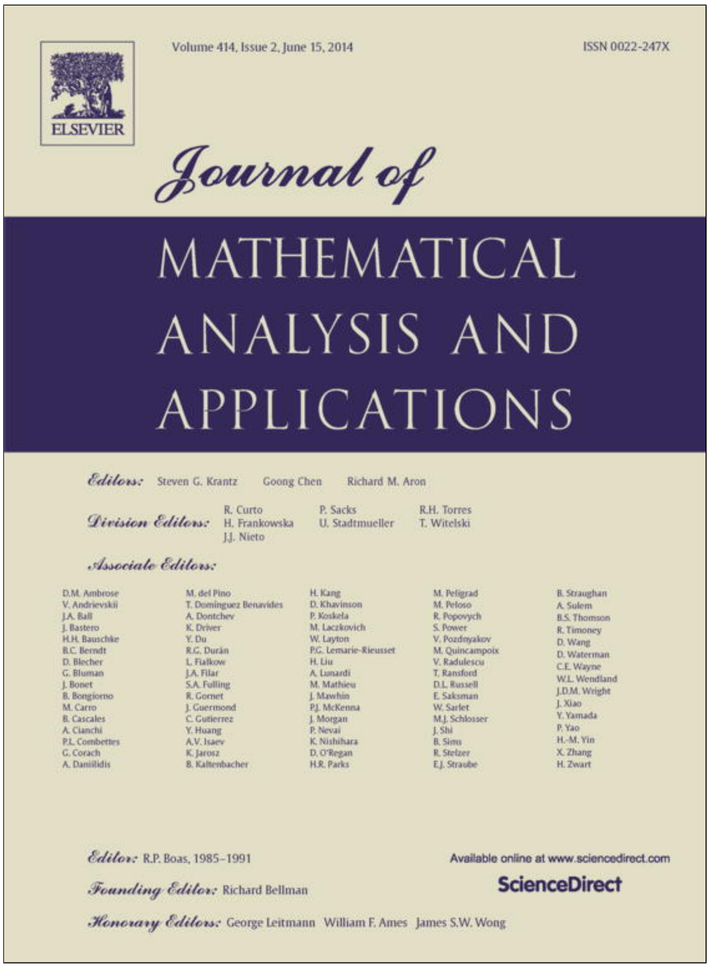
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On the existence of strong Nash equilibria [☆]

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ABSTRACT

This paper investigates the existence of strong Nash equilibria (SNE) in continuous and concave games. It is shown that the coalition consistency property introduced in the paper, together with concavity and continuity of payoffs, permits the existence of SNE in games with compact and convex strategy spaces. We also characterize the existence of SNE by providing necessary and sufficient conditions. We suggest an algorithm for computing SNE. The results are illustrated with applications to economies with multilateral environmental externalities and to the static oligopoly model.

1. Introduction

This paper studies the existence of strong Nash equilibrium (SNE) in general economic games. Although Nash equilibrium is probably the most important central behavioral solution concept in game theory, it has some drawbacks. The main drawback is that Nash equilibrium is a strictly noncooperative notion and is only concerned with unilateral deviations from which no one can be improved. No cooperation among agents is allowed. As such, although a Nash equilibrium may be easy to reach, it may not be stable in the sense that there may exist a group of agents that can be improved by forming a coalition. Then it is natural to call for an equilibrium concept that allows possible cooperation or coalitions among agents.

The solution concept of strong Nash equilibrium introduced by Aumann [4] overcomes this shortcoming. SNE is defined as a strategy profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies. A SNE is then not only immune to unilateral deviations, but also to deviations by coalitions.

However, the existence of SNE is a largely unsolved problem. Ichiishi [13] introduced the notion of social coalitional equilibrium and proved its existence under a set of assumptions. The concept of social coalitional equilibrium extends the notion of social equilibrium introduced by Debreu [8] to prevent deviations by

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coalitions. It can also be specialized to SNE. Then, the sufficient conditions for the existence of social coalitional equilibria are also sufficient for the existence of SNE. However, the assumptions imposed in [13] are difficult to verify. Although there are several other studies on the existence of SNE in various specific environments such as those in [12,11,9,14,15], there is no general theorem on the existence of SNE.

This paper fills the gap by proposing some existence results on SNE in general games. We show that the coalition consistency property introduced in the paper, together with concavity and continuity of payoffs, permits the existence of SNE in games with compact and convex strategy spaces. We also characterize the existence of SNE by providing necessary and sufficient conditions. Moreover, we suggest an algorithm that can be used to compute SNE. The results are illustrated with applications to economies with multilateral environmental externalities and to the simple static oligopoly model.

The remainder of the paper is organized as follows. Section 2 presents the notions, definitions, and some properties. Section 3 establishes sufficient conditions for the existence of a strong Nash equilibrium. Section 4 provides characterization for the existence of strong Nash equilibrium and also a method for its computation. Section 5 is dedicated to the applications of the main new results to economies with multilateral environmental externalities and the simple static oligopoly model. Section 6 concludes.

2. Preliminaries

Consider a game in normal form $G = \langle X_i, u_i\rangle_{i\in I}$ where $I = \{1,\ldots,n\}$ is the finite set of players, X_i is the set of strategies of player i which is a subset of a Hausdorff locally convex topological vector space, and u_i is player i's payoff function from the set of strategy profiles $X = \prod_{i\in I} X_i$ to $\mathbb{R}$. Denote by $u = (u_1, u_2, \ldots, u_n)$ the profile of utility functions.

Let $\Im$ denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $S \in \Im$, denote by $-S = \{i \in I\colon\ i \notin S\}$ the remaining of coalition S. If S is reduced to a singleton $\{i\}$, we denote simply by $-i$ all other players rather than player i. We also denote by $X_S = \prod_{i\in S} X_i$ the set of strategies of players in coalition S.

We say that a game $G = (X_i, u_i)_{i\in I}$ is compact, concave, and continuous, respectively if, for all $i \in I$, X_i is compact and convex, and u_i is concave and continuous on X, respectively.

We say that a strategy profile $x^* \in X$ is a *Nash equilibrium* of a game G if for all $i \in I$,

$$u_i\big(x^*\big) \geqslant u_i\big(y_i, x^*_{-i}\big) \quad \text{for all } y_i \in X_i.$$

Definition 2.1. A strategy profile $\bar{x} \in X$ is said to be a *strong Nash equilibrium* (SNE) of a game G if $\forall S \in \Im$, there does not exist any $y_S \in X_S$ such that

$$u_i(y_S, \bar{x}_{-S}) > u_i(\bar{x}) \quad \text{for all } i \in S. \tag{2.1}$$

Definition 2.2. A strategy profile $\bar{x} \in X$ of a game G is said to be *weakly Pareto efficient* if there does not exist any $y \in X$ such that $u_i(y) > u_i(\bar{x})$ for all $i \in I$.

A strategy profile is a SNE means no coalition (including the grand coalition, *i.e.*, all players collectively) can profitably deviate from the prescribed profile. This immediately implies that any SNE is both weakly Pareto efficient and a Nash equilibrium. Also, it is stable with regard to the deviation of any coalition.

It is worth pointing out that all the following solution concepts are implied by SNE.

Definition 2.3 *(The weak α-core).* A strategy profile $\bar{x} \in X$ is in the weak α-core of a game G if for all $S \in \Im$ and all $x_S \in X_S$, there exists a $y_{-S} \in X_{-S}$ such that

$$u_i(\bar{x}) \geqslant u_i(x_S, y_{-S}) \quad \text{for at least some } i \in S.$$

A strategy profile $\bar{x}$ is in the weak α-core means that for any coalition S and any deviation x_S of $\bar{x}_S$, the coalition of the remaining players $(-S)$ can find a strategy y_{-S} such as in the new strategy (x_S, y_{-S}), the payoffs of at least one player in coalition S cannot be better than those in the strategy $\bar{x}$ (for all the players of the coalition S at the same time).

Definition 2.4 *(The weak β-core).* A strategy profile $\bar{x} \in X$ is in the weak β-core of a game G if for all $S \in \Im$, there exists a $y_{-S} \in X_{-S}$ such that for every $x_S \in X_S$,

$$u_i(\bar{x}) \geqslant u_i(x_S, y_{-S}) \quad \text{for at least some } i \in S.$$

A strategy profile $\bar{x}$ is in the weak β-core means that for any coalition S, the coalition of players $-S$ possesses a strategy y_{-S} which prevents all deviations of the coalition S of the strategy $\bar{x}$. Thus the stability property of an outcome in the weak β-core is stronger than that of the weak α-core: a deviating coalition S can be countered by the complement coalition $-S$ even if the players of S keep secret their joint strategy X_S.

Definition 2.5 *(The k-equilibrium).* A strategy profile $\bar{x} \in X$ is said to be a k-equilibrium ($k \in I$) of a game G if for all coalitions S with $|S| = k$, there does not exist any $y_S \in X_S$ such that

$$u_i(y_S, \bar{x}_{-S}) > u_i(\bar{x}) \quad \text{for all } i \in S.$$

No k-players' coalition can make all these players better off at the same time by deviating from the strategy $\bar{x}$.

The following lemma characterizes SNE of a game G.

Lemma 2.1. *The strategy profile $\bar{x} \in X$ is a SNE of a game $G = \langle X_i, u_i\rangle_{i\in I}$ if and only if for each $S \in \Im$, the strategy $\bar{x}_S \in X_S$ is weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \bar{x}_{-S})\rangle_{j\in S}$ which is obtained by fixing $\bar{x}_{-S}$.*

Proof. It is a straightforward consequence of Definition 2.1. □

3. Existence results

In this section we investigate the existence of strong Nash equilibria in general games. We first provide some sufficient conditions for the existence of SNE. To do so, we use the following g-fixed point theorem given by Nessah and Chu [17].

Denote by $\mathrm{cl}(A)$ the closure of set A and by ∂A its boundary. Letting Y_0 be a nonempty convex subset of a convex set Y in a vector space and $y \in Y_0$, we denote by $Z_{Y_0}(y)$ the following set: $Z_{Y_0}(y) = [\mathrm{cl}(\bigcup_{h>0}[Y_0 - \{y\}]/h) + \{y\}] \cap Y$. Note that $\mathrm{cl}(\bigcup_{h>0}[Y_0 - \{y\}]/h)$ is called tangent cone to Y_0 at the point y.

A correspondence $F : X \to 2^Y$ is *upper hemi-continuous* at x if for each open set U containing $F(x)$, there is an open set $N(x)$ containing x such that if $x' \in N(x)$, then $F(x') \subset U$. A correspondence $F : X \to 2^Y$ is *upper hemi-continuous* if it is upper hemi-continuous at every $x \in X$, or equivalently, if the set $\{x \in X\colon F(x) \subset V\}$ is open in X for every open subset V of Y. The definition of hemi-continuity for a correspondence was introduced by Browder [6] and often used in the study of non-linear set-valued analysis (e.g., see Aubin and Frankowska [3], Aliprantis and Chakrabarti [1,2] and related references therein).

Lemma 3.1. *(See Nessah and Chu [17].) Let X be a nonempty compact set in a metric space E, and Y a nonempty convex and compact set in a locally convex Hausdorff vector space F. Let $g : X \to Y$ be a continuous function and $C : X \to 2^Y$ an upper hemi-continuous correspondence with nonempty closed and convex values. Suppose that the following conditions are met:*

(a) *$g(X)$ is convex in Y;*
(b) *for each $g(x) \in \partial g(X)$, $C(x) \cap Z_{g(X)}(g(x)) \neq \emptyset$.*

Then, there exists $\bar{x} \in X$ such that $g(\bar{x}) \in C(\bar{x})$.

Let

$$\Delta_S = \left\{ \lambda_S = (\lambda_1, \dots, \lambda_{|S|}) \in \mathbb{R}_+^{|S|}\colon \sum_{j \in S} \lambda_j = 1 \right\}$$

be the unit simplex of $\mathbb{R}^{|S|}$ ($S \in \Im$), and let

$$\Delta = \prod_{S \in \Im} \Delta_S \quad \text{and} \quad \hat{X} = \prod_{S \in \Im} X_S.$$

For each coalition S, define the S-weighted best-reply correspondence $C_S : X_{-S} \times \Delta_S \to 2^{X_S}$ by

$$C_S(x_{-S}, \lambda_S) = \left\{ z_S \in X_S\colon \sup_{y_S \in X_S} \sum_{i \in S} \lambda_{i,S} u_i(y_S, x_{-S}) \leqslant \sum_{i \in S} \lambda_{i,S} u_i(z_S, x_{-S}) \right\},$$

and then the $\Im$-weighted best-reply correspondence $C : X \times \Delta \to 2^{\hat{X}}$ by

$$x \mapsto C(x, \lambda) = \left\{ \hat{z} = \prod_{S \in \Im} z_S \in \hat{X}\colon\ z_S \in C_S(x_{-S}, \lambda_S) \right\},$$

where $\prod_{S \in \Im} z_S$ is the Cartesian product of z_S over $\Im$ for the notational convenience.[2]

Define the function $\phi : X \to \hat{X}$ by

$$\phi(x) = \prod_{S \in \Im} x_S.$$

We then have the following lemma.

Lemma 3.2. *Suppose that for all $i \in I$, X_i is convex and compact. Then we have*:

(a) *The function ϕ is continuous on X.*
(b) *The set $\phi(X)$ is convex and compact.*

Proof. The continuity of function ϕ is a consequence of its definition and the construction of the set $\hat{X}$. Also, by the Weierstrass Theorem, we know that $\phi(X)$ is compact if ϕ is continuous and X is compact (cf. Tian and Zhou [19]). The convexity of $\phi(X)$ is a consequence of the linearity of ϕ, which is easily verified. □

To show the existence of strong Nash equilibrium, we assume the $\Im$-weighted best-reply correspondence $C(x, \lambda)$ satisfies the following coalition consistency property:

Definition 3.1 *(Coalition consistency property).* A game $G = (X_i, u_i)_{i \in I}$ is said to satisfy the coalition consistency property if there exists $\lambda \in \Delta$ such that for each $x \in X$, there exists $z \in X$ such that

$$z_S \in C_S(x_{-S}, \lambda_S) \quad \text{for all } S \in \Im. \tag{3.1}$$

[2] We can do so by imaging z_S as a single-element set.

The coalition consistency property implies the existence of $\lambda \in \Delta$ such that, for each $x \in X$, there is a point $z \in X$ with z_S being the best reply of every coalition S, given strategies of players in $-S$. In particular, when the point z turns out to be a fixed point of $\Im$-weighted best-reply correspondence $C(\cdot, \lambda)$, it is a strong Nash equilibrium. We then need to provide conditions so that a fixed-point theorem can be applied. Theorem 3.1 below will provide such conditions.

Remark 3.1. The coalition consistency property is relatively easy to check, much easier than those given in [13]. Indeed, by the definition of the $\Im$-weighted best-reply correspondence $C(x, \lambda)$, $z_S \in C_S(x_{-S}, \lambda_S)$ for all $S \in \Im$ implies that z_S is the maximum of utilitarian social welfare function, *i.e.*, the weighted average of payoff functions, of individuals in S for every $S \in \Im$, and consequently, is weakly Pareto efficient to the sub-game $\langle X_j, u_j(., x_{-S})\rangle_{j\in S}$ for all $S \in \Im$. As such, when u_i are differentiable for all i, to guarantee that the first order conditions for the social maximization are also sufficient, we need to assume that payoff functions of players are concave, which is also needed to guarantee the existence of strong Nash equilibrium as shown in Theorem 3.1 below. Then, to check if the coalition consistency property is satisfied is reduced to checking if there exists a suitable weight $\lambda \in \Delta$ such that every component $z_{i,S}$ of z_S is equal to $z_{\{i\}}$ that is obtained for singleton coalition $S = \{i\}$. If so, z is a strategy profile as required in (3.1), *i.e.*, $z \in X$ and $z_S \in C_S(x_{-S}, \lambda_S)$ for all $S \in \Im$, which means the coalition consistency property is satisfied.

We now establish the following existence theorem on SNE.

Theorem 3.1. *Suppose the game $G = (X_i, u_i)_{i\in I}$ is compact, concave, continuous, and satisfies the coalition consistency property. Then, it possesses a strong Nash equilibrium.*

Proof. We prove step by step that the functions ϕ and C defined by $\phi(x) = \prod_{S\in\Im} x_S$ and $C(x, \lambda) = \{\hat{z} = \prod_{S\in\Im} z_S \in \hat{X}\colon\ z_S \in C_S(x_{-S}, \lambda_S)\}$, respectively, satisfy the conditions of Lemma 3.1:

(1) For all $x \in X$ and $\lambda \in \Delta$, $C(x, \lambda) \neq \emptyset$. Indeed, for any $x \in X$, the function $y_S \mapsto \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S})$, $S \in \Im$ is continuous on compact set X_S and then by the Weierstrass Theorem, there exists $\bar{z}_S \in X_S$ such that

$$\max_{y_S\in X_S} \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S}) = \sum_{j\in S} \lambda_{j,S} u_j(\bar{z}_S, x_{-S}), \quad i.e., \quad \bar{z}_S \in C_S(x_{-S}, \lambda_S).$$

Hence $\hat{z} = \prod_{S\in\Im} \bar{z}_S \in C(x, \lambda)$ and consequently $C(x, \lambda)$ is nonempty and further compact for all $x \in X$ and $\lambda \in \Delta$ by the Weierstrass Theorem.

(2) For all $x \in X$ and $\lambda \in \Delta$, $C(x, \lambda)$ is convex in $\hat{X}$. Indeed, let $x \in X$, $\lambda \in \Delta$, $\bar{z} = \prod_{S\in\Im} \bar{z}_S$ and $\bar{\bar{z}} = \prod_{S\in\Im} \bar{\bar{z}}$ be two elements of $C(x, \lambda)$ and $\theta \in [0, 1]$. We want to prove that $\theta\bar{z} + (1-\theta)\bar{\bar{z}} \in C(x, \lambda)$. Since $\bar{z}_S$ and $\bar{\bar{z}}_S$ are both the maximum of $\sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S})$, we must have: $\sum_{j\in S} \lambda_{j,S} u_j(\bar{z}_S, x_{-S}) = \sum_{j\in S} \lambda_{j,S} u_j(\bar{\bar{z}}_S, x_{-S})$ and thus, by the concavity of function u_i, we have

$$\max_{y_S\in X_S} \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S}) \leqslant \sum_{j\in S} \lambda_{j,S} u_j(\bar{z}_S, x_{-S}) = \sum_{j\in S} \lambda_{j,S} u_j(\bar{\bar{z}}_S, x_{-S}) \tag{3.2}$$

$$\leqslant \sum_{j\in S} \lambda_{j,S} u_j\big(\theta\bar{z}_S + (1-\theta)\bar{\bar{z}}_S, x_{-S}\big), \quad \theta \in [0, 1]. \tag{3.3}$$

Therefore, $\theta\bar{z} + (1-\theta)\bar{\bar{z}} \in C(x, \lambda)$.

(3) C is upper hemi-continuous over X. Note that X is compact, and thus $\hat{X}$ is compact (Tychonoff Theorem). Thus, to prove that C is upper hemi-continuous on X, it suffices to prove that *Graph*$(C) \subset X \times \hat{X}$ is closed.

To see this, let $(x,\hat{z}) \in \mathrm{cl}(Graph(C))$. Then there exists a sequence $\{(x^p,\hat{z}^p)\}_{p\geqslant 1}$ in $Graph(C)$ that converges to $(x,\hat{z})$.
Thus, we have $\hat{z}^p \in C(x^p,\lambda)$ for all $p \geqslant 1$, *i.e.*,

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j\left(y_S,x^p_{-S}\right) \leqslant \sum_{j\in S}\lambda_{j,S}u_j\left(z^p_S,x^p_{-S}\right) \quad \text{for all } S\in\Im.$$

Then, by the continuity of functions u_i, as $p\to\infty$, we have

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S}) \leqslant \sum_{j\in S}\lambda_{j,S}u_j(z_S,x_{-S}) \quad \text{for all } S\in\Im,$$

i.e., $\hat{z}\in C(x,\lambda)$, hence $(x,\hat{z})\in Graph(C)$, which means that $Graph(C)$ is closed in $X\times\hat{X}$. Thus the function C is upper hemi-continuous on X.

(4) For each $\phi(x)\in\partial\phi(X)$, $C(x,\lambda)\cap Z_{\phi(X)}(\phi(x))\neq\emptyset$ where $Z_{\phi(X)}(\phi(x)) = [\mathrm{cl}(\bigcup_{h>0}\frac{\phi(X)-\{\phi(x)\}}{h}) + \{\phi(x)\}]\cap\hat{X} = [\mathrm{cl}(\bigcup_{h>0}\{h[\phi(u)-\phi(x)],\ u\in X\}) + \{\phi(x)\}]\cap\hat{X}$.
Indeed, by the coalition consistency property, there exists $\lambda\in\Delta$ such that for each $x\in X$ with $\phi(x)\in\partial\phi(X)$, there exists $z\in X$ such that

$$z_S\in C_S(x_{-S},\lambda_S) \quad \text{for all } S\in\Im.$$

For each $a\geqslant 1$, let $y_S = \frac{1}{a}z_S + \frac{a-1}{a}x_s$. Since $\frac{1}{a}>0$, $\frac{a-1}{a}\geqslant 0$ and $\frac{1}{a}+\frac{a-1}{a}=1$, we have $y_S\in X_S$ by the convexity of X, and $ay_S+(1-a)x_S = z_S\in C_S(x_{-S})$ for all S. Thus, $\phi(ay+(1-a)x) = a\phi(y)+(1-a)\phi(x)\in C(x,\lambda)$ (because ϕ is linear). Since $a[\phi(y)-\phi(x)]\in\frac{\phi(X)-\{\phi(x)\}}{1/a}\subset\mathrm{cl}(\bigcup_{h>0}\frac{\phi(X)-\{\phi(x)\}}{h})$, then $a\phi(y)+(1-a)\phi(x) = a[\phi(y)-\phi(x)]+\phi(x)\in Z_{\phi(X)}(\phi(x))$. Therefore, $a\phi(y)+(1-a)\phi(x)\in C(x,\lambda)\cap Z_{\phi(X)}(\phi(x))$, *i.e.*, $C(x,\lambda)\cap Z_{\phi(X)}(\phi(x))\neq\emptyset$.

Also, by Lemma 3.2, ϕ is continuous on X and $\phi(X)$ is convex and compact. Thus, all the conditions of Lemma 3.1 are satisfied. Consequently, there exists $\bar{x}\in X$ such that $\phi(\bar{x})\in C(\bar{x},\lambda)$, *i.e.*, for all $S\in\Im$, $\bar{x}_S\in C_S(\bar{x}_{-S},\lambda_S)$. Therefore, for all $S\in\Im$ and $y_S\in X_S$, we have:

$$\sum_{j\in S}\lambda_{j,S}u_j(y_S,\bar{x}_{-S}) \leqslant \sum_{j\in S}\lambda_{j,S}u_j(\bar{x}_S,\bar{x}_{-S}) = \sum_{j\in S}\lambda_{j,S}u_j(\bar{x}). \tag{3.4}$$

Now we prove that $\bar{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_j,u_j(.,\bar{x}_{-S})\rangle_{j\in S}$ for all $S\in\Im$.

Suppose that there exists $S_0\in\Im$ such that $\bar{x}_{S_0}$ is not weakly Pareto efficient to the sub-game $\langle X_j,u_j(.,\bar{x}_{-S_0})\rangle_{j\in S_0}$. Then, there exists $\tilde{y}_{S_0}\in X_{S_0}$ such that:

$$u_j(\tilde{y}_{S_0},\bar{x}_{-S_0}) > u_j(\bar{x}) \quad \text{for all } j\in S_0. \tag{3.5}$$

System (3.5), together with $\lambda\in\Delta$, implies that $\sum_{j\in S_0}\lambda_{j,S}u_j(\tilde{y}_{S_0},\bar{x}_{-S_0}) > \sum_{j\in S_0}\lambda_{j,S}u_j(\bar{x})$. This contradicts inequality (3.4) for $S=S_0$ and $y_S=\tilde{y}_{S_0}$. Hence $\bar{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_j,u_j(.,\bar{x}_{-S})\rangle_{j\in S}$ for all $S\in\Im$, and consequently, by Lemma 2.1 it is a strong Nash equilibrium. The proof is completed. □

Remark 3.2. The above theorem on the existence of SNE requires much stronger conditions than the existence of Nash equilibrium does. First, note that, in order to apply a fixed-point theorem, we need to impose the quasi-concavity of weighted individual payoff functions so that the $\Im$-weighted best-reply correspondence is convex-valued. This can be ensured by the concavity of individual payoffs. Of course, we can slightly weaken the condition to that: for each coalition $S\in\Im$ and each weight $\lambda_S\in\Delta_S$, the function

$y_S \mapsto \sum_{i \in S} \lambda_{i,S} u_i(y_S, x_{-S})$ is quasi-concave on X_S, for each $x_{-S} \in X_{-S}$. Unfortunately, we cannot further weaken u_i to be quasi-concave on X since a weighted average of payoff functions may not be quasi-concave. Thus, to apply a fixed-point theorem, one may have to impose the concavity of individual payoffs, and thus it is an appealing condition for the existence of SNE. This is different from the case of Nash equilibrium, in which the quasi-concavity of payoffs is an appealing condition for the existence of Nash equilibrium.

Remark 3.3. The coalition consistency property is a condition that cannot be dispensed with for the existence of SNE, which requires that for every strategy profile x and every coalition S, there is a z such that z_s is a weighted best-reply strategy in coalition S. As such, the coalition consistency property can be checked, say, by using the same methods for finding the maximum of utilitarian social welfare function for every coalition and then checking if there exists a suitable weight such that every component of such coalitions is the same as those obtained from single individual deviations. Nevertheless, this condition imposes a significant restriction on the existence of SNE, and in fact, as argued by Bernheim et al. [5] and Dubey [10], the solution concept of SNE is "too strong", which requires to be weakly Pareto efficient. As a result, SNE does not exist for general economic games. However, as Peleg [18] indicated, certain important economic games such as voting games do possess SNE.

The following example shows that a game without satisfying the coalition consistency condition may not possess a strong Nash equilibrium even if it is compact, continuous and concave.

Example 3.1. Consider a game with $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$ and

$$u_1(x) = -x_1 + 2x_2,$$
$$u_2(x) = 2x_1 - x_2.$$

It can be easily seen that the game is compact, continuous and concave. Moreover, it possesses a unique Nash equilibrium that is $(0, 0)$. However, there is no strong Nash equilibrium. One can see this by showing the failure of coalition consistency. Indeed, notice that the efficient profile is $(1, 0)$ if the weight λ to player 1 in the coalition $(1, 2)$ is less than $1/3$, is $(1, 1)$ if $\lambda \in (1/3, 2/3)$, and is $(0, 1)$ if $\lambda = 2/3$. Also, if $\lambda = 1/3$, the set of efficient profiles is the convex hull of $(1, 0)$ and $(1, 1)$ and if $\lambda = 2/3$, it is the convex hull of $(0, 1)$ and $(1, 1)$. Thus, for $x = (0, 0)$, the coalition consistency property cannot be satisfied. As such, the failure of coalition consistency leads to the non-existence of SNE.

As such, an additional condition, such as the coalition consistency property, should be imposed. The following example shows this.

Example 3.2. Consider a game with $n = 2$, $I = \{1, 2\}$, $X_1 = [1/3, 2]$, $X_2 = [3/4, 2]$, and

$$u_1(x) = -x_1^2 + x_2 + 1,$$
$$u_2(x) = x_1 - x_2^2 + 1.$$

Since X is compact and convex and payoff functions are continuous and concave on X, we only need to show that the coalition consistency property is also satisfied so that we know there exists a strong Nash equilibrium by Theorem 3.1.

Thus, to check the coalition consistency property, we need to find a $\lambda \in \Delta$ such that, for each $x \in X$, there exists $z \in X$ such that $z_S \in C_S(x_{-S}, \lambda_S)$ for all $S \in \Im$. Indeed, for $\Im = \{\{1\}, \{2\}, \{1, 2\}\}$, letting $\lambda = (1, 1, (0.6, 0.4))$, we have:

(1) for $S = \{1\}$ and $\lambda_S = 1$, $\max_{y_1 \in X_1} u_1(y_1, x_2) = \max_{y_1 \in X_1}(-y_1^2 + x_2 + 1) = -(1/3)^2 + x_2 + 1$, which means that $z_1 = 1/3$ is the maximum;
(2) for $S = \{2\}$ and $\lambda_S = 1$, $\max_{y_2 \in X_2} u_2(x_1, y_2) = \max_{y_2 \in X_2}(x_1 - y_2^2 + 1) = x_1 - (3/4)^2 + 1$, which means that $z_2 = 3/4$ is the maximum;
(3) for $S = \{1, 2\}$ and $\lambda_S = (0.6, 0.4)$, $\max_{(y_1,y_2) \in X}[0.6u_1(y_1, y_2) + 0.4u_2(y_1, y_2)] = \max_{(y_1,y_2) \in X}[-0.6y_1^2 + 0.4y_1 - 0.4y_2^2 + 0.6y_2 + 1] = [-0.6(1/3)^2 + 0.4(1/3) - 0.4(3/4)^2 + 0.6(3/4) + 1]$, which means that $z = (z_1, z_2) = (1/3, 3/4)$ is the maximum.

Thus, for all $x \in X$, there exists $z = (1/3, 3/4) \in X$ such that

$$z_S \in C_S(x_{-S}, \lambda_S) \quad \text{for all } S \in \Im.$$

Therefore, the coalition consistency property is satisfied, and thus by Theorem 3.1, the game has a strong Nash equilibrium.

Example 3.3. Let $I_0 = \{1, 2, \dots, n-1\}$ be the set of agents. The set of all coalitions of I_0 is denoted by $\aleph$. There are m commodities. For each agent i, his strategy space is X^i, a subset of $\mathbb{R}^m \times \mathbb{R}^m \times E^i$ where E^i is a vector space over $\mathbb{R}^m$, and $u_i : \prod_{h \in I_0} X^h \to \mathbb{R}$ is the expected utility function of the i-th agent. A generic element $x^i \in X^i$ is denoted by (x_1^i, x_2^i, x_3^i) with $x_1^i, x_2^i \in \mathbb{R}^m$ and $x_3^i \in E^i$. The total excess demand for the marketed commodities is $\sum_{i \in I_0}(x_1^i + x_2^i)$.[3]

$\bar{x} \in \prod_{h \in I_0} X^h$ is an equilibrium for this market economy $\mathcal{E} = (X_i, u_i)_{i \in I_0}$ if

(i) $\bar{x}$ is a strong Nash equilibrium of $\mathcal{E}$;
(ii) $\sum_{i \in I_0}(\bar{x}_1^i + \bar{x}_2^i) \leqslant 0$.

Let P be the market price domain $\{p \in \mathbb{R}_+^m\colon \sum_{h=1}^m p_h = 1\}$. Also, let $I = I_0 \cup \{n\}$, $X^n = P$, $X = \prod_{i \in I} X^i$ and $u_n(x, p) = p.\sum_{i=1}^{n-1}(x_1^i + x_2^i)$, where $x \in \prod_{i \in I_0} X^i$.

The market economy $\mathcal{E}$ is said to satisfy the weak form of Walras' law if

$$\text{for every } (x, p) \in \left(\prod_{i \in I_0} X^i\right) \times P, \quad p.\sum_{i=1}^{n-1}(x_1^i + x_2^i) \leqslant 0.$$

Corollary 3.1. *Suppose that the market economy game $\mathcal{E} = (X_i, u_i)_{i \in I_0}$ is convex, compact, continuous, concave and satisfies the weak form of Walras' law. If the game $G' = (X_i, u_i)_{i \in I}$ satisfies the coalition consistency property, then $\mathcal{E}$ has an equilibrium.*

4. Characterization of strong Nash equilibria

In the following, we characterize the existence of SNE by providing a necessary and sufficient condition. To do so, define a function $\digamma : X \times \Delta \times \hat{X} \to \mathbb{R}$ by

$$\digamma(x, \lambda, \hat{y}) = \sum_{S \in \Im} \sum_{i \in S} \lambda_i \{u_i(y_S, x_{-S}) - u_i(x)\},$$

where $\hat{X} = \prod_{S \in \Im} X_S$.

[3] For more details, see the book of Ichiishi (Ref. [14]).

Note that, by the definition of $\digamma$, we have

$$\max_{\hat{y}\in\hat{X}} \digamma(x,\lambda,\hat{y}) \geqslant 0 \quad \text{for all } x \in X \text{ and } \lambda \in \Delta. \tag{4.1}$$

Indeed, for $x \in X$ and $\lambda \in \Delta$, letting $\hat{y} = \phi(x) = (x_S,\ S \in \Im)$,[4] we have $\digamma(x,\lambda,\hat{y}) = 0$, and consequently, $\max_{\hat{y}\in\hat{X}} \digamma(x,\lambda,\hat{y}) \geqslant 0$ for all $(x,\lambda) \in X \times \Delta$.

Let

$$\alpha = \inf_{\lambda\in\Delta} \inf_{x\in X} \sup_{\hat{y}\in\hat{X}} \digamma(x,\lambda,\hat{y}).$$

We will use the following result given by Moulin et al. [16, p. 162].

Lemma 4.1. *Suppose that X is convex in a vector space and the functions u_i, $i \in I$, are concave on X. Then, $\bar{x} \in X$ is a weakly Pareto efficient strategy profile of the game $G = \langle X_i, u_i\rangle_{i\in I}$ if and only if there exists $\lambda \in \Delta_I$ such that $\sup_{y\in X} \sum_{i\in I} \lambda_{i,I} u_i(y) = \sum_{i\in I} \lambda_{i,I} u_i(\bar{x})$.*

We then have the following theorem.

Theorem 4.1 *(Necessity Theorem). Suppose that X_i is a nonempty convex subset of a topological vector space and u_i is concave on X for all $i \in I$. If the game $G = \langle X_i, u_i\rangle_{i\in I}$ has a strong Nash equilibrium, then $\alpha = 0$.*

Proof. Let $\bar{x} \in X$ be a strong Nash equilibrium of the game $G = \langle X_i, u_i\rangle_{i\in I}$. According to Lemma 2.1, $\bar{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_j, u_j(., \bar{x}_{-S})\rangle_{j\in S}$ for all $S \in \Im$. Since X_i is nonempty and convex, and u_i is concave on X for all $i \in I$, then by Lemma 4.1, there exists $\bar{\lambda}_S \in \Delta_S$ such as $\sup_{y_S\in X_S} \sum_{i\in S} \bar{\lambda}_{i,S}\{u_i(y_S, \bar{x}_{-S}) - u_i(\bar{x})\} = 0$ for all $S \in \Im$. This equality implies:

$$\sup_{\hat{y}\in\hat{X}} \digamma(\bar{x},\bar{\lambda},\hat{y}) = 0.$$

Thus, we have:

$$\alpha = \inf_{x\in X} \inf_{\lambda\in\Delta} \sup_{\hat{y}\in\hat{X}} \digamma(x,\lambda,\hat{y}) \leqslant \sup_{\hat{y}\in\hat{X}} \digamma(\bar{x},\bar{\lambda},\hat{y}) = 0. \tag{4.2}$$

Inequalities (4.1) and (4.2) imply $\alpha = 0$. This proves the necessity. □

Theorem 4.2 *(Sufficiency Theorem). Suppose that for all $i \in I$, X_i is a nonempty compact subset of a Hausdorff topological space, and u_i is continuous on X. If $\alpha = 0$, then the game $G = \langle X_i, u_i\rangle_{i\in I}$ possesses a strong Nash equilibrium.*

Proof. By the assumptions of Theorem 4.2, for all $x \in X$ and $\lambda \in \Delta$, the maximum of the function $\digamma(x,\lambda,.)$ over $\hat{X}$ and $\min_{x\in X} \min_{\lambda\in\Delta} \max_{\hat{y}\in\hat{X}} \digamma(x,\lambda,\hat{y})$ exist.

Suppose that $\alpha = 0$. Since the functions $x \mapsto \digamma(x,\lambda,\hat{y})$ and $\lambda \mapsto \digamma(x,\lambda,\hat{y})$ are continuous over compact X and Δ, respectively, then the Weierstrass Theorem implies that there exist $\bar{x} \in X$ and $\bar{\lambda} \in \Delta$ such that $\alpha = \max_{\hat{y}\in\hat{X}} \digamma(\bar{x},\bar{\lambda},\hat{y}) = 0$, and this equality implies $\digamma(\bar{x},\bar{\lambda},\hat{y}) = \sum_{S\in\Im} \sum_{i\in S} \bar{\lambda}_{i,S}\{u_i(y_S, \bar{x}_{-S}) - u_i(\bar{x})\} \leqslant 0$ for all $\hat{y} \in \hat{X}$.

[4] The function ϕ is defined in the last section.

For any arbitrarily fixed $S \in \Im$, we have for all $\hat{y} \in \hat{X}$,

$$F(\bar{x}, \bar{\lambda}, \hat{y}) = \sum_{i \in S} \bar{\lambda}_{i,S}\{u_i(y_S, \bar{x}_{-S}) - u_i(\bar{x})\} + \sum_{K \in \Im,\, K \neq S} \sum_{i \in K} \bar{\lambda}_{i,S}\{u_i(y_K, \bar{x}_{-K}) - u_i(\bar{x})\} \leqslant 0.$$

For $\hat{y} \in \hat{X}$ such that y_S is arbitrarily chosen in X_S and $y_K = \bar{x}_K$ for all $K \neq S$, we have $\sum_{K \in \Im,\, K \neq S} \sum_{i \in K} \bar{\lambda}_{i,S}\{u_i(y_K, \bar{x}_{-K}) - u_i(\bar{x})\} = 0$. Then, from the last inequality, we deduce that $\sum_{i \in S} \bar{\lambda}_{i,S} u_i(y_S, \bar{x}_{-S}) \leqslant \sum_{i \in S} \bar{\lambda}_{i,S} u_i(\bar{x})$ for all $y_S \in X_S$. Since S is arbitrarily chosen in $\Im$, then for all $y_S \in X_S$,

$$\sum_{i \in S} \bar{\lambda}_{i,S} u_i(y_S, \bar{x}_{-S}) \leqslant \sum_{i \in S} \bar{\lambda}_{i,S} u_i(\bar{x}) \quad \text{for all } S \in \Im. \tag{4.3}$$

Now we prove that $\bar{x}_S$ is weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \bar{x}_{-S})\rangle_{j \in S}$ for all $S \in \Im$.

Suppose that there exists $S_0 \in \Im$ such that $\bar{x}_{S_0}$ is not weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \bar{x}_{-S_0})\rangle_{j \in S_0}$. Then, there exists $\tilde{y}_{S_0} \in X_{S_0}$ such that:

$$u_j(\tilde{y}_{S_0}, \bar{x}_{-S_0}) > u_j(\bar{x}) \quad \text{for all } j \in S_0. \tag{4.4}$$

System (4.4) implies that $\sum_{j \in S_0} \bar{\lambda}_{j,S_0} u_j(\tilde{y}_{S_0}, \bar{x}_{-S_0}) > \sum_{j \in S_0} \bar{\lambda}_{j,S_0} u_j(\bar{x})$ with $\bar{\lambda}_{j,S_0} \geqslant 0$ and $\sum_{j \in S_0} \bar{\lambda}_{j,S_0} = 1$. This contradicts inequality (4.3) for $S = S_0$ and $y_S = \tilde{y}_{S_0}$. Hence, $\bar{x}_S$ is weakly Pareto efficient for the sub-game $\langle X_j, u_j(., \bar{x}_{-S})\rangle_{j \in S}$ for all $S \in \Im$. Consequently, by Lemma 2.1, $\bar{x}_S$ is a strong Nash equilibrium. □

Theorems 4.1 and 4.2 actually provide a method of finding a SNE of a game under certain conditions (see Algorithm 1).

Algorithm 1 Procedure for determining SNE

Require: Suppose that all the conditions of Theorems 4.1 and 4.2 are satisfied.
Require: Calculate the value $\alpha = \min_{x \in X} \min_{\lambda \in \Delta} \max_{\hat{y} \in \hat{X}} F(x, \lambda, \hat{y})$.
if $\alpha > 0$, **then**
the game $G = (X_i, u_i)_{i \in I}$ has no SNE.
else
any strategy profile $\bar{x} \in X$ such that $\min_{\lambda \in \Delta} \max_{\hat{y} \in \hat{X}} F(\bar{x}, \lambda, \hat{y}) = 0$ is a SNE of the game $G = \langle X_i, u_i\rangle_{i \in I}$.
end if

The following example illustrates the application of Algorithm 1.

Example 4.1. Consider a game with $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, $x = (x_1, x_2)$, and

$$u_1(x) = 3x_1 - x_2^2 + 4x_2,$$
$$u_2(x) = -x_1^2 + x_1 - 2x_2.$$

It is obvious to see that the functions u_i are concave over the convex X, $i = 1, 2$.

In this example, we have $\hat{X} = X_1 \times X_2 \times (X_1 \times X_2)$, and put $\hat{y} = (a, b, (c, d)) \in X_1 \times X_2 \times (X_1 \times X_2)$ and $x = (u, v)$.

We have

$$\begin{aligned}
\alpha &= \min_{(x,\lambda) \in X \times \Delta} \max_{\hat{y} \in \hat{X}} F(x, \hat{y}) = \min_{\lambda \in [0,1]} \min_{u,v \in [-1,1]} \max_{a,b,c,d \in [-1,1]} \big\{ \big[u_1(a, v) - u_1(u, v)\big] \\
&\quad + \big[u_2(u, b) - u_2(u, v)\big] + \big[\lambda\big(u_1(c, d) - u_1(u, v)\big) + (1 - \lambda)\big(u_2(c, d) - u_2(u, v)\big)\big] \big\} \\
&= \min_{u,v \in [-1,1]} \min_{\lambda \in [0,1]} \max_{a,b,c,d \in [-1,1]} \big\{ [3a - 2b] + \big[-(1 - \lambda)c^2 + (1 + 2\lambda)c\big] + \big[-\lambda d^2 + 2(3\lambda - 1)d\big] \\
&\quad + \big[(1 - \lambda)u^2 - 2(2 + \lambda)u\big] + \big[\lambda v^2 + (4 - 6\lambda)v\big] \big\}.
\end{aligned}$$

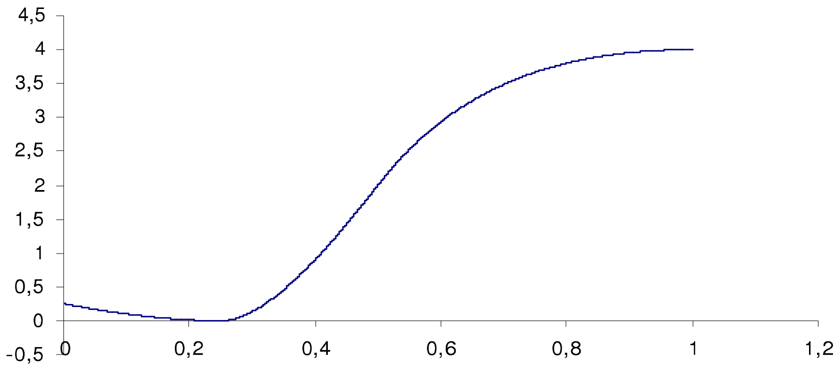

Fig. 1. The graph of function h.

Let us consider the following function:

$$h : [0, 1] \to \mathbb{R}$$

defined by $\lambda \mapsto h(\lambda) = \min_{u,v\in[-1,1]} \max_{a,b,c,d\in[-1,1]}\{[3a - 2b] + [-(1-\lambda)c^2 + (1+2\lambda)c] + [-\lambda d^2 + 2(3\lambda - 1)d] + [(1-\lambda)u^2 - 2(2+\lambda)u] + [\lambda v^2 + (4-6\lambda)v]\}$.

We recall that $\alpha = \min_{\lambda\in[0,1]} h(\lambda)$.

The minimum and maximum of function F are attained respectively from: $\tilde{a} = \tilde{u} = 1$, $\tilde{b} = -1$,

$$\tilde{c} = \begin{cases} \frac{1+2\lambda}{2(1-\lambda)}, & \text{if } 0 \leqslant \lambda \leqslant 1/4, \\ 1, & \text{if } 1/4 \leqslant \lambda \leqslant 1, \end{cases} \qquad \tilde{d} = \begin{cases} -1, & \text{if } 0 \leqslant \lambda \leqslant 1/4, \\ \frac{3\lambda-1}{\lambda}, & \text{if } 1/4 \leqslant \lambda \leqslant 1/2, \\ 1, & \text{if } 1/2 \leqslant \lambda \leqslant 1 \end{cases} \quad \text{and}$$

$$\tilde{v} = \begin{cases} -1, & \text{if } 0 \leqslant \lambda \leqslant 1/2, \\ \frac{3\lambda-1}{\lambda}, & \text{if } 1/2 \leqslant \lambda \leqslant 1/2. \end{cases}$$

We then obtain:

$$h(\lambda) = \begin{cases} \frac{16\lambda^2-8\lambda+1}{4(1-\lambda)}, & \text{if } 0 \leqslant \lambda \leqslant 1/4, \\ \frac{16\lambda^2-8\lambda+1}{\lambda}, & \text{if } 1/4 \leqslant \lambda \leqslant 1/2, \\ \frac{-4\lambda^2+12\lambda-4}{\lambda}, & \text{if } 1/2 \leqslant \lambda \leqslant 1. \end{cases}$$

We see that $\alpha = \min_{\lambda\in[0,1]} h(\lambda) = h(1/4) = 0$ (Fig. 1). According to Algorithm 1, the considered game has a strong Nash equilibrium which is $\bar{x} = (\tilde{u}, \tilde{v}) = (1, -1)$.

5. Applications

In this section we show how our main existence result is applied to some important economic games. We provide two applications: one is to games in an economy with multilateral environmental externalities, which is intensively studied by Chander and Tulkens [7], and the other is to a simple oligopoly game.

5.1. Games in an economy with multilateral environmental externalities

Consider an economy with multilateral externalities and n agents, indexed by $i \in I = \{1, \dots, n\}$. A consumption good $y_i \geqslant 0$ is produced from an input $e_i \in [0, e_i^0]$. The technology is described by a production function $y_i = g_i(e_i)$, and each agent's preference is presented by a quasilinear utility function $u_i(y_i, z) = y_i - v_i(z)$ where $v_i(z)$ is i's disutility function of the externality given by $z = \sum_{h\in I} e_h$.

Define an n-person noncooperative game $G = \langle X_i, u_i \rangle_{i\in I}$ as follows. Let

$$X_i = \{e_i \in \mathbb{R}\colon\ 0 \leqslant e_i \leqslant e_i^0\}$$

be the strategy set of each player i, and X_S the space of joint strategies of players in $S \in \Im$. Let X denote the space of joint strategies of all players, *i.e.*, $X = X_I$. For a strategy profile $[(e_1, \dots, e_n)] \in X$, we choose $u_i(y_i, z) = y_i - v_i(z)$ with $z = \sum_{i\in I} e_i$ as the payoff for player i. Let $u = (u_1, \dots, u_n)$.

By Lemma 2.1, we know that $\bar{e} \in X$ is a strong Nash equilibrium of the game $G = \langle X_i, u_i \rangle_{i\in I}$ if and only if $\bar{e}_S \in X_S$ is weakly Pareto efficient for the sub-game $G_S(\bar{e}) = \langle X_j, u_j(., \bar{e}_{-S})\rangle_{j\in S}$. By Lemma 4.1, $\bar{e}_S \in X_S$ is weakly Pareto efficient for the sub-game $G_S(\bar{e})$ if and only if there exists $\lambda_S \in \Delta_S$ such that

$$\sup_{d_S\in X_S} \sum_{i\in S} \lambda_{i,S}\bigl[g_i(d_i) - v_i(d_S + \bar{e}_{-S})\bigr] = \sum_{i\in S} \lambda_{i,S}\bigl[g_i(\bar{e}_i) - v_i(\bar{e})\bigr],$$

where $d_S + \bar{e}_{-S} = \sum_{j\in S} d_j + \sum_{j\in -S} \bar{e}_j$ and $\bar{e} = \sum_{j\in I} \bar{e}_j$.

To characterize weak Pareto efficiency for the sub-game $G_S(e)$, we get the first order conditions

$$\lambda_{j,S} g_j'(d_j) = \sum_{h\in S} \lambda_{h,S} v_h'\Biggl(\sum_{i\in S} d_i + \sum_{i\in -S} e_i\Biggr), \quad j \in S,\ \lambda_S \in \Delta_S. \tag{5.1}$$

Consider two coalitions $S_1, S_2 \in \Im$ and player j such that $j \in S_1 \cap S_2$. Then, (5.1) implies:

$$\begin{cases} (1)\ \lambda_{j,S_1} g_j'(d_j^1) = \sum_{h\in S_1} \lambda_{h,S_1} v_h'\Bigl(\sum_{i\in S_1} d_i^1 + \sum_{i\in -S_1} e_i\Bigr), & \lambda_{S_1} \in \Delta_{S_1}; \\ (2)\ \lambda_{j,S_2} g_j'(d_j^2) = \sum_{h\in S_2} \lambda_{h,S_2} v_h'\Bigl(\sum_{i\in S_2} d_i^2 + \sum_{i\in -S_2} e_i\Bigr), & \lambda_{S_2} \in \Delta_{S_2}. \end{cases} \tag{5.2}$$

For $e \in X$ to be a strong Nash equilibrium, it is necessary that $d_j^1 = d_j^2 = \cdots = d_j^k = e_j$, for each $j \in S_1 \cap S_2 \cap \cdots \cap S_k$.

While we can use Theorems 4.1 and 4.2 to provide necessary and sufficient conditions for the existence of strong Nash equilibrium for this problem, here we provide sufficient conditions for the existence of strong Nash equilibrium by applying Theorem 3.1. To do so, we make the following assumptions.

Assumption 1. $g_i(e_i) - v_i(z)$ is concave and differentiable over an interval $[0, e_i^0]$.

Assumption 2. There exist $\lambda \in \Delta$ and $e \in X$ such that

$$\lambda_{j,S} g_j'(e_j) = \sum_{h\in S} \lambda_{h,S} v_h'\Biggl(\sum_{i\in I} e_i\Biggr) \quad \text{for all } j \in S \text{ and } S \in \Im. \tag{5.3}$$

Then, by Theorem 3.1, we have the following result.

Proposition 5.1. *Suppose Assumptions 1 and 2 are satisfied.*[5] *Then, the game $G = \langle X_i, u_i \rangle_{i\in I}$ possesses a strong Nash equilibrium.*

[5] The solutions of the following system are within the set $\prod_{i\in S}[0, e_i^0]$, $j \in S$ and $\lambda_S \in \Delta_S$:

$$\lambda_{j,S} g_j'(e_j) = \sum_{h\in S} \lambda_{h,S} v_h'\Bigl(\sum_{i\in I} e_i\Bigr).$$

Example 5.1. Consider the game $G = \langle X_i, u_i\rangle_{i\in I}$ with $I = \{1, 2, \ldots, n\}$, $e = (e_1, \ldots, e_n)$, $z = \sum_{i=1}^n e_i$, and

$$g_i(e_i) = a_i e_i^2 - b_i e_i + c_i, \qquad u_i(y_i, z) = y_i - v_i(z),$$
$$v_i(z) = az^2 - bz + c \quad \text{with } a_i, b_i, a, b > 0,\ c \geqslant 0 \text{ and } b_i^2 - 4a_i c_i < 0.$$

Assume that $u_i(e) = g_i(e_i) - v_i(z)$ is concave over $\prod_{i\in I}[0, e_i^0]$ with $e_i^0 \geqslant \frac{b_i}{2a_i}$. We now show that Assumption 2 is satisfied. Consider $\lambda \in \Delta$ and $\bar{e} \in X$ defined as follows:

$$\lambda_{i,S} = \frac{1}{|S|} \quad \text{for all } S \in \Im \quad \text{and} \quad \bar{e}_i = \frac{b_i}{2a_i} \quad \text{for all } i \in I.$$

If $\bar{z} = \sum_{i=1}^n \frac{b_i}{2a_i} = \frac{b}{2a}$, then $\bar{e} = (\frac{b_1}{2a_1}, \ldots, \frac{b_n}{2a_n})$ is a strong Nash equilibrium. Indeed, we have $g_i(e_i) = a_i e_i^2 - b_i e_i + c_i$ and $v_i(z) = az^2 - bz + c$, then $g_i'(\bar{e}_i) = 0$ and $v_i'(\bar{z}) = 0$. Thus (5.3) holds.

5.2. Simple static oligopoly game

This subsection is dedicated to examining a simple oligopoly game. We first recall the Cournot model in which the firms are quantity choosers producing a homogeneous good.

Let p be the market price of a perfectly homogeneous good produced by the n firms of an industry ($I = \{1, \ldots, n\}$), q_i be the sales of the i-th firm, $q = (q_1, \ldots, q_n)$, and $Q = \sum_{i=1}^n q_i$ be the total sales in the market. The inverse demand function is $p = F(Q)$. The cost for the i-th firm is given by $C_i(q_i)$. The profit of the i-th firm is then given by $\psi_i(q) = q_i F(Q) - C_i(q_i)$.

Define a noncooperative game $G = \langle X_i, \psi_i\rangle_{i\in I}$ as follows. Let $X_i = [0, \bar{q}_i]$, $X = \prod_{i\in I}[0, \bar{q}_i]$, $X_S = \prod_{i\in S}[0, \bar{q}_i]$, for each $S \in \Im$, and $\psi = (\psi_1, \ldots, \psi_n)$.

Again, we want to provide some sufficient conditions that guarantee the existence of SNE. To do so, we make the following assumptions.

Assumption 3. $F(Q)$ and $C_i(q_i)$ are continuous and nonnegative on $Q \in [0, +\infty)$ and $q_i \in [0, +\infty)$, respectively.

Assumption 4. There exists $\bar{q}_i > 0$, $i = 1, \ldots, n$ such that $\psi_i(q)$ is concave over $\prod_{i\in I}[0, \bar{q}_i]$.

Assumption 5. There exist $\lambda \in \Delta$ and $q \in X$ such that

$$\lambda_{j,S} C_j'(q_j) = \lambda_{j,S} F\left(\sum_{i\in I} q_i\right) + F'\left(\sum_{i\in I} q_i\right) \sum_{h\in S} \lambda_{h,S} q_h \quad \text{for all } j \in S \text{ and } S \in \Im. \tag{5.4}$$

Then, by Theorem 3.1, we have the following proposition.

Proposition 5.2. *Suppose Assumptions 3, 4 and 5 are satisfied.*[6] *Then, the game $G = \langle X_i, \psi_i\rangle_{i\in I}$ possesses a strong Nash equilibrium.*

[6] The solutions of the following system are within the set $\prod_{i\in S}[0, e_i^0]$, $j \in S$ and $\lambda_S \in \Delta_S$:

$$\lambda_{j,S} C_j'(q_j) = \lambda_{j,S} F\Big(\sum_{i\in I} q_i\Big) + F'\Big(\sum_{i\in I} q_i\Big) \sum_{h\in S} \lambda_{h,S} q_h.$$

Example 5.2. Consider a game with $I = \{1, 2, \ldots, n\}$, $q = (q_1, \ldots, q_n)$, $Q = \sum_{i=1}^{n} q_i$, and

$$F(Q) = \begin{cases} aQ^2 - bQ + c, & \text{if } 0 \leqslant Q \leqslant \frac{b}{2a}, \\ \frac{-b^2+2b+4ac}{4a} - Q, & \text{if } \frac{b}{2a} < Q \leqslant \frac{-b^2+2b+4ac}{4a}, \\ 0 & \text{if } Q > \frac{-b^2+2b+4ac}{4a}, \end{cases}$$

and

$$C_i(q_i) = \theta_i q_i^2 \quad \text{for } i = 1, \ldots, n,$$

where $b^2 - 4ac < 0$ and $a, b, \theta_i > 0$ for $i = 1, \ldots, n$, and the inverse demand $F(Q)$ is non-increasing in Q.

Suppose that $\psi_i(q) = q_i F(Q) - C_i(q_i)$ is concave over $\prod_{i \in I} [0, q_i^0]$ with $q_i^0 \geqslant \frac{4ac-b^2}{8a\theta_i}$.

If $(4ac - b^2) \sum_{i=1}^{n} \frac{1}{\theta_i} = 4b$, then there exists $\bar{q} = (\frac{4ac-b^2}{8a\theta_1}, \ldots, \frac{4ac-b^2}{8a\theta_n})$ such that Assumption 5 is satisfied. To see this, let $\lambda_{i,S} = \frac{1}{|S|}$ for all $S \in \Im$ and $\bar{q}_i = \frac{4ac-b^2}{8a\theta_i}$ for all $i \in I$. Then $Q = \sum_{i=1}^{n} \bar{q}_i = \frac{b}{2a}$, *i.e.*, $F'(\bar{q}) = 0$.

Since $F'(\bar{q}) = 0$, then system (5.4) becomes:

$$2\theta_i \bar{q}_i = \frac{4ac - b^2}{4a} \quad \text{for all } i \in I.$$

Thus, $\bar{q}_i = \frac{4ac-b^2}{8a\theta_i}$, $i \in I$ such that $F'(\bar{q}) = 0$. This condition is equivalent to $(4ac - b^2) \sum_{i=1}^{n} \frac{1}{\theta_i} = 4b$. Therefore, $\bar{q} = (\frac{4ac-b^2}{8a\theta_1}, \ldots, \frac{4ac-b^2}{8a\theta_n})$ is a strong Nash equilibrium.

6. Conclusion

In the present paper we provide some existence results on strong Nash equilibria in general games. We introduce a condition, called coalition consistency property which, together with the concavity and continuity of payoffs, permits the existence of strong Nash equilibria in games with compact and convex strategy spaces. The coalition consistency property is a general condition that cannot be dispensed with for the existence of strong Nash equilibrium. It is satisfied in many economic games and relatively easy to check.

We also characterize the existence of strong Nash equilibria by providing a necessary and sufficient condition. Moreover, we suggest a procedure that can be used to compute strong Nash equilibrium. Our results would be useful for solving theoretical and practical problems from various domains. The results are illustrated with applications to economies with multilateral environmental externalities and to the simple static oligopoly model.

Strong Nash equilibrium implicitly considers only pure strategies, excluding mixed/correlated strategies. However, by making reasonable restrictions, the set of all probability measures over product of pure strategies can satisfy the conditions imposed in our existence results. For instance, compactness would obtain if one assumes the weak* topology and each pure strategy set X_i is a compact metric space, endowing each product of sets with the product topology.

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