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Responsive affirmative action in school choice: A comparison study

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HIGHLIGHTS

- This note investigates the responsive affirmative action in school choice.
- We provide a comparison study on the responsiveness of DA and TTC mechanisms.
- We show that the DA mechanism has an advantage over the TTC mechanism.

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ABSTRACT

This note provides a comparison study on responsiveness of two extensively used mechanisms to affirmative action in school choice. For priority-based affirmative action, we show that, if a stronger priority-based affirmative action favors minority students by *giving full priority to the minority*, then such a policy makes *each minority student* weakly better off under the student-proposing deferred acceptance (henceforth, DA) mechanism. However, the top trading cycles (henceforth, TTC) mechanism does not satisfy this property. Under the DA mechanism, if the original problem gives full priority to the minority, then the assignment of minority students *does not change* when the problem moves to a higher level of affirmative action. On the contrary, this property does not hold under the TTC mechanism.

1. Introduction

In school choice programs, affirmative action policies have been playing an important role in favoring minority students to attend their desired schools in the United States and many other countries. There are three popular types of affirmative action policies in school choice: *the quota-based, the reserve-based, and the priority-based*.

The *quota-based* affirmative action policy in school choice gives minority students higher chances to attend more preferred schools by *limiting the number of admitted majority students* at some schools. Abdulkadiroğlu and Sönmez (2003) discuss the fairness and strategic properties for school choice with quota-based affirmative action. More generally, Ehlers et al. (2014) study the quota-based affirmative action policy when there are both upper and lower type-specific bounds, and allowing for more than two types of students.

The *reserve-based* affirmative action policy is to *reserve some seats at each school for the minority students*, and to require that a reserved seat at a school be assigned to a majority student only if no minority student prefers that school to her assignment. Hafalir et al. (2013) show that, in the efficiency aspect, the reserve-based policy has an advantage over the quota-based policy.

The *priority-based* affirmative action favors minority students by means of *promoting their priority ranking at schools*. In Chinese college admissions, the minority students are favored by a priority-based affirmative action policy that awards bonus points to minority students in the national college entrance examination. Chen and Kesten (2017) point out that, in recent years, there are about 10 million high school seniors who compete for 6 million seats at universities in China each year.

For school choice with affirmative action, it is comparatively ideal to find a stable or efficient and responsive assignment mechanism from the mechanism-design perspective. It is well known that the DA mechanism is stable and the TTC mechanism

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is efficient. A mechanism is "responsive" to a kind of affirmative action if *all minority students become weakly better off when the level of affirmative action is strengthened*. Since the affirmative action in school choice aims to improve the welfare of the minority students, the responsiveness is seemingly a natural requirement. However, it is known from Kojima (2012) and Hafalir et al. (2013) that, on the full domain of school choice, neither DA mechanism nor the TTC mechanism is responsive to any type of the three prominent affirmative action policies. That is, in general, both DA and TTC mechanism suffer from the following difficulty: a higher level of affirmative action may not (weakly) benefit all minority students.[1]

These impossibility results are based on the full domain of school choice problems. One may have a positive result if the domain of school choice problems is restricted in certain ways. We consider the comparison on the responsiveness of DA and TTC mechanisms on a restricted school priority domain. Doğan (2016) considers a restricted domain of school choice where *full priority is given to the minority* (in the sense that, at each school c, either each minority student is one of its q_c highest-priority students, or each minority student has higher priority than all majority students), and shows that, for school choice with *quota-based* affirmative action, the DA mechanism is *minimally responsive* (a weaker concept than responsiveness) *if and only if* the given problem gives full priority to the minority.

We first consider the priority-based affirmative action problem in which the level of affirmative action is sufficiently increased such that the new problem gives full priority to the minority. We show that, if a stronger priority-based affirmative action favors minority students by way of *giving full priority to the minority*, then such a policy makes *each minority student* weakly better off under the DA mechanism (Proposition 1). This property is undoubtedly attractive when an affirmative action aims to improve the Pareto welfare of the minority students. However, the TTC mechanism does not have such a desirable property as that for the DA mechanism (Example 1).

We then show that when the original problem gives full priority to the minority, under the DA mechanism, the original problem and a stronger priority-based and a stronger reserve-based affirmative action produce the *same* assignment outcome. The original problem and a stronger quota-based affirmative action produce the *same* assignment outcome for the *minority* students, while the stronger quota-based affirmative action makes the majority students weakly worse off (Theorem 1). Thus, a quota-based affirmative action would result in an avoidable efficiency loss and both priority-based and reserve-based affirmative action policies do not play an actual role under the DA mechanism. Compared with the DA, we find that, under the TTC mechanism, even if the original problem gives full priority to the minority, each type of stronger affirmative action policy may produce a worse assignment for at least one minority student (Example 2).

The remainder of the paper is organized as follows. We present some preliminaries on the formal model in the next section. Section 3 presents the comparison on the responsiveness of DA and TTC mechanisms.

2. The model

2.1. Settings

Let S and C be finite and disjoint sets of **students** and **schools**. There are two types of students: **minority students** and **majority students**. Let S^m and S^M denote the sets of minority and majority students, respectively. They are nonempty sets such that $S^m \cup S^M = S$ and $S^m \cap S^M = \emptyset$. Suppose that $|C|, |S| \geq 2$.

For each student $s \in S$, P_s is a strict (i.e., complete, transitive, and anti-symmetric) preference relation over $C \cup \{s\}$, where s denotes the outside option, which can be attending a private school or being home-schooled. School c is **acceptable** to student s if $cP_s s$. The **preference profile** for a group of students S' is denoted by $P_{S'} = (P_s)_{s \in S'}$. For any $c, c' \in C$ and $s \in S$, $cR_s c'$ denotes either $cP_s c'$ or $c = c'$. For each school $c \in C$, $\succ_c$ is a strict priority order over S.[2] The **priority profile** for a group of schools C' is denoted by $\succ_{C'} = (\succ_c)_{c \in C'}$. For any $s, s' \in S$ and $c \in C$, $s \succeq_c s'$ denotes either $s \succ_c s'$ or $s = s'$.

For each $c \in C$, q_c is the capacity of c or the number of seats in c. We assume that there are enough seats for all students, so $\sum_{c \in C} q_c \geq |S|$. Let $q = (q_c)_{c \in Cy}$ be the **capacity profile**.

For each school $c \in C$, there is a **majority quota** affirmative action parameter q_c^M such that $q_c^M \leq q_c$ and $q_c^M \in \mathbb{Z}_+$,[3] and q_c^M denotes the majority type-specific quota of school c. Let $q^M \equiv (q_c^M)_{c \in C}$ be the **majority quota profile**.

For each school $c \in C$, there is also a **minority reserve** affirmative action parameter r_c^m with $r_c^m \in \mathbb{Z}_+$ and $r_c^m \leq q_c$, and r_c^m denotes the number of seats at c at which the minority students are "favored". Let $r^m \equiv (r_c^m)_{c \in C} \in$ be the **minority reserve profile**.

A **school choice problem with affirmative action**, or simply a **problem**, is a tuple $G \equiv (S, C, P_S, \succ_C, q, q^M, r^m)$. Since S, C and q are fixed throughout this paper, unless otherwise noted, a problem is simply a quadruple $\boldsymbol{G \equiv (P_S, \succ_C, q^M, r^m)}$.

A matching is an assignment of students to schools such that each student is assigned to a school or to her outside option, no school admits more students than its capacity, and no school admits more majority students than its majority type-specific quota. Formally, a **matching** μ is a mapping from $C \cup S$ to the subsets of $C \cup S$ such that

(i) for each $s \in S$, $\mu(s) \in C \cup \{s\}$,
(ii) for each $c \in C$ and $s \in S$, $\mu(s) = c$ if and only if $s \in \mu(c)$,
(iii) for each $c \in C$, $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$, and
(iv) for each $c \in C$, $|\mu(c) \cap S^M| \leq q_c^M$.

A **mechanism** is a mapping ϕ that, for each school choice problem G, associates a matching $\phi(G)$.

As a restricted domain of school choice, we specify a condition proposed by Doğan (2016) as follows:

Definition 1. A problem $G = (P_S, \succ_C, q^M, r^m)$ **gives full priority to the minority** if there are no $m \in S^m$, $M \in S^M$, and $c \in C$ such that $M \succ_c m$ and $|\{s \in S : s \succeq_c m\}| > q_c$.

In other words, a problem $G = (P_S, \succ_C, q^M, r^m)$ giving full priority to the minority means that, at each school c, either each minority student is ranked above each majority student, or each minority student is one of the q_c highest-priority students.

2.2. Affirmative action policies

In this subsection, we will introduce the three kinds of affirmative action policies: the quota-based type, the reserve-based type and the priority-based type. For a problem $G = (P_S, \succ_C, q^M, r^m)$, the **quota-based affirmative action policy** is implemented by prohibiting each school c to admit more students than it is majority type-specific quota q_c^M and setting $r_c^m = 0$ for all $c \in C$.

[1] Kojima (2012) and Hafalir et al. (2013) investigate a weaker responsiveness requirement and obtain impossibility result for the DA and TTC mechanisms. For another extensively used mechanism, Boston mechanism, Afacan and Salman (2016) obtain that it is not responsive on the full domain of school choice.

[2] Echenique and Yenmez (2015) study the stability in school choice with affirmative action under the framework of abstract school choice rule.

[3] $\mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}$ is the set of nonnegative integers.

A problem $\tilde{G} = (P_S, \succ_C, \tilde{q}^M, r^m)$ is said to **have a stronger quota-based affirmative action policy than** $G = (P_S, \succ_C, q^M, r^m)$ if, for all $c \in C$, $\tilde{q}_c^M \leq q_c^M$ and $r_c^m = 0$.

For a problem $G = (P_S, \succ_C, q^M, r^m)$, the **reserve-based affirmative action policy** is implemented by giving priority to minority students at each school c up to the minority reserve r_c^m and setting $q_c^M = q_c$ for all $c \in C$. For a school c, if the number of minority students admitted to it is less than r_c^m, then any minority applicant is given priority over any majority applicant at c. If there are not enough minority students to fill up the reserves, majority students can still be assigned to school c's reserved seats. A problem $\tilde{G} = (P_S, \succ_C, q^M, \tilde{r}^m)$ is said to **have a stronger reserve-based affirmative action policy than** $G = (P_S, \succ_C, q^M, r^m)$ if, for all $c \in C$, $\tilde{r}_c^m \geq r_c^m$ and $q_c^M = q_c$.

Definition 2. For a problem $G = (P_S, \succ_C, q^M, r^m)$ and any $c \in C$, a priority $\succ_c'$ **is an improvement for minority students over** $\succ_c$ if

(i) $s \succ_c s'$ and $s \in S^m$ imply $s \tilde{\succ}_c s'$, and

(ii) $s, s' \in S^M$ and $s \succ_c s'$ imply $s \tilde{\succ}_c s'$.

A priority profile $\succ_C'$ **is an improvement for minority students over** $\succ_C$ if, for all $c \in C$, $\succ_c'$ is an improvement for minority students over $\succ_c$.

In other words, if we change $\succ_c$ to $\succ_c'$ by means of promoting the ranking of some minority students at schools relative to majority students while keeping the relative ranking of each student within her own group fixed, then $\succ_c'$ is an improvement for minority students over $\succ_c$.

For a problem $G = (P_S, \succ_C, q^M, r^m)$, the **priority-based affirmative action policy** is implemented by improving the rankings of minority students and setting $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$. A problem $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ is said to **have a stronger priority-based affirmative action policy than** $G = (P_S, \succ_C, q^M, r^m)$ if, for all $c \in C$, (i) $\tilde{\succ}_C$ is an improvement for minority students over $\succ_C$, and (ii) $q_c^M = q_c$ and $r_c^m = 0$.

2.3. DA mechanism

For each problem $(P_S, \succ_C, q^M, r^m)$, the DA mechanism is defined through the following **deferred acceptance algorithm**[4]:

• **Step** 1: Start with a matching in which no student is matched. Each student applies to her most preferred acceptable school. Each school c first considers minority applicants and tentatively accepts them up to its minority reserve r_c^m one at a time according to its priority order if there are enough minority applicants. School c then considers all the applicants who are yet to be accepted and it tentatively accepts them, one at a time according to its priority order, until its capacity is filled or the applicants are exhausted, while not admitting more majority students than q_c^M. The rest of the applicants, if any remain, are rejected by c.

In general, at

• **Step** k, $k \geq 2$: Start with the tentative matching obtained at the end of step $k-1$. Each student who got rejected at Step $k-1$ applies to her next preferred acceptable school. Each school c considers the new applicants and students admitted tentatively at step $k-1$. Among these students, school c first tentatively accepts minority students up to its minority reserve r_c^m one at a time according to its priority order. School c then considers all the applicants who are yet to be accepted, and one at a time according to its priority order, it tentatively accepts as many students as up to its capacity while not admitting more majority students than the remaining majority-type specific quota. The rest of the students, if any remain, are rejected by c. If there are no rejections, then stop.

The algorithm terminates when no rejection occurs and the tentative matching at that step is finalized. Since no student reapplies to a school that has rejected her and at least one rejection occurs in each step, the algorithm stops in finite time. For a problem $(P_S, \succ_C, q^M, r^m)$, the DA matching is the one reached at the termination of the deferred acceptance algorithm and is denoted by $\mathbf{DA}(\boldsymbol{P_S}, \boldsymbol{\succ_C}, \boldsymbol{q^M}, \boldsymbol{r^m})$.

For a problem $(P_S, \succ_C, q^M, r^m)$, (i) if $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the original version by Gale and Shapley (1962), or the version proposed by Abdulkadiroğlu and Sönmez (2003) for school choice problem without affirmative action. (ii) If $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the version, proposed by Abdulkadiroğlu and Sönmez (2003), for controlled school choice problems (or for problems with only quota-based affirmative action, see for instance Kojima (2012)). (iii) If $q_c^M = q_c$ for all $c \in C$, then the above algorithm reduces to the version, proposed by Hafalir et al. (2013), for problems with only reserve-based affirmative action.

2.4. TTC mechanism

For each problem $(P_S, \succ_C, q^M, r^m)$, the TTC mechanism is defined through the following **top trading cycles algorithm**[5]:

• **Step** 1: Start with a matching in which no student is matched. For schools, if a school has minority reserves, then it points to a minority student who has the highest priority at that school among the minority students; otherwise it points to a student who has the highest priority at that school among all students. For students, each student s points to her most preferred school that is acceptable and still has a seat for her (if there is such a school; otherwise she points to herself), that is, an acceptable school whose capacity is strictly positive and, if $s \in S^M$, its majority type-specific quota is strictly positive. Since the number of students and schools are finite, there exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity of each school in a cycle is reduced by one. If the assigned student is in S^M and the school, say c, matched to s has majority quota at this step, then the school matched to that student reduces its majority-specific quota by one. If the assigned student, say s, is in S^m and the school, say c, matched to s has minority reserves at this step, then school c reduces its minority reserves by one. If no student remains, terminate. Otherwise, proceed to the next step.

In general, at

• **Step** k, $k \geq 2$: For each remaining school $c \in C$, if it has minority reserves, then c points to a minority student who has the highest priority at c among all remaining minority students; otherwise it points to a student who has the highest priority at c among all remaining students. Each remaining student s points to her most preferred school (among the remaining schools) that is acceptable and still has a seat for her (if there is such a school; otherwise she points to herself), that is, an acceptable school whose remaining capacity is strictly positive and, if $s \in S^M$, its remaining majority-specific quota is strictly positive. There exists at least one cycle. Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity of each school in a cycle is reduced by one. If the assigned student is in S^M and

4 The original deferred acceptance algorithm was proposed by Gale and Shapley (1962).

5 The original top trading cycles algorithm was proposed for housing markets and is attributed to David Gale by Shapley and Scarf (1974).

the school, say c, matched to s has majority quota at this step, then the school matched to that student reduces its majority-specific quota by one. If the assigned student, say s, is in S^m and the school, say c, matched to s has minority reserves at this step, then school c reduces its minority reserves by one. If no student remains, terminate. Otherwise, proceed to the next step.

This algorithm terminates in a finite number of steps because at least one student is matched at each step as long as the algorithm has not terminated and there are a finite number of students. For a problem $(P_S, \succ_C, q^M, r^m)$, the TTC matching is the one reached at the termination of the top trading cycles algorithm and is denoted by $\mathbf{TTC}(\boldsymbol{P_S}, \boldsymbol{\succ_C}, \boldsymbol{q^M}, \boldsymbol{r^m})$.

For a problem $(P_S, \succ_C, q^M, r^m)$, (i) if $q_c^M = q_c$ and $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the version, proposed by Abdulkadiroğlu and Sönmez (2003), for school choice problem without affirmative action. (ii) If $r_c^m = 0$ for all $c \in C$, then the above algorithm reduces to the version for problems with only quota-based affirmative action, see for instance Kojima (2012). (iii) If $q_c^M = q_c$ for all $c \in C$, then the above algorithm reduces to the version, proposed by Hafalir et al. (2013), for problems with only reserve-based affirmative action.

3. Results

In this section we consider whether increasing the level of affirmative action can weakly Pareto improve the welfare of minority students. For priority-based affirmative action, we obtain a desirable result which says that, if we increase the level of affirmative action such that the new problem gives full priority to the minority, then the stronger affirmative action policy makes each minority student weakly better off under the DA algorithm. Formally, we have the following result.

Proposition 1. *Let $G = (P_S, \succ_C, q^M, r^m)$ and $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ be two problems such that $\tilde{G}$ has stronger priority-based affirmative action than G. If $\tilde{G} = (P_S, \tilde{\succ}_C, q^M, r^m)$ gives full priority to the minority, then $DA(\tilde{G}) \succeq_s DA(G)$ for each minority student $s \in S^m$.*

Proof. We argue by contradiction. Let $\mu = DA(G)$ and $\tilde{\mu} = DA(\tilde{G})$. Suppose that there exists some minority student $s_0 \in S^m$ such that $\mu(s_0)P_{s_0}\tilde{\mu}(s_0)$. We denote $\mu(s_0) \equiv c_0$. Consider the DA algorithm for $\tilde{G}$, $\mu(s_0)P_{s_0}\tilde{\mu}(s_0)$ implies that s_0 must ever propose to c_0 and c_0 finally rejects her at some step, say Step k_n. Then it is easy to see that $|\tilde{\mu}(c_0)| = q_{c_0}$ and each member in $\tilde{\mu}(c_0)$ has higher priority than s_0 at c_0 (with respect to $\tilde{\succ}_{c_0}$). We obtain that s_0 is not in the set of the q_{c_0} highest-priority students of c_0 (with respect to $\tilde{\succ}_{c_0}$). Since problem $\tilde{G}$ gives full priority to the minority, by definition it must be the case that each minority student is ranked above each majority student under $\tilde{\succ}_{c_0}$. Then there is no majority student who has higher priority than s_0 with respect to $\tilde{\succ}_{c_0}$. Therefore, one can infer that c_0 tentatively accepts q_{c_0} students when c_0 rejects s_0 at Step k_n, and each of the q_{c_0} students is in S^m and has higher priority than s_0 at c_0 (with respect to $\tilde{\succ}_{c_0}$). Since $|\mu(c_0)| \leq q_{c_0}$ and $s_0 \in \mu(c_0)$, it is easy to see that there exists at least one student, say s_1 (in S^m), among the q_{c_0} students tentatively accepted by c_0 at Step k_n such that $s_1 \notin \mu(c_0)$. We can obtain that $s_1 \tilde{\succ}_{c_0} s_0$ is equivalent to $s_1 \succ_{c_0} s_0$, as both s_0 and s_1 are minority students. Combining $s_0 \in \mu(c_0)$, $s_1 \notin \mu(c_0)$ and $s_1 \succ_{c_0} s_0$, one can infer that s_1 has never proposed to c_0 in the process of the DA algorithm for G. Let $c_1 \equiv \mu(s_1)$. Then we get $c_1 P_{s_1} c_0$.

Since $c_1 P_{s_1} c_0$ and s_1 ever proposed to c_0 at some step, say Step $\tilde{k}_n (\leq k_n)$, in the DA process for $\tilde{G}$, one can infer that s_1 must have proposed to c_1 and c_1 rejected her at another step, say Step $k_{n-1} (< \tilde{k}_n)$, in the DA process of $\tilde{\mu}$. Then it is exactly similar to the analysis above, and one can obtain that $|\tilde{\mu}(c_1)| = q_{c_1}$ and each member in $\tilde{\mu}(c_1)$ has higher priority than s_1 at c_1 (with respect to $\tilde{\succ}_{c_1}$). Then s_1 is not in the set of the q_{c_1} highest-priority students of c_1 (with respect to $\tilde{\succ}_{c_1}$). Since problem $\tilde{G}$ gives full priority to the minority, by definition it must be the case that each minority student is ranked above each majority student under $\tilde{\succ}_{c_1}$. Then there is no majority student who has higher priority than s_1 under $\tilde{\succ}_{c_1}$. Therefore, one can infer that c_1 tentatively accepts q_{c_1} students when c_1 rejects s_1 at Step k_{n-1}, and each of the q_{c_1} students is in S^m and has higher priority than s_1 at c_1 (with respect to $\tilde{\succ}_{c_1}$). Since $|\mu(c_1)| \leq q_{c_1}$ and $s_1 \in \mu(c_1)$, it is easy to see that there exists at least one student, say s_2 (in S^m), among the q_{c_1} students tentatively accepted by c_1 at Step k_{n-1} such that $s_2 \notin \mu(c_1)$. We can infer that $s_2 \tilde{\succ}_{c_1} s_1$ implies $s_2 \succ_{c_1} s_1$, as both s_1 and s_2 are minority students. Combining $s_1 \in \mu(c_1)$, $s_2 \notin \mu(c_1)$ and $s_2 \succ_{c_1} s_1$, one can infer that s_2 has never proposed to c_1 in the process of the DA algorithm under G. Let $c_2 \equiv \mu(s_2)$. Then $c_2 P_{s_2} c_1$.

Taking a repeated argument procedure as above, we can obtain a sequence of students and schools $s_1, c_1, \ldots, s_i, c_i \cdots$ and a infinite sequence of steps of DA algorithm process $k_n, k_{n-1}, \ldots, k_{n+1-i}, \ldots$ such that $k_m > k_{m-1}$ for all $m \leq n$. Then $k_{n+1-i} < 0$ when i is sufficiently large. Since k_{n+1-i} is some step of the DA algorithm process under $\tilde{G}$, $k_{n+1-i} \geq 1$. We reach a contradiction and complete the proof. □

We note that, for the TTC mechanism, one cannot expect to obtain a desirable result as that for DA mechanism in Proposition 1. Specifically, we consider the following example.

Example 1. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $q_{c_i} = q_{c_i}^M = 2$ for $i = 1, 2$, $q_{c_3} = q_{c_3}^M = 1$ and $r_{c_i}^m = 0$ for $i = 1, 2, 3$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
c_1	c_3	c_2	c_3	s_4	s_3	s_1
s_1	s_2	s_3	s_4	s_2	s_4	s_2
				s_3	s_2	s_3
				s_1	s_1	s_4

For $(P, \succ, q^M, r^m)$, the outcome of the TTC mechanism is

$$\text{TTC}(P, \succ, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_2 \\ s_1 & s_3 & s_4 & s_2 \end{pmatrix}.$$

Let the priority of c_3 be changed to $\tilde{\succ}_{c_3}: s_2, s_3, s_4, s_1$, and $\tilde{\succ}_{c_i} = \succ_{c_i}$ for $i = 1, 2$. For $(P, \tilde{\succ}, q^M, r^m)$, the outcome of the TTC mechanism is

$$\text{TTC}(P, \tilde{\succ}, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_1 & s_3 & s_2 & s_4 \end{pmatrix}.$$

It is easy to check that $(P, \tilde{\succ}, q^M, r^m)$ has a stronger priority-based affirmative action policy than $(P, \succ, q^M, r^m)$ and $(P, \tilde{\succ}, q^M, r^m)$ gives full priority to the minority. One can see that, the minority student s_4 is strictly worse off under $\text{TTC}(P, \tilde{\succ}, q^M, r^m)$ than under $\text{TTC}(P, \succ, q^M, r^m)$.

Next we study the comparison between different affirmative action policies when the original problem $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority. Let $(P_S, \tilde{\succ}_C, q^M, r^m)$, $(P_S, \succ_C, q^M, \tilde{r}^m)$ and $(P_S, \succ_C, \tilde{q}^M, r^m)$ be three problems that are different from $(P_S, \succ_C, q^M, r^m)$. For a given problem $(P_S, \succ_C, q^M, r^m)$, if all of the majority students leave the market, then we denote the corresponding small problem by $(S^m, C, P_{S^m}, \succ_C|_{S^m}, q^M, r^m)$. Denote $\mu \equiv \text{DA}(P_S, \succ_C, q^M, r^m)$, $\mu' \equiv \text{DA}(S^m, C, P_{S^m}, \succ_C|_{S^m}, q^M, r^m)$, $\mu^p \equiv \text{DA}(P_S, \tilde{\succ}_C, q^M, r^m)$, $\mu^q \equiv \text{DA}(P_S, \succ_C, \tilde{q}^M, r^m)$, and $\mu^r \equiv \text{DA}(P_S, \succ_C, q^M, \tilde{r}^m)$. Then we present our last result.

Theorem 1. *Suppose that the original problem $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority students. Then*

(1) $\mu(s) = \mu'(s)$ for every $s \in S^m$;

(2) If $(P_S, \succ_C, \tilde{q}^M, r^m)$ has stronger quota-based affirmative action than $(P_S, \succ_C, q^M, r^m)$, then $\mu(s) = \mu^q(s)$ for every $s \in S^m$ and $\mu(s) \succeq_s \mu^q(s)$ for all $s \in S^M$;

(3) If $(P_S, \tilde{\succ}_C, q^M, r^m)$ and $(P_S, \succ_C, q^M, \tilde{r}^m)$ are respectively stronger priority-based and reserve-based affirmative action policies than $(P_S, \succ_C, q^M, r^m)$, then $\mu(s) = \mu^p(s) = \mu^r(s)$ for all $s \in S$.

Proof. (1) By a classical result of Gale and Sotomayor (1985) (see also Theorem 5.35 in Roth and Sotomayor (1990)), one can obtain that $\mu'(s) R_s \mu(s)$ for each $s \in S^m$. We only need to show that $\mu(s) R_s \mu'(s)$ for each $s \in S^m$. Suppose not, then there exists some $s \in S^m$ such that $\mu'(s) P_s \mu(s)$. One can repeat a procedure as in the proof of Proposition 1 and complete the proof.

(2) We first show that $\mu(s) R_s \mu^q(s)$ for each $s \in S^m$. Suppose not. Then there exists some $s \in S^m$ such that $\mu^q(s) P_s \mu(s)$. One can take a similar procedure as in the proof of Proposition 1 and reach a contradiction. Symmetrically, we can show that $\mu^q(s) R_s \mu(s)$. Then $\mu(s) = \mu^q(s)$ for each $s \in S^m$. For the second part, we suppose there exists some $s \in S^M$ such that $\mu^q(s) P_s \mu(s)$. We denote $\mu^q(s) \equiv c_0$. According to the DA algorithm under G, $c_0 P_s \mu(s)$ implies that s must ever propose to c_0 and c_0 finally rejects her at some step, say step k_n. Then one can infer that c_0 tentatively accepts q_{c_0} students when c_0 rejects s at step k_n, and each of the q_{c_0} students has higher priority than s_0 at c_0 (with respect to $\succ_{c_0}$). As $\mu(s) = \mu^q(s)$ for each $s \in S^m$, we obtain $\{s \in \mu^q(c_0) : s \in S^m\} = \{s \in \mu(c_0) : s \in S^m\}$. Since this market gives full priority to the minority, c_0 tentatively accepts no more than $|\{s \in \mu^q(c_0) : s \in S^m\}|$ minority students at step k_n. Otherwise, it will result in $|\{s \in \mu^q(c_0) : s \in S^m\}| < |\{s \in \mu(c_0) : s \in S^m\}|$. Since c_0 rejects s at step k_n, one can infer that there exists at least one student, say $s_1 \in S^M$, among the q_{c_0} students tentatively accepted by c_0 at step k_n such that $s_1 \notin \mu^q(c_0)$. Then $s_1 \succ_{c_0} s$. Combining $s \in \mu^q(c_0)$, $s_1 \notin \mu^q(c_0)$ and $s_1 \succ_{c_0} s$, we can infer that s_1 has never proposed to c_0 in the process of the DA algorithm for μ^q. Then $\mu^q(s_1) \equiv c_1 P_{s_1} c_0$.

Since $c_1 P_{s_1} c_0$ and s_1 ever proposes to c_0 at some step, say step $k'_n (\le k_n)$, in the DA process under G, one can infer that s_1 must ever propose to c_1 and c_1 rejects her at another step, say step $k_{n-1} (< k'_n)$, in the DA process of G. Then it is exactly similar to the analysis given above, and one can infer that c_1 tentatively accepts q_{c_1} students when c_1 rejects s_1 at step k_{n-1}, and each of the q_{c_1} students has higher priority than s_1 at c_1. As $\mu(s) = \mu^q(s)$ for each $s \in S^m$, we obtain $\{s \in \mu^q(c_1) : s \in S^m\} = \{s \in \mu(c_1) : s \in S^m\}$. Since this market gives full priority to the minority, c_1 tentatively accepts no more than $|\{s \in \mu^q(c_1) : s \in S^m\}|$ minority students at step k_{n-1}. Otherwise, it will result in $|\{s \in \mu^q(c_1) : s \in S^m\}| < |\{s \in \mu(c_1) : s \in S^m\}|$. Since c_1 rejects s_1 at step k_{n-1}, one can infer that there exists at least one student, say $s_2 \in S^M$, among the q_{c_1} students tentatively accepted by c_1 at step k_{n-1} such that $s_2 \notin \mu^q(c_1)$. Then $s_2 \succ_{c_1} s_1$. Combining $s_1 \in \mu^q(c_1)$, $s_2 \notin \mu^q(c_1)$ and $s_2 \succ_{c_1} s_1$, we can infer that s_2 has never proposed to c_1 in the process of the DA algorithm for μ^q. Then $\mu^q(s_2) \equiv c_2 P_{s_2} c_1$.

Taking a repeated argument process as above, we can obtain a sequence of students and schools $s_1, c_1, \ldots, s_i, c_i \cdots$ and a sequence of steps of DA algorithm procedure $k_n, k_{n-1}, \ldots, k_{n+1-i}, \ldots$ such that $k_m > k_{m-1}$ for all $m \le n$. Then $k_{n+1-i} < 0$ when i is sufficiently large. Since k_{n+1-i} is some step of the DA algorithm process for μ^q, $k_{n+1-i} \ge 1$. We reach a contradiction and complete the proof.

(3) It is easy to see that the stronger policy problem $(P_S, \tilde{\succ}_C, q^M, r^m)$ also gives full priority to the minority. We only need to show that, for each $s \in S$, $\mu(s) = \mu^p(s)$ and $\mu(s) = \mu^r(s)$, respectively. For the case of minority students we can take a similar argument as in the proof of Proposition 1, and for the case of majority students we can take a similar argument as in the proof of part (2) of this theorem. □

According to Theorem 1, one can see that, if a market gives full priority to the minority, stronger priority-based and reserve-based affirmative action policies do not play a role under the DA algorithm. Moreover, a stronger quota-based affirmative action probably makes majority students worse off and results in avoidable efficiency loss.

We note that all of the corresponding results for the DA mechanism given in Theorem 1 fail to hold for the TTC mechanism. Specifically, we consider the following example.

Example 2. Let $S^M = \{s_1\}$, $S^m = \{s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $q_{c_i} = q^M_{c_i} = 1$ for $i = 1, 2$, $q_{c_3} = q^M_{c_3} = 4$ and $r^m_{c_i} = 0$ for $i = 1, 2, 3$. Students' preferences and schools' priorities are given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
c_2	c_3	c_1	c_1	s_2	s_4	s_1
s_1	s_2	s_3	s_4	s_4	s_3	s_3
				s_3	s_2	s_2
				s_1	s_1	s_4

For $(P_S, \succ_C, q^M, r^m)$, the outcome of the TTC mechanism is

$$\mathrm{TTC}(P_S, \succ_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_3 \\ s_4 & s_1 & s_2 & s_3 \end{pmatrix}.$$

For a small problem with all majority students leaving the market, the outcome of the TTC mechanism for minority students is

$$\mathrm{TTC}(S^m, C, P_{S^m}, \succ_C |_{S^m}, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_3 & \varnothing & s_2 & s_4 \end{pmatrix}.$$

It is easy to see that $(P_S, \succ_C, q^M, r^m)$ gives full priority to the minority. One can see that student s_4 becomes strictly worse off and student s_3 becomes strictly better off under the small market, while student s_2 keeps unchanged.

For the quota-based affirmative action, we choose $\tilde{q}^M_{c_2} = 0$ and $\tilde{q}^M_{c_i} = q^M_{c_i}$ for $i = 1, 3$. Then $(P_S, \succ_C, \tilde{q}^M, r^m)$ has a stronger quota-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\mathrm{TTC}(P_S, \succ_C, \tilde{q}^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_1 & s_4 \\ s_3 & \varnothing & s_2 & s_1 & s_4 \end{pmatrix}.$$

Then comparing with $\mathrm{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\mathrm{TTC}(P_S, \succ_C, \tilde{q}^M, r^m)$, student s_4 becomes strictly worse off, student s_3 becomes strictly better off and student s_2 keeps unchanged.

For the reserve-based affirmative action, we choose $\tilde{r}^m_{c_3} = 1$ and $\tilde{r}^m_{c_i} = r^m_{c_i}$ for $i = 1, 2$. Then $(P_S, \succ_C, q^M, \tilde{r}^m)$ has a stronger reserve-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\mathrm{TTC}(P_S, \succ_C, q^M, \tilde{r}^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_3 & s_1 & s_2 & s_4 \end{pmatrix}.$$

Then comparing with $\mathrm{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\mathrm{TTC}(P_S, \succ_C, q^M, \tilde{r}^m)$, student s_4 becomes strictly worse off, student s_3 becomes strictly better off and student s_2 keeps unchanged.

For the priority-based affirmative action, let $\tilde{\succ}_{c_3} : s_3, s_1, s_2, s_4$ and $\tilde{\succ}_{c_i} = \succ_{c_i}$ for $i = 1, 2$. Then $(P_S, \tilde{\succ}_C, q^M, \tilde{r}^m)$ has a stronger priority-based affirmative action policy than $(P_S, \succ_C, q^M, r^m)$. It is easy to check that

$$\mathrm{TTC}(P_S, \tilde{\succ}_C, q^M, r^m) = \begin{pmatrix} c_1 & c_2 & c_3 & s_4 \\ s_3 & s_1 & s_2 & s_4 \end{pmatrix}.$$

Then comparing with $\mathrm{TTC}(P_S, \succ_C, q^M, r^m)$, one can see that, under $\mathrm{TTC}(P_S, \tilde{\succ}_C, q^M, r^m)$, student s_4 becomes strictly worse off, student s_3 becomes strictly better off and student s_2 keeps unchanged.

Finally, we note that, when schools give full priority to the minority, the DA mechanism has an advantage over the TTC mechanism from the perspective of responsiveness to affirmative action.

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