On Responsiveness of Two Competing Mechanisms to Affirmative Action in School Choice

Zhenhua Jiao and Guoqiang Tian*

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Abstract

This paper investigates the responsiveness of two competing mechanisms to affirmative action in school choice. We show that the top trading cycles (henceforth, TTC) mechanism is minimally responsive to the priority-based and reserve-based affirmative action policies provided all schools have *common priority within student type*, while it is not true for the student-proposing deferred acceptance (henceforth, DA) mechanism. We also fully characterize the priority structures of the DA mechanism. It is shown that the DA mechanism is *minimally responsive* to the priority-based affirmative action *if and only if* schools' priority structure satisfies a stronger version of Doğan's acyclicity. Moreover, we show that the TTC mechanism is *not minimally responsive* to *any of the three popular types* of affirmative action policies even if full priority is given to the minority, while the DA mechanism *is minimally responsive* to *all of the three types* of affirmative action policies under such scope.

Keywords: School choice; Affirmative action; Minimal responsiveness; Deferred acceptance algorithm; Top trading cycles mechanism
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1 Introduction

School choice programs aim to give students the option to choose their school. At the same time, underrepresented students should be favored to close the opportunity gap.

*Jiao: School of Economics, Shanghai University of Finance and Economics, Shanghai, 200433, China, jiao.zhenhua@mail.shufe.edu.cn. Tian: Department of Economics, Texas A&M University, College Station, TX 77843, USA, gtian@tamu.edu. Jiao acknowledges financial support from the Zhejiang Provincial Natural Science Foundation of China (Grant No. LY16G010008). Tian acknowledges financial support from the National Natural Science Foundation of China (NSFC-71371117) and the Key Laboratory of Mathematical Economics (SUFE) at Ministry of Education of China.

Affirmative action policies that intend to give minority students higher chances to attend their desired schools have been playing an important role in achieving this goal in the United States and many other countries. As such, how to favor minority students while achieving other desiderata has been an important concern.

There are three popular types of affirmative action policies in school choice: *the quota-based, the reserve-based, and the priority-based*. The *quota-based* affirmative action policy in school choice gives minority students higher chances to attend more preferred schools by *limiting the number of admitted majority students* at some schools. There are many examples of majority-quota type public school admission policies in the United States.[1] For school choice with quota-based affirmative action, many authors (for instance, Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Bó (2016), Echenique and Yenmez (2015), Ehlers *et al.* (2014) and Fragiadakis and Troyan (2017)) study the stability, efficiency and proofness issues. The *reserve-based* affirmative action policy is to *reserve some seats at each school for the minority students*, and to require that a reserved seat at a school be assigned to a majority student only if no minority student prefers that school to her assignment. Hafalir *et al.* (2013) show that, in the efficiency aspect, the reserve-based policy has an advantage over the quota-based policy. The *priority-based* affirmative action favors minority students by means of *promoting their priority ranking at schools*. In Chinese college admissions, the minority students are favored by a priority-based affirmative action policy that awards bonus points to minority students in the national college entrance examination.[2]

A measure of how an assignment mechanism performs in terms of affirmative action is the so-called minimal responsiveness.[3] A mechanism is "minimally responsive" to an affirmative action if *increasing the level of affirmative action (ceteris paribus) does not result in a Pareto inferior assignment for the minority students*. Since the affirmative action in school choice aims to improve the welfare of the minority students, this property is seemingly the *minimal* requirement for a satisfactory assignment mechanism.

For school choice with affirmative action, it is comparatively ideal to find a stable or efficient mechanism that is minimally responsive from the mechanism-design perspective.

[1]See, for instance, Hafalir *et al.* (2013), Ehlers *et al.* (2014) and Doğan (2016) for detailed discussions, including disadvantages of the policy, such as avoidable inefficiency.

[2]As Chen and Kesten (2017) point out, in recent years, there are about 10 million high school seniors who compete for 6 million seats at universities in China each year. In the college admissions, students' priority orders at colleges are determined by students' scores in the National College Entrance Examination.

[3]Kojima (2012) first introduces this concept. He calls it "respecting the spirit of affirmative action". Recently, Doğan (2016) starts to use the notion of "minimal responsiveness".

It is well known that the DA mechanism is stable and the TTC mechanism is efficient. Although the minimal responsiveness is a seemingly mild requirement, it is known from the literature Kojima (2012) and Hafalir *et al.* (2013) that, on the full domain of school choice, neither DA mechanism nor TTC mechanism is minimally responsive to any type of the three prominent affirmative action policies. That is, in general, both DA and TTC mechanisms suffer from the following difficulty: a higher level of affirmative action may not benefit any minority student but hurt some minority students.

These impossibility results exist on the full domain of school choice problems. One may have positive results if the domain of school choice problems is restricted in certain ways. Indeed, Doğan (2016) shows that the DA mechanism is minimally responsive to the *reserve-based* affirmative action if and only if a kind of acyclicity condition is satisfied.[4] Loosely speaking, Doğan's acyclicity condition prevents situations in which both a priority-loop condition and some kind of scarcity condition hold simultaneously.[5]

We first consider the minimal responsiveness of the TTC mechanism on a restricted school priority domain and obtain a desirable result. To the best of our knowledge, at present there exists no related positive result on the responsiveness of the TTC mechanism in the literature. Our first main result is concerned with problems in which the priority rankings among the same type of students are exactly the same between any two schools, and the rankings between different types of students can differ between these schools in an arbitrary way. Under such a priority restriction, we show that the TTC mechanism is *minimally responsive to the priority-based and reserve-based* affirmative action policies (Theorem 1). However, the DA mechanism does not satisfy such property under this kind of priority structure (Example 1).

In order to give a full characterization on the minimal responsiveness of the DA mechanism to the *priority-based* affirmative action, we introduce a new acyclicity condition, which is stronger than Doğan's acyclicity, and then show that the DA mechanism is minimally responsive to the *priority-based* affirmative action if and only if our acyclicity condition is satisfied (Theorem 2). It turns out that the acyclicity is a very

[4] In the literature, there are many types of acyclicity conditions having a similar spirit. See Ergin (2002), Kesten (2006), Haeringer and Klijn (2009), Kojima (2011), Klaus and Klijn (2013) and Hatfield *et al.* (2016) for details.

[5] The scarcity of desirable school seat resource is probably a source of the failure of minimal responsiveness to affirmative action. If the school seat resources are not scarce, for instance, we consider an extreme situation: $q_c \geq |S|$ for each $c \in C$. For such matching problems, we can see that the resources (desirable school seats) are abundant enough to assign each minority student her most preferred school seat. Thus, running DA or TTC, every minority student is matched with her most preferred school under any affirmative action policy. Then both DA and TTC mechanisms are minimally responsive.

restrictive condition, so Theorem 2 may be regarded as an impossibility result for the responsiveness of the DA mechanism.

Doğan (2016) also considers a restricted domain of school choice where *full priority is given to the minority* (in the sense that, at each school c, either each minority student is one of its q_c highest-priority students, or each minority student has higher priority than all majority students), and shows that, for school choice with *quota-based* affirmative action, a stable and minimally responsive mechanism exists *if and only if* the given problem gives full priority to the minority. As pointed out by Doğan (2016), the condition of giving full priority to the minority is very restrictive. In fact, under such a restricted domain, the DA mechanism is minimally responsive to *all of the three popular types* of affirmative action policies. However, for the TTC mechanism, we obtain that it is *not* minimally responsive to *any type of the three* affirmative action policies even when full priority is given to the minority (Theorem 3).

We also deal with the priority-based affirmative action problem in which the level of affirmative action is increased sufficiently enough such that the new problem gives full priority to the minority. We show that, if a stronger priority-based affirmative action favors minority students by way of *giving full priority to the minority*, then such a policy makes *each minority student* weakly better off under the DA mechanism. That is, under the DA mechanism, a stronger affirmative action policy makes no minority student worse off if full priority is given to the minority under this policy (Proposition 2). This property is undoubtedly attractive when an affirmative action aims to improve the Pareto welfare of the minority students. However, the TTC mechanism does not have such a desirable property as that for the DA mechanism (Example 3).

Our last result shows that when the original problem gives full priority to the minority, under the DA mechanism, the original problem and a stronger priority-based/reserve-based affirmative action produce the *same* assignment outcome. The original problem and a stronger quota-based affirmative action produce the *same* assignment outcome for the *minority* students, while the stronger quota-based affirmative action makes the majority students weakly worse off (Theorem 4). Thus, a quota-based affirmative action would result in avoidable efficiency loss and, both priority-based and reserve-based affirmative action policies do not play an actual role under the DA mechanism. Comparing with the DA, we find that, under the TTC mechanism, even if the original problem gives full priority to the minority, each type of stronger affirmative action policy may result in a Pareto inferior assignment for minority students (Example 4).

The remainder of the paper is organized as follows. We present some preliminaries on the formal model in the next section. Section 3 investigates the minimal responsiveness of the TTC mechanism. Section 4 provides two impossibility results respectively on the minimal responsiveness of DA and TTC mechanism. Section 5 presents discussion on weak Pareto improvement for the minority. Section 6 concludes. All technical proofs are provided in the Appendix.

2 The Model

2.1 Settings

Let S and C be finite and disjoint sets of **students** and **schools**. There are two types of students: **minority students** and **majority students**. Let S^m and S^M denote the sets of minority and majority students, respectively. They are nonempty sets such that $S^m \cup S^M = S$ and $S^m \cap S^M = \emptyset$. Suppose that $|C|, |S| \geq 2$.

For each student $s \in S$, P_s is a strict (i.e., complete, transitive, and anti-symmetric) preference relation over $C \cup \{s\}$, where s denotes the outside option, which can be attending a private school or being home-schooled. School c is **acceptable** to student s if $cP_s s$. The **preference profile** for a group of students S' is denoted by $P_{S'} = (P_s)_{s \in S'}$. For any $c, c' \in C$ and $s \in S$, $cR_s c'$ denotes either $cP_s c'$ or $c = c'$. For each school $c \in C$, $\succ_c$ is a strict priority order over S. The **priority profile** for a group of schools C' is denoted by $\succ_{C'} = (\succ_c)_{c \in C'}$. For any $s, s' \in S$ and $c \in C$, $s \succeq_c s'$ denotes either $s \succ_c s'$ or $s = s'$.

For each $c \in C$, q_c is the capacity of c or the number of seats in c. We assume that there are enough seats for all students, so $\sum_{c \in C} q_c \geq |S|$. Let $q = (q_c)_{c \in C}$ be the **capacity profile**.

For each school $c \in C$, there is a **majority quota** affirmative action parameter q_c^M such that $q_c^M \leq q_c$ and $q_c^M \in \mathbb{Z}_+$[6], and q_c^M denotes the majority type-specific quota of school c. Let $q^M \equiv (q_c^M)_{c \in C}$ be the **majority quota profile**.

For each school $c \in C$, there is also a **minority reserve** affirmative action parameter r_c^m with $r_c^m \in \mathbb{Z}_+$ and $r_c^m \leq q_c$, and r_c^m denotes the number of seats at c at which the minority students are "favored". Let $r^m \equiv (r_c^m)_{c \in C} \in$ be the **minority reserve profile**.

A **school choice problem with affirmative action**, or simply a **problem**, is a tuple $G \equiv (S, C, P_S, \succ_C, q, q^M, r^m)$. Since S, C and q are fixed throughout this paper, unless otherwise noted, a problem is simply a quadruple $\boldsymbol{G \equiv (P_S, \succ_C, q^M, r^m)}$.

A matching is an assignment of students to schools such that each student is assigned

[6] $\mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}$is the set of nonnegative integers.

to a school or to her outside option, no school admits more students than its capacity, and no school admits more majority students than its majority type-specific quota. Formally, a **matching** μ is a mapping from $C \cup S$ to the subsets of $C \cup S$ such that

(i) for each $s \in S$, $\mu(s) \in C \cup \{s\}$,

(ii) for each $c \in C$ and $s \in S$, $\mu(s) = c$ if and only if $s \in \mu(c)$,

(iii) for each $c \in C$, $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$, and

(iv) for each $c \in C$, $|\mu(c) \cap S^M| \leq q_c^M$.

A matching μ' **is Pareto inferior to** μ **for the minority** if (i) $\mu(s) R_s \mu'(s)$ for all $s \in S^m$; (ii) $\mu(s') P_{s'} \mu'(s')$ for at least one $s' \in S^m$. A **mechanism** is a mapping ϕ that, for each school choice problem G, associates a matching $\phi(G)$.

2.2 Affirmative Action Policies

In this subsection, we will introduce the three kinds of affirmative action policies: the quota-based type, the reserve-based type and the priority-based type, sequentially. For a problem $G = (P_S, \succ_C, q^M, r^m)$, the **quota-based affirmative action policy** is implemented by prohibiting each school c to admit more students than its majority type-specific quota q_c^M and setting $r_c^m = 0$ for all $c \in C$. A problem $\tilde{G} = (P_S, \succ_C, \tilde{q}^M, r^m)$ is said to **have a stronger quota-based affirmative action policy than** $G = (P_S, \succ_C, q^M, r^m)$ if, for all $c \in C$, $\tilde{q}_c^M \leq q_c^M$ and $r_c^m = 0$.

Definition 1. A matching mechanism ϕ is said to **be minimally responsive to the quota-based affirmative action** if there are no problems G and $\tilde{G}$ such that $\tilde{G}$ has a stronger quota-based affirmative action policy than G and $\phi(\tilde{G})$ is Pareto inferior to $\phi(G)$ for the minority.

For a problem $G = (P_S, \succ_C, q^M, r^m)$, the **reserve-based affirmative action policy** is implemented by giving priority to minority students at each school c up to the minority reserve r_c^m and setting $q_c^M = q_c$ for all $c \in C$. For a school c, if the number of minority students admitted to it is less than r_c^m, then any minority applicant is given priority over any majority applicant at c. If there are not enough minority students to fill up the reserves, majority students can still be assigned to school c's reserved seats. A problem $\tilde{G} = (P_S, \succ_C, q^M, \tilde{r}^m)$ is said to **have a stronger reserve-based affirmative action policy than** $G = (P_S, \succ_C, q^M, r^m)$ if, for all $c \in C$, $\tilde{r}_c^m \geq r_c^m$ and $q_c^M = q_c$.

Definition 2. A matching mechanism ϕ is said to **be minimally responsive to the reserve-based affirmative action** if there are no problems G and $\tilde{G}$ such that $\tilde{G}$ has a stronger reserve-based affirmative action policy than G and $\phi(\tilde{G})$ is Pareto inferior to $\phi(G)$ for the minority.

Definition 3. For a problem $G = (P_S, \succ_C, q^M, r^m)$ and any $c \in C$, a priority $\succ'_c$ **is an improvement for minority students over** $\succ_c$ if

(i) $s \succ_c s'$ and $s \in S^m$ imply $s \tilde{\succ}_c s'$, and

(ii) $s, s' \in S^M$ and $s \succ_c s'$ imply $s \tilde{\succ}_c s'$.

A priority profile $\succ'_C$ **is an improvement for minority students over** $\succ_C$ if, for all $c \in C$, $\succ'_c$ is an improvement for minority students over $\succ_c$.

In other words, if we change $\succ_c$ to $\succ'_c$ by means of promoting the rankings of some minority students at schools relative to majority students while keeping the relative ranking of each student within her own group fixed, then $\succ'_c$ is an improvement for minority students over $\succ_c$.

For a problem $G = (P_S, \succ_C, q^M, r^m)$, the **priority-based affirmative action policy** is implemented by improving the rankings of minority students and setting $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$. A problem $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ is said to **have a stronger priority-based affirmative action policy than** $G = (P_S, \succ_C, q^M, r^m)$ if, for all $c \in C$, (i) $\tilde{\succ}_C$ is an improvement for minority students over $\succ_C$, and (ii) $q_c^M = q_c$ and $r_c^m = 0$.

Definition 4. A matching mechanism ϕ is said to **be minimally responsive to the priority-based affirmative action** if there are no problems G and $\tilde{G}$ such that $\tilde{G}$ has a stronger priority-based affirmative action policy than G and $\phi(\tilde{G})$ is Pareto inferior to $\phi(G)$ for the minority.

2.3 DA Mechanism

For each problem $(P_S, \succ_C, q^M, r^m)$, the DA mechanism is defined through the following **deferred acceptance algorithm**:[7]

• **Step** 1: Start with a matching in which no student is matched. Each student applies to her most preferred acceptable school. Each school c first considers minority applicants and tentatively accepts them up to its minority reserve r_c^m one at a time according to its priority order if there are enough minority applicants. School c then considers all the applicants who are yet to be accepted and tentatively accepts them, one at a time

[7]The original deferred acceptance algorithm was proposed by Gale and Shapley (1962). Now it has been the most important mechanism in matching theory and applications.

according to its priority order, until its capacity is filled or the applicants are exhausted, while not admitting more majority students than q_c^M. The rest of the applicants, if any remain, are rejected by c.

In general, at

• **Step k,** $k \geq 2$: Start with the tentative matching obtained at the end of Step $k-1$. Each student who got rejected at Step $k-1$ applies to her next preferred acceptable school. Each school c considers the new applicants and students admitted tentatively at Step $k-1$. Among these students, school c first tentatively accepts minority students up to its minority reserve r_c^m one at a time according to its priority order. School c then considers all the applicants who are yet to be accepted, and one at a time according to its priority order, it tentatively accepts as many students as up to its capacity while not admitting more majority students than the remaining majority-type specific quota. The rest of the students, if any remain, are rejected by c. If there are no rejections, then stop.

The algorithm terminates when no rejection occurs and the tentative matching at that step is finalized. Since no student reapplies to a school that has rejected her and at least one rejection occurs in each step, the algorithm stops in finite time. For a problem $(P_S, \succ_C, q^M, r^m)$, the DA matching is the one reached at the termination of the deferred acceptance algorithm and is denoted by $\mathbf{DA}(\boldsymbol{P_S}, \boldsymbol{\succ_C}, \boldsymbol{q^M}, \boldsymbol{r^m})$.

For a problem $(P_S, \succ_C, q^M, r^m)$, (i) if $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the original version by Gale and Shapley (1962), or the version proposed by Abdulkadiroğlu and Sönmez (2003) for school choice problem without affirmative action. (ii) If $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the version proposed by Abdulkadiroğlu and Sönmez (2003) for controlled school choice problems (or for problems with only quota-based affirmative action, see for instance Kojima (2012)). (iii) If $q_c^M = q_c$ for all $c \in C$, then the above algorithm reduces to the version proposed by Hafalir *et al.* (2013) for problems with only reserve-based affirmative action.

2.4 TTC Mechanism

For each problem $(P_S, \succ_C, q^M, r^m)$, the TTC mechanism is defined through the following **top trading cycles algorithm**:[8]

[8]The original top trading cycles algorithm was proposed for housing markets and attributed to David Gale by Shapley and Scarf (1974). It is extensively used in matching theory and applications.

• **Step** 1: Start with a matching in which no student is matched. For schools, if a school has minority reserves, then it points to a minority student who has the highest priority at that school among the minority students; otherwise it points to a student who has the highest priority at that school among all students. For students, each student s points to her most preferred school that is acceptable and still has a seat for her (if there is such a school; otherwise she points to herself), that is, an acceptable school whose capacity is strictly positive and, if $s \in S^M$, its majority type-specific quota is strictly positive. Since the number of students and schools is finite, there exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity of each school in a cycle is reduced by one. If the assigned student is in S^M and the school, say c, matched to s has majority quota at this step, then the school matched to that student reduces its majority-specific quota by one. If the assigned student, say s, is in S^m and the school, say c, matched to s has minority reserves at this step, then school c reduces its minority reserves by one. If no student remains, terminate. Otherwise, proceed to the next step.

In general, at

• **Step** k**,** $k \geq 2$: For each remaining school $c \in C$, if it has minority reserves, then c points to a minority student who has the highest priority at c among all remaining minority students; otherwise it points to a student who has the highest priority at c among all remaining students. Each remaining student s points to her most preferred school (among the remaining schools) that is acceptable and still has a seat for her (if there is such a school; otherwise she points to herself), that is, an acceptable school whose remaining capacity is strictly positive and, if $s \in S^M$, its remaining majority-specific quota is strictly positive. There exists at least one cycle. Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity of each school in a cycle is reduced by one. If the assigned student is in S^M and the school, say c, matched to s has majority quota at this step, then the school matched to that student reduces its majority-specific quota by one. If the assigned student, say s, is in S^m and the school, say c, matched to s has minority reserves at this step, then school c reduces its minority reserves by one. If no student remains, terminate. Otherwise, proceed to the next step.

This algorithm terminates in a finite number of steps because at least one student is matched at each step as long as the algorithm has not terminated and there are a finite number of students. For a problem $(P_S, \succ_C, q^M, r^m)$, the TTC matching is the one reached at the termination of the top trading cycles algorithm and is denoted by **TTC($P_S, \succ_C, q^M, r^m$)**.

For a problem $(P_S, \succ_C, q^M, r^m)$, (i) if $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the version proposed by Abdulkadiroğlu and Sönmez (2003) for school choice problem without affirmative action. (ii) If $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the version proposed by Abdulkadiroğlu and Sönmez (2003) for controlled school choice problems (or for problems with only quota-based affirmative action, see for instance Kojima (2012)). (iii) If $q_c^M = q_c$ for all $c \in C$, then the above algorithm reduces to the version proposed by Hafalir *et al.* (2013) for problems with only reserve-based affirmative action.

3 On Minimal Responsiveness of TTC Mechanism

According to Hafalir *et al.* (2013) and Kojima (2012), we know that, on the full domain of school choice, the TTC mechanism is not minimally responsive to any of the three types of affirmative action policies. In this section, we investigate the minimal responsiveness of TTC mechanism under some restricted school priority condition. Our first main result is concerned with the priority-based and reserve-based affirmative action policies on a restricted domain where all schools have the common priority within student type. This kind of priority structure is very similar to that studied by Kentaro (2018). We specify the following concept on schools' priority structure.

Definition 5. For a problem $G = (P_S, \succ_C, q^M, r^m)$, **all schools have the common priority within student type** if, for any $c, c' \in C$, $s_1^m, s_2^m \in S^m$ and $s_1^M, s_2^M \in S^M$, we have $s_1^m \succ_c s_2^m \Leftrightarrow s_1^m \succ_{c'} s_2^m$ and $s_1^M \succ_c s_2^M \Leftrightarrow s_1^M \succ_{c'} s_2^M$.

The condition of common priority within type is a kind of acyclicity priority condition which says that the priority rankings among the same type of students are exactly the same between any two given schools. For example, suppose that there are four schools $c_1, \cdots, c_4$ and four students $S^M = \{s_1, s_2\}, S^m = \{s_3, s_4\}$. The priorities of four schools are given by

$\succ_{c_1}$: s_1, s_2, s_3, s_4;

$\succ_{c_2}$: s_1, s_3, s_2, s_4;

$\succ_{c_3}$: s_3, s_4, s_1, s_2;

$\succ_{c_4}$: s_3, s_1, s_2, s_4.

Then the school priority profile $(\succ_c)_{c \in C}$ is common within type.

From the above example, one can see that the condition of common priority within type is much weaker than homogeneous priority condition. Under such a priority restriction, we obtain that the TTC mechanism is minimally responsive to priority-based and reserve-based affirmative action. Specifically, we have the following result.

Theorem 1. *Let $G = (P_S, \succ_C, q^M, r^m)$ be a problem such that all schools have the common priority within student type. Then*

(i). The TTC mechanism is minimally responsive to the priority-based affirmative action.

(ii). The TTC mechanism is minimally responsive to the reserve-based affirmative action.

Proof. See Appendix A.1.

For the DA mechanism, we cannot obtain a desirable result as that for the TTC mechanism in Theorem 1. For the reserve-based affirmative action, Example 1 in Hafalir *et al.* (2013) shows that there exists matching environment in which $\tilde{G} = (P_S, \succ_C, q^M, \tilde{r}^m)$ has a stronger reserve-based affirmative action policy than $G = (P_S, \succ_C, q^M, r^m)$ and all schools have the homogeneous priority, but $\text{DA}(\tilde{G})$ is Pareto inferior to $\text{DA}(G)$ for the minority. For the priority-based affirmative action, we consider the following example.

Example 1. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$, $q_c = q_c^M = 1$, for all $c \in C$, $r^m = (0, 0, 0)$. Students' preferences and schools' priorities are given by the table.

P_{s_1}	P_{s_2}	P_{s_3}	$\succ_{c_1} = \succ_{c_2} = \succ_{c_3}$	$\tilde{\succ}_{c_1}$	$\tilde{\succ}_{c_2}$	$\tilde{\succ}_{c_3}$
c_1	c_3	c_1	s_1	s_2	s_1	s_1
c_3	c_1	c_2	s_2	s_3	s_2	s_2
c_2	c_2	c_3	s_3	s_1	s_3	s_3

For $(P_S, \succ_C, q^M, r^m)$, the outcome of the DA mechanism is

$$\text{DA}(P_S, \succ_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix}.$$

For $(P_S, \tilde{\succ}_C, q^M, r^m)$, the outcome of the DA mechanism is

$$\text{DA}(P_S, \tilde{\succ}_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_3 & s_1 \end{pmatrix}.$$

It is easy to see that all schools have the common priority within student type under both $(P_S, \succ_C, q^M, r^m)$ and $(P_S, \tilde{\succ}_C, q^M, r^m)$, and $(P_S, \tilde{\succ}_C, q^M, r^m)$ has a stronger priority-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. One can see that, the minority student s_2 is strictly worse off under DA$(P_S, \tilde{\succ}_C, q^M, r^m)$ than under DA$(P_S, \succ_C, q^M, r^m)$, while s_3 is matched to the same school under both matchings. Then DA$(P_S, \tilde{\succ}_C, q^M, r^m)$ is Pareto inferior to DA$(P_S, \succ_C, q^M, r^m)$ for the minority.

From the perspective of minimal responsiveness to the reserve-based or priority-based affirmative action, we can see that the TTC mechanism has an advantage over the DA mechanism when all schools have the common priority within student type.

Moreover, for the quota-based affirmative action, one cannot expect a desirable result as that for priority-based and reserve-based affirmative action policies. Specifically, we consider the following example.

Example 2. Let $C = \{c_1, c_2\}, S^M = \{s_1\}$, $S^m = \{s_2\}$, $q_{c_i} = 1$ and $r^m_{c_i} = 0$ for $i = 1, 2$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	$\succ_{c_1} = \succ_{c_2}$
c_1	c_2	s_1
c_2	c_1	s_2

Let $q^M = (1, 1)$ and $\tilde{q}^M = (0, 1)$ be different schools' majority quota profiles. One can see that both schools have the homogeneous priority over students and $\tilde{G} = (P_S, \succ_C, \tilde{q}^M, r^m)$ has a stronger quota-based affirmative action policy than $G = (P_S, \succ_C, q^M, r^m)$. The TTC matchings under G and $\tilde{G}$ are respectively

$$\mu = \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix},$$

and

$$\tilde{\mu} = \begin{pmatrix} c_1 & c_2 \\ s_2 & s_1 \end{pmatrix}.$$

One can see that under the stronger affirmative action policy $\tilde{G}$, the only minority student s_2 becomes strictly worse off than under G. That is, $\tilde{\mu}$ is Pareto inferior to μ for the minority. Thus, the TTC mechanism is not minimally responsive to the quota-based affirmative action even under the homogeneous school priority condition.

As an immediate consequence of Theorem 1, we have the following result.

Corollary 1. *Let $G = (P_S, \succ_C, q^M, r^m)$ be a problem such that all schools have homogeneous priority, that is, $\succ_c = \succ_{c'}$ for all $c, c' \in C$. Then the TTC mechanism is minimally responsive to the priority-based and reserve-based affirmative action policies.*

4 Further Impossibilities for Affirmative Action

In this section, we provide two impossibility results respectively on the minimal responsiveness of DA and TTC mechanism. For the DA mechanism, according to Kojima (2012), we know that, on the full domain of school choice, it is not minimally responsive to the priority-based affirmative action. For this type of affirmative action, it is interesting to characterize the domain where the DA mechanism is minimally responsive. Our next result completes the task.

For the reserve-based affirmative action, Doğan (2016) proposes an acyclicity condition which characterizes exactly the domain where the DA mechanism is minimally responsive. Now we introduce a new acyclicity condition, which is a stronger requirement than Doğan's acyclicity. We will show that it is necessary and sufficient for the DA mechanism to be minimally responsive to the priority-based affirmative action. Specifically, we first present the following notions introduced by Doğan (2016).

For school $c \in C$ and student $s \in S$, let $U_c^{\succ}(s) \equiv \{s' \in S : s' \succ_c s\}$. Let $s_1, s_2 \in S$, $N \subseteq S$, $c \in C$. The pair **(s_1, N) is a threat to s_2 at c and $(q_c, \succ_c)$** if $|N| = q_c - 1$ and

either: $s_2 \in S^m$, $s_1 \succ_c s_2$ and $N \subseteq U_c^{\succ}(s_2) \backslash \{s_1\}$,

or: $s_2 \in S^M$, $N \cap S^M \subseteq U_c^{\succ}(s_2) \backslash \{s_1\}$, and if $s_1 \in S^M$ then $s_1 \succ_c s_2$.

The intuition is that if (s_1, N) is a threat to s_2 at c and $(q_c, \succ_c)$, then there exists $\succ_c'$ being an improvement for minority students over $\succ_c$ such that the students s_1, s_2, and N compete for a seat at c and s_2 does not get a seat at c. Let **$T(s, c)$ denote the set of threats to s at c and $(q_c, \succ_c)$**.

A problem **G has a cycle** if there is a minority student $m \in S^m$, a majority student $M \in S^M$, a list of students $s_1, \cdots, s_{k-1} \in S$, a list of schools $c_1, \cdots, c_k \in C$, and a list of disjoint sets of students $N_1, \cdots, N_k \subseteq S$ such that

1) $s_{k-1} \succ_{c_1} m$, $M \succ_{c_1} m$, $M \succ_{c_2} s_1$, $\{s_1, \cdots, s_{k-1}\} \bigcap S^m \neq \emptyset$ and

2) $(m, N_1) \in T(M, c_1)$, $(M, N_2) \in T(s_1, c_2)$, $(s_{k-1}, N_1) \in T(m, c_1)$, and for each $t \in \{2, \cdots, k-1\}$, $(s_{t-1}, N_{t+1}) \in T(s_t, c_{t+1})$.

A list $(m, M, s_1, \cdots, s_{k-1}, c_1, \cdots, c_k, N_1, \cdots, N_k)$ satisfying 1 and 2 is a cycle of length

k.

Definition 6. A problem ***G* is acyclic** if it has no cycle.

A cycle is a *Doğan's cycle* if we also require that $s_{k-1} \in S^m$. A problem G satisfies *Doğan's acyclicity* if it has no Doğan's cycle. Obviously, the acyclicity condition is a stronger version of Doğan's acyclicity.

The acyclicity introduced above fully characterizes the domain where the DA mechanism is minimally responsive to the priority-based affirmative action. Specifically, we have the following result.

Theorem 2. *The DA mechanism is minimally responsive to the priority-based affirmative action if and only if the problem G is acyclic.*

Proof. See Appendix A.3.

As pointed out by Doğan (2016), Doğan's acyclicity is a restrictive condition. Since the acyclicity condition is even stronger than Doğan's acyclicity, Theorem 2 can be regarded as an impossibility result.

For the minimal responsiveness to quota-based affirmative action, Doğan (2016) introduces priority structures called "giving full priority to the minority" and shows that DA is minimally responsive to the quota-based affirmative action if and only if the problem gives full priority to the minority. Loosely speaking, a problem giving full priority to the minority means that, at each school c, either each minority student is ranked above each majority student, or each minority student is one of the q_c highest-priority students. Formally, a problem $(P_S, \succ_C, q^M, r^m)$ **gives full priority to the minority** if there are no $m \in S^m$, $M \in S^M$, and $c \in C$ such that $M \succ_c m$ and $|\{s \in S : s \succeq_c m\}| > q_c$.

One can see that the condition of giving full priority to the minority is very restrictive. Under such a restriction, we obtain much more desirable responsiveness results for the DA mechanism in Section 5. Additionally, it is easy to see that, if a problem gives full priority to the minority, then the problem satisfies the acyclicity condition and Doğan's acyclicity (see also Appendix A.7). Thus, giving full priority to the minority is sufficient for the DA mechanism to be minimally responsive to the priority-based and reserve-based affirmative action. However, for the responsiveness of another extensively used mechanism—TTC mechanism, we have a negative result as follows.

Theorem 3. *The TTC mechanism is not minimally responsive to the quota-based, reserve-based or priority-based affirmative action even when schools give full priority to the minority.*

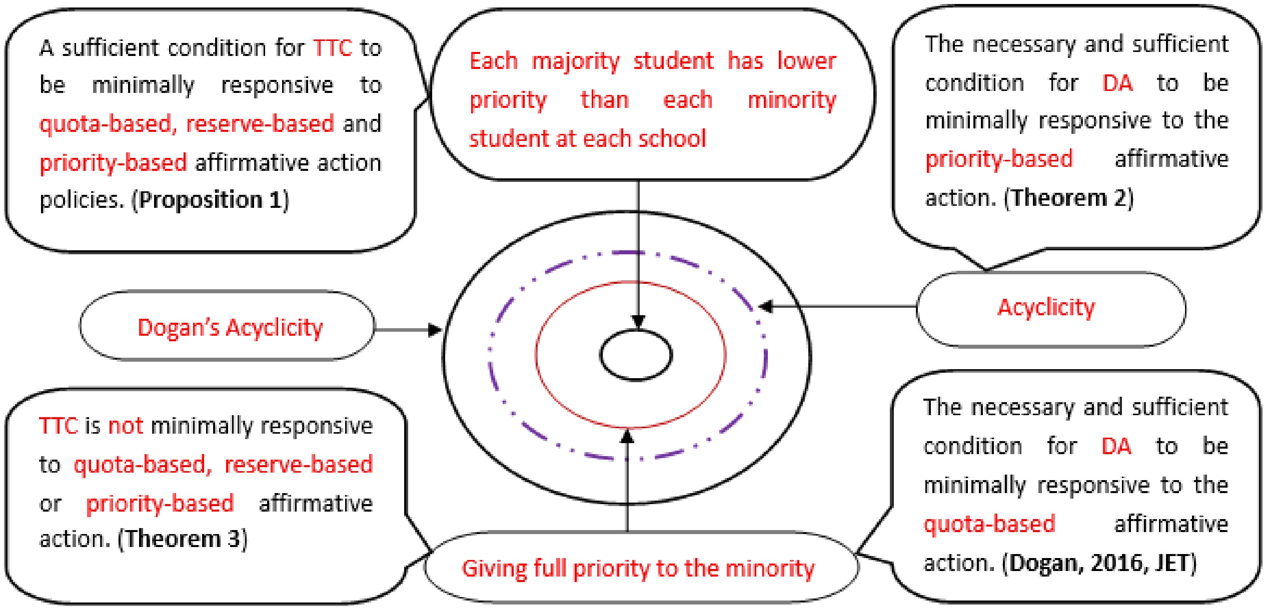

Figure 1: Impossibilities and Relationship Between Different Domains

Proof. See Appendix A.4.

Although the condition of giving full priority to the minority is very restrictive, it is not sufficient for the TTC mechanism to be minimally responsive to any of the three popular types of affirmative action policies. Then Theorem 3 can be regarded as an impossibility result.

At the end of this section, we provide two sufficient conditions for the TTC mechanism to be minimally responsive to all of the three types of affirmative action.

Proposition 1. *(i). If there are no* $m \in S^m$, $M \in S^M$ *and* $c \in C$ *such that* $M \succ_c m$, *that is, at each school* $c \in C$, *each majority student has lower priority than each minority student, then the TTC mechanism is minimally responsive to the quota-based, reserve-based and priority-based affirmative action policies.*[9]

(ii). If there are no $m \in S^m$ *and* $c \in C$ *such that* $|\{s \in S : s \succeq_c m\}| > q_c$, *that is, at each school* $c \in C$, *every minority student is one of the* q_c *highest-priority students, then the TTC mechanism is minimally responsive to the quota-based, reserve-based and priority-based affirmative action policies.*[10]

[9] On this restricted domain, another famous mechanism, the Boston mechanism is not minimally responsive to the quota-based affirmative action. See Appendix A.8.

[10] It is easy to show that, on this restricted domain, the Boston mechanism is also minimally responsive to the quota-based affirmative action.

Proof. The proof is trivial, so we omit it.

According to Theorem 1 in Doğan (2016) and Theorem 3 above, we can see that, for the minimal responsiveness to affirmative action, the DA mechanism has an advantage over TTC mechanism when schools give full priority to the minority. As a simple summary, we describe the main results of this section by an intuitive figure as above (Figure 1).

5 Weak Pareto Improvement for the Minority

Now we consider a much stronger responsiveness requirement that increasing the level of affirmative action can weakly Pareto improve the welfare of minority students. For priority-based affirmative action, we can obtain a more desirable result which says that, if we increase the level of affirmative action sufficiently enough such that the new problem gives full priority to the minority, then the stronger affirmative action policy makes each minority student weakly better off under the DA algorithm. Formally, we have the following result.

Proposition 2. *Let $G = (P_S, \succ_C, q^M, r^m)$ and $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ be two problems such that $\tilde{G}$ has stronger priority-based affirmative action than G. If $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ gives full priority to the minority, then $DA(\tilde{G}) \succeq_s DA(G)$ for each minority student $s \in S^m$.*

Proof. See Appendix A.5.

We note that, for the TTC mechanism, one cannot expect to obtain a desirable result as that for DA mechanism in Proposition 2. Specifically, we consider the following example.

Example 3. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $q_{c_i} = q^M_{c_i} = 2$ for $i = 1, 2$, $q_{c_3} = q^M_{c_3} = 1$ and $r^m_{c_i} = 0$ for $i = 1, 2, 3$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
c_1	c_3	c_2	c_3	s_4	s_3	s_1
s_1	s_2	s_3	s_4	s_2	s_4	s_2
				s_3	s_2	s_3
				s_1	s_1	s_4

For $(P, \succ, q^M, r^m)$, the outcome of the TTC is

$$\text{TTC}(P, \succ, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_2 \\ s_1 & s_3 & s_4 & s_2 \end{pmatrix}.$$

Let the priority of c_3 be changed to $\tilde{\succ}_{c_3}$: s_2, s_3, s_4, s_1, and $\tilde{\succ}_{c_i} = \succ_{c_i}$ for $i = 1, 2$. For $(P, \tilde{\succ}, q^M, r^m)$, the outcome of the TTC is

$$\text{TTC}(P, \tilde{\succ}, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_1 & s_3 & s_2 & s_4 \end{pmatrix}.$$

It is easy to check that $(P, \tilde{\succ}, q^M, r^m)$ has a stronger priority-based affirmative action policy than $(P, \succ, q^M, r^m)$ and $(P, \tilde{\succ}, q^M, r^m)$ gives full priority to the minority. One can see that, the minority student s_4 is strictly worse off under $\text{TTC}(P, \tilde{\succ}, q^M, r^m)$ than under $\text{TTC}(P, \succ, q^M, r^m)$. That is, $\text{TTC}(P, \tilde{\succ}, q^M, r^m)$ is not a weak Pareto improvement over $\text{TTC}(P, \succ, q^M, r^m)$ for minority students.

Next we study the comparison between different affirmative action policies when the original problem $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority. Let $(P_S, \tilde{\succ}_C, q^M, r^m)$, $(P_S, \succ_C, q^M, \tilde{r}^m)$ and $(P_S, \succ_C, \tilde{q}^M, r^m)$ be three problems that are different from $(P_S, \succ_C, q^M, r^m)$. For a given problem $(P_S, \succ_C, q^M, r^m)$, if all of the majority students leave the market, then we denote the corresponding small problem by $(S^m, C, P_{S^m}, \succ_C |_{S^m}, q^M, r^m)$. Denote $\mu \equiv \text{DA}(P_S, \succ_C, q^M, r^m)$, $\mu' \equiv \text{DA}(S^m, C, P_{S^m}, \succ_C |_{S^m}, q^M, r^m)$, $\mu^p \equiv \text{DA}(P_S, \tilde{\succ}_C, q^M, r^m)$, $\mu^q \equiv \text{DA}(P_S, \succ_C, \tilde{q}^M, r^m)$, and $\mu^r \equiv \text{DA}(P_S, \succ_C, q^M, \tilde{r}^m)$. Then we present our last result.

Theorem 4. *Suppose that the original problem $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority students. Then*

(1). $\mu(s) = \mu'(s)$ for every $s \in S^m$;

(2). If $(P_S, \succ_C, \tilde{q}^M, r^m)$ has stronger quota-based affirmative action than $(P_S, \succ_C, q^M, r^m)$, then $\mu(s) = \mu^q(s)$ for every $s \in S^m$ and $\mu(s) \succeq_s \mu^q(s)$ for all $s \in S^M$;

(3). If $(P_S, \tilde{\succ}_C, q^M, r^m)$ and $(P_S, \succ_C, q^M, \tilde{r}^m)$ are respectively stronger priority-based and reserve-based affirmative action policies than $(P_S, \succ_C, q^M, r^m)$, then $\mu(s) = \mu^p(s) = \mu^r(s)$ for all $s \in S$.

Proof. See Appendix A.6.

According to Theorem 4, one can see that, if a market gives full priority to the minority, stronger priority-based and reserve-based affirmative action policies do not play an actual role under the DA algorithm. Moreover, a stronger quota-based affirmative action just probably makes majority students worse off and results in avoidable efficiency loss.

We note that all of the corresponding results for the DA mechanism given in Theorem 4 fail to hold for the TTC mechanism. Specifically, we consider the following example.

Example 4. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $q_{c_i} = q^M_{c_i} = 1$ for $i = 1, 2$, $q_{c_3} = q^M_{c_3} = 4$ and $r^m_{c_i} = 0$ for $i = 1, 2, 3$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
c_2	c_3	c_1	c_1	s_2	s_4	s_1
s_1	s_2	s_3	s_4	s_4	s_3	s_3
				s_3	s_2	s_2
				s_1	s_1	s_4

For $(P_S, \succ_C, q^M, r^m)$, the outcome of the TTC is

$$\text{TTC}(P_S, \succ_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_3 \\ s_4 & s_1 & s_2 & s_3 \end{pmatrix}.$$

For a small problem with all majority students leaving the market, the outcome of the TTC for minority students is

$$\text{TTC}(S^m, C, P_{S^m}, \succ_C |_{S^m}, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_3 & \varnothing & s_2 & s_4 \end{pmatrix}.$$

It is easy to see that $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority. One can also see that student s_4 becomes strictly worse off and student s_3 becomes strictly better off under the small market, while student s_2 remains unchanged.

For the quota-based affirmative action, we choose $\tilde{q}^M_{c_2} = 0$ and $\tilde{q}^M_{c_i} = q^M_{c_i}$ for $i = 1, 3$. Then $(P_S, \succ_C, \tilde{q}^M, r^m)$ has a stronger quota-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\text{TTC}(P_S, \succ_C, \tilde{q}^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_1 & s_4 \\ s_3 & \varnothing & s_2 & s_1 & s_4 \end{pmatrix}.$$

Then comparing with $\text{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\text{TTC}(P_S, \succ_C, \tilde{q}^M, r^m)$, student s_4 becomes strictly worse off, student s_3 becomes strictly better off and student s_2 remains unchanged.

For the reserve-based affirmative action, we choose $\tilde{r}^m_{c_3} = 1$ and $\tilde{r}^m_{c_i} = r^m_{c_i}$ for $i = 1, 2$. Then $(P_S, \succ_C, q^M, \tilde{r}^m)$ has a stronger reserve-based affirmative action policy than

$(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\text{TTC}(P_S, \succ_C, q^M, \tilde{r}^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_3 & s_1 & s_2 & s_4 \end{pmatrix}.$$

Then comparing with $\text{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\text{TTC}(P_S, \succ_C, q^M, \tilde{r}^m)$, student s_4 becomes strictly worse off, student s_3 becomes strictly better off and student s_2 remains unchanged.

For the priority-based affirmative action, let $\tilde{\succ}_{c_3} : s_3, s_1, s_2, s_4$ and $\tilde{\succ}_{c_i} = \succ_{c_i}$ for $i = 1, 2$. Then $(P_S, \tilde{\succ}_C, q^M, \tilde{r}^m)$ has a stronger priority-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\text{TTC}(P_S, \tilde{\succ}_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_3 & s_1 & s_2 & s_4 \end{pmatrix}.$$

Then comparing with $\text{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\text{TTC}(P_S, \tilde{\succ}_C, q^M, r^m)$, student s_4 becomes strictly worse off, student s_3 becomes strictly better off and student s_2 remains unchanged.

Finally, we note that, when schools give full priority to the minority, the DA mechanism has an advantage over the TTC mechanism from the perspective of improving Pareto welfare of minority students.

6 Conclusion

Affirmative action policies in school choice aim to improve the welfare of the minority students. This requires that a given matching mechanism should satisfy the minimal responsiveness. In view of fairness and efficiency issues, it is comparatively ideal to find a stable or efficient and minimally responsive assignment mechanism from the mechanism-design perspective. The DA mechanism is stable while TTC mechanism is efficient. Unfortunately, neither DA nor TTC mechanism is minimally responsive to the popular affirmative action policies on the full domain of school choice problems.

We studied the responsiveness of the DA and TTC mechanism to affirmative actions in school choice on restricted domains. We first obtained that the TTC is minimally responsive to the priority-based and reserve-based affirmative action policies if all schools

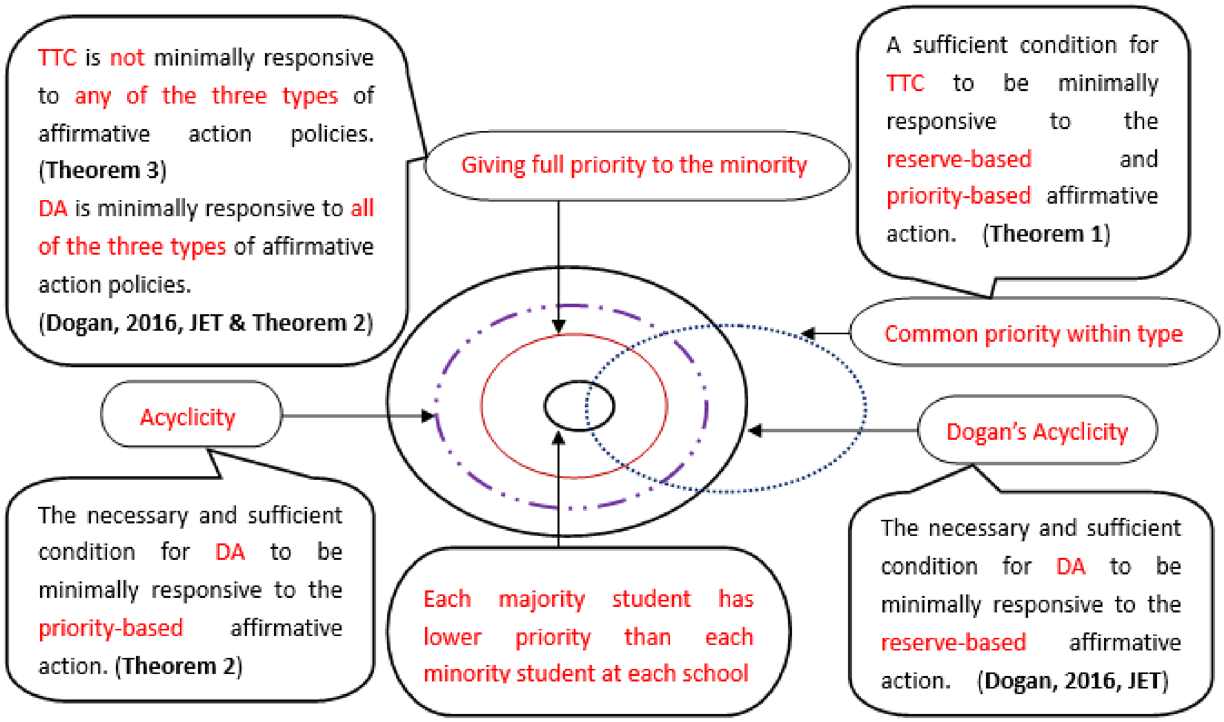

Figure 2: Main Results and Relationship Between Different Domains

have the common priority within student type. We then gave a full characterization of the priority structures for which the DA mechanism is minimally responsive to the priority-based affirmative action. We also showed that the TTC mechanism is not minimally responsive to any type of the three popular affirmative action policies even when full priority is given to the minority. It is a surprising result, as giving full priority to the minority is so restrictive that it guarantees the DA is minimally responsive to all of the three types of affirmative action policies. Part of our results can be described by a figure as above (Figure 2).

Doğan (2016) obtains that giving full priority to the minority and Doğan's acyclic priority structure respectively characterize exactly the domain where the DA mechanism is minimally responsive to quota-based and reserve-based affirmative action. Theorem 2 of the present paper shows that an acyclic priority condition, which is a restricted domain between Doğan's acyclicity and the condition of giving full priority to the minority, exactly characterizes the domain where the DA mechanism is minimally responsive to priority-based affirmative action. For future investigation, it is interesting to give an exact characterization of the priority structures (respectively for three types of affirmative action) for which the TTC mechanism is minimally responsive.

A Appendix

A.1 Proof of Theorem 1. (i) Let $G = (P_S, \succ_C, q^M, r^m)$ and $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ be two problems with $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$. Suppose that $\tilde{G}$ has a stronger priority-based affirmative action policy than G. We want to show that $\tilde{\mu}$ =TTC$(\tilde{G})$ is not Pareto inferior to μ =TTC(G) for minority students.

Let $S^M = \{s_1^M, s_2^M, \cdots, s_l^M\}$ such that $s_i^M \succ_c s_j^M$ for all $c \in C$ and $1 \le i < j \le l$, and $S^m = \{s_1^m, s_2^m, \cdots, s_n^m\}$ such that $s_i^m \succ_c s_j^m$ for all $c \in C$ and $1 \le i < j \le n$.

Step I. We show that $\tilde{\mu}(s_1^m) R_{s_1^m} \mu(s_1^m)$.

Consider the procedure of the top trading cycles algorithm for problem $\tilde{G}$. Suppose that s_1^m is assigned to school $\tilde{\mu}(s_1^m)$ at Step k_1 of the top trading cycles algorithm for problem $\tilde{G}$. We discuss two cases:

Case 1: $k_1 = 1$, that is, s_1^m is assigned to school $\tilde{\mu}(s_1^m)$ at the first step of the top trading cycles algorithm of $\tilde{\mu}$. If this happens, then $\tilde{\mu}(s_1^m)$ is her most preferred school. Thus we infer that $\tilde{\mu}(s_1^m) R_{s_1^m} \mu(s_1^m)$.

Case 2: $k_1 > 1$. If this happens, then we can infer that all students who are matched before Step k_1 are majority students. Specifically, at Step 1 of the TTC algorithm, each school $c \in C$ points to either s_1^M or s_1^m (See Figure 3 (a)).

Let $\tilde{C}_1^m \equiv \{c \in C : c$ points to s_1^mat Step 1 of the TTC algorithm for $\tilde{G}\}$ and

$\tilde{C}_1^M \equiv \{c \in C : c$ points to s_1^Mat Step 1 of the TTC algorithm for $\tilde{G}\}$.

We have $\tilde{C}_1^m \cup \tilde{C}_1^M = C$. There are four possible cases:

(1) At Step 1 of the TTC algorithm for $\tilde{G}$, both s_1^m and s_1^M point to a (same or different) school in $\tilde{C}_1^m$, respectively (See Figure 3 (b)). For this case, it is easy to see that s_1^m can be matched with her favorite school at Step 1. We get a contradiction. Then this case is impossible.

(2) At Step 1 of the TTC algorithm for $\tilde{G}$, both s_1^m and s_1^M point to a (same or different) school in $\tilde{C}_1^M$, respectively (See Figure 3 (c)). For this case, it is easy to see that s_1^M is assigned to her favorite school at Step 1 and s_1^m is not assigned to any school.

(3) At Step 1 of the TTC algorithm for $\tilde{G}$, s_1^m points to a school in $\tilde{C}_1^m$ and s_1^M points to a school in $\tilde{C}_1^M$ (See Figure 3 (d)). For this case, it is easy to see that both s_1^M and s_1^m are assigned to their favorite school at Step 1. We get a contradiction. Then this case is impossible.

(4) At Step 1 of the TTC algorithm for $\tilde{G}$, s_1^m points to a school in $\tilde{C}_1^M$ and s_1^M points

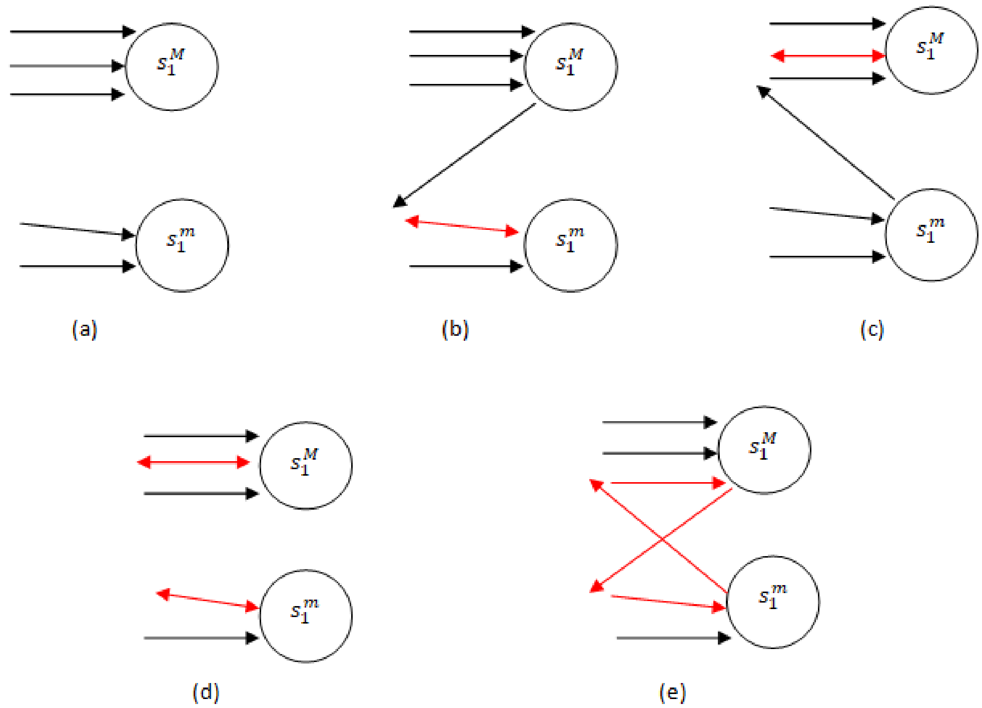

Figure 3: Matching Cycles in the Procedure of the TTC Algorithm

to a school in $\tilde{C}_1^m$ (See Figure 3 (e)). For this case, it is easy to see that both s_1^M and s_1^m are assigned to their favorite school at Step 1. We get a contradiction. Then this case is impossible.

Therefore, we show that, at Step 1 of the TTC algorithm for $\tilde{G}$, s_1^M is assigned to her favorite school $\tilde{\mu}(s_1^M)$. Since $\tilde{\mu}(s_1^M)$ points to s_1^M at this step, one can infer that s_1^M has the highest priority at school $\tilde{\mu}(s_1^M)$ under the priority profile $\tilde{\succ}$.

Since $\tilde{G}$ has a stronger priority-based affirmative action policy than G, we can infer that s_1^M has the highest priority at school $\tilde{\mu}(s_1^M)$ under the priority profile $\succ$. Thus, in the procedure of the TTC algorithm for G, s_1^M and $\tilde{\mu}(s_1^M)$ point to each other and s_1^M is assigned to $\tilde{\mu}(s_1^M)$ at the first step. We have $\tilde{\mu}(s_1^M) = \mu(s_1^M)$.

Let $C_1^m \equiv \{c \in C : c \text{ points to } s_1^m \text{at Step 1 of the TTC algorithm for } G\}$ and

$C_1^M \equiv \{c \in C : c \text{ points to } s_1^M \text{at Step 1 of the TTC algorithm for } G\}$.

We have $C_1^m \cup C_1^M = C$, $\tilde{C}_1^M \subseteq C_1^M$ and $C_1^m \subseteq \tilde{C}_1^m$. Since the most preferred school of s_1^m is in $\tilde{C}_1^M$, we obtain that the school is in C_1^M. Thus, we infer that, s_1^m is not assigned to any school at Step 1 of the TTC algorithm for G. Therefore, s_1^M is the only student who successfully attends a school at Step 1 of the TTC algorithm for both G and $\tilde{G}$.

By a similar argument, we can show that s_i^M ($\forall 1 < i < k_1$) is the only student who successfully attends a school at Step i of the TTC algorithm for both G and $\tilde{G}$, and $\tilde{\mu}(s_i^M) = \mu(s_i^M)$ for all $1 < i < k_1$. Then we obtain that, s_i^M ($\forall 1 \leq i < k_1$) is assigned to $\tilde{\mu}(s_i^M)$ at Step i during the procedure of TTC algorithm for $\tilde{G}$, while s_i^M ($\forall 1 \leq i < k_1$) is assigned to $\mu(s_i^M) = \tilde{\mu}(s_i^M)$ at Step i during the procedure of TTC algorithm for G.

Let $\bar{C}_1 \equiv \{c \in C :$ school c has at least one available seat at the end of Step $k_1 - 1$ during the TTC algorithm for $\tilde{G}\}$. Since s_1^m is assigned to school $\tilde{\mu}(s_1^m)$ at Step k_1 of the top trading cycles algorithm of $\tilde{\mu}$, it implies that s_1^m points to $\tilde{\mu}(s_1^m)$ at this step. Consequently, we have $\tilde{\mu}(s_1^m) R_{s_1^m} c$ for all $c \in \bar{C}_1$. Since s_i^M ($\forall 1 \leq i < k_1$) is assigned to $\mu(s_i^M) = \tilde{\mu}(s_i^M)$ at Step i during the procedure of TTC algorithm for G, which happens before s_1^m is assigned to $\mu(s_1^m)$, it is easy to see that $\mu(s_1^m) \in \bar{C}_1$. Then we get $\tilde{\mu}(s_1^m) R_{s_1^m} \mu(s_1^m)$. Moreover, by an argument in the contrary direction, one can show that $\mu(s_1^M) R_{s_1^M} \tilde{\mu}(s_1^M)$.

If $\tilde{\mu}(s_1^m) P_{s_1^m} \mu(s_1^m)$, we are done. If $\tilde{\mu}(s_1^m) = \mu(s_1^m)$, we consider s_2^m and proceed to the next step.

Step II. We show that $\tilde{\mu}(s_2^m) R_{s_2^m} \mu(s_2^m)$.

Suppose that s_2^m is assigned to school $\tilde{\mu}(s_2^m)$ at Step k_2 ($k_2 \geq 2$) of the top trading cycles algorithm for $\tilde{G}$, and that the set of students who are matched from Step 1 to Step $(k_2 - 1)$ is $S_1 = \{s_1^m\} \cup \{s_1^M, s_2^M, \cdots, s_{t_1}^M\}$ for some $0 \leq t_1 \leq k_2 - 1$. We discuss two cases:

Case 1: $t_1 = 0$, i.e., $S_1 = \{s_1^m\}$, then s_2^m is matched to her most preferred seat in the set of all seats with one seat of school $\tilde{\mu}(s_1^m)$ being removed. By $\tilde{\mu}(s_1^m) = \mu(s_1^m)$, we can infer that the removed seat is not available to s_2^m in the top trading cycles algorithm for G. Then we have $\tilde{\mu}(s_2^m) R_{s_2^m} \mu(s_2^m)$.

Case 2: $t_1 > 0$, i.e., $s_1^M \in S_1$, then we consider the following two subcases.

Subcase (i): s_1^M is matched to $\tilde{\mu}(s_1^M)$ at the first step of the top trading cycles algorithm of $\tilde{\mu}$. If this is true, then we can infer that $\tilde{\mu}(s_1^M)$ is her most preferred school and $\tilde{\mu}(s_1^M) = \mu(s_1^M)$.

Subcase (ii): s_1^M is matched to $\tilde{\mu}(s_1^M)$ at the second step of the top trading cycles algorithm of $\tilde{\mu}$. If this is true, then we can infer that s_1^M is matched to her most preferred seat in the set of all seats with one seat of school $\tilde{\mu}(s_1^m)$ removed. By $\tilde{\mu}(s_1^m) = \mu(s_1^m)$, we obtain that the removed seat is not available to s_1^M during the top trading cycles algorithm of μ. Then we have $\tilde{\mu}(s_1^M) R_{s_1^M} \mu(s_1^M)$. Together with $\mu(s_1^M) R_{s_1^M} \tilde{\mu}(s_1^M)$, we obtain $\mu(s_1^M) = \tilde{\mu}(s_1^M)$.

By a repeated similar argument as above for s_1^M [in Case 2], we can infer that $\mu(s_i^M) = \tilde{\mu}(s_i^M)$ for all $1 \leq i \leq t_1$. At Step k_2 of the top trading cycles algorithm of $\tilde{\mu}$, student s_2^m is matched to $\tilde{\mu}(s_2^m)$. Then $\tilde{\mu}(s_2^m)$ is her most preferred school among all remaining schools at Step k_2 of the top trading cycles algorithm of $\tilde{\mu}$. Since $\mu(s_1^m) = \tilde{\mu}(s_1^m)$ and $\mu(s_i^M) = \tilde{\mu}(s_i^M)$ for all $1 \leq i \leq t_1$, and in the procedure of the TTC algorithm for G, s_i^M ($\forall 1 \leq i \leq t_1$) is assigned to $\mu(s_i^M)$ before s_2^m is assigned to $\mu(s_2^m)$, it is easy to see that student s_2^m has a (weakly) smaller set of remaining schools when she is matched at some step of the top trading cycles algorithm of μ than that for $\tilde{\mu}$. Then we have $\tilde{\mu}(s_2^m) R_{s_2^m} \mu(s_2^m)$.

If $\tilde{\mu}(s_2^m) P_{s_2^m} \mu(s_2^m)$, we are done. If $\tilde{\mu}(s_2^m) = \mu(s_2^m)$, we consider s_3^m and proceed to the next step.

Step III. We show that $\tilde{\mu}(s_3^m) R_{s_3^m} \mu(s_3^m)$.

The argument is exactly similar to that for s_2^m. We can obtain $\tilde{\mu}(s_3^m) R_{s_3^m} \mu(s_3^m)$. If $\tilde{\mu}(s_3^m) P_{s_3^m} \mu(s_3^m)$, the proof is done. If $\tilde{\mu}(s_3^m) = \mu(s_3^m)$, we consider the remaining minority students sequentially, such as s_4^m, s_5^m, and so on.

Since the number of students is finite, we will finally reach the conclusion that either $\tilde{\mu}(s_i^m) P_{s_i^m} \mu(s_i^m)$ for some $1 \leq i \leq n$ or $\tilde{\mu}(s_i^m) = \mu(s_i^m)$ for all $1 \leq i \leq n$. Then the TTC$(\tilde{G})$ is not Pareto inferior to TTC(G) for the minority. Thus, the proof for the priority-based affirmative action part is done.

(ii) Let $G = (P_S, \succ_C, q^M, r^m)$ and $\tilde{G} = (P_S, \succ_C, q^M, \tilde{r}^m)$ be two problems with $q_c^M = q_c$ and $\tilde{r}_c^m \geq r_c^m$ for all $c \in C$. Then $\tilde{G}$ has a stronger reserve-based affirmative action policy than G. We want to show that $\tilde{\mu} =$TTC$(\tilde{G})$ is not Pareto inferior to $\mu =$TTC(G) for minority students.

Let $S^M = \{s_1^M, s_2^M, \cdots, s_l^M\}$ such that $s_i^M \succ_c s_j^M$ for all $c \in C$ and $1 \leq i < j \leq l$, and $S^m = \{s_1^m, s_2^m, \cdots, s_n^m\}$ such that $s_i^m \succ_c s_j^m$ for all $c \in C$ and $1 \leq i < j \leq n$.

One can complete the proof by repeating the arguments in part (i) step by step. □

A.2 Four Lemmas.

In order to prove Theorem 2, we first show the following lemmas.

Lemma 1. *If G' has a stronger priority-based affirmative action policy than G and $\mu' = DA(G')$ is Pareto inferior to $\mu = DA(G)$ for the minority, then μ' is Pareto inferior to μ for all students.*

Proof of Lemma 1. The proof is inspired by Doğan (2016). We argue by contradiction. Suppose not, then there exists some $M \in S^M$ such that $\mu'(M) P_M \mu(M)$. Let $c \equiv \mu(M)$ and $c_1 \equiv \mu'(M)$.

Step 1. Since $c_1 P_M c$ and $\mu(M) = c$, there is a step of the DA algorithm for problem G, say k_1, at which M is rejected by c_1. Note that at this step the capacity of c_1 is exhausted. Since $M \in \mu'(c_1)$, there is a student $s \in S \setminus \mu'(c_1)$ who is temporarily accepted by c_1 at Step

k_1 of the DA algorithm for problem G.

We claim that all of such students are majority students. To see this, suppose that $s \in S^m$. Note that since μ' is Pareto inferior to μ for the minority, $\mu(s) R_s \mu'(s)$. Also, since s is temporarily accepted by c_1 at a step of the DA algorithm for problem G, we have $c_1 R_s \mu(s)$. Thus, we infer $c_1 R_s \mu'(s)$, which, together with $s \notin \mu'(c_1)$, implies $c_1 P_s \mu'(s)$. According to $c_1 P_s \mu'(s)$, $M \in \mu'(c_1)$ and the stability of μ', one can obtain that $M \succ'_{c_1} s$. Since $\succ'_{c_1}$ is an improvement for minority students over $\succ_{c_1}$, we get $M \succ_{c_1} s$, which contradicts that c_1 rejects M and temporarily accepts s at Step k_1 of the DA algorithm for problem G.

We choose one such student, say, M_1. Then $M_1 \succ_{c_1} M$ and $M_1 \notin \mu'(c_1)$. By $M_1 \succ_{c_1} M$ we have $M_1 \succ'_{c_1} M$, which, together with $M_1 \notin \mu'(c_1)$ and $M \in \mu'(c_1)$, implies $\mu'(M_1) P_{M_1} c_1$. It is easy to see that $c_1 R_{M_1} \mu(M_1)$. Thus, we have $\mu'(M_1) P_{M_1} \mu(M_1)$.

Step 2. Let $c_2 \equiv \mu'(M_1)$. By the same arguments as in Step 1, there is $M_2 \in S^M \setminus \mu'(c_2)$ such that there is a step of the DA algorithm for problem G, say k_2, at which M_1 is rejected by c_2 and M_2 is temporarily accepted by c_2 at the same step. Now, since $c_2 P_{M_1} c_1$, $k_2 < k_1$. Continuing in this way, eventually we have $c_n \in C$ and $M_{n-1}, M_n \in S^M$ such that at Step 1 of the DA algorithm for problem G, M_{n-1} is rejected by c_n, M_n is temporarily accepted by c_n, and $M_n \notin \mu'(c_n), M_{n-1} \in \mu'(c_n)$. Note that c_n is the most preferred school of M_n, and $M_n \succ_{c_n} M_{n-1}$ implies $M_n \succ'_{c_n} M_{n-1}$. Yet $M_n \notin \mu'(c_n), M_{n-1} \in \mu'(c_n)$, contradicting the stability of μ'. The proof is completed. □

Lemma 2. *Suppose that G' has stronger priority-based affirmative action policy than G and $\mu' = DA(G')$ is Pareto inferior to $\mu = DA(G)$ for the minority. For each $c \in C$, if $\mu(c) \neq \mu'(c)$, then $|\mu(c)| = |\mu'(c)| = q_c$.*

Proof of Lemma 2. By Lemma 1, we have $\mu(s) R_s \mu'(s)$ for all $s \in S$. According to $\mu(s) R_s \mu'(s)$ for all $s \in S$ and the stability of μ', if there is $s \in \mu(c) \setminus \mu'(c)$, then $|\mu'(c)| = q_c$. Thus for each $c \in C$, if $\mu(c) \setminus \mu'(c) \neq \emptyset$, then $|\mu'(c) \setminus \mu(c)| \geq |\mu(c) \setminus \mu'(c)|$. For $c \in C$, if $\mu(c) \setminus \mu'(c) = \emptyset$, it is obvious to infer $|\mu'(c) \setminus \mu(c)| \geq |\mu(c) \setminus \mu'(c)|$. Thus we have $|\mu'(c) \setminus \mu(c)| \geq |\mu(c) \setminus \mu'(c)|$ for all $c \in C$. Suppose that there exists $c \in C$ such that $|\mu'(c) \setminus \mu(c)| > |\mu(c) \setminus \mu'(c)|$. Then there exists at least one student $s \in S$ such that $\mu(s) = s$ and $\mu'(s) \in C$. Since we have obtained that $\mu(s') R_{s'} \mu'(s')$ for all $s' \in S$, one can infer that $s P_s \mu'(s)$, which contradicts the individual rationality of μ'. Therefore, we have $|\mu'(c) \setminus \mu(c)| = |\mu(c) \setminus \mu'(c)|$ for all $c \in C$. If $\mu(c) \neq \mu'(c)$, then one can infer that $|\mu'(c) \setminus \mu(c)| = |\mu(c) \setminus \mu'(c)| \geq 1$, and consequently $\mu(c) \setminus \mu'(c) \neq \emptyset$. Then we have $|\mu'(c)| = q_c$. By $|\mu'(c) \setminus \mu(c)| = |\mu(c) \setminus \mu'(c)|$ we obtain $|\mu(c)| = q_c$. □

Lemma 3. *If G' has stronger priority-based affirmative action policy than G and $\mu' = DA(G')$ is*

Pareto inferior to $\mu = DA(G)$ for the minority, then $\sum_{c\in C}|\mu(c)| = \sum_{c\in C}|\mu'(c)|$.

Proof of Lemma 3. For each $c \in C$, either $\mu(c) = \mu'(c)$ or $\mu(c) \neq \mu'(c)$. If $\mu(c) = \mu'(c)$, then $|\mu(c)| = |\mu'(c)|$. If $\mu(c) \neq \mu'(c)$, by Lemma 2 we get $|\mu'(c)| = |\mu'(c)| = q_c$. The proof is completed. □

Lemma 4. *Suppose that G' has stronger priority-based affirmative action policy than G. If $\mu' = DA(G')$ is Pareto inferior to $\mu = DA(G)$ for the minority, then $\cup_{c\in C}\mu'(c) = \cup_{c\in C}\mu(c)$.*

Proof of Lemma 4. For each $s \in S$, if $\mu'(s) \in C$, then by Lemma 1 we get $\mu(s)R_s\mu'(s)$. Since μ' is individually rational, we have $\mu'(s)P_s s$. Thus, we infer that $\mu(s)P_s s$ and $\mu(s) \in C$. On the other hand, for each $s \in S$, if $\mu(s) \in C$, we claim that $\mu'(s) \in C$. Specifically, if $\mu(s) \in C$ but $\mu'(s) \notin C$, by Lemma 3 one can infer that there is at least one student, say $s' \in S$, such that $\mu'(s') \in C$, but $\mu(s') = \{s'\}$. Since μ' is individually rational, we have $\mu'(s')P_s\mu(s')$, which contradicts the conclusion of Lemma 1. The proof is completed. □

A.3 Proof of Theorem 2.

Only if part. Suppose that G has a cycle $(m, M, s_1, \cdots, s_{k-1}, c_1, \cdots, c_k, N_1, \cdots, N_k)$. Let the students' preferences be given as follows:

(1) For each $s \in N_i$ $(i = 1, \cdots, k)$, $c_i P_s s$ and $s P_s c$ for all $c \in C \setminus \{c_i\}$;

(2) For each $s \in S \setminus [\{m, M, s_1, \cdots, s_{k-1}\} \cup N_1 \cup \cdots \cup N_k]$, $s P_s c$ for all $c \in C$;

(3) For each $c \in C \setminus \{c_1, c_k\}$, $s_{k-1} P_{s_{k-1}} c$ and $c_k P_{s_{k-1}} c_1 P_{s_{k-1}} s_{k-1}$;

(4) For each $c \in C \setminus \{c_1\}$, $m P_m c$ and $c_1 P_m m$;

(5) For each $c \in C \setminus \{c_1, c_2\}$, $M P_M c$ and $c_1 P_M c_2 P_M M$;

(6) For each $i \in \{1, 2, \cdots, k-2\}$, $c_{i+1} P_{s_i} c_{i+2} P_{s_i} s_i$, and for each $c \in C \setminus \{c_{i+1}, c_{i+2}\}$, $s_i P_{s_i} c$.

$P_{s_{k-1}}$	P_m	P_M	P_{s_1}	$\cdots$	P_{s_i}	$\cdots$	$P_{s_{k-2}}$
c_k	c_1	c_1	c_2	$\cdots$	c_{i+1}	$\cdots$	c_{k-1}
c_1	m	c_2	c_3	$\cdots$	c_{i+2}	$\cdots$	c_k
s_{k-1}	$\vdots$	M	s_1	$\cdots$	s_i	$\cdots$	s_{k-2}
$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$

For $\succ_C$ satisfying $s_{k-1} \succ_{c_1} m$, $M \succ_{c_1} m$ and $M \succ_{c_2} s_1$, we assume that $s_{k-1} \succ_{c_1} M$ and define $\succ'_C$ as an improvement for minority students over $\succ_C$ such that

(1) $s \succ'_{c_1} m \succ'_{c_1} M$ for all $s \in \{s_{k-1}\} \cup N_1$;

(2) $s \succ'_{c_2} s_1$ for all $s \in \{M\} \cup N_2$;

(3) For each $i \in \{3, \cdots, k-1\}$, $s \succ'_{c_i} s_{i-1}$ for all $s \in \{s_{i-2}\} \cup N_i$;

(4) $s \succ'_{c_k} s_{k-1}$ for all $s \in \{s_{k-2}\} \cup N_k$.

Let $G = (P_S, \succ_C, q^M, r^m)$, $G' = (P_S, \succ'_C, q^M, r^m)$ and G' have a stronger priority-based affirmative action policy than G. Denote $\mu = \text{DA}(G)$ and $\mu' = \text{DA}(G')$. Then it is easy to see that $\mu(c_1) = \{M\} \cup N_1$, $\mu(c_i) = \{s_{i-1}\} \cup N_i$ for all $i \in \{2, \cdots, k\}$, and $\mu(c) = \emptyset$ for all $c \in C \setminus \{c_1, \cdots, c_k\}$. For μ', we have $\mu'(c_1) = \{s_{k-1}\} \cup N_1$, $\mu'(c_2) = \{M\} \cup N_2$, $\mu'(c_i) = \{s_{i-2}\} \cup N_i$ for all $i \in \{3, \cdots, k\}$, and $\mu'(c) = \emptyset$ for all $c \in C \setminus \{c_1, \cdots, c_k\}$. Obviously, $\mu(s) P_s \mu'(s)$ for all $s \in \{s_1, s_2, \cdots, s_{k-1}\}$ and $\mu(s) R_s \mu'(s)$ for all $s \in S$. Thus, although G' have a stronger priority-based affirmative action policy than G, $\text{DA}(G')$ is Pareto inferior to $\text{DA}(G)$ for the minority, as $\{s_1, \cdots, s_{k-1}\} \cap S^m \neq \emptyset$. Then the DA mechanism is not minimally responsive to the priority-based affirmative action. This part is completed.

If part. Suppose that schools' priority structure is acyclic. We want to show that the DA mechanism is minimally responsive to the priority-based affirmative action. We argue by contradiction. Suppose that the DA mechanism is not minimally responsive to the priority-based affirmative action. Then there are two problems $G = (P_S, \succ_C, q^M, r^m)$ and $G' = (P_S, \succ'_C, q^M, r^m)$ such that G' has a stronger priority-based affirmative action policy than G and $\mu' = \text{DA}(G')$ is Pareto inferior to $\mu = \text{DA}(G)$ for the minority. Then by Lemma 1, we have $\mu(s) R_s \mu'(s)$ for all $s \in S$.

Step 1. Since μ' is Pareto inferior to μ for the minority, there exists some minority student, say s^m, such that $\mu(s^m) P_{s^m} \mu'(s^m)$. By Lemma 4, we have $\mu(s^m) \in C$ and $\mu'(s^m) \in C$. Let $c_0 \equiv \mu(s^m)$ and $c'_1 \equiv \mu'(s^m)$. Since $s^m \in \mu(c_0)$ and $s^m \in \mu'(c'_1)$, one can infer that $\mu'(c_0) \neq \mu(c_0)$ and $\mu'(c'_1) \neq \mu(c'_1)$. By Lemma 2 we have $|\mu'(c_0)| = |\mu(c_0)| = q_{c_0}$ and $|\mu'(c'_1)| = |\mu(c'_1)| = q_{c'_1}$. Suppose that, at Step $\tilde{k}_1$ of the DA algorithm for G', student s^m proposes to school c'_1. There are two possible cases:

Case I. There exists some step of the DA algorithm for G', say $k'_1 \geq \tilde{k}_1$, such that at this step school c'_1 rejects some student, say s'_1, satisfying $\mu(s'_1) R_{s'_1} c'_1$.

If this case happens, then suppose that c'_1 rejects s'_1 at Step k'_1 of the DA algorithm for G' and $\mu(s'_1) R_{s'_1} c'_1$. Let $N'_1 \equiv \mu'(c'_1) \setminus \{s^m\}$. Then it is easy to see that $(s^m, N'_1) \in T(s'_1, c'_1)$ and $|N'_1| = q_{c'_1} - 1$.

Since $\mu(s'_1) R_{s'_1} c'_1$ and s'_1 is rejected by c'_1 during the DA algorithm for G', it implies $c'_1 P_{s'_1} \mu'(s'_1)$ and $\mu(s'_1) P_{s'_1} \mu'(s'_1)$. Then by Lemma 4 we can see that $\mu'(s'_1) \in C$. Let $c'_2 \equiv \mu'(s'_1)$. Suppose that, at Step $\tilde{k}_2 > k'_1$ of the DA algorithm for G', student s'_1 proposes to school c'_2.

Case II. From Step $\tilde{k}_1$ to the last step of the DA algorithm for G', school c'_1 does not reject any student $s \in S$ satisfying $\mu(s) R_s c'_1$.

If this case happens, then by $|\mu'(c'_1)| = |\mu(c'_1)| = q_{c'_1}$ one can infer that, during the DA algorithm for G', c'_1 rejects at least one student, say s', in $\mu(c'_1)$ before Step $\tilde{k}_1$. Thus, it implies that c'_1 has already temporarily accepted $q_{c'_1}$ students at the end of Step $\tilde{k}_1 - 1$. We denote by $\tilde{S} \equiv \{s \in S\colon c'_1$ temporarily accepts s at the end of Step $\tilde{k}_1 - 1\}$. By $s^m \in \mu'(c'_1)$, we infer that, at Step $\tilde{k}_1$ of the DA algorithm for G', c'_1 temporarily accepts s^m and rejects at least one student $s'_1 \in \tilde{S}$ satisfying $c'_1 P_{s'_1} \mu(s'_1)$. Let $N'_1 \equiv \mu'(c'_1) \setminus \{s^m\}$. Then it is easy to see that $(s^m, N'_1) \in T(s'_1, c'_1)$ and $|N'_1| = q_{c'_1} - 1$.

Since $c'_1 P_{s'_1} \mu(s'_1)$ and μ is stable, we infer that $s \succ_{c'_1} s'_1$ for all $s \in \mu(c'_1)$ and $s' \succ_{c'_1} s'_1$. However, by the above arguments we can get $s'_1 \succ'_{c'_1} s'$. Then we obtain that $\succ'_{c'_1}$ improves the priority of s'_1 over $\succ_{c'_1}$. Furthermore, we have $s'_1 \in S^m$ and $s' \in S^M$.

Since the DA algorithm for G' stops in finite time, we can infer that there exists a finite list $(c_0, s^m, c'_1, s'_1, c'_2, s'_2, \cdots, c'_t, s'_t, s'_{t+1}; N'_1, \cdots, N'_t)$ and $\tilde{k}_1 \leq k'_1 < \tilde{k}_2 \leq k'_2 < \cdots < \tilde{k}_t$, such that

(1) $\mu(s^m) = c_0$, $\mu'(s^m) = c'_1$, $c_0 P_{s^m} c'_1$; $\mu'(s'_i) = c'_{i+1}$ and $c'_i P_{s'_i} c'_{i+1}$ for $i = 1, \cdots, t-1$.

(2) $\tilde{k}_1 \leq k'_1 < \tilde{k}_2 \leq k'_2 < \cdots < \tilde{k}_t$, and for $1 < i < t$, student s'_{i-1} proposes to school c'_i at Step $\tilde{k}_i$ of the DA algorithm for G'. For $1 < i < t$, school c'_i rejects student s'_i at Step k'_i of the DA algorithm for G' and $\mu(s'_i) P_{s'_i} c'_i$.

(3) $N'_1 \equiv \mu'(c'_1) \setminus \{s^m\}$ and $N'_i \equiv \mu'(c'_i) \setminus \{s'_{i-1}\}$ for $i \in \{2, \cdots, t\}$; $(s^m, N'_1) \in T(s'_1, c'_1)$ and $(s'_i, N'_{i+1}) \in T(s'_{i+1}, c'_{i+1})$ for $i \in \{1, \cdots, t-1\}$, $|N'_i| = q_{c'_i} - 1$ for $i \in \{1, \cdots, t\}$.

(4) Student s'_{t-1} proposes to school c'_t at Step $\tilde{k}_t$ of the DA algorithm for G'. From Step $\tilde{k}_t$ to the last step of the DA algorithm for G', school c'_t does not reject any student $s \in S$ satisfying $\mu(s) R_s c'_t$. During the DA algorithm for G', c'_t rejects $s'_t \in \mu(c'_t)$ before Step $\tilde{k}_t$. At Step $\tilde{k}_t$ of the DA algorithm for G', c'_t temporarily accepts s'_{t-1} and rejects student $s'_{t+1} \in S$ satisfying $c'_t P_{s'_{t+1}} \mu(s'_{t+1})$. For $N'_t = \mu'(c'_t) \setminus \{s'_{t-1}\}$, it is easy to see that $(s'_{t-1}, N'_t) \in T(s'_{t+1}, c'_t)$. Hence we obtain $s'_{t+1} \succ'_{c'_t} s'_t$. By the stability of μ, $c'_t P_{s'_{t+1}} \mu(s'_{t+1})$ and $s'_t \in \mu(c'_t)$, we infer that $s'_t \succ_{c'_t} s'_{t+1}$. Thus, one can see that $\succ'_{c'_t}$ improves the priority of s'_{t+1} over $\succ_{c'_t}$. Furthermore, we have $s'_{t+1} \in S^m$ and $s'_t \in S^M$.

Under the DA algorithm for G, s'_{t+1} ever proposes to c'_t, but c'_t rejects s'_{t+1}. $\succ'_{c_t}$ improves the priority of s'_{t+1} over $\succ_{c_t}$ such that, under the DA algorithm for G', c'_t ever temporarily accepts s'_{t+1} and rejects $s'_t \in \mu(c'_t)$. At Step $\tilde{k}_t$ of the DA algorithm for G', c'_t accepts s'_{t-1} and rejects s'_{t+1}. We call student s'_{t+1} *an interrupter at school* c'_t[11].

[11] This concept has the same spirit as that defined by Kesten (2010) and Doğan (2016).

Step 2. Since $s^m \in \mu(c_0) \setminus \mu'(c_0)$ and $c_0 P_{s^m} \mu'(s^m)$, there exists a student, say s_1'', such that $s_1'' \notin \mu(c_0)$ and c_0 temporarily accepts s_1'' and rejects s^m at some step, say k_0 ($k_0 < \tilde{k}_1$), of the DA algorithm for G'. It is easy to see that $s_1'' \succ'_{c_0} s^m$. If $s_1'' \in \mu'(c_0)$, let $\tilde{s}_1'' \equiv s_1''$. If $s_1'' \notin \mu'(c_0)$, choose any student, say $\tilde{s}_1''$, in $\mu'(c_0) \setminus \mu(c_0)$. One can infer that $\tilde{s}_1'' \succ'_{c_0} s^m$. Since s^m is a minority student, we have $s_1'' \succ_{c_0} s^m$ and $\tilde{s}_1'' \succ_{c_0} s^m$. Then by $s_1'' \notin \mu(c_0)$, $\tilde{s}_1'' \notin \mu(c_0)$, $s^m \in \mu(c_0)$ and the stability of μ, we can infer that $\mu(s_1'') P_{s_1''} c_0$ and $\mu(\tilde{s}_1'') P_{\tilde{s}_1''} c_0$. Let $c_1'' \equiv \mu(s_1'')$ and $N_1'' \equiv \mu'(c_0) \setminus \{\tilde{s}_1''\}$. It is easy to see that $(s_1'', N_1'') \in T(s^m, c_0)$.

Since $c_1'' P_{s_1''} c_0$ and $s_1'' \notin \mu'(c_1'')$, there exists some step, say Step k_1'' ($k_1'' < k_0$), such that s_1'' is rejected by c_1'' at Step k_1'' of the DA algorithm for G'. By a similar argument as above, we infer that there exists some student, say $s_2'' \in S$, such that $s_2'' \notin \mu(c_1'')$ and c_1'' temporarily accepts s_2'' at Step k_1'' of the DA algorithm for G'. If $s_2'' \in \mu'(c_1'')$, let $\tilde{s}_2'' \equiv s_2''$. If $s_2'' \notin \mu'(c_1'')$, choose any student, say $\tilde{s}_2''$, in $\mu'(c_1'') \setminus \mu(c_1'')$. One can infer that $\tilde{s}_2'' \succ'_{c_1''} s_1''$ and $s_2'' \succ'_{c_1''} s_1''$. Let $c_2'' \equiv \mu(s_2'')$ and $N_2'' \equiv \mu'(c_1'') \setminus \{\tilde{s}_2''\}$. It is easy to see that $(s_2'', N_2'') \in T(s_1'', c_1'')$.

There are two possible cases:

Case I: $c_2'' P_{s_2''} c_1''$. Since $c_2'' P_{s_2''} c_1''$ and c_1'' temporarily accepts s_2'' at Step k_1'' of the DA algorithm for G', there exists some step, say Step k_2'' ($k_2'' < k_1''$), such that s_2'' is rejected by c_2'' at Step k_2'' of the DA algorithm for G', and there exists some student, say $s_3'' \in S$, such that $s_3'' \notin \mu(c_2'')$ and c_2'' temporarily accepts s_3'' at Step k_2'' of the DA algorithm for G'. One can infer that $s_3'' \succ'_{c_2''} s_2''$. If $s_3'' \in \mu'(c_2'')$, let $\tilde{s}_3'' \equiv s_3''$. If $s_3'' \notin \mu'(c_2'')$, choose any student, say $\tilde{s}_3''$, in $\mu'(c_2'') \setminus \mu(c_2'')$. One can infer that $\tilde{s}_3'' \succ'_{c_2''} s_2''$. Let $c_3'' \equiv \mu(s_3'')$ and $N_3'' \equiv \mu'(c_2'') \setminus \{\tilde{s}_3''\}$. It is easy to see that $(s_3'', N_3'') \in T(s_2'', c_2'')$.

Case II: $c_1'' P_{s_2''} c_2''$. By $\mu(s_2'') R_{s_2''} \mu'(s_2'')$, we obtain $c_1'' P_{s_2''} \mu'(s_2'')$. Since $s_1'' \in \mu(c_1'')$, $s_2'' \notin \mu(c_1'')$ and $c_1'' P_{s_2''} \mu(s_2'')$, we obtain $s_1'' \succ_{c_1''} s_2''$ by the stability of μ. Then we can see that $\succ'_{c_1''}$ improves the priority of s_2'' over $\succ_{c_1''}$. Thus, $s_2'' \in S^m$ and $s_1'' \in S^M$. Furthermore, we can see that s_2'' is an interrupter at school c_1''. That is, during the DA algorithm for G', c_1'' temporarily accepts s_2'' and rejects $s_1'' \in \mu(c_1'')$ at Step k_1'' of the DA algorithm for G', and finally s_2'' is rejected by c_1''.

Since there are finite steps from Step k_0 to the starting of the DA algorithm for G', we can infer that there exists a finite list $(c_0, s_1'', c_1'', s_2'', c_2'', \cdots, c_{r-1}'', s_r''; \tilde{s}_1'', \cdots, \tilde{s}_r''; N_1'', \cdots, N_r'')$ and $\tilde{k}_1 > k_1'' > k_2'' > \cdots > k_r''$ such that

(1) $c_i'' = \mu(s_i'')$ for $1 \le i < r$. $s_{i+1}'' \succ'_{c_i''} s_i''$ for $1 \le i \le r-1$. $c_i'' P_{s_i''} c_{i-1}''$ for $1 < i < r$. During the DA algorithm for G', c_i'' temporarily accepts s_{i+1}'' and rejects s_i'' at Step k_i'' of the DA algorithm for G' (for $1 \le i \le r-1$).

(2) $N_i'' \equiv \mu'(s_{i-1}'') \setminus \{\tilde{s}_i''\}$, $(s_i'', N_i'') \in T(s_{i-1}'', c_{i-1}'')$ for $1 < i \le r$.

(3) $c''_{r-1} P_{s''_r} \mu(s''_r)$, then by $\mu(s''_r) R_{s''_r} \mu'(s''_r)$, we obtain $c''_{r-1} P_{s''_r} \mu'(s''_r)$. Since $s''_{r-1} \in \mu(c''_{r-1})$, $s''_r \notin \mu(c''_{r-1})$ and $c''_{r-1} P_{s''_r} \mu(s''_r)$, we obtain $s''_{r-1} \succ_{c''_{r-1}} s''_r$ by the stability of μ. Then we can see that $\succ'_{c''_{r-1}}$ improves the priority of s''_r over $\succ_{c''_{r-1}}$. Thus, $s''_r \in S^m$ and $s''_{r-1} \in S^M$. Furthermore, we can see that s''_r is an interrupter at school c''_{r-1}. That is, during the DA algorithm for G', c''_{r-1} temporarily accepts s''_r and rejects s''_{r-1} at Step k''_{r-1} of the DA algorithm for G', and finally s''_r is rejected by c''_{r-1}.

Let $m = s''_r$ and $M = s''_{r-1}$. Then we get a rejection-chain

$$(m, N''_r, c''_{r-1}, M, N''_{r-1}, c''_{r-2}, s''_{r-2}, \cdots, s''_1, N''_1, c_0, s^m, N'_1, c'_1, s'_1, N'_2, c'_2, s'_2, \cdots, s'_{t-1}, N'_t, c'_t, s'_t, s'_{t+1}),$$

where $(m, N''_r) \in T(M, c''_{r-1})$, $(M, N''_{r-1}) \in T(s''_{r-2}, c''_{r-2})$, $(s''_{r-2}, N''_{r-2}) \in T(s''_{r-3}, c''_{r-3})$, $\cdots$, $(s''_1, N''_1) \in T(s^m, c_0)$, $(s^m, N'_1) \in T(s'_1, c'_1)$,$\cdots$,$(s'_{t-1}, N'_t) \in T(s'_{t+1}, c'_t)$.

By the above arguments, we know that $m \in S^m$, $M \in S^M$ and m is an interrupter at c''_{r-1}, while $s'_{t+1} \in S^m$, $s'_t \in S^M$ and s'_{t+1} is an interrupter at c'_t. We call the pair (m, c''_{r-1}) the starting point of the above rejection-chain, and the pair (s'_{t+1}, c'_t) the ending point of the rejection-chain. If $m = s'_{t+1}$, $M = s'_t$ and $c'_t = c''_{r-1}$, then we get a cycle and achieve a contradiction. The proof is done. If the ending point does not coincide with the starting point, since $\mu(s) R_s \mu'(s)$ for all $s \in S$, there exists another rejection-chain from (s'_{t+1}, c'_t) to (m, c''_{r-1}). Thus, we get a desirable cycle and achieve a contradiction. The proof is completed. □

A.4 Proof of Theorem 3. We complete the proof by considering the following example. Let $S^M = \{s_1, s_2, s_3\}$, $S^m = \{s_4, s_5, s_6, s_7\}$, $C = \{c_1, \cdots, c_6\}$, $q_{c_i} = q^M_{c_i} = 1$ for $i = 1, 2, 3$, $q_{c_i} = q^M_{c_i} = 7$ for $i = 4, 5, 6$, and $r^m_{c_i} = 0$ for all $i \in \{1, \cdots, 6\}$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$	$\succ_{c_5}$	$\succ_{c_6}$
c_1	c_2	c_3	c_1	c_5	c_4	c_6	s_5	s_4	s_6	s_7	s_2	s_3
s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_6	s_7	s_7	s_1	s_5	s_7
							s_7	s_6	s_4	$\vdots$	$\vdots$	$\vdots$
							s_4	s_5	s_5			
							$\vdots$	s_2	s_3			
								$\vdots$	$\vdots$			

For $(P_S, \succ_C, q^M, r^m)$, the outcome of the TTC is

$$\text{TTC}(P_S, \succ_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & s_1 \\ s_4 & s_2 & s_3 & s_6 & s_5 & s_7 & s_1 \end{pmatrix}.$$

It is easy to see that $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority. For the quota-based affirmative action, we choose $\tilde{q}^M_{c_2} = \tilde{q}^M_{c_3} = 0$ and $\tilde{q}^M_{c_i} = q^M_{c_i}$ for $i = 1, 4, 5, 6$. Then $(P_S, \succ_C, \tilde{q}^M, r^m)$ has a stronger quota-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\text{TTC}(P_S, \succ_C, \tilde{q}^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & s_2 & s_3 & s_4 \\ s_1 & \varnothing & \varnothing & s_6 & s_5 & s_7 & s_2 & s_3 & s_4 \end{pmatrix}.$$

Then comparing with $\text{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\text{TTC}(P_S, \succ_C, \tilde{q}^M, r^m)$, student s_4 becomes strictly worse off and other minority students remain unchanged. That is, increasing the level of affirmative action results in a Pareto inferior assignment for the minority. Thus, the TTC mechanism is not minimally responsive to the quota-based affirmative action even when schools give full priority to the minority.

For the reserve-based affirmative action, we choose $\tilde{r}^m_{c_5} = \tilde{r}^m_{c_6} = 1$ and $\tilde{r}^m_{c_i} = r^m_{c_i}$ for $i = 1, 2, 3, 4$. Then $(P_S, \succ_C, q^M, \tilde{r}^m)$ has a stronger reserve-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\text{TTC}(P_S, \succ_C, q^M, \tilde{r}^m) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & s_4 \\ s_1 & s_2 & s_3 & s_6 & s_5 & s_7 & s_4 \end{pmatrix}.$$

Then comparing with $\text{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\text{TTC}(P_S, \succ_C, q^M, \tilde{r}^m)$, student s_4 becomes strictly worse off and other minority students remain unchanged. That is, increasing the level of affirmative action results in a Pareto inferior assignment for the minority. Thus, the TTC mechanism is not minimally responsive to the reserve-based affirmative action even when schools give full priority to the minority.

For the priority-based affirmative action, we choose $\tilde{\succ}_{c_5} : s_5, s_2, \cdots$, as an improvement for the minority over $\succ_{c_5}$, and $\tilde{\succ}_{c_6} : s_7, s_3, \cdots$, as an improvement for the minority over $\succ_{c_6}$, and $\tilde{\succ}_{c_i} = \succ_{c_i}$ for $i = 1, 2, 3, 4$. Then $(P_S, \tilde{\succ}_C, q^M, r^m)$ has a stronger priority-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is similar to the case for reserve-based affirmative action, and one can check that

$$\text{TTC}(P_S, \tilde{\succ}_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & s_4 \\ s_1 & s_2 & s_3 & s_6 & s_5 & s_7 & s_4 \end{pmatrix}.$$

Then comparing with $\text{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\text{TTC}(P_S, \tilde{\succ}_C, q^M, r^m)$, student s_4 becomes strictly worse off and other minority students remain unchanged. That is, increasing the level of affirmative action results in a Pareto inferior assignment for the minority. Thus, the TTC mechanism is not minimally responsive to the priority-based affirmative action even when schools give full priority to the minority. □

A.5 Proof of Proposition 2. We argue by contradiction. Suppose that there exists some minority student $s_0 \in S^m$ such that $DA(G) \equiv \mu P_{s_0} \mu' \equiv DA(G')$. We denote $\mu(s_0) \equiv c_0$. Consider the DA algorithm for G', $\mu(s_0) P_{s_0} \mu'(s_0)$ implies that s_0 must have propose to c_0 and c_0 finally rejects her at some step, say Step k_n. Then it is easy to see that $|\mu'(c_0)| = q_{c_0}$ and each member in $\mu'(c_0)$ has higher priority than s_0 at c_0 (with respect to $\succ'_{c_0}$). We obtain that s_0 is not in the set of the q_{c_0} highest-priority students of c_0 (with respect to $\succ'_{c_0}$). Since problem G' gives full priority to the minority, by definition it must be the case that each minority student is ranked above each majority student under $\succ'_{c_0}$. Then there is no majority student who has higher priority than s_0 with respect to $\succ'_{c_0}$. Therefore, one can infer that c_0 tentatively accepts q_{c_0} students when c_0 rejects s_0 at Step k_n, and each of the q_{c_0} students is in S^m and has higher priority than s_0 at c_0 (with respect to $\succ'_{c_0}$). Since $|\mu(c_0)| \leq q_{c_0}$ and $s_0 \in \mu(c_0)$, it is easy to see that there exists at least one student, say s_1 (in S^m), among the q_{c_0} students tentatively accepted by c_0 at Step k_n such that $s_1 \notin \mu(c_0)$. We can obtain that $s_1 \succ'_{c_0} s_0$ is equivalent to $s_1 \succ_{c_0} s_0$, as both s_0 and s_1 are minority students. Combining $s_0 \in \mu(c_0)$, $s_1 \notin \mu(c_0)$ and $s_1 \succ_{c_0} s_0$, one can infer that s_1 has never proposed to c_0 in the process of the DA algorithm for G. Let $c_1 \equiv \mu(s_1)$. Then we get $c_1 P_{s_1} c_0$.

Since $c_1 P_{s_1} c_0$ and s_1 has proposed to c_0 at some step, say Step $k'_n (\leq k_n)$, in the DA process for G', one can infer that s_1 must have proposed to c_1 and c_1 rejected her at another step, say Step $k_{n-1} (< k'_n)$, in the DA process of μ'. Then it is exactly similar to the analysis above, and one can obtain that $|\mu'(c_1)| = q_{c_1}$ and each member in $\mu'(c_1)$ has higher priority than s_1 at c_1 (with respect to $\succ'_{c_1}$). Then s_1 is not in the set of the q_{c_1} highest-priority students of c_1 (with respect to $\succ'_{c_1}$). Since problem G' gives full priority to the minority, by definition it must be the case that each minority student is ranked above each majority student under $\succ'_{c_1}$. Then there is no majority student who has higher priority than s_1 under $\succ'_{c_1}$. Therefore, one can infer that c_1 tentatively accepts q_{c_1} students when c_1 rejects s_1 at Step k_{n-1}, and each of the q_{c_1} students is in S^m and has higher priority than s_1 at c_1 (with respect to $\succ'_{c_1}$). Since $|\mu(c_1)| \leq q_{c_1}$ and $s_1 \in \mu(c_1)$, it is easy to see that there exists at least one student, say s_2 (in S^m), among the q_{c_1} students tentatively accepted by c_1 at Step k_{n-1} such that $s_2 \notin \mu(c_1)$. We can infer that $s_2 \succ'_{c_1} s_1$ implies $s_2 \succ_{c_1} s_1$, as both s_1 and s_2 are minority students. Combining $s_1 \in \mu(c_1)$, $s_2 \notin \mu(c_1)$ and $s_2 \succ_{c_1} s_1$, one can infer that s_2 has never proposed to c_1 in the process of the DA algorithm under G. Let $c_2 \equiv \mu(s_2)$. Then $c_2 P_{s_2} c_1$.

Taking a repeated argument procedure as above, we can obtain a sequence of students and schools $s_1, c_1, \cdots, s_i, c_i \cdots$ and an infinite sequence of steps of DA algorithm process $k_n, k_{n-1}, \cdots, k_{n+1-i}, \cdots$ such that $k_m > k_{m-1}$ for all $m \leq n$. Then $k_{n+1-i} < 0$ when i is sufficiently large. Since k_{n+1-i} is some step of the DA algorithm process under G', $k_{n+1-i} \geq 1$. We reach a contradiction and complete the proof. □

A.6 Proof of Theorem 4. (1). By a classical result of Gale and Sotomayor (1985) (see also Theorem 5.35 in Roth and Sotomayor (1990)), one can obtain that $\mu'(s) R_s \mu(s)$ for each $s \in S^m$. We only need to show that $\mu(s) R_s \mu'(s)$ for each $s \in S^m$. Suppose not, then there exists some $s \in S^m$ such that $\mu'(s) P_s \mu(s)$. One can repeat a procedure as in the proof of Proposition 2 and complete the proof.

(2). We first show that $\mu(s) R_s \mu^q(s)$ for each $s \in S^m$. Suppose not. Then there exists some $s \in S^m$ such that $\mu^q(s) P_s \mu(s)$. One can take a similar procedure as in the proof of Theorem 5 and reach a contradiction. Symmetrically, we can show that $\mu^q(s) R_s \mu(s)$. Then $\mu(s) = \mu^q(s)$ for each $s \in S^m$. For the second part, we suppose there exists some $s \in S^M$ such that $\mu^q(s) P_s \mu(s)$. We denote $\mu^q(s) \equiv c_0$. According to the DA algorithm under G, $c_0 P_s \mu(s)$ implies that s must have propose to c_0 and c_0 finally rejects her at some step, say Step k_n. Then one can infer that c_0 tentatively accepts q_{c_0} students when c_0 rejects s at Step k_n, and each of the q_{c_0} students has higher priority than s_0 at c_0 (with respect to $\succ_{c_0}$). As $\mu(s) = \mu^q(s)$ for each $s \in S^m$, we obtain $\{s \in \mu^q(c_0) : s \in S^m\} = \{s \in \mu(c_0) : s \in S^m\}$. Since this market gives full priority to the minority, c_0 tentatively accepts no more than $|\{s \in \mu^q(c_0) : s \in S^m\}|$ minority students at Step k_n. Otherwise, it will result in $|\{s \in \mu^q(c_0) : s \in S^m\}| < |\{s \in \mu(c_0) : s \in S^m\}|$. Since c_0 rejects s at Step k_n, one can infer that there exists at least one student, say $s_1 \in S^M$, among the q_{c_0} students tentatively accepted by c_0 at Step k_n such that $s_1 \notin \mu^q(c_0)$. Then $s_1 \succ_{c_0} s$. Combining $s \in \mu^q(c_0)$, $s_1 \notin \mu^q(c_0)$ and $s_1 \succ_{c_0} s$, we can infer that s_1 has never proposed to c_0 in the process of the DA algorithm for μ^q. Then $\mu^q(s_1) \equiv c_1 P_{s_1} c_0$.

Since $c_1 P_{s_1} c_0$ and s_1 has proposes to c_0 at some step, say Step $k'_n (\leq k_n)$, in the DA process under G, one can infer that s_1 must have propose to c_1 and c_1 rejects her at another step, say Step $k_{n-1}(< k'_n)$, in the DA process of G. Then it is exactly similar to the analysis given above, and one can infer that c_1 tentatively accepts q_{c_1} students when c_1 rejects s_1 at Step k_{n-1}, and each of the q_{c_1} students has higher priority than s_1 at c_1. As $\mu(s) = \mu^q(s)$ for each $s \in S^m$, we obtain $\{s \in \mu^q(c_1) : s \in S^m\} = \{s \in \mu(c_1) : s \in S^m\}$. Since this market gives full priority to the minority, c_1 tentatively accepts no more than $|\{s \in \mu^q(c_1) : s \in S^m\}|$ minority students at Step k_{n-1}. Otherwise, it will result in $|\{s \in \mu^q(c_1) : s \in S^m\}| < |\{s \in \mu(c_1) : s \in S^m\}|$. Since c_1 rejects s_1 at Step k_{n-1}, one can infer that there exists at least one student, say $s_2 \in S^M$, among the q_{c_1} students tentatively accepted by c_1 at Step k_{n-1} such that $s_2 \notin \mu^q(c_1)$. Then $s_2 \succ_{c_1} s_1$. Combining $s_1 \in \mu^q(c_1)$, $s_2 \notin \mu^q(c_1)$ and $s_2 \succ_{c_1} s_1$, we can infer that s_2 has never proposed to c_1 in the process of the DA algorithm for μ^q. Then $\mu^q(s_2) \equiv c_2 P_{s_2} c_1$.

Taking a repeated argument process as above, we can obtain a sequence of students and schools $s_1, c_1, \cdots, s_i, c_i \cdots$ and a sequence of steps of DA algorithm procedure $k_n, k_{n-1}, \cdots, k_{n+1-i}, \cdots$ such that $k_m > k_{m-1}$ for all $m \leq n$. Then $k_{n+1-i} < 0$ when i is sufficiently large. Since k_{n+1-i} is some step of the DA algorithm process for μ^q, $k_{n+1-i} \geq 1$. We reach a contradiction and complete the proof.

(3). It is easy to see that the stronger policy problem $(P_S, \succsim_C, q^M, r^m)$ also gives full priority to the minority. We only need to show that, for each $s \in S$, $\mu(s) = \mu^p(s)$ and $\mu(s) = \mu^r(s)$, respectively. For the case of minority students, we can take a similar argument as in the proof of Proposition 2, and for the case of majority students, we can take a similar argument as in the proof of (2) of this theorem. □

A.7 Relationships among different restricted domains

(a) We consider an example such that each majority student has lower priority than each minority student at each school. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3\}$, $C = \{c_1, c_2\}$, $q_{c_1} = 2$ and $q_{c_2} = 1$. Schools' priorities are given by the following table.

$\succ_{c_1}$	$\succ_{c_2}$
s_2	s_3
s_3	s_2
s_1	s_1

It is easy to check that each majority student has lower priority than each minority student at each school. However, schools don't have the common priority within student type.

(b) The condition of common priority within type does not necessarily imply Doğan's acyclicity. We consider the following example. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$, $q_c = 1$, for all $c \in C$, $r^m = (0, 0, 0)$. Students' preferences and schools' priorities are given by the following table.

$\succ_{c_1}=\succ_{c_2}=\succ_{c_3}$	$\tilde{\succ}_{c_1}$	$\tilde{\succ}_{c_2}$	$\tilde{\succ}_{c_3}$
s_1	s_2	s_1	s_1
s_2	s_3	s_2	s_2
s_3	s_1	s_3	s_3

One can see that $\succ_C$ satisfies the common priority within type condition. However, the identical priority does not satisfy Doğan's acyclicity. Specifically, we can see that, with respect to $\tilde{r}^m = (1, 0, 0)$ or priority profile $\tilde{\succ}_C$, $(s_3, \emptyset)$ is a threat to s_1 at c_1; $(s_1, \emptyset)$ is a threat to s_2 at c_3; $(s_2, \emptyset)$ is a threat to s_3 at c_1; then we get a Doğan's cycle.

(c) It is easy to check that, if each majority student has lower priority than each minority student at each school, then full priority is given to the minority students. By definition, it turns out that the acyclicity condition is stronger than Doğan's acyclicity, and giving full priority to the minority implies the acyclicity. We further consider the following example. Let $C = \{c_1, c_2\}$, $S^M = \{s_1\}$, $S^m = \{s_2\}$, $q_{c_i} = 1$ for $i = 1, 2$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	$\succ_{c_1}=\succ_{c_2}$
c_1	c_2	s_1
c_2	c_1	s_2

It is easy to check that the priority structure satisfies the acyclicity condition, but schools do not give full priority to the minority students. Thus, giving full priority to the minority is stronger than the acyclicity condition.

In summary, the relationship among different restricted domains can be described by a Venn diagram in Figure 4.

A.8 An impossibility result for the Boston mechanism

On the full domain of school choice, Afacan and Salman (2016) show that the Boston mechanism[12] is not minimally responsive to the quota-based affirmative action, while it

[12]The Boston mechanism finds a matching for each problem $(P, \succ, q^M, r^m)$ through the following immediate acceptance algorithm: *Step* 1 Each student applies to her most preferred acceptable school. Each offer-receiving school c first considers minority applicants and permanently accepts them up to its minority reserve r_c^m one at a time according to its priority order. School c then considers all the applicants who are yet to be accepted, and one at a time according to its priority order, it permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than q_c^M, and rejects the rest. *Step* k, $k \geq 2$ Each rejected student in the previous step applies to her next preferred acceptable school. Each offer-receiving school c which still has available seats first considers minority applicants and permanently accepts

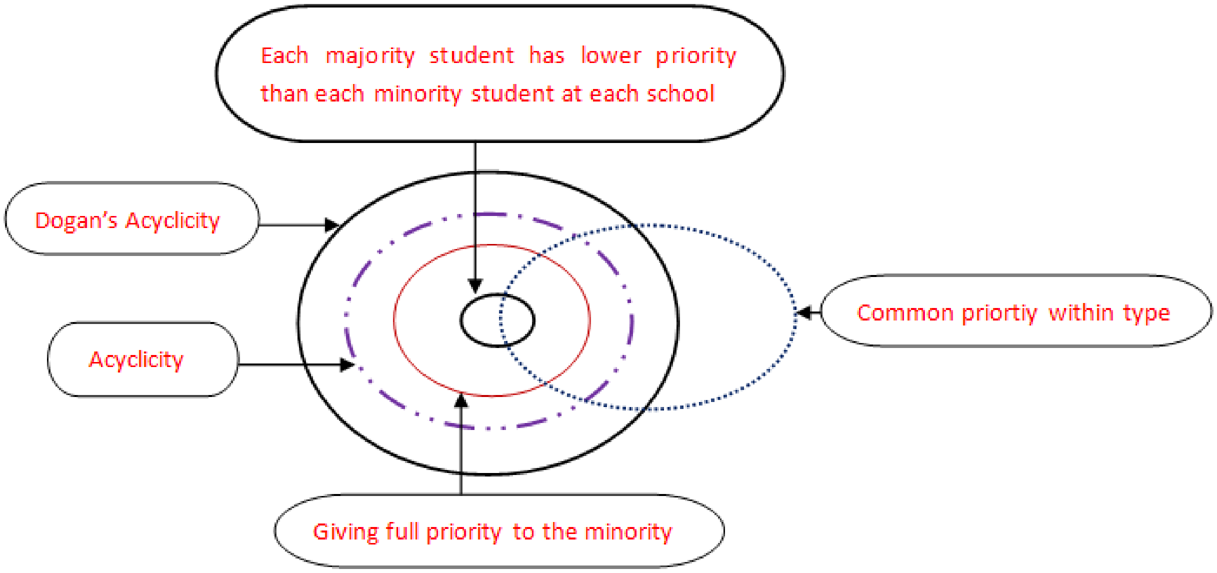

Figure 4: Relationship Between Different Restricted Domains

is minimally responsive to reserve-based and priority-based affirmative action policies. Furthermore, if each majority student has lower priority than each minority student at each school, we still have an impossibility result. Specifically, we consider an example as follows.

Example 5. Let $S^M = \{s_1\}, S^m = \{s_2, s_3, s_4\}, C = \{c_1, \cdots, c_4\}, q_{c_i} = q_{c_i}^M = 1$ and $r_{c_i}^m = 0$ for all $i \in \{1, \cdots, 4\}$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	$\succ_{c_1} = \succ_{c_2} = \succ_{c_3} = \succ_{c_4}$
c_1	c_2	c_2	c_2	s_2
c_4	c_1	c_3	c_3	s_3
c_2	c_3	c_1	c_4	s_4
c_3	c_4	c_4	c_1	s_1

them up to its left minority reserve one at a time according to its priority order. School c then considers all the applicants who are yet to be accepted, and one at a time according to its priority order, it permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than the remaining majority-type specific quota. The algorithm terminates whenever each student is accepted by a school or there is no remaining school seat for students. The assignments at the terminal step realize as the final matching.

For $(P_S, \succ_C, q^M, r^m)$, the outcome of the Boston mechanism is

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_1 & s_2 & s_3 & s_4 \end{pmatrix}.$$

It is easy to see that, under $\succ_C$, each majority student has lower priority than each minority student at each school. For the quota-based affirmative action, we choose $\tilde{q}^M_{c_1} = 0$ and $\tilde{q}^M_{c_i} = q^M_{c_i}$ for $i = 2, 3, 4$. Then $(P_S, \succ_C, \tilde{q}^M, r^m)$ has a stronger quota-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that, for $(P_S, \succ_C, \tilde{q}^M, r^m)$, the outcome of the Boston mechanism is

$$\tilde{\mu} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_4 & s_2 & s_3 & s_1 \end{pmatrix}.$$

Then comparing with μ, one can see that, under $\tilde{\mu}$, student s_4 becomes strictly worse off and other minority students remain unchanged. That is, increasing the level of affirmative action results in a Pareto inferior assignment for the minority. Thus, the Boston mechanism is not minimally responsive to the quota-based affirmative action even if each majority student has lower priority than each minority student at each school.

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