results in Nessah (2011), Prokopovych (2011) and Carmona (2009) cannot be applied.

However, it is diagonally quasi-weak transfer continuous. Indeed, let (x, x) be a nonequilibrium strategy profile. By nonequilibrium of (x, x), we have $\frac{1}{2} \leq x \leq 1$. Then, there exists a player $i = 1$, some neighborhood $\mathcal{N} \subset (x - \epsilon, x + \epsilon)$ (ϵ is sufficiently small) of x and $\overline{y} = 0$ such that for all $z \in \mathcal{N}$ and $\mathcal{V}_z \subset \mathcal{N}$, we have

$\underline{u}_i(\overline{y}, t) = 2 > -2 = \underline{u}_i(z, z)$. Since the game is also quasi-symmetric, compact, and quasiconcave, by Theorem 2, it possesses a symmetric Nash equilibrium.

Quasiconcavity is still a strong assumption for many economic games. For instance, the classic Bertrand model typically results in nonquasiconcave and discontinuous payoffs. Thus, a general existence result for nonquasiconcave and discontinuous games is called for. In the following, we provide an existence result for general nonquasiconcave and discontinuous games. First, recall the following definition of diagonal transfer quasiconcavity introduced by Baye et al. (1993).

Definition 7 A symmetric game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally transfer quasiconcave* if X is convex, and for every player i, any finite subset of $\{y^1, y^2, \dots, y^m\} \subseteq X$, there is a corresponding finite subset $\{x^1, \dots, x^m\} \subseteq X$ such that for any subset J of $\{1, \dots, m\}$ and every $\overline{x} \in \text{co}\{x^j,\ j \in J\}$, we have

$$u_i(\overline{x}, \dots, \overline{x}) \geq \min_{k \in J} u_i(\overline{x}, \dots, \overline{x}, y^k, \overline{x}, \dots, \overline{x}).$$

While diagonal transfer quasiconcavity is weaker than diagonal quasiconcavity,[5] and consequently weaker than quasiconcavity, the following notion of diagonal weak transfer continuity is stronger than the diagonal quasi-weak transfer continuity.

Definition 8 A symmetric game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally weak transfer continuous* if whenever $(x^*, \dots, x^*) \in X^n$ is not an equilibrium, there exists a player i, strategy $\overline{y} \in X$ and neighborhood $\mathcal{N}$ of x^* such that $u_i(z, \dots z, \overline{y}, z, \dots, z) > u_i(z, \dots, z)$ for all $z \in \mathcal{N}$.

We now state an existence result for nonquasiconcave and discontinuous games.

Theorem 3 *Suppose that $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, diagonally transfer quasiconcave, and diagonally weak transfer continuous. Then, it has a symmetric pure strategy Nash equilibrium.*

Remark 1 Theorem 3 strictly generalizes Proposition 5.2 in Reny (2013).

As an illustration, we will use Bertrand model to show the usefulness of Theorem 3 in Sect. 5. Similar to the previous section, we can provide some sets of sufficient conditions for diagonal quasi-weak transfer continuity by introducing various notions of diagonal upper/lower semicontinuity, such as diagonal quasi-weak upper semicontinuity, diagonal quasi upper semicontinuity, and diagonal weak transfer lower semicontinuity.[6]

4 Further extensions and improvements

We can further improve our main result in two directions. One is to weaken continuity, and the other is to weaken quasiconcavity.

[5] A game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally quasiconcave* if X is convex, and for every player i, all $x^1, \dots, x^m \in X$ and all $\overline{x} \in co\{x^1, \dots, x^m\}$, $u_i(\overline{x}, \dots, \overline{x}) \geq \min_{k=1,\dots,m} u_i(\overline{x}, \dots, \overline{x}, x^k, \overline{x}, \dots, \overline{x})$.

[6] See Nessah and Tian (2008).

4.1 Weakening quasiconcavities

For each set Z, denote by co$Z$ the convex hull of Z. Let $I = \{1, \dots, n\}$ be the set of finite players, X_i the set of pure strategies for player i that is a nonempty subset of a Hausdorff locally convex topological vector space E_i, $X = \prod_{i \in I} X_i$ the set of pure strategies profiles, and $\succeq_i$ a binary relation on X of player i. A game $G = (X_i, \succeq_i)_{i \in I}$ is compact and convex if, for all $i \in I$, X_i is compact and convex. We say that a strategy profile $\overline{x} \in X$ is a *pure strategy Nash equilibrium* of a game G if, for each $i \in I$, and $y_i \in X_i$, $(\overline{x}_i, \overline{x}_{-i}) \succeq_i (y_i, \overline{x}_{-i})$.

A correspondence $C : X \times X \rightrightarrows X \times X$ is said to be ***inherited*** if it satisfies:

(a) (Reflexivity:) for all $x, y \in X$, $(x, y) \in C(x, y)$.
(b) (Transitivity:) for all $x^1, x^2, x^3, y^1, y^2, y^3 \in X$, $(x^1, y^1) \in C(x^2, y^2)$ and $(x^2, y^2) \in C(x^3, y^3)$ imply $(x^1, y^1) \in C(x^3, y^3)$.
(c) (Completeness:) for all $x^1, x^2, y^1, y^2 \in X$, either $(x^1, y^1) \in C(x^2, y^2)$ or $(x^2, y^2) \in C(x^1, y^1)$.
(d) (Smallest Finite Element:) for all $x, x^1, \dots, x^m, y^1, \dots, y^m, z^1, \dots, z^m \in X$, whenever $(x, y^h) \in C(x^h, z^h)$, $h = 1, \dots, m$, there is $h_0 = 1, \dots, m$ such that $(x, y^h) \in C(x^{h_0}, z^{h_0})$ for $h = 1, \dots, m$.

Let us consider the following definition.

Definition 9 A game $G = (X_i, \succeq_i)_{i \in I}$ is said to be $P^{\succeq}$*-correspondence secure* (CS) if for every $i \in I$, there exists an inherited correspondence $P_i^{\succeq} : X \times X \rightrightarrows X \times X$ such that whenever $\overline{x} \in X$ is not an equilibrium of G, there exists a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \rightrightarrows X$ satisfying that for every nonequilibrium $z \in \mathcal{V}$ and for some player j, we have

$$z \notin co\{t \in X \text{ such that } (z, t) \in P_j^{\succeq}(x, y)\} \text{ holds for each } (x, y) \in \text{Graph}(\phi_{\overline{x}}).$$

Theorem 4 *If the game $G = (X_i, \succeq_i)_{i \in I}$ is compact, convex, and $P^{\succeq}$-correspondence secure, then it has a pure strategy Nash equilibrium.*

Remark 2 If for each $x, y, z \in X$, the set $\{t \in X \text{ such that } (z, t) \in P_j^{\succeq}(x, y)\}$ is convex, then the condition $z \notin co\{t \in X \text{ such that } (z, t) \in P_j^{\succeq}(x, y)\}$ becomes $(z, z) \notin P_j^{\succeq}(x, y)$.

Example 6 Consider the two-player game on the square $[0, 1] \times [0, 1]$.

$$u_i(p) = \begin{cases} 4p_i & \text{if } p_{-i} \geq \frac{1}{2} \\ p_i & \text{otherwise.} \end{cases}$$

It is clear that this game is bounded, compact, and quasiconcave, but it is not continuously secure. To see this, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$. Then p is not an equilibrium. For every neighborhood $\mathcal{V}$ of p, correspondence $\varphi : \mathcal{V} \rightrightarrows X$, and $\alpha \in \mathbb{R}^n$, if $\alpha_i \leq 1$, $i = 1, 2$, choosing $z \in \mathcal{V}$ with $z_i > \frac{1}{2}$, we have $u_i(z) = 4z_i \geq 2 > \alpha_i$; if $\alpha_j > 1$ for some j, choosing $z \in \mathcal{V}$ with $z_i < \frac{1}{2}$, $i = 1, 2$, we have $u_j(t_j, z_{-j}) = t_j \leq 1 < \alpha_j$ for all $t_j \in \varphi_j(z)$. Thus, this game is not continuously secure, so Theorem 2.2 and Proposition 2.4 in Barelli and Meneghel (2013) cannot be applied. Since the continuous security condition is weaker than the C-security, Proposition 2.7 of McLennan et al. (2011) cannot be applied either.

This game does not have the generalized deviation property (it is not correspondence secure when $\succeq_i$ can be represented by a utility function u_i). Indeed, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$. Then, p is not an equilibrium. For every neighborhood $\mathcal{V}$ of p and correspondence $\varphi \in \prod_{i \in I} W_{\mathcal{V}}(p_i, p_{-i})$, choosing $z \in \mathcal{V}$ with $z = (\frac{1}{2}, \frac{1}{2})$, and $z' \in \mathcal{V}$ with $z'_{-i} < \frac{1}{2}$, $i \in I$, we have $u_i^{\varphi}(y'_i, z'_{-i}) \leq u_i(y'_i, z'_{-i}) = y'_i \leq 2 = u_i(z)$ for all $y'_i \in \varphi_i(z'_{-i})$. Hence, Theorem 5.6 of Reny (2013) and Theorem 5.8 of Bich and Laraki (2012) cannot be applied.

This game does not have the single lower deviation property (it is not correspondence secure when $\succeq_i$ can be represented by a utility function u_i).[7] Indeed, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$, then p is not an equilibrium. For any neighborhood $\mathcal{V}$ of p and all $y \in X$, choosing $z \in \mathcal{V}$ with $z_j > \frac{1}{2}$ for $j = 1, 2$ and $t \in \mathcal{V}$ with $t_{-i} < \frac{1}{2}$ for $i \in I$, then there exists a neighborhood $\mathcal{V}_z \subset (\frac{1}{2}, 1]^2$ satisfying $\underline{u}_i(z) \geq \inf_{z' \in \mathcal{V}_z} u_i(z_i, z'_{-i}) = 4z_i > 2 > y_i = \underline{u}_i(y_i, t_{-i})$. Thus, this game does not have the single lower deviation property. As such, Theorem 2.2 in Reny (2009) cannot be applied.

This game is not (generalized) better-reply secure. Indeed, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (2, 2)$. Then, (p, u) is in the closure of the graph of its vector function, and p is not an equilibrium. Each player i cannot obtain a payoff strictly above $u_i = 2$. For any neighborhood $\mathcal{V} \subset [0, 1]$ of p_{-i} and for all well-behaved correspondence $\varphi_i : \mathcal{V} \rightrightarrows X_i = [0, 1]$, choosing $p'_{-i} \in \mathcal{V}$ with $p'_{-i} < \frac{1}{2}$, we then have $u_i(q_i, p'_{-i}) = q_i \leq 2 = u_i$ for all $q_i \in \varphi_i(p'_{-i})$. Thus, this game is not (generalized) better-reply secure. Therefore, Corollary 4.5 of Barelli and Meneghel (2013), Theorem 1 of Carmona (2011) and Theorem 3.1 in Reny (1999) cannot be applied.

This game is not generalized payoff secure. Indeed, let $i = 1$, $x = (1, \frac{1}{2})$, $\epsilon = \frac{1}{2}$. Then for any neighborhood $\mathcal{V} \subset [0, 1]$ of $\frac{1}{2}$ and for all well-behaved correspondence $\varphi_i : \mathcal{V} \rightrightarrows [0, 1]$, choosing $p'_2 \in \mathcal{V}$ with $p'_2 < \frac{1}{2}$, we have $u_1(q_1, p'_2) = q_1 \leq 1 < 4 - \epsilon$, for each $q_1 \in \varphi_1(p'_2)$. Thus, this game is not generalized payoff secure, and consequently Theorem 4 in Prokopovych (2011) and Corollary 2 in Carmona (2009) cannot be applied.

However, it is $P^{\succeq}$-correspondence secure where the correspondence $P_i^{\succeq} : X \times X \rightrightarrows X \times X$ is defined as

$$(z, t) \in P_i^{\succeq}(x, y) \text{ iff } u_i(t_i, z_{-i}) - u_i(z) \geq u_i(y_i, x_{-i}) - u_i(x).$$

It is clear that for each $i = 1, 2$, $P_i^{\succeq}$ is inherited. Let $p = (p_1, p_2)$ be a nonequilibrium strategy profile with at least one non-one coordinate. Then, there exists $i \in I$ with

[7] A game $G = (X_i, u_i)_{i \in I}$ has the single lower deviation property if whenever x^* is not an equilibrium, there is a player, a neighborhood $\mathcal{V}$ of x^* and a strategy $\overline{y} \in X$ such that for each $z \in \mathcal{V}$, there is a player j so as $\underline{u}_j(\overline{y}_j, t_{-j}) > \underline{u}_j(z)$ for all $t \in \mathcal{V}$.

$p_i < 1$. Therefore, there exists a neighborhood $\mathcal{V}$ of p, $\epsilon > 0$ with $p_i^{'} + \epsilon < 1$ for all $p^{'} \in \mathcal{V}$ and a well-behaved correspondence $\phi_p : \mathcal{V} \rightrightarrows X$ defined by $\phi_p(p^{'}) = \{(1, 1)\}$, for each $p^{'} \in \mathcal{V}$ such that for each $z \in \mathcal{V}$, there exists a player $j = i$ so as for each $(t, y) \in \text{Graph}(\phi_p)$, we have $u_i(y_i, t_{-i}) = \begin{cases} 4, & \text{if } t_{-i} \geq \frac{1}{2} \\ 1, & \text{otherwise} \end{cases}$ and $u_i(t) = \begin{cases} 4t_i, & \text{if } t_{-i} \geq \frac{1}{2} \\ t_i, & \text{otherwise.} \end{cases}$ Thus, for each $(t, y) \in \text{Graph}(\phi_p)$ we have $u_j(y_j, t_{-j}) - u_j(t) > 0 = u_j(z_j, z_{-j}) - u_j(z)$. As the game is compact and convex, by Theorem 4, the considered game possesses a Nash equilibrium.

Remark 3 The considered game in Example 2 is $P^{\succeq}$-correspondence secure and consequently by Theorem 4 it has a Nash equilibrium.

The following definition extends Definition 1 to a nonquasiconcave game.

Definition 10 A game $G = (X_i, u_i)_{i \in I}$ is said to be *correspondence quasi-weakly transfer continuous* if whenever $\overline{x} \in X$ is not an equilibrium, there exists a neighborhood $\mathcal{N}$ of $\overline{x}$ and a well-behaved correspondence $\phi : \mathcal{N} \rightrightarrows X$ satisfying that for every nonequilibrium $z \in \mathcal{N}$ and for some player j, we have

$$z_j \notin co\{t_j \in X_j \text{ such that } \underline{u}_j(t_j, z_{-j}) - \underline{u}_j(z) \geq \underline{u}_j(y_j, x_{-j}) - \underline{u}_j(x)\}$$

holds for each $(x, y) \in \text{Graph}(\phi)$.

We then have the following corollary which is a strict generalization of Theorem 1.

Corollary 1 *If $G = (X_i, u_i)_{i \in I}$ is compact, convex, and correspondence quasi-weakly transfer continuous, then it has a pure strategy Nash equilibrium.*

Proof It is sufficient to consider in Theorem 4 the following inherited correspondence $P_i^{\succeq} : X \times X \rightrightarrows X \times X$ $(j = 1, \ldots, n)$ defined by:

$$P_j^{\succeq}(x, y) = \{(t, z) \in X \times X : \underline{u}_j(t_j, z_{-j}) - \underline{u}_j(z) \geq \underline{u}_j(y_j, x_{-j}) - \underline{u}_j(x)\}.$$

□

The weak transfer continuity was first introduced in our previously circulated Nessah and Tian (2008), which is the same as the single player deviation property introduced in Prokopovych (2013).

Definition 11 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer continuous* if whenever $x \in X$ is not an equilibrium, there exists a player i, $\bar{y}_i \in X_i$ and some neighborhood $\mathcal{N}_x$ of x such that $u_i(\bar{y}_i, z_{-i}) > u_i(z)$ for all $z \in \mathcal{N}_x$.

The following definition extends Definition 11 to a nonquasiconcave game.

Definition 12 A game $G = (X_i, u_i)_{i \in I}$ is said to be *correspondence weakly transfer continuous* if whenever $\overline{x} \in X$ is not an equilibrium, there exists a neighborhood

$\mathcal{N}$ of $\overline{x}$ and a well-behaved correspondence $\phi : \mathcal{N} \rightrightarrows X$ satisfying that for every nonequilibrium $z \in \mathcal{N}$ and for some player j, we have

$$z_j \notin co\{t_j \in X_j \text{ such that } u_j(t_j, z_{-j}) - u_j(z) \geq u_j(y_j, x_{-j}) - u_j(x)\}$$

holds for each $(x, y) \in \text{Graph}(\phi)$.

Consequently, we obtain the following corollary which is a strict generalization of Nessah and Tian (2008) and Nessah (2011).

Corollary 2 *If $G = (X_i, u_i)_{i \in I}$ is compact, convex, and correspondence weakly transfer continuous, then it has a pure strategy Nash equilibrium.*

Proof It is sufficient to consider in Theorem 4 the following inherited correspondence $P_j^{\succeq} : X \times X \rightrightarrows X \times X$ $(j = 1, \ldots, n)$ defined as follows:

$$P_j^{\succeq}(x, y) = \{(t, z) \in X \times X : \; u_j(t_j, z_{-j}) - u_j(z) \geq u_j(y_j, x_{-j}) - u_j(x)\}.$$

□

Reny (2013) introduced the following notion. A game $G = (X_i, \succeq_i)_{i \in I}$ is said to be *correspondence secure* if whenever $\overline{x} \in X$ is not an equilibrium, there exists a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi : \mathcal{V} \rightrightarrows X$ so that for each $z \in \mathcal{V}$, there exists a player j for whom $z_j \notin co\{t_j \in X_j : (t_j, z_{-j}) \succeq_j (y_j, x_{-j})\}$ holds for each $(x, y) \in \text{Graph}(\phi)$.

For the game $G = (X_i, \succeq_i)_{i \in I}$, we associate the following correspondence $P_i^{\succeq} : X \times X \rightrightarrows X \times X$ $(i = 1, \ldots, n)$ as follows:

$$P_i^{\succeq}(x, y) = \{(t, z) \in X \times X : \; (t_i, z_{-i}) \succeq_i (y_i, x_{-i})\}.$$

Obviously, if the relation $\succeq_i$ is complete, reflexive and transitive, then $P_i^{\succeq}$ is inherited and consequently we have the following corollaries:

Corollary 3 (Prokopovych 2015) *If $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave, and correspondence secure, then it has a pure strategy Nash equilibrium.*

Corollary 4 (Reny 2013) *If $G = (X_i, u_i)_{i \in I}$ is compact, convex, complete, reflexive, transitive, and correspondence secure, then it has a pure strategy Nash equilibrium.*

Corollary 5 (Barelli and Meneghel 2013) *If $G = (X_i, u_i)_{i \in I}$ is compact, convex, and continuously secure, then it has a pure strategy Nash equilibrium.*

Corollary 6 (Barelli and Meneghel 2013) *If $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave and generalized better-reply secure, then it has a pure strategy Nash equilibrium.*

Corollary 7 (McLennan et al. 2011) *If $G = (X_i, u_i)_{i \in I}$ is compact, convex, and C-secure, then it has a pure strategy Nash equilibrium. If the restriction operator $\mathcal{X}$ is universal, then Theorem 4 is a strict generalization of Theorem 3.4 of McLennan et al. (2011).*

4.2 Further weakening continuity

Definition 13 A game $G = (X_i, u_i)_{i \in I}$ is said to be *pseudo quasi-weakly transfer continuous* if whenever $x^* \in X$ is not an equilibrium, there exists a neighborhood $\mathcal{N}$ of x^*, a player i, and a strategy $\overline{y}_i \in X_i$ such that $\inf_{z' \in \mathcal{N}} u_i(\overline{y}_i, z'_{-i}) > \inf_{z'' \in \mathcal{N}} u_i(z_i, z''_{-i})$, for each $z \in \mathcal{N}$.

The difference between pseudo quasi-weak transfer continuity and quasi-weak transfer continuity is that the former takes the neighborhood $\mathcal{N}_z$ equal to $\mathcal{N}_x$ so that quasi-weak transfer continuity implies pseudo quasi-weak transfer continuity.

Definition 14 A game $G = (X_i, u_i)_{i \in I}$ is said to be *strongly quasiconcave* if for each $i \in I$, two sets $\mathcal{N}^1$ and $\mathcal{N}^2$ with $\mathcal{N}^1 \cap \mathcal{N}^2 \neq \emptyset$ and $y_i^1, y_i^2 \in X_i$ with $\inf_{z' \in \mathcal{N}} u_i(y_i^j, z'_{-i}) > \inf_{z'' \in \mathcal{N}^j} u_i(z_i, z''_{-i})$ for each $z \in \mathcal{N}^j$, $j = 1, 2$, $\inf_{z' \in \mathcal{N}} u_i(\tilde{y}_i, z'_{-i}) > \inf_{z'' \in \tilde{\mathcal{N}}} u_i(z_i, z''_{-i})$ for each $z \in \tilde{\mathcal{N}} = \mathcal{N}^1 \cap \mathcal{N}^2$ and $\tilde{y}_i \in co(y_i^1, y_i^2)$.

We then have the following result.

Theorem 5 *If the game $G = (X_i, u_i)_{i \in I}$ is compact, convex, strongly quasiconcave and pseudo quasi-weakly transfer continuous, then it possesses a pure strategy Nash equilibrium.*

The following proposition shows that pseudo quasi-weak transfer continuity is also weaker than better-reply security, and consequently Theorem 5 extends Theorem 3.1 in Reny (1999) by weakening better-reply security.

Proposition 2 *If $G = (X_i, u_i)_{i \in I}$ is better-reply secure, then it is pseudo quasi-weakly transfer continuous.*

While in terms of continuity, Theorem 5 is more interesting than Theorem 1 as well as Theorem 3.1 in Reny (1999), the strong quasiconcavity of G is more complicated to check. A question is then whether strong quasiconcavity of G can be replaced by quasiconcavity of $u_i(x_i, x_{-i})$ in x_i. Unfortunately, the answer is negative.

Example 7 Consider, on the unit square, the following game that has no pure strategy Nash equilibrium.

$$u_1(x) = \begin{cases} x_1, & \text{if } x_2 = 0 \\ 2 - x_1, & \text{otherwise} \end{cases}$$

$$u_2(x) = \begin{cases} x_2, & \text{if } x_1 = 1 \\ 2 - x_2, & \text{otherwise} \end{cases}$$

The considered game is pseudo quasi-weakly transfer continuous and quasiconcave, but it is not strongly quasiconcave. Indeed, let $\mathcal{N}^1 = [0, \frac{1}{2} + \epsilon)$, $\mathcal{N}^2 = (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$ (for sufficiently small ϵ) and $y_1^1 = \frac{1}{2} + \epsilon$, $y_1^2 = 0$. Then, we have $u_1(y_1^j, z_2) > \inf_{z' \in \mathcal{N}^j} u_1(z_1, z'_2)$, for each $z \in \mathcal{N}^j$, $j = 1, 2$. Or if we consider $z = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{N}^2 = \mathcal{N}^1 \cap \mathcal{N}^2$ and $\tilde{y}_1 = \frac{1}{2} \in co(y_1^1, y_1^2)$, we obtain $u_1(\tilde{y}_1, z_2) = \inf_{z' \in \mathcal{N}^2} u_1(z_1, z'_2) = \frac{3}{2}$.

Our main result can be further improved by introducing the following notions of transfer quasi-continuity and strong diagonal transfer quasiconcavity.

Definition 15 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer quasi-continuous* if whenever $x \in X$ is not an equilibrium, there exists a $y \in X$ and a neighborhood $\mathcal{N}_x$ of x such that for every $x' \in \mathcal{N}_x$, there exists a player i satisfying $u_i(y_i, x'_{-i}) > u_i(x')$.

Weak transfer quasi-continuity, which was first introduced in our previously circulated Nessah and Tian (2008) and then called *single-deviation property* in Reny (2009), only requires that each strategy profile in a neighborhood of x be upset by one, but not all players. Thus, it is a very weak notion of continuity so that it is a form of *quasi-continuity*.

Definition 16 A game $G = (X_i, u_i)_{i \in I}$ is said to be *strongly diagonal transfer quasiconcave* if for any finite subset $\{y^1, \ldots, y^m\} \subseteq X$, there exists a corresponding finite subset $\{x^1, \ldots, x^m\} \subseteq X$ such that for any subset $\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} \subseteq \{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\}$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\}$, there exists $y \in \{y^{k^1}, \ldots, y^{k^s}\}$ so that

$$u_i(y_i, x_{-i}) \leq u_i(x) \quad \forall i \in I. \tag{1}$$

It is clear that a game is diagonally transfer quasiconcave if it is strongly diagonal transfer quasiconcave.[8] We then have the following result that generalizes Corollary 1 in Prokopovych (2013).

Theorem 6 *Suppose that a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, weakly transfer quasi-continuous. Then, the game possesses a pure strategy Nash equilibrium if and only if it is strongly diagonal transfer quasiconcave.*

It may be remarked that weak transfer quasi-continuity and quasi-weak transfer continuity are not implied by nor imply each other. The game considered in Example 2 is quasi-weakly transfer continuous, but it does not have the weak single-deviation property which in turn does not satisfy the single-deviation property/weak transfer quasi-continuity. On the other hand, the game in Example 3.1 in Reny (2009) is weakly transfer quasi-continuous, but it is not quasi-weakly transfer continuous.

While weak transfer quasi-continuity in Theorem 6 is weaker than the better-reply security and diagonal transfer continuity, it requires that the game be strongly diagonal transfer quasiconcave. Can strong diagonal transfer quasiconcavity in Theorem 6 be replaced by conventional quasiconcavity? Unfortunately, the answer is no. Reny (2009) showed this by giving a counterexample (Example 3.1 in his paper) where a game $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave, and weakly transfer quasi-continuous, but it may not possess a pure strategy Nash equilibrium.

Thus, Theorems 3 and 6 both show that there is a trade-off between continuity condition and quasiconcavity condition.

[8] Indeed, summing up (1) and denoting $U(x, y) = \sum_{i \in I} u_i(y_i, x_i)$, we have $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$, which is the condition for diagonal transfer quasiconcavity.

5 Applications

In this section, we show how our main existence results are applied to some important economic games. We provide two applications: one is the shared resource games intensively studied by Rothstein (2007), and the other is the classic Bertrand price competition games studied first by Bertrand (1883).

5.1 The shared resource games

The shared resource games that usually result in discontinuous payoffs include a wide class of games such as the canonical game of fiscal competition for mobile capital. In these games, players compete for a share of a resource that is in fixed total supply, except perhaps at certain joint strategies. Each player's payoff depends on her opponents' strategies only through the effect those strategies have on the amount of the shared resource that the player obtains. As Rothstein (2007) argued, when ad valorem taxes instead of unit taxes are adopted and the aggregate amount of mobile capital is fixed instead of variable, it will typically result in at least one, and possibly many, discontinuity points.

Formally, for such a game $G = (X_i, u_i)_{i\in I}$, each player i has a convex and compact strategy space $X_i \subset \mathbb{R}^l$ and a payoff function u_i that depends on other players' strategies only through the *sharing rule* defined by $S_i : X \to [0, \bar{s}]$ with $\bar{s} \in (0, +\infty)$. That is to say, each player has a payoff function $u_i : X \to \mathbb{R}$ with the form $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$ where $F_i : X_i \times [0, \bar{s}] \to \mathbb{R}$ and u_i is bounded.[9]

Let $D_i \subseteq X$ be the set of joint strategies at which S_i is discontinuous and let the set $\Delta = \bigcup_{i\in I} D_i$ be all of the joint strategies at which one or more of the sharing rules are discontinuous. The set $X\backslash\Delta$ is then all of the joint strategies at which the sharing rules are continuous.

Rothstein (2007) showed a shared resource game possesses a pure strategy Nash equilibrium if the following conditions are satisfied: (1) X is compact and convex; (2) u_i is continuous on X and quasiconcave in x_i; (3) S_i satisfies: (3.i) $\sum_{i=1}^n S_i(x) = \bar{s}$ for all $x \in X \setminus \Delta$; (3.ii) there exists $\underline{s} \in [0, \bar{s}]$ such that $\sum_{i=1}^n S_i(x) = \underline{s}$ for all $x \in \Delta$; (3.iii) for all i, $(x_i, x_{-i}) \in D_i$ and every neighborhood $\mathcal{V}(x_i)$ of x_i, there exists $x_i' \in \mathcal{V}(x_i)$ such that $(x_i', x_{-i}) \in X \setminus D_i$; (3.iv) there exists a constant $\tilde{s}_i$ satisfying $\bar{s} \geq \tilde{s}_i > \bar{s}/n$ such that for all i, all $(x_i, x_{-i}) \in \Delta$, and all $(x_i', x_{-i}) \in X \setminus D_i$, $S_i(x_i', x_{-i}) \geq \tilde{s}_i \geq S_i(x_i, x_{-i})$; (4) F_i is continuous, nondecreasing in s_i, and satisfies $\max_{x_i\in X_i} F_i(x_i, s_i) > \max_{x_i\in X_i} F_i(x_i, \bar{s}/n)$ for any $s_i > \bar{s}/n$.

In the following, we will give an existence result with much simpler conditions and its proof is also much easier:

Assumption 1: The game is compact and quasiconcave.

Assumption 2: If $(y_i, x_{-i}) \in D_i$ and $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ for player i, then there exists some player $j \in I$ and $\bar{y}_j$ such that $(\bar{y}_j, x_{-j}) \in X\backslash D_j$ and $F_j(\bar{y}_j, S_j(\bar{y}_j, x_{-j})) > F_j(x_j, S_j(x))$.

[9] For more details on this model, see Rothstein (2007).

Assumption 3: If $(y_i, x_{-i}) \in X \backslash D_i$ and $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ for player i, then there exists a player $j \in I$, a deviation strategy profile $\bar{y}_i$ and a neighborhood $\mathcal{N}_x$ of x such that for every $z \in \mathcal{N}_x$, we have $\underline{F}_j(\bar{y}_j, S_j(\bar{y}_j, z_{-j})) > \underline{F}_j(z_j, S_j(z_j, z_{-j}))$.

Assumption 1 is standard. A well-known sufficient condition for a composite function $u_i = F_i[x_i, S_i(x_i, x_{-i})]$ to be quasiconcave is that F_i is quasiconcave and nondecreasing in s_i, and S_i is concave. Assumption 2 means that if x is not an equilibrium and can be improved at a discontinuous strategy profile (y_i, x_{-i}) when player i uses the deviation strategy y_i, then there exists a player j such that it must also be improved by a continuous strategy profile $(\bar{y}_j, x_{-j})$ when player j uses the deviation strategy $\bar{y}_j$. Assumption 3 means that if a strategy profile x is not an equilibrium and can be improved by a continuous strategy profile (y_i, x_{-i}) when player i uses a deviation strategy y_i, then there exists a securing strategy profile $\bar{y}$ and a neighborhood of x such that all points in the neighborhood cannot be equilibria. We then have the following result.

Proposition 3 *A shared resource game possesses a pure strategy Nash equilibrium if it satisfies Assumptions 1–3.*

5.2 The Bertrand price competition games

It is well known that Bertrand competition typically results in discontinuous and nonquasiconcave payoffs. It is a normal form game in which each of $n \geq 2$ firms, $i = 1, 2, \ldots, n$, simultaneously sets a price $p_i \in P = [0, \overline{p}]$. Under the assumption of profit maximization, the payoff to each firm i is

$$\pi_i(p_i, p_{-i}) = p_i D_i(p_i, p_{-i}) - C_i(D_i(p_i, p_{-i})),$$

where p_{-i} denotes the vector of prices charged by all firms other than i, $D_i(p_i, p_{-i})$ represents the total demand for firm i's product at prices (p_i, p_{-i}), and $C_i(D_i(p_i, p_{-i}))$ is firm i's total cost of producing the output $D_i(p_i, p_{-i})$. A Bertrand equilibrium is a Nash equilibrium of this game.

Let $A_i \subseteq P^n$ be the set of joint strategies at which π_i is discontinuous, $\Delta = \bigcup_{i \in I} A_i$ be the set of all of the joint strategies at which one or more of the payoffs are discontinuous, and $X \backslash \Delta$ be the set of all joint strategies at which all of the payoffs are continuous.

We make the following assumptions:

Assumption $1'$: The game is compact, convex, and strongly diagonal transfer quasiconcave.

Assumption $2'$: If $(q_i, p_{-i}) \in A_i$ and $\pi_i(q_i, p_{-i}) > \pi_i(p)$ for $i \in I$, then there exists a firm $j \in I$, and $\bar{q}_j$ such that $(\bar{q}_j, p_{-i}) \in P^n \backslash A_j$ and $\pi_j(\bar{q}_j, p_{-i}) > \pi_i(p)$.

Assumption $3'$: If $(q_i, p_{-i}) \in P^n \backslash A_i$ and $\pi_i(q_i, p_{-i}) > \pi_i(p)$ for player i, then there exists a player $j \in I$, a deviation strategy profile $\bar{q}_j$ and a neighborhood $\mathcal{N}_p$ of p such that for every $r \in \mathcal{N}_p$, we have $\underline{\pi}_j(\bar{q}_j, z_{-j}) > \underline{\pi}_j(r)$.

We then have the following result.

Proposition 4 *Each Bertrand price competition game has a pure strategy Nash equilibrium if it satisfies Assumptions* $1'$–$3'$.

Example 8 Consider a quasi-symmetric two-player Bertrand price competition game on the square $[0, a] \times [0, a]$ with $a > 0$. Assume that the demand function is discontinuous and is defined by

$$D_i(p_i, p_{-i}) = \begin{cases} \alpha f(p_i) & \text{if } p_i < p_{-i} \\ \beta f(p_i) & \text{if } p_i = p_{-i} \\ \gamma f(p_i) & \text{if } p_i > p_{-i} \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nonincreasing function and $\alpha > \beta > \gamma \geq 0$. Suppose that the total cost of production is zero for each firm. Then, the payoff function for each firm i becomes

$$\pi_i(p_i, p_{-i}) = \begin{cases} \alpha p_i f(p_i) & \text{if } p_i < p_{-i} \\ \beta p_i f(p_i) & \text{if } p_i = p_{-i} \\ \gamma p_i f(p_i) & \text{if } p_i > p_{-i} \end{cases}.$$

The game is quasi-symmetric and compact. Since $\alpha > \beta > \gamma$ and f is nonincreasing, it is clearly diagonally quasiconcave. Thus, Assumption $1'$ is satisfied. Note that the set of discontinuity points is given by $A = A_1 = A_2 = \{(p_1, p_2) \colon p_1 = p_2\}$. Assumption $2'$ is obviously satisfied. Indeed, if $\pi_i(q_i, p) > \pi_i(p, p)$, we must have $q_i \neq p$, *i.e.*, $(q_i, p) \notin A$.

We show that Assumption $3'$ is also satisfied. Let $(q_i, p) \notin A$ and $\pi_i(q_i, p) > \pi_i(p, p) + 3\epsilon$. Thus, there exists a neighborhood $\mathcal{N}$ of p with $q_i \notin \mathcal{N}$ such that $\pi_i(q_i, p) - \epsilon \leq \pi_i(q_i, p')$ for all $p' \in \mathcal{N}$ by the continuity of f. We also have $\pi_i(p, p) + \epsilon \geq \pi_i(p', p')$ for all $p' \in \mathcal{N}$. Therefore, $\underline{\pi}_i(q_i, p') > \pi_i(p', p') \geq \underline{\pi}_i(p', p')$ for every $p' \in \mathcal{N}$. Then, by Proposition 4, the considered game possesses a symmetric pure strategy Nash equilibrium.

6 Conclusion

In this paper, we investigate the existence of Nash equilibria in games that may be discontinuous and/or nonquasiconcave. We offer new existence results on Nash equilibrium for a large class of discontinuous games by introducing new notions of weak continuity, such as quasi-weak transfer continuity, strong quasi-weak transfer continuity, pseudo quasi-weak transfer continuity, weak transfer quasi-continuity, $P^{\succeq}$-correspondence security, etc. We prove our main result—Theorem 1 by constructing a suitable game with an additional player and then applying Theorem 3.4 of Reny (2013). We prove Theorem 4 in a similar manner by applying Theorem 5.6 in Reny

(2013). An advantage of such an approach is that these results are nicely connected to the results obtained in Reny (2013). Since this literature is becoming filled with many different results and proofs, we should make connections whenever possible so that the literature can begin to become somewhat more unified.

Our equilibrium existence results neither imply nor are implied by those results in Baye et al. (1993), Reny (1999, 2009), Carmona (2009, 2011), Prokopovych (2011, 2013), Nessah (2011), McLennan et al. (2011), and Barelli and Meneghel (2013). These results permit us to significantly weaken the continuity conditions for the existence of Nash equilibria. We also provide examples and economic applications where our general results are applicable.

The approach developed in the paper can be similarly used to study the existence of mixed strategy Nash and Bayesian Nash equilibria in general discontinuous games. For details, see our earlier version of this paper (cf. Nessah and Tian (2008)). By imposing diagonal transfer continuity defined in Baye et al. (1993) and transfer lower semicontinuity introduced in Tian (1992a) and applying the KKM lemma properly, Prokopovych and Yannelis (2014) proved the existence of the mixed strategy equilibrium. Our conjecture is that, by replacing diagonal transfer continuity with weak transfer quasi-continuity, one may also similarly prove the existence of the mixed strategy equilibrium.

7 Appendix

Proof of Theorem 1 Define a new game G' that includes all of the original players $i \in I$ and a new player $i = 0 \notin I$. Thus, the new set of players is $N = I \cup \{0\}$. Player 0's strategy set is X and the strategy set of player $i \in I$ is X_i. If player 0 chooses $z \in X$ and each player $i \in I$ chooses $x_i \in X_i$, then the payoff to player 0 is

$$v_0(z, x) = \begin{cases} 1, & \text{if } z = x \\ 0, & \text{otherwise} \end{cases}$$

and the payoff to player $i \in I$ is

$$v_i(z, x) = \underline{u}_i(x) - \underline{u}_i(z).$$

Suppose that the game G does not have an equilibrium in X. Let $\overline{x} \in X$ be a non-equilibrium of G'. Then, by quasi-weak transfer continuity, there exists a player i, a strategy $\overline{y}_i \in X_i$ and some neighborhood $\mathcal{N}_{\overline{x}}$ of $\overline{x}$ such that for every $z \in \mathcal{N}_{\overline{x}}$, we have $\underline{u}_i(\overline{y}_i, z_{-i}) > \underline{u}_i(z)$. Therefore, for each $t \in \mathcal{N}_{\overline{x}}$, there is a player $j = i \in I$ for whom, we have for each $z \in \mathcal{N}_{\overline{x}}$,

$$v_j(z, (\overline{y}_j, z_{-j})) = \underline{u}_i(\overline{y}_i, z_{-i}) - \underline{u}_i(z) > 0 = v_j(t, t).$$

Hence G' is point secure with respect to I. Consequently, since $\underline{u}_i(x_i, x_{-i})$ is quasi-concave in x_i, then this new game satisfies the hypotheses of Theorem 3.4 in Reny (2013) and therefore possesses a pure strategy Nash equilibrium $(\overline{z}, \overline{x})$. Payoff of player 0 implies that $\overline{z} = \overline{x}$. For each player $j \in I$, his payoff implies that $\overline{x}_j$ maximizes $\underline{u}_j(y_j, \overline{x}_{-j})$. By quasi-weak transfer continuity, there exists a player i and a strategy $\overline{y}_i \in X_i$ such that $\underline{u}_i(\overline{y}_i, \overline{x}_{-i}) > \underline{u}_i(\overline{x})$, which contradicts that $\overline{x}_i$ maximizes $\underline{u}_i(y_i, \overline{x}_{-i})$. □

Proof of Proposition 1 (a) Suppose that G is strongly quasi-weakly transfer continuous. Then, if $x \in X$ is not an equilibrium, there exists a player i, $\overline{y}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{V}$ of x such that for every $z \in \mathcal{V}$, we have $u_i(\overline{y}_i, z_{-i}) \geq \underline{u}_i(z) + \epsilon$. Therefore, for each $z \in \mathcal{V}$ and $\mathcal{N}_z \subset \mathcal{V}$ as a neighborhood of z, we have

$$\inf_{z' \in \mathcal{N}_z} u_i(\overline{y}_i, z'_{-i}) \geq \inf_{z' \in \mathcal{N}_z} \underline{u}_i(z_i, z'_{-i}) + \epsilon. \tag{2}$$

Indeed, if (2) is false for some $z \in \mathcal{V}$ and $\mathcal{N}_z \subset \mathcal{V}$, then there is $\tilde{z} \in \mathcal{N}_z$ so that

$$u_i(\overline{y}_i, \tilde{z}_{-i}) < \inf_{z' \in \mathcal{N}_z} \underline{u}_i(z_i, z'_{-i}) + \epsilon \leq \underline{u}_i(z_i, \tilde{z}_{-i}) + \epsilon \leq u_i(\overline{y}_i, \tilde{z}_{-i}),$$

which is impossible. Obviously, for each $z \in X$ and each neighborhood $\mathcal{N}_z$ of z, we have

$$\inf_{z' \in \mathcal{N}_z} u_i(t_i, z'_{-i}) = \inf_{z' \in \mathcal{N}_z} \underline{u}_i(t_i, z'_{-i}), \quad \text{for each } t_i \in X_i. \tag{3}$$

Combining (2) and (3), we obtain for each $\mathcal{N}_z \subset \mathcal{V}$,

$$\underline{u}_i(\overline{y}_i, z_{-i}) \geq \inf_{z' \in \mathcal{N}_z} u_i(\overline{y}_i, z'_{-i}) \geq \inf_{z' \in \mathcal{N}_z} \underline{u}_i(z_i, z'_{-i}) + \epsilon = \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + \epsilon.$$

Consequently for each $z \in \mathcal{V}$, we have $\underline{u}_i(\overline{y}_i, z_{-i}) \geq \underline{u}_i(z) + \epsilon > \underline{u}_i(z)$, which means G is quasi-weakly transfer continuous.

(b) Suppose that G is *QWUSC* and payoff secure. If $\overline{x} \in X$ is not a Nash equilibrium, then by quasi-weak upper semicontinuity, some player i has a strategy $\hat{x}_i \in X_i, \epsilon > 0$ and a neighborhood $\mathcal{N}^1$ of $\overline{x}$ such that for every $z \in \mathcal{N}^1$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}^1$ of z, $u_i(\hat{x}_i, \overline{x}_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{N}_z$. The payoff security of G implies that there exists a strategy y_i and a neighborhood $\mathcal{N}^2$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) \geq u_i(\hat{x}_i, \overline{x}_{-i}) - \frac{\epsilon}{2}$ for all $z \in \mathcal{N}^2$. Thus, for every $z \in \mathcal{N} = \mathcal{N}^1 \cap \mathcal{N}^2$ and its neighborhood $\mathcal{N}_z \subseteq \mathcal{N}$, there exists $z' \in \mathcal{N}_z$ such that $u_i(y_i, z_{-i}) > u_i(z_i, z'_{-i}) + \frac{\epsilon}{2}$.

(c) Suppose that G is *WTLSC* and *QUSC*. Then, if $\overline{x} \in X$ is not a Nash equilibrium, by *WTLSC*, there exists a player i, $y_i \in X_i$, $\epsilon > 0$ and a neighborhood $\mathcal{N}^1$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) > u_i(\overline{x}) + \epsilon$ for all $z \in \mathcal{N}^1$. The QUSC implies that for $i \in I, \overline{x} \in X$, and $\epsilon > 0$, there exists a neighborhood $\mathcal{N}^2$ of $\overline{x}$ such that for every $z \in \mathcal{N}^2$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}^2$ of z, $u_i(\overline{x}) \geq u_i(z_i, z'_{-i}) - \frac{\epsilon}{2}$ for some $z' \in \mathcal{N}_z$. Thus, for every $z \in \mathcal{N} = \mathcal{N}^1 \cap \mathcal{N}^2$ and its neighborhood $\mathcal{N}_z \subseteq \mathcal{N}$, there exists $z' \in \mathcal{N}_z$ such that $u_i(y_i, z_{-i}) > u_i(z_i, z'_{-i}) + \frac{\epsilon}{2}$.

Thus, by Theorem 1, the game possesses a pure strategy Nash equilibrium. □

Proof of Theorem 2 Define a new two-player game G' by a simple player's strategy set Z. If player 1 chooses $t \in Z$ and player 2 chooses $z \in Z$, player 1's preferences are presented by

$$v_1(t, z) = \begin{cases} 1, & \text{if } z = t \\ 0, & \text{otherwise} \end{cases}$$

and player 2's preference relation is defined by $v_2(t, z) = \underline{f}(z, t) - \underline{f}(t, t)$. The game G' is point secure with respect to $I' = \{2\}$. Then, by Theorem 3.4 in Reny (2013), G' possesses a Nash equilibrium $(\overline{x}, \overline{x})$ which is also a symmetric Nash equilibrium of G (because G is diagonally quasi-weak transfer continuous). □

Proof of Theorem 3 Suppose that the considered game does not have a symmetric Nash equilibrium. Then, by diagonal weak transfer continuity, for each $x \in X$, there exists an open neighborhood $\mathcal{N}_x$ and $y \in X$ so that $f(z, y) > f(z, z)$, for each $z \in \mathcal{N}_x$. Thus, we obtain a collection $\{(\mathcal{N}_x, y^x)\}_{x \in X}$ where $\{\mathcal{N}_x\}_{x \in X}$ forms an open cover of X. Since X is compact, then one can extract a finite subcollection $\{(\mathcal{N}_{x^k}, y^k)\}_{k \in K}$ (K is a finite set), so as for each $k \in K$ and for each $z \in \mathcal{N}_{x^k}$, $f(z, y^k) > f(z, z)$ for each $z \in \mathcal{N}_{x^k}$. Let $\{\beta_k\}_{k \in K}$ be a partition of the unity subordinate to $\{\mathcal{N}_{x^k}\}_{k \in K}$. By the diagonal transfer quasiconcavity, for the finite set $\{y^k,\ k \in K\} \subseteq X$, there is a corresponding finite subset $\{\tilde{x}^k,\ k \in K\}$ such that for any subset J of K and every $\overline{x} \in \text{co}\{\tilde{x}^j,\ j \in J\}$, we have

$$f(\overline{x}, \overline{x}) \geq \min_{j \in J} f(\overline{x}, y^k). \tag{4}$$

Let us consider the following function $g : X \to X$ defined by $g(z) = \sum_{k \in K} \beta_k(z) \tilde{x}^k$. Brouwer fixed point theorem implies that there is $\overline{x} \in X$ so that $\overline{x} = g(\overline{x}) = \sum_{j \in J} \beta_j(\overline{x}) \tilde{x}^j$ where $J = \{j \in K :\ \beta_j(\overline{x}) > 0\}$. Then, for each $j \in J$, $\overline{x} \in \mathcal{N}_{x^j}$ and consequently $f(\overline{x}, y^j) > f(\overline{x}, \overline{x})$. Hence $f(\overline{x}, \overline{x}) \geq \min_{j \in J} f(\overline{x}, y^k) > f(\overline{x}, \overline{x})$, which is impossible. □

Proof of Theorem 4 Suppose that there is no equilibrium in X. Define the following two-player game by a single player's strategy set X. If player 1 chooses $t \in X$ and player 2 chooses $z \in X$, player 1's preferences are presented by

$$u_1(t, z) = \begin{cases} 1, & \text{if } z = t \\ 0, & \text{otherwise} \end{cases}$$

and player 2's preference relation is defined by $(t, z) \succeq (t', z')$ if and only if $(t, z) \in P_i^{\succeq}(t', z')$, for each $i \in I$.

The two-player game is correspondence secure with respect to $I' = \{2\}$. Indeed, assume that it is not correspondence secure with respect to $I' = \{2\}$ in nonequilibrium $\overline{x}$. Then combining it with the $P^{\succeq}$-correspondence security, there exists a neighborhood $\mathcal{N} \subseteq X$ of $\overline{x}$, a well-behaved correspondence $\varphi : \mathcal{N} \rightrightarrows X$, $\tilde{z} \in \mathcal{N}$, $\tilde{y} \in \varphi(\tilde{x})$ and a player j such that

$$\tilde{z} \in co\,\{w:\ (\tilde{z}, w) \succeq (\tilde{x}, \tilde{y})\} \tag{5}$$

$$\tilde{z} \notin co\left\{w \in X \text{ such that } (\tilde{z}, w) \in P_j^{\succeq}(\tilde{x}, \tilde{y})\right\}. \tag{6}$$

By definition of $\succeq$, we obtain

$$\begin{aligned}\tilde{z} &\in co\,\{w:\ (\tilde{z}, w) \in P_i^{\succeq}(\tilde{x}, \tilde{y}),\ \text{for each } i \in I\}\\ &= co\left(\bigcap_{i\in I}\{w:\ (\tilde{z}, w) \in P_i^{\succeq}(\tilde{x}, \tilde{y})\}\right).\end{aligned}$$

Since $co\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I}(coA_i)$, then for each $i \in I$,

$$\tilde{z} \in co\left\{w \in X \text{ such that } (\tilde{z}, w) \in P_i^{\succeq}(\tilde{x}, \tilde{y})\right\},$$

which contradicts (5). It is straightforward to show that the preference $\succeq$ is complete, reflexive, and transitive. Then, by Theorem 5.6 in Reny (2013), this game has an equilibrium $(\overline{x}, \overline{x})$, i.e., for each $y \in X$, we have $(\overline{x}, \overline{x}) \succeq (\overline{x}, y)$ which is equivalent to

$$(\overline{x}, \overline{x}) \in P_i^{\succeq}(\overline{x}, y), \quad \text{for each } i \in I. \tag{7}$$

Or $\overline{x}$ is not a Nash equilibrium of G. Then by $P^{\succeq}$-correspondence security of G, there is a neighborhood $\mathcal{N} \subseteq X$ of $\overline{x}$, a well-behaved correspondence $\varphi : \mathcal{N} \rightrightarrows X$ and a player $j \in I$ so that

$$\overline{x} \notin co\left\{t \in X:\ (\overline{x}, t) \in P_j^{\succeq}(\overline{x}, y)\right\}$$

holds for each $y \in \varphi(\overline{x})$, which contradicts (7). □

Proof of Theorem 5 Let $\Omega(x)$ be the set of all open neighborhoods $\mathcal{N}$ of x. For each player $i \in I$ and every $x \in X$, define the following correspondence $C_i : X \rightrightarrows X_i$ by

$$C_i(x) = \{y_i \in X_i:\ \exists \mathcal{N} \in \Omega(x),\ \forall z \in \mathcal{N},\ \inf_{z' \in \mathcal{N}} u_i(y_i, z'_{-i}) > \inf_{z'' \in \mathcal{N}} u_i(z_i, z''_{-i})\}.$$

By the strong quasiconcavity, $C_i(x)$ is convex for all $x \in X$ and $i \in I$. The correspondence C_i has the lower open section. Now suppose, by way of contradiction, that for each $x \in X$, there exists a player $i \in I$ such that $C_i(x) \neq \emptyset$. Then, by Theorem 3a in Deguire and Lassonde (1995), there exists a point $\widetilde{x} \in X$ and $i \in I$ such that $\widetilde{x}_i \in C_i(\widetilde{x})$, which is impossible. Thus, there exists $\overline{x} \in X$ such that for each $i \in I$, we have $C_i(\overline{x}) = \emptyset$. If $\overline{x}$ is not a Nash equilibrium, then as the game G is quasi-weakly transfer continuous, there exists a player i, a strategy $\overline{y}_i \in X_i$ and some neighborhood $\mathcal{N}_{\overline{x}}$ of $\overline{x}$ such that for every $z \in \mathcal{N}_{\overline{x}}$, we have $\inf_{z' \in \mathcal{N}} u_i(\overline{y}_i, z'_{-i}) > \inf_{z'' \in \mathcal{N}} u_i(z_i, z''_{-i})$. Then, $\overline{y}_i \in C_i(\overline{x})$, which is a contradiction to $C_i(\overline{x}) = \emptyset$. □

Proof of Proposition 2 Let $G = (X_i, u_i)_{i \in I}$ be better-reply secure. Suppose, by way of contradiction, that the game is not pseudo quasi-weakly transfer continuous. Then, there exists a nonequilibrium $x^* \in X$ such that for all player j, $\epsilon > 0$, every neighborhood $\mathcal{N}$ of x^*, and all y_j, there exists $x' \in \mathcal{N}$ satisfying

$$u_j(y_j, x'_{-j}) \leq u_j(x'_j, x''_{-j}) + \epsilon, \quad \text{for all } x'' \in \mathcal{N}.$$

Letting $\overline{u}$ be the limit of the vector of payoffs corresponding to some sequence of strategies converging to x^*, and U^* be the set of all such points, which is a compact set by the boundedness of payoffs, we have $(x^*, \overline{u}) \in \text{cl}(\Gamma)$ for all $\overline{u} \in U^*$. Then, for each $(x^*, \overline{u}) \in \text{cl}(\Gamma)$ with $\overline{u} \in U^*$, there exists a player i, a strategy $\overline{y}_i$, $\epsilon > 0$ and a neighborhood $\overline{\mathcal{N}}$ of x^* such that $u_i(\overline{y}_i, x'_{-i}) > \overline{u}_i + \epsilon$ for all $x' \in \overline{\mathcal{N}}$. Then $\inf_{x' \in \overline{\mathcal{N}}} u_i(\overline{y}_i, x'_{-i}) \geq \overline{u}_i + \epsilon$. Let U^*_i be the projection of U^* to coordinate i and

$$u^*_i = \sup \left\{ \overline{u}_i \in U^*_i : \inf_{x' \in \overline{\mathcal{N}}} u_i(\overline{y}_i, x'_{-i}) \geq \overline{u}_i + \epsilon \right\}.$$

Then, for $\epsilon/2 > 0$, there is a neighborhood $\mathcal{N}^{i,*}$ of x^* and a strategy y^*_i such that

$$\inf_{x' \in \mathcal{N}^{i,*}} u_i(y^*_i, x'_{-i}) \geq (u^*_i + \epsilon) - \epsilon/2 = u^*_i + \epsilon/2. \tag{8}$$

Now, since the game is not pseudo quasi-weakly transfer continuous, then for a directed system of neighborhoods $\{\mathcal{N}^k\}_k$ of x^*, a sequence $\{\epsilon^k\}_k$ converging to 0, and every $j \in I$, there exists a sequence $\{x^{j,k}\}_k$ with $x^{j,k} \in \mathcal{N}^k$ so that $\{x^{j,k}\}_k$ converges to x^* and

$$u_j(y^*_j, x^{j,k}_{-j}) \leq u_j(x^{j,k}_j, x'_{-j}) + \epsilon^k, \quad \text{for each } x' \in \mathcal{N}^k. \tag{9}$$

Consider the following sequence: for each k, let $\widetilde{x}^k = (x^{1,k}_1, \ldots, x^{n,k}_n)$. Since for each $j \in I$, $x^{j,k} \in \mathcal{N}^k$ and $\{x^{j,k}\}_k$ converges to x^*, then $\widetilde{x}^k \in \mathcal{N}^k$ and the sequence $\{\widetilde{x}^k\}_k$ converges to x^*. Therefore, inequality (9) becomes

$$u_j\left(y^*_j, x^{j,k}_{-j}\right) \leq u_j\left(x^{j,k}_j, \widetilde{x}^k_{-j}\right) = u_j(\widetilde{x}^k) + \epsilon^k, \quad \text{for each } k,\ j \in I. \tag{10}$$

Assume that $\{u(\widetilde{x}^k)\}_k$ converges and $\widetilde{u} = \lim_{k \to \infty} u(\widetilde{x}^k)$. Hence, $(x^*, \widetilde{u}) \in \text{cl}(\Gamma)$ with $\widetilde{u} \in U^*$, then there exists a player $i \in I$ such that $\widetilde{u}_i \leq u^*_i$. Thus, for $\epsilon/3 > 0$, there exists k_1 such that whenever $k > k_1$, we have $u_i(y^*_i, x^{i,k}_{-i}) \leq u^*_i + \epsilon/3 \leq \inf_{x' \in \mathcal{N}^{i,*}} u_i(y^*_i, x'_{-i}) - \epsilon/6$. Then for $k > k_1$, we obtain

$$u_i(y^*_i, x^{i,k}_{-i}) \leq u_i(y^*_i, x'_{-i}) - \epsilon/6, \quad \text{for each } x' \in \mathcal{N}^{i,*}. \tag{11}$$

Since the sequence $\{x^{i,k}\}_k$ converges to x^*, then for $\mathcal{N}^{i,*}$, there exists k_2 such that for $k > k_2$, we have $x^{i,k} \in \mathcal{N}^{i,*}$. Thus, by (11) for $k > \max(k_1, k_2)$, we have

$u_i(y_i^*, x_{-i}^{i,k}) \leq u_i(y_i^*, x_{-i}^{i,k}) - \epsilon/6$, which is impossible. Hence, the game must be pseudo quasi-weakly transfer continuous. □

Proof of Theorem 6 **Sufficiency**. For each $y \in X$, let

$$F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x),\ \forall i \in I\}.$$

It is clear that G is weakly transfer quasi-continuous if and only if F is transfer closed-valued. For $y \in X$, let $\bar{F}(y) = \text{cl } F(y)$. Then $\bar{F}(y)$ is closed, and by the strong diagonal transfer quasiconcavity, it is also transfer FS-convex. By Lemma 1 in Tian (1993), we know that $\bigcap_{y \in X} F(y) = \bigcap_{y \in X} \bar{F}(y) \neq \emptyset$. Thus, there exists a strategy profile $\overline{x} \in X$ such that

$$u_i(y_i, \overline{x}_{-i}) \leq u_i(\overline{x}), \quad \text{for all } y \in X \text{ and } i \in I.$$

Thus $\overline{x}$ is a pure strategy Nash equilibrium of the game G.

Necessity: Suppose the game Γ has a pure strategy Nash equilibrium $x^* \in X$. We want to show that it is strongly diagonal transfer quasiconcave in y. Indeed, for any finite subset $\{y^1, \ldots, y^m\} \subset X$, let the corresponding finite subset $X^m = \{x^1, \ldots, x^m\} = \{x^*\}$. Then, for any subset $\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} \subset X^m = \{x^*\}$, $1 \leq s \leq m$, $x \in co\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} = \{x^*\}$, and $y \in \{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\}$, we have

$$u_i(y_i, x_{-i}) = u_i(y_i, x_{-i}^*) \leq u_i(x_i^*, x_{-i}^*) = u_i(x_i, x_{-i}).$$

Hence U is strongly diagonal transfer quasiconcave in x. □

Proof of Proposition 3 Suppose x is not an equilibrium. Then, some player i has a strategy y_i such that $u_i(y_i, x_{-i}) > u_i(x)$, *i.e.*, $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$. If $(y_i, x_{-i}) \in X \backslash D_i$, then by Assumption 3, there exists a player $j \in I$, a deviation strategy profile $\bar{y}$, and a neighborhood $\mathcal{V}$ of x such that for every $z \in \mathcal{V}$, we have $\underline{F}_j(\bar{y}_j, S_j(\bar{y}_j, z_{-j})) > \underline{F}_j(z_j, S_j(z))$, *i.e.*, $\underline{u}_j(y_j', z_{-j}) > \underline{u}_j(z)$. If $(y_i, x_{-i}) \in D_i$, then by Assumption 2, there exists a player $j \in I$ and $\bar{y}_j$ such that $(\bar{y}_j, x_{-j}) \in X \backslash D_j$ and $F_j(\bar{y}_j, S_j(\bar{y}_j, x_{-j})) > F_j(x_j, S_j(x))$. Thus, by Assumption 3, there exists a player $k \in I$, a deviation strategy profile $\tilde{y}$, and a neighborhood $\mathcal{V}$ of x such that for every $z \in \mathcal{V}$, we have $\underline{F}_k(\tilde{y}_k, S_k(\tilde{y}_k, z_{-k})) > \underline{F}_k(z_k, S_k(z))$, *i.e.*, $\underline{u}_k(\tilde{y}_k, z_{-k}) > \underline{u}_k(z)$. Therefore, the game is quasi-weakly transfer continuous. Since it is also compact and quasiconcave, by Theorem 1, it has a pure strategy Nash equilibrium. □

Proof of Proposition 4 Suppose p is not an equilibrium. Then, some player i has a strategy q_i such that $\pi_i(q_i, p_{-i}) > \pi_i(p)$. If $(q_i, p_{-i}) \in P^n \backslash A_i$, then by Assumption 3, there exists a player $j \in I$, a deviation strategy profile $\bar{q}_j$ and a neighborhood $\mathcal{N}$ of p such that for every $r \in \mathcal{N}_p$, $\underline{\pi}_j(\bar{q}_j, r_{-i}) > \underline{\pi}_j(r)$. If $(q_i, p_{-i}) \in A_i$, then by Assumption 2, there exists a firm $j \in I$, and $\bar{q}_j$ such that $(\bar{q}_j, p_{-i}) \in P^n \backslash A_j$ and $\pi_j(\bar{q}_j, p_{-i}) > \pi_i(p)$. Thus, by Assumption 3, there exists a player $j \in I$, a deviation strategy profile $\bar{q}_j$ and a neighborhood $\mathcal{N}$ of p such that for every $r \in \mathcal{N}_p$,

$\underline{\pi}_j(\bar{q}_j, r_{-i}) > \underline{\pi}_j(r)$. Therefore, the game is weakly transfer quasi-continuous. Since the game is also compact, convex, and quasiconcave, by Theorem 6, it has a pure strategy Nash equilibrium. □

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